

Directional Derivatives

Directional derivative

Like all derivatives the *directional derivative* can be thought of as a ratio. Fix a unit vector \mathbf{u} and a point P_0 in the *plane*. The **directional derivative** of w at P_0 in the direction \mathbf{u} is defined as

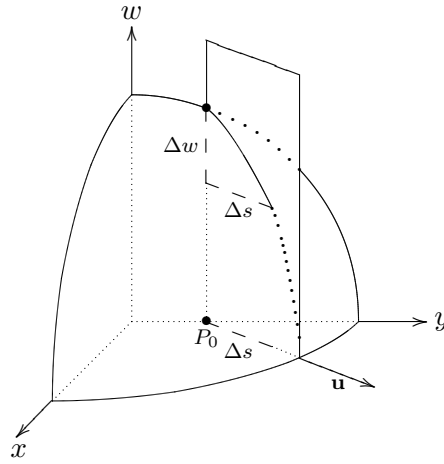
$$\left. \frac{dw}{ds} \right|_{P_0, \mathbf{u}} = \lim_{\Delta s \rightarrow 0} \frac{\Delta w}{\Delta s}.$$

Here Δw is the change in w caused by a step of length Δs in the direction of \mathbf{u} (all in the xy -plane).

Below we will show that

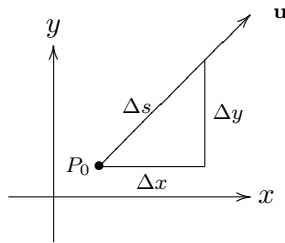
$$\left. \frac{dw}{ds} \right|_{P_0, \mathbf{u}} = \nabla w(P_0) \cdot \mathbf{u}. \quad (1)$$

We illustrate this with a figure showing the graph of $w = f(x, y)$. Notice that Δs is measured in the plane and Δw is the change of w on the graph.



Proof of equation 1

The figure below represents the change in position from P_0 resulting from taking a step of size Δs in the \mathbf{u} direction.



Since $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$ we have that $\left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s} \right\rangle$ is a unit vector, so

$$\mathbf{u} = \left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s} \right\rangle.$$

The tangent plane approximation at P_0 is

$$\Delta w \approx \left. \frac{\partial w}{\partial x} \right|_{P_0} \Delta x + \left. \frac{\partial w}{\partial y} \right|_{P_0} \Delta y$$

Dividing this approximation by Δs gives

$$\frac{\Delta w}{\Delta s} \approx \left. \frac{\partial w}{\partial x} \right|_{P_0} \frac{\Delta x}{\Delta s} + \left. \frac{\partial w}{\partial y} \right|_{P_0} \frac{\Delta y}{\Delta s}.$$

We can rewrite this as a dot product

$$\frac{\Delta w}{\Delta s} \approx \left\langle \left. \frac{\partial w}{\partial x} \right|_{P_0}, \left. \frac{\partial w}{\partial y} \right|_{P_0} \right\rangle \cdot \left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s} \right\rangle.$$

In the dot product the first term is $\nabla w|_{P_0}$ and the second is just \mathbf{u} , so,

$$\frac{\Delta w}{\Delta s} \approx \nabla w|_{P_0} \cdot \mathbf{u}.$$

Now taking the limit we get equation (1).

Example: (Algebraic example) Let $w = x^3 + 3y^2$.

Compute $\frac{dw}{ds}$ at $P_0 = (1, 2)$ in the direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

Answer: We compute all the necessary pieces:

i) $\nabla w = \langle 3x^2, 6y \rangle \Rightarrow \nabla w|_{(1,2)} = \langle 3, 12 \rangle.$

ii) \mathbf{u} must be a unit vector, so $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$

iii) $\left. \frac{dw}{ds} \right|_{P_0, \mathbf{u}} = \nabla w|_{(1,2)} \cdot \mathbf{u} = \langle 3, 12 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \boxed{\frac{57}{5}}.$

Example: (Geometric example) Let \mathbf{u} be the direction of $\langle 1, -1 \rangle$.

Using the picture at right estimate $\left. \frac{\partial w}{\partial x} \right|_P$, $\left. \frac{\partial w}{\partial y} \right|_P$, and $\left. \frac{dw}{ds} \right|_{P, \mathbf{u}}.$

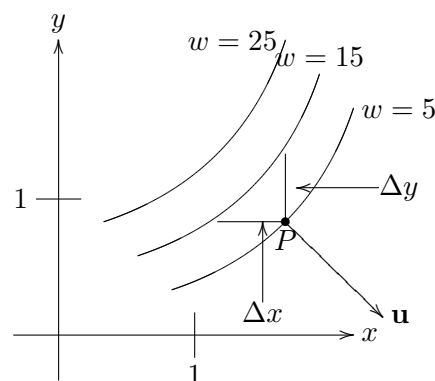
By measuring from P to the next in level curve in the x direction we see that $\Delta x \approx -.5$.

$$\Rightarrow \left. \frac{\partial w}{\partial x} \right|_P \approx \frac{\Delta w}{\Delta x} \approx \frac{10}{-.5} = -20.$$

Similarly, we get $\left. \frac{\partial w}{\partial y} \right|_P \approx 20.$

Measuring in the \mathbf{u} direction we get $\Delta s \approx -.3$

$$\Rightarrow \left. \frac{dw}{ds} \right|_{P, \mathbf{u}} \approx \frac{\Delta w}{\Delta s} \approx \frac{10}{.3} = -33.3.$$



Direction of maximum change:

The direction that gives the maximum rate of change is in the same direction as ∇w . The proof of this uses equation (1). Let θ be the angle between ∇w and \mathbf{u} . Then the geometric form of the dot product says

$$\left. \frac{dw}{ds} \right|_{\mathbf{u}} = \nabla w \cdot \mathbf{u} = |\nabla w| |\mathbf{u}| \cos \theta = |\nabla w| \cos \theta.$$

(In the last equation we dropped the $|\mathbf{u}|$ because it equals 1.) Now it is obvious that this is greatest when $\theta = 0$. That is, when ∇w and \mathbf{u} are in the same direction.

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18.02SC Multivariable Calculus
Fall 2010

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