A Strong Tracking Extended Kalman Observer for Nonlinear Discrete-Time Systems

M. Boutayeb and D. Aubry

***Abstract*— In this contribution the authors show how the extended Kalman filter (EKF), used as an observer for nonlinear discrete-time systems or extended Kalman observer (EKO), becomes a useful state estimator when the arbitrary matrices, namely R_k and Q_k in the paper, are adequately chosen. As a first step, they use the linearization technique in [4] which consists of introducing unknown diagonal matrices to take the approximation errors into account. It is shown that the decreasing Lyapunov function condition leads to a linear matrix inequality (LMI) problem, which points out the connection between a good convergence behavior of the EKO and the instrumental matrices R_k and Q_k. In order to satisfy the obtained LMI, a particular design of Q_k is given. High performances of the proposed technique will be shown through numerical examples under the worst conditions.**

***Index Terms*—Extended Kalman observers, nonlinear systems, stability analysis.**

I. Introduction

Observer design for nonlinear continuous-time systems has received particular attention in the last few years. Most of the proposed techniques are based on coordinate changes to bring the original system into canonical forms which, in general, necessitate conservative assumptions; therefore, only a restricted class of nonlinear systems can be dealt with. However, when these conditions are satisfied, global and exponential convergence are ensured; we have in mind the recent work of Deza *et al.* [7] and Gauthier *et al.* [10]. Few efforts, however, have been done to design state observers for nonlinear discrete-time systems. Recent results show that the use of simultaneous nonlinear equations may improve the local convergence significantly [5], [12].

An alternative approach for state observers design that can be applied to a large class of nonlinear continuous and/or discrete-time systems consists of using approximate models. No doubt, one of the most popular techniques is the celebrated extended Kalman observer (EKO). If the convergence mechanisms in the linear case are well understood by now, the stability analysis is far from being solved when nonlinear models are considered. The main results in this field were developed earlier by Deyst *et al.* [6], Baras *et al.* [1], and well clarified by Song *et al.* [15]. In particular, it was shown that, under the observability rank condition, boundedness of the error "covariance" matrix P_k is assured. This strong property will be exploited later in our developments. Using a nice transformation technique, Moraal *et al.* [12] set up how the Newton observer is related to the extended Kalman filter (EKF) (see also the work of Bell *et al.* [2]).

In order to improve stability and rate of convergence, slight modifications of the standard EKF were recently performed in [8] and [13]. Based on the incremental norm approach, Fromion *et al.* [8] proposed a globally exponentially stable and bounded modified EKF while Reif *et al.* [13] have shown that the introduction of a fictitious additive term of instability in the Riccati equation may significantly increase performances of the EKF.

In a recent paper, Zhou *et al.* [17] have shown through numerical examples that for a particular design of the error covariance matrix they obtain efficient results with the EKF applied to the Friedland's algorithm. Few theoretical developments, however, were established while a large number of design parameters have to be selected.

It should be noticed that the influence of the arbitrary matrices R_k and Q_k on the convergence mechanisms of the EKO was neglected in most of the works established by now. Indeed, when the above matrices are arbitrarily chosen, in general $R_k = \eta I_p$ and $Q_k = \sigma I_n$ with $0 \leq \eta,\, \sigma \leq 1$, the EKO fails to converge if it is not initialized closely enough to the actual states.

Our interest in this problem is, in fact, motivated by the encouraging results recently developed in [3] and [4]. Only the influence of R_k, while Q_k is setting to zero, on the EKO behavior was analyzed. On the other hand, a large class of nonlinear systems may be considered, under weak conditions, for stability and without the need of coordinate changes.

The aim of this contribution is to relax sufficient conditions established in [4] and to analyze in depth the role of both instrumental matrices mentioned above in studying their connection with behavior of the EKO. The technique that we propose here consists of first introducing unknown diagonal matrices to quantify linearization errors. With the aid of the Lyapunov approach, we show that the stability analysis of the EKO comes to solving a linear matrix inequality (LMI) problem where R_k and Q_k play a central role to improve the convergence of the EKO. The design of the above matrix is given so that strong tracking, under the observability rank condition, of the observer is assured even if the state error vector belongs to a very large convex compact set. In order to illustrate the accuracy and high performances of the modified EKO two numerical examples, which can be found in [9]–[11], [14], and [16], under bad initial conditions are provided.

II. Problem Formulation

Let us consider the general form of nonlinear discrete-time systems

$$x_{k+1} = f(x_k,\, u_k) = f_{u_k}(x_k) \tag{1a}$$

$$y_k = h(x_k,\, u_k) = h_{u_k}(x_k). \tag{1b}$$

$x_k \in R^n$, $u_k \in R^r$, and $y_k \in R^p$ denote the state, input, and output vectors at time instant k, respectively. The nonlinear maps $f_{u_k}(x_k)$ and $h_{u_k}(x_k)$ are assumed to be continuously differentiable with respect to x_k. The EKO that we use here as a state observer is given by

$$\hat{x}_{k+1} = \hat{x}_{k+1/k} + K_{k+1}e_{k+1} \tag{2}$$

$$P_{k+1} = (I_n - K_{k+1}H_{k+1})P_{k+1/k} \tag{3}$$

$$\hat{x}_{k+1/k} = f_{u_k}(\hat{x}_k) \tag{4}$$

$$P_{k+1/k} = F_k P_k F_k^T + Q_k \tag{5}$$

where

$$K_{k+1} = P_{k+1/k}H_{k+1}^T\left(H_{k+1}P_{k+1/k}H_{k+1}^T + R_{k+1}\right)^{-1} \tag{6}$$

$$e_{k+1} = y_{k+1} - h_{u_{k+1}}(\hat{x}_{k+1/k}) \tag{7}$$

$$F_k = F_{u_k}(\hat{x}_k) = \left.\frac{\partial f_{u_k}(x_k)}{\partial x_k}\right|_{x_k = \hat{x}_k} \tag{8}$$

$$H_{k+1} = H_{u_{k+1}}(\hat{x}_{k+1/k}) = \left.\frac{\partial h_{u_{k+1}}(x_{k+1})}{\partial x_{k+1}}\right|_{x_{k+1} = \hat{x}_{k+1/k}}. \tag{9}$$

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The authors are with CRAN-CNRS UPRES-A 7039, University of Henri Poincaré, Nancy I 186, 54400 Cosnes et Romain, France (e-mail: boutayeb@iut-longwy.u-nancy.fr).

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In what follows, we point out the connection between convergence of the state observer (2)–(9) and the instrumental positive definite matrices R_k and Q_k.

III. Convergence Analysis

In this section, the convergence analysis of the EKO (2)–(9) will be performed by the Lyapunov approach in which unknown diagonal matrices to model linearization errors are introduced. In the following, we show that a strictly decreasing Lyapunov function means resolving an LMI problem whose solution depends closely on designing the matrices R_k and Q_k. First of all, we need to define the error vectors $\tilde{x}_{k+1}$ and $\tilde{x}_{k+1/k}$, respectively, by

$$\tilde{x}_{k+1} = x_{k+1} - \hat{x}_{k+1} \tag{10}$$

$$\tilde{x}_{k+1/k} = x_{k+1} - \hat{x}_{k+1/k} \tag{11}$$

and a candidate Lyapunov function V_{k+1} as follows:

$$V_{k+1} = \tilde{x}_{k+1}^T P_{k+1}^{-1} \tilde{x}_{k+1}. \tag{12}$$

By subtracting both sides of (2) from x_{k+1}, we obtain

$$\tilde{x}_{k+1} = \tilde{x}_{k+1/k} - K_{k+1} e_{k+1} \tag{13}$$

instead of using the following classical approximations:

$$\tilde{x}_{k+1/k} \approx F_k \tilde{x}_k \tag{14}$$

and

$$e_{k+1} \approx H_{k+1} \tilde{x}_{k+1/k}. \tag{15}$$

We introduce unknown diagonal matrices $\beta_k = \mathrm{diag}(\beta_{1k} \cdots \beta_{nk})$ and $\alpha_{k+1} = \mathrm{diag}(\alpha_{1k+1} \cdots \alpha_{pk+1})$, to model errors due to the first-order linearization technique, so that we obtain the following exact equalities:

$$\tilde{x}_{k+1/k} = \beta_k F_k \tilde{x}_k \tag{16}$$

$$\alpha_{k+1} e_{k+1} = H_{k+1} \tilde{x}_{k+1/k}. \tag{17}$$

Remark 1: It is well known that the usual EKO works very well when the state estimation $\hat{x}_k$ is very close to the actual state x_k (i.e., β_k and α_{k+1} in (16) and (17) are close to identity matrices); this is because the approximation (14), (15) is satisfied and therefore the convergence analysis is similar to the linear case. However, in the case of large initial state estimation errors and also in the case of high nonlinearities, the usual EKO fails to converge. The main goal behind the use of the parameterization in (16) and (17) is to perform a rigorous analysis concerning the convergence of the EKO without any approximation and to point out the role of the instrumental matrices R_k and Q_k so that the proposed modified EKO can tolerate, for example, very bad initial conditions.

The aim, in what follows, is to determine conditions for which $\{V_k\}_{k=1,\cdots}$ is a decreasing sequence taking β_k and α_{k+1} into account. This problem returns, in fact, to show what limitations are in using the EKO and at the same time the influence of R_k and Q_k on the convergence behavior. Next, from (3) and (6), we have

$$\begin{aligned} K_{k+1} &= P_{k+1} H_{k+1}^T R_{k+1}^{-1} \\ &= P_{k+1/k} H_{k+1}^T \Big(H_{k+1} P_{k+1/k} H_{k+1}^T + R_{k+1}\Big)^{-1} \end{aligned} \tag{18}$$

and

$$P_{k+1}^{-1} = P_{k+1/k}^{-1} + H_{k+1}^T R_{k+1}^{-1} H_{k+1}. \tag{19}$$

Substituting (18) into (13) and (13) into (12), the Lyapunov function V_{k+1} becomes

$$\begin{aligned} V_{k+1} = {}& \tilde{x}_{k+1/k}^T P_{k+1}^{-1} \tilde{x}_{k+1/k} - \tilde{x}_{k+1/k}^T H_{k+1}^T R_{k+1}^{-1} e_{k+1} \\ & - e_{k+1}^T R_{k+1}^{-1} H_{k+1} \tilde{x}_{k+1/k} \\ & + e_{k+1}^T R_{k+1}^{-1} H_{k+1} P_{k+1} H_{k+1}^T R_{k+1}^{-1} e_{k+1} \end{aligned} \tag{20}$$

and (19) into (20)

$$\begin{aligned} V_{k+1} = {}& V_{k+1/k} + \tilde{x}_{k+1/k}^T H_{k+1}^T R_{k+1}^{-1} H_{k+1} \tilde{x}_{k+1/k} \\ & - \tilde{x}_{k+1/k}^T H_{k+1}^T R_{k+1}^{-1} e_{k+1} - e_{k+1}^T R_{k+1}^{-1} H_{k+1} \tilde{x}_{k+1/k} \\ & + e_{k+1}^T R_{k+1}^{-1} H_{k+1} P_{k+1} H_{k+1}^T R_{k+1}^{-1} e_{k+1} \end{aligned} \tag{21}$$

with

$$V_{k+1/k} = \tilde{x}_{k+1/k}^T P_{k+1/k}^{-1} \tilde{x}_{k+1/k}. \tag{22}$$

From (16) and (17), (21) becomes

$$\begin{aligned} V_{k+1} = {}& V_{k+1/k} + e_{k+1}^T (\alpha_{k+1} R_{k+1}^{-1} \alpha_{k+1} - \alpha_{k+1} R_{k+1}^{-1} \\ & - R_{k+1}^{-1} \alpha_{k+1} + R_{k+1}^{-1} H_{k+1} P_{k+1} H_{k+1}^T R_{k+1}^{-1}) e_{k+1}. \end{aligned} \tag{23}$$

On the other hand, $V_{k+1/k}$ may be written as

$$V_{k+1/k} = \tilde{x}_k^T F_k^T \beta_k (F_k P_k F_k^T + Q_k)^{-1} \beta_k F_k \tilde{x}_k. \tag{24}$$

A decreasing sequence $\{V_k\}_{k=1,\cdots}$ means that there exists a positive scalar $0 < \zeta < 1$ so that

$$V_{k+1} - V_k \leq -\zeta V_k \tag{25}$$

or equivalently

$$\begin{aligned} & V_{k+1} - (1-\zeta) V_k \\ & \quad = e_{k+1}^T \Big(\alpha_{k+1} R_{k+1}^{-1} \alpha_{k+1} - \alpha_{k+1} R_{k+1}^{-1} \\ & \qquad - R_{k+1}^{-1} \alpha_{k+1} + R_{k+1}^{-1} H_{k+1} P_{k+1} H_{k+1}^T R_{k+1}^{-1}\Big) e_{k+1} \\ & \qquad + \tilde{x}_k^T \Big(F_k^T \beta_k (F_k P_k F_k^T + Q_k)^{-1} \\ & \qquad \cdot \beta_k F_k - (1-\zeta) P_k^{-1}\Big) \tilde{x}_k \leq 0. \end{aligned} \tag{26}$$

A sufficient condition to ensure (26) leads to the following couple of LMI's:

$$\begin{aligned} & \alpha_{k+1} R_{k+1}^{-1} \alpha_{k+1} - \alpha_{k+1} R_{k+1}^{-1} - R_{k+1}^{-1} \alpha_{k+1} \\ & \quad + R_{k+1}^{-1} H_{k+1} P_{k+1} H_{k+1}^T R_{k+1}^{-1} \leq 0 \end{aligned} \tag{27}$$

and

$$F_k^T \beta_k \Big(F_k P_k F_k^T + Q_k\Big)^{-1} \beta_k F_k - (1-\zeta) P_k^{-1} \leq 0. \tag{28}$$

We notice that (27) may be written into an equivalent form. Indeed, by the use of (18) and a simple factorization technique, we obtain

$$\begin{aligned} & (\alpha_{k+1} - I_p) R_{k+1}^{-1} (\alpha_{k+1} - I_p) - R_{k+1}^{-1} \\ & \quad + R_{k+1}^{-1} H_{k+1} P_{k+1/k} H_{k+1}^T \\ & \quad \cdot \Big(H_{k+1} P_{k+1/k} H_{k+1}^T + R_{k+1}\Big)^{-1} \leq 0 \end{aligned} \tag{29}$$

$$\begin{aligned} \Leftrightarrow {}& (\alpha_{k+1} - I_p) R_{k+1}^{-1} (\alpha_{k+1} - I_p) - R_{k+1}^{-1} \\ & \cdot \Big(I_p - H_{k+1} P_{k+1/k} H_{k+1}^T \\ & \quad \Big(H_{k+1} P_{k+1/k} H_{k+1}^T + R_{k+1}\Big)^{-1}\Big) \leq 0. \end{aligned} \tag{30}$$

Using the identity in (30)

$$\begin{aligned} I_p = {}& \Big(H_{k+1} P_{k+1/k} H_{k+1}^T + R_{k+1}\Big) \\ & \cdot \Big(H_{k+1} P_{k+1/k} H_{k+1}^T + R_{k+1}\Big)^{-1}. \end{aligned} \tag{31}$$

Equations (27) and (28) then become

$$\begin{aligned} & (\alpha_{k+1} - I_p) R_{k+1}^{-1} (\alpha_{k+1} - I_p) \\ & \quad - \Big(H_{k+1} P_{k+1/k} H_{k+1}^T + R_{k+1}\Big)^{-1} \leq 0 \end{aligned} \tag{32}$$

and

$$F_k^T \beta_k \Big(F_k P_k F_k^T + Q_k\Big)^{-1} \beta_k F_k - (1-\zeta) P_k^{-1} \leq 0. \tag{33}$$

Before we deliver the main result of this paper, let us determine domains in which (32) and (33) are satisfied. It can be shown that under the following assumptions:

$$\left|\alpha_{i(k+1)} - 1\right| \le \overline{\alpha}_{k+1} = \sup_i \left|\alpha_{i(k+1)} - 1\right| \le \left(\frac{\underline{\sigma}(R_{k+1})}{\overline{\sigma}\left(H_{k+1}P_{k+1/k}H_{k+1}^T + R_{k+1}\right)}\right)^{1/2}, \qquad \text{for } i = 1, \cdots, p \tag{34}$$

$$|\beta_{jk}| \le \overline{\beta}_k = \sup_j |\beta_{jk}| \le \left(\frac{(1-\zeta)\underline{\sigma}\left(F_k P_k F_k^T + Q_k\right)}{\overline{\sigma}(F_k^T)\overline{\sigma}(P_k)\overline{\sigma}(F_k)}\right)^{1/2}, \quad \text{for } j = 1, \cdots, n. \tag{35}$$

$\{V_k\}_{k=1,\cdots}$ is a decreasing sequence. $\overline{\sigma}$ and $\underline{\sigma}$ denote the maximum and minimum singular values, respectively.

Indeed, under (34) and (35) and using the fact that $(\alpha_{k+1} - I_p)$ and β_k are diagonal matrices, we have

$$[\overline{\sigma}(\alpha_{k+1} - I_p)]^2 \le \frac{\underline{\sigma}(R_{k+1})}{\overline{\sigma}\left(\left(H_{k+1}P_{k+1/k}H_{k+1}^T + R_{k+1}\right)\right)} \tag{36}$$

$$[\overline{\sigma}(\beta_k)]^2 \le \frac{(1-\zeta)\underline{\sigma}\left(F_k P_k F_k^T + Q_k\right)}{\overline{\sigma}(F_k^T)\overline{\sigma}(P_k)\overline{\sigma}(F_k)} \tag{37}$$

$$\Leftrightarrow [\overline{\sigma}(\alpha_{k+1} - I_p)]^2 \overline{\sigma}\left(R_{k+1}^{-1}\right) \le \underline{\sigma}\left(\left(H_{k+1}P_{k+1/k}H_{k+1}^T + R_{k+1}\right)^{-1}\right) \tag{38}$$

$$[\overline{\sigma}(\beta_k)]^2 \le \frac{(1-\zeta)\underline{\sigma}\left(P_k^{-1}\right)}{\overline{\sigma}(F_k^T)\overline{\sigma}\left((F_k P_k F_k^T + Q_k)^{-1}\right)\overline{\sigma}(F_k)} \tag{39}$$

as

$$\overline{\sigma}\left((\alpha_{k+1} - I_p)R_{k+1}^{-1}(\alpha_{k+1} - I_p)\right) \le [\overline{\sigma}(\alpha_{k+1} - I_p)]^2 \overline{\sigma}\left(R_{k+1}^{-1}\right) \tag{40}$$

and

$$\overline{\sigma}\left(F_k^T \beta_k \left(F_k P_k F_k^T + Q_k\right)^{-1} \beta_k F_k\right) \le [\overline{\sigma}(\beta_k)]^2 \overline{\sigma}\left(F_k^T\right)\overline{\sigma}\left(\left(F_k P_k F_k^T + Q_k\right)^{-1}\right)\overline{\sigma}(F_k). \tag{41}$$

We then have

$$\begin{aligned}\overline{\sigma}\left((\alpha_{k+1} - I_p)R_{k+1}^{-1}(\alpha_{k+1} - I_p)\right) &\le [\overline{\sigma}(\alpha_{k+1} - I_p)]^2 \overline{\sigma}\left(R_{k+1}^{-1}\right) \\ &\le \underline{\sigma}\left(\left(H_{k+1}P_{k+1/k}H_{k+1}^T + R_{k+1}\right)^{-1}\right)\end{aligned} \tag{42}$$

$$\begin{aligned}\overline{\sigma}\left(F_k^T \beta_k \left(F_k P_k F_k^T + Q_k\right)^{-1} \beta_k F_k\right) &\le [\overline{\sigma}(\beta_k)]^2 \overline{\sigma}\left(F_k^T\right)\overline{\sigma}\left(\left(F_k P_k F_k^T + Q_k\right)^{-1}\right)\overline{\sigma}(F_k) \\ &\le (1-\zeta)\underline{\sigma}\left(P_k^{-1}\right)\end{aligned} \tag{43}$$

which induce that (32) and (33) are satisfied, and consequently V_k is a strictly decreasing sequence.

Remark 2:

1) Most certainly, (34) and (35) cannot be checked (in the global sense) before the process is started, since the parameters α_{ik} and β_{jk} are unknown; however, a qualitative analysis from the sufficient conditions (34) and (35) is of great interest since they determine domains to which α_{ik} and β_{jk} should belong so that V_k is a decreasing sequence. At the same time we point out the role of the arbitrary matrices R_k and Q_k to satisfy (34) and (35) and consequently to enlarge the attraction domain. For example, one intuitive idea consists of setting, in particular, the matrix Q_k sufficiently large so that (35) is satisfied. This means that the proposed state observer can tolerate arbitrary large initial state estimation errors, i.e., β_k and α_{k+1} may be large and not necessarily very close to identity matrices. On the other hand, it should be noticed that as long as (34) and (35) are satisfied, $\hat{x}_k$ converges to x_k and consequently β_k and α_{k+1} go to identity matrices. Along this reasoning, we propose in the following a relevant design of R_k and Q_k to make the EKO a useful state estimator.

2) Now let us give some remarks on how to put in practice the proposed modified EKO. It is reasonable to assume, from the local property inherent to the used state observer, that the unknown parameters α_{ik} and β_{jk} are bounded. Moreover, the upper bounds $\overline{\alpha}_k$ and $\overline{\beta}_k$ [used in (34) and (35)] may be estimated before the process is started. Indeed, in practice, we have in general a reasonable estimation of the state vector of the physical process to be observed (for example, concentrations in chemical reactors, fluxes, and speed in induction motors, ...) and by the use of the analytic functions $h_u(.)$ and $f_u(.)$ of the system we can evaluate $\overline{\alpha}_k$ and $\overline{\beta}_k$. As we mentioned before, with a relevant design of R_k and Q_k that ensures (34) and (35), in particular when the algorithm starts, $\hat{x}_k$ converges to x_k; consequently β_k and α_{k+1} go to identity matrices, or equivalently $\overline{\beta}_k$ and $\overline{\alpha}_k$ decrease to one and zero, respectively. This is why bounds in (34) and (35) should be as large as possible before the process is started. On the other hand, we should keep in mind that there is a large class of nonlinear physical processes where the measurement equation is linear, i.e., $y_k = Hx_k$, in which case we have $\alpha_k = I_p$. In the same manner when $x_{k+1} = Ax_k$, we have $\beta_k = I_n$.

The main result of this paper is summarized in the following theorem.

Theorem: We assume the following.

1) (1a) and (1b) is N-locally uniformly rank observable, i.e., there exists an integer $N \ge 1$ such that

$$\operatorname{rank} \frac{\partial}{\partial x}\left.\begin{pmatrix} h_{u_k}(x) \\ h_{u_{k+1}} \circ f_{u_k}(x) \\ \vdots \\ h_{u_{k+N-1}} \circ f_{u_{k+N-2}} \circ \cdots \circ f_{u_k}(x)\end{pmatrix}\right|_{x=x_k} = n \tag{44}$$

for all $x_k \in K$ and N-tuple of controls $(u_k, \cdots, u_{k+N-1}) \in U$ (K and U are two compact subsets of R^n and $(R^r)^N$, respectively).

2) F_k, H_k are uniformly bounded matrices and F_k^{-1} exists.

3) The instrumental matrices R_k and Q_k are chosen so that the bounds

$$\left(\frac{\underline{\sigma}(R_{k+1})}{\overline{\sigma}\left(H_{k+1}P_{k+1/k}H_{k+1}^T + R_{k+1}\right)}\right)^{1/2} \tag{45}$$

and

$$\left(\frac{(1-\zeta)\underline{\sigma}\left(F_k P_k F_k^T + Q_k\right)}{\overline{\sigma}(F_k^T)\overline{\sigma}(P_k)\overline{\sigma}(F_k)}\right)^{1/2} \tag{46}$$

are sufficiently large and greater than $\overline{\alpha}_{k+1}$ and $\overline{\beta}_k$, defined in (34) and (35), respectively.

Then the proposed modified EKO ensures local asymptotic convergence

$$\lim_{k\to\infty}(x_k - \hat{x}_k) = 0. \tag{47}$$

■

Proof: First of all, under assumptions (45) and (46) where the upper bounds $\overline{\alpha}_{k+1}$ and $\overline{\beta}_k$ may be estimated using Remark 2, it has been shown that V_k is a decreasing sequence. However, in order to ensure $\lim_{k\to\infty}(x_k - \hat{x}_k) = 0$, we need to introduce the local observability condition (44) to assure that P_k is a bounded matrix from above and below (see Remark 3-2) with [6] and [15]).

Indeed, since V_k is a strictly decreasing sequence and P_k is a bounded matrix, it follows that

$$0 \le \mu \tilde{x}_k^T \tilde{x}_k \le V_k \le (1-\zeta)^k V_0 \tag{48}$$

$$\Rightarrow 0 \le \mu \lim_{k\to\infty} \left(\tilde{x}_k^T \tilde{x}_k\right) \le \lim_{k\to\infty} (V_k) \le V_0 \lim_{k\to\infty} (1-\zeta)^k = 0 \tag{49}$$

with

$$0 < \mu I_n \le P_k^{-1}. \tag{50}$$

Therefore, convergence of the error dynamics to zero follows. □

Remark 3:

1) While the lower and upper bounds on α_{ik}, for any choice of R_k, are zero and two, respectively, $i = 1, \cdots, p$, those of β_{jk} depend on Q_k which allows us to obtain a very large value of

$$\left(\frac{(1-\zeta)\underline{\sigma}\left(F_k P_k F_k^T + Q_k\right)}{\overline{\sigma}(F_k^T)\overline{\sigma}(P_k)\overline{\sigma}(F_k)}\right)^{1/2}$$

in (35). A judicious choice of Q_k that we propose here is

$$Q_k = \gamma e_k^T e_k I_n + \delta I_n \tag{51}$$

where $e_k = y_k - h_{u_k}(\hat{x}_{k/k-1})$.

γ have to be chosen sufficiently large and positive, in particular for bad initial conditions, while δ is a positive scalar which is small enough.

The reason for this particular choice is to ensure the sufficient condition (35). Indeed, we notice that when $x_k \neq \hat{x}_k$, the term $\gamma e_k^T e_k$ in Q_k and then the upper bound

$$\left(\frac{(1-\zeta)\underline{\sigma}\left(F_k P_k F_k^T + Q_k\right)}{\overline{\sigma}(F_k^T)\overline{\sigma}(P_k)\overline{\sigma}(F_k)}\right)^{1/2}$$

is large, the value depends on the nonlinearities of $f_{u_k}(x_k)$, $h_{u_k}(x_k)$ and the arbitrary positive scalar γ, and consequently we may enlarge the attraction domain. On the other hand, when x_k is close to $\hat{x}_k$, i.e., $\overline{\beta}_k$ and $\overline{\alpha}_k$ decrease to one and zero, respectively, the term $e_k^T e_k$ goes to zero and the proposed observer becomes the classical EKO.

In the same way, the choice of R_k is also important. Indeed, it is easy to verify that the choice proposed in both examples, i.e., $R_{k+1} = \lambda H_{k+1} P_{k+1/k} H_{k+1}^T + \delta I_p$, where λ is a positive scalar fixed by the user, may enlarge the interval

$$\left]-\frac{\underline{\sigma}(R_{k+1})}{\overline{\sigma}\left(\left(H_{k+1}P_{k+1/k}H_{k+1}^T + R_{k+1}\right)\right)}, \frac{\underline{\sigma}(R_{k+1})}{\overline{\sigma}\left(\left(H_{k+1}P_{k+1/k}H_{k+1}^T + R_{k+1}\right)\right)}\right[$$

in order to satisfy (34).

2) The relation between the N-observability rank condition (44) and the observability of the linearized system along the EKO's trajectory is described in [15] in the case of uncontrolled nonlinear systems. We notice that when inputs are handled, the observability properties of both the original system and the linearized system along the EKO's trajectory become input-dependent so we have to introduce the concept of uniform observability as in [12].

Following [15], if conditions (44) and (45) are verified, the linearized system along the EKO's trajectory is locally uniformly observable, and therefore there exist positive real numbers η_1 and η_2 so that for all $k \ge N-1$ we have

$$\eta_1 I_n \le O_e^T(k-N+1, k)\ \mathrm{diag}\left(R_{k-N+1}^{-1}, \cdots, R_k^{-1}\right) \cdot O_e(k-N+1, k) \le \eta_2 I_n \tag{52}$$

where

$$O_e(k-N+1, k) = \begin{bmatrix} H_{k-N+1} \\ H_{k-N+2}F_{k-N+1} \\ \cdot \\ \cdot \\ H_k F_{k-1} F_{k-2} \cdots F_{k-N+1} \end{bmatrix}.$$

Under uniform observability condition (52), the boundedness of the covariance matrix P_k follows from [6].

3) We notice that in practice, the local uniform observability of system (1a) and (1b) may be checked by a literal validation of condition (44) using derivatives of $f_{u_k}(x_k)$ and $h_{u_k}(x_k)$ or at least by a numerical rank test on the observability matrix $O_e(k-N+1, k)$ given in (52).

4) The study proposed by Deza *et al.* [7] concerns the class of nonlinear multi-input/single-output systems affine in the input. They use a nonlinear change of coordinates to bring the system into a canonical form. This transformation needs strong assumptions under which global and exponential stability is guaranteed. The approach that we consider here concerns a large class of nonlinear multi-input/multi-output (MIMO) systems, under weak conditions and not necessarily affine in the input, but only local asymptotic convergence is ensured. It should be noticed that the examples proposed in our paper cannot be treated by the approach in [7]. We also have in mind the problem of combined states and parameters estimation.

IV. Simulation Results

In order to show accuracy and high performances of the proposed approach, a comparison to the usual EKO is made through the following examples.

Example 1: The first numerical example that we consider in this section is a fifth-order two-phase nonlinear model of an induction motor which was already the object of a large number of applications, especially in control designs (see [11] and the references inside). It could be mentioned that, unlike most of the works on induction motors where the rotor speed is assumed to be known, only the stator currents are needed to provide an estimate of both rotor fluxes and angular speed.

Using an Euler discretization of step size h, the complete discrete-time model in stator fixed (a, b) reference frame is given by

$$\begin{cases} x_{1k+1} = x_{1k} + h\left(-\gamma x_{1k} + \dfrac{K}{T_r} x_{3k} + Kp x_{5k} x_{4k} + \dfrac{1}{\sigma L_s} u_{1k}\right) \\ x_{2k+1} = x_{2k} + h\left(-\gamma x_{2k} - Kp x_{5k} x_{3k} + \dfrac{K}{T_r} x_{4k} + \dfrac{1}{\sigma L_s} u_{2k}\right) \\ x_{3k+1} = x_{3k} + h\left(\dfrac{M}{T_r} x_{1k} - \dfrac{1}{T_r} x_{3k} - p x_{5k} x_{4k}\right) \\ x_{4k+1} = x_{4k} + h\left(\dfrac{M}{T_r} x_{2k} + p x_{5k} x_{3k} - \dfrac{1}{T_r} x_{4k}\right) \\ x_{5k+1} = x_{5k} + h\left(\dfrac{pM}{JL_r}(x_{3k}x_{2k} - x_{4k}x_{1k}) - \dfrac{T_L}{J}\right) \\ \quad y_{1k} = x_{1k} \\ \quad y_{2k} = x_{2k} \end{cases}$$

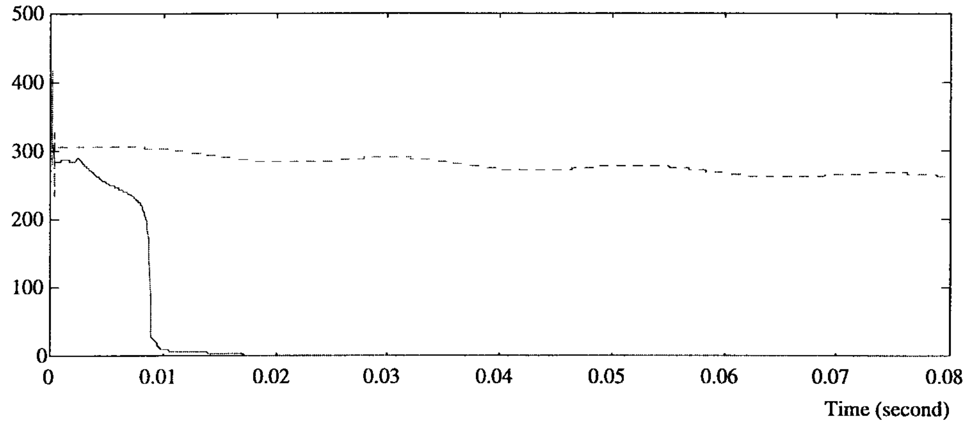

Fig. 1. State estimation error norm $\|\tilde{x}\|_2$. Example 1, modified EKO (solid), and usual EKO (dashed).

where $x_k^T = (x_{1k}\ x_{2k}\ x_{3k}\ x_{4k}\ x_{5k}) = (i_{sak}\ i_{sbk}\ \psi_{rak}\ \psi_{rbk}\ \omega_k)$ represents the stator currents, the rotor fluxes, and the angular speed, respectively, $u_k^T = (u_{1k}\ u_{2k}) = (u_{sak}\ u_{sbk})$ is the stator voltages control vector, p is the number of pair of poles, and T_L is the load torque.

The rotor time constant T_r and the parameters σ, K, and γ are defined as follows:

$$T_r = \frac{L_r}{R_r}, \qquad \sigma = 1 - \frac{M^2}{L_s L_r}, \qquad K = \frac{M}{\sigma L_s L_r}$$

and

$$\gamma = \frac{R_s}{\sigma L_s} + \frac{R_r M^2}{\sigma L_s L_r^2}$$

where R_s, R_r denote stator and rotor per-phase resistances; L_s, L_r are stator and rotor per-phase inductances and J is the rotor moment of inertia.

The simulations are performed with the same numerical values as in [11], i.e., $R_s = 0.18\ \Omega$, $R_r = 0.15\ \Omega$, $M = 0.068$ H, $L_s = 0.0699$ H, $L_r = 0.0699$ H, $J = 0.0586$ kgm^2, $T_L = 10$ Nm, $p = 1$, and $h = 0.1$ ms.

As explained in the previous section, the design of the instrumental matrices R_k and Q_k is crucial to control stability and rate of convergence of the proposed algorithm. An accurate choice is

$$R_{k+1} = 0.1 H_{k+1} P_{k+1/k} H_{k+1}^T + 10^{-3} I_2$$

and

$$Q_k = 10^{10} e_k^T e_k I_5 + 10^{-3} I_5.$$

The initial conditions for the observer and the system are taken as

$$\hat{x}_{10} = 200, \qquad \hat{x}_{20} = 200, \qquad \hat{x}_{30} = 50, \qquad \hat{x}_{40} = 50$$
$$\hat{x}_{50} = 300, \qquad \text{and} \qquad P_0 = 10^8 I_5$$

and

$$x_{10} = x_{20} = x_{30} = x_{40} = x_{50} = 0.$$

The input signals are $u_{1k} = 350\cos(0.03k)$ and $u_{2k} = 300\sin(0.03k)$.

Example 2: The second numerical example concerns the famous bearings-only measurement problem [9]–[14], [16] for which special observer structure forms were introduced in order to enhance the convergence behavior of the EKF. It should be noticed that for a rigorous comparison with the approaches developed in [9] and [14], extension of the proposed technique to stochastic models will be the topic of the next paper.

A two-dimensional discrete-time model is described as follows:

$$\begin{cases} x_{1k+1} = x_{1k} + h x_{3k} - 0.5 h^2 u_{1k} \\ x_{2k+1} = x_{2k} + h x_{4k} - 0.5 h^2 u_{2k} \\ x_{3k+1} = x_{3k} - h u_{1k} \\ x_{4k+1} = x_{4k} - h u_{2k} \\ \quad y_k = \tan^{-1}\left(\dfrac{x_{2k}}{x_{1k}}\right) \end{cases}$$

where x_{1k}, x_{2k} and x_{3k}, x_{4k} denote target positions and velocities at time instant k, respectively, and $u_k^T = (u_{1k}\ u_{2k})$ is the two-dimensional missile acceleration used as the control vector. $h = 5$ ms is the sampling period. The guidance law u_k [16] is used to generate nominal trajectories from which the observer performances are tested. We suppose that, starting with a constant speed, the target adopts a new constant velocity trajectory, to avoid the missile for example, after 4 s. By this abrupt change (that can be seen as a perturbation), we want to test if the modified EKO remains to be a convergent observer.

Initializations 1:

$$\hat{x}_{10} = 7000 \text{ m}, \qquad \hat{x}_{20} = 5000 \text{ m}, \qquad \hat{x}_{30} = 100 \text{ ms}^{-1}$$
$$\hat{x}_{30} = 200 \text{ ms}^{-1}, \qquad \text{and} \qquad P_0 = 10^{15} I_4.$$

Initializations 2—very bad:

$$\hat{x}_{10} = 15\,000 \text{ m}, \qquad \hat{x}_{20} = 10\,000 \text{ m}, \qquad \hat{x}_{30} = 500 \text{ ms}^{-1}$$
$$\hat{x}_{30} = 500 \text{ ms}^{-1}, \qquad \text{and} \qquad P_0 = 10^{15} I_4$$

while the true values are: $x_{10} = 5000$ m, $x_{20} = 3000$ m, $x_{30} = -100$ ms^{-1}, $x_{40} = 50$ ms^{-1}.

The instrumental matrices R_k and Q_k are chosen as

$$R_{k+1} = 3 H_{k+1} P_{k+1/k} H_{k+1}^T + 10^{-5}$$

and

$$Q_k = 10^{15} e_k^2 I_4 + 10^{-5} I_4.$$

Simulation results are given in Figs. 1–3. The time evolution of the state estimation error norm $\|\tilde{x}\|_2$ is plotted for both examples and initializations. A comparison with the usual EKO, i.e., when R_k and Q_k have the form $R_k = I_p$ and $Q_k = 0.1\, I_n$, is provided.

The plots confirm our findings that with a relevant choice of instrumental matrices Q_k and R_k, the modified EKO remains extremely robust not only when observer initializations are very far from the actual values (Figs. 1 and 3) but also when abrupt changes in the

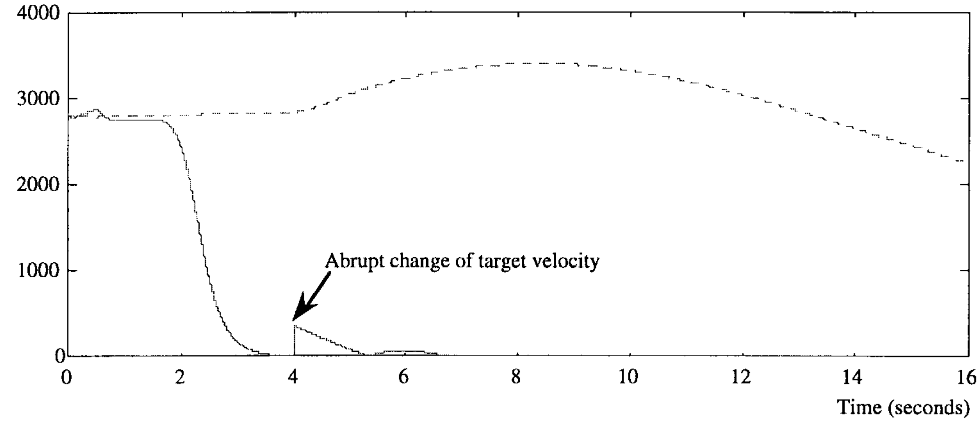

Fig. 2. State estimation error norm $\|\tilde{x}\|_2$. Example 2, Initialization 1, modified EKO (solid), and usual EKO (dashed).

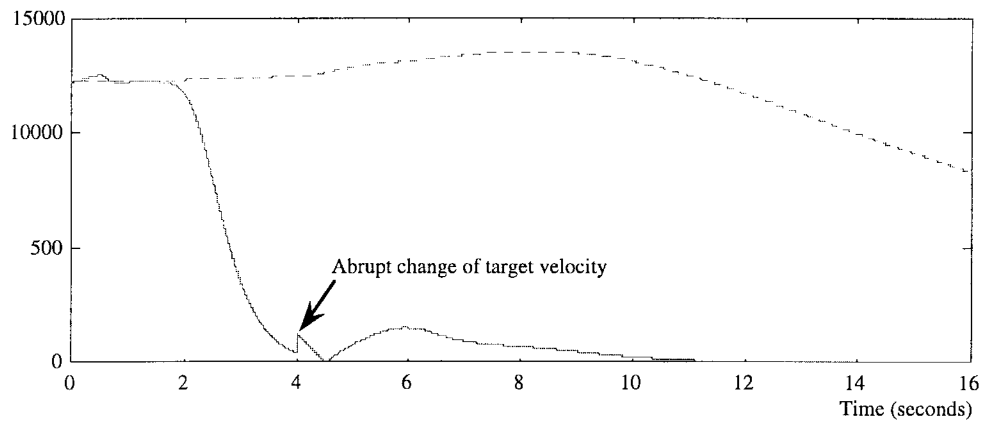

Fig. 3. State estimation error norm $\|\tilde{x}\|_2$. Example 2, Initialization 2, modified EKO (solid), and usual EKO (dashed).

target speed occur (Figs. 2 and 3) while the usual EKO fails to converge.

This approach has also been successfully applied to several examples treated in the literature even with initializations outside the physically feasible set of initial values.

V. Conclusion

From the above study we point out that design of R_k, and more particularly Q_k, makes the EKO one of the efficient techniques for state estimation of a large class of MIMO nonlinear discrete-time systems. Perhaps the main feature in this paper is that weak conditions, in comparison with those established in the literature, are needed to ensure asymptotic convergence without the need of any preliminary transformations. Furthermore, few parameters have to be selected in order to design the above matrices; this has the advantage of easy implementation in real time applications. Finally, the proposed technique was successfully applied to a large number of numerical examples treated in the literature, and two of them are detailed in this paper.

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Robustness with Respect to Disturbance Model Uncertainty—Analysis and Design

Daniel E. Davison, Pierre T. Kabamba, and Semyon M. Meerkov

***Abstract*—Motivated by an aircraft control problem where the airplane encounters turbulence with uncertain characteristics, this paper studies the analysis and design of controllers that are robust with respect to uncertainty in the disturbance intensity and bandwidth, where disturbance rejection is measured by the output variance.**

***Index Terms*— Aircraft control, disturbance rejection, disturbance model uncertainty, performance limitations.**

I. Introduction

Consider the single-input/single-output linear time-invariant (SISO LTI) system shown in Fig. 1. The plant is assumed to be composed of two fixed proper transfer functions, P_1 and P_2, and it is assumed that there exists at least one proper internally stabilizing LTI controller C. The disturbance d is assumed to be generated by filtering white noise of unity power spectral density (PSD) through the LTI filter KF, where K is a scaling parameter and F is the stable transfer function

$$F(s) = \frac{\sqrt{2\omega_b}}{s+\omega_b}. \tag{1}$$

This model is appealing since two meaningful parameters, namely the *intensity* ($K = \|KF\|_2$) and *bandwidth* ($\omega_b = 3$ dB breakpoint of KF), are captured explicitly. For simplicity, this paper will deal only with filter (1), although some of the results are readily extended to more general filters. The choice of this filter arose out of a study on aircraft control in turbulence [1].

The performance measure of interest in this paper is the (steady-state) variance of $\overline{y}$, where $\overline{y}$ is generated by filtering y through H, as shown in Fig. 1. Variance has useful interpretations in many engineering applications. For example, there is an empirical relationship between the variance of acceleration and the comfort of passengers in airplanes [2]. In terms of the closed-loop transfer function from d to $\overline{y}$, $G \triangleq P_2H/(1+P_1P_2C)$, the output variance is

$$\operatorname{var}(\overline{y}) = \|KFG\|_2^2 = \frac{K^2}{\pi}\int_0^\infty |F(j\omega)G(j\omega)|^2\,d\omega. \tag{2}$$

To ensure that the variance is finite, it is assumed throughout that $G \in \mathcal{RH}_\infty$, i.e., G is a stable proper transfer function. Note that the quantity in (2) also has deterministic interpretations as (the square of) the L_2 to L_∞ gain and, assuming the white noise signal is replaced with an impulse, as the energy of $\overline{y}$.

The problem addressed in this paper is to investigate the following questions.

- Q1: How does disturbance rejection performance vary with ω_b and K?
- Q2: How can disturbance rejection robustness be quantified?
- Q3: Are there fundamental limitations on disturbance rejection robustness?
- Q4: Is there a tradeoff between *nominal* and *robust* disturbance rejection?
- Q5: Can C be designed to achieve a desired level of disturbance rejection robustness?

The five questions are considered in, respectively, Sections II–VI, and conclusions are provided in Section VII. For brevity, proofs are omitted but can be found in [1].

II. The V-Transform

Although expression (2) provides an answer to Question Q1, a more useful formula relating the output variance to the uncertain disturbance model parameters can be derived using the results of [3].

Theorem 1: The output variance is a rational function of ω_b and K. Specifically, if $G \in \mathcal{RH}_\infty$ has the state-space realization

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) \triangleq D + C(sI-A)^{-1}B$$

then the variance of $\overline{y}$, explicitly as a function of ω_b and K, is

$$\begin{aligned}\operatorname{var}(\overline{\mathrm{y}}) = &\left[\begin{array}{c|c} A & B \\ \hline 2K^2(B^TL_0 + DC) & K^2D^2 \end{array}\right](\omega_b)\\ \triangleq\ & K^2D^2 + 2K^2(B^TL_0+DC)(\omega_bI - A)^{-1}B\end{aligned} \tag{3}$$

where L_0 is the observability Gramian of (C, A), i.e., the unique solution to $A^TL_0 + L_0A + C^TC = 0$. □

Note that only a *single* Lyapunov equation has to be solved to determine the output variance *for all* ω_b and K. It is convenient to denote the variance expression in (3) by $V(KG)(\omega_b)$, i.e., explicitly show the dependence of the output variance on G, ω_b, and K. The "V" denotes "variance," and we refer to $V(KG)$ as the "V-transform" of KG since it transforms KG, a function of ω, into $V(KG)$, a function of ω_b. This definition depends on the structure of F, but since that structure is restricted to (1) here, this dependence is not explicitly shown. It is possible to prove that, for any $G \in \mathcal{RH}_\infty$ and $K > 0$, $V(KG)$ is an analytic, nonnegative, and bounded function of ω_b (see [1]).

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D. E. Davison and S. M. Meerkov are with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109 USA (e-mail: davisod@eecs.umich.edu).

P. T. Kabamba is with the Department of Electrical Engineering and Computer Science and the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109-2140 USA.

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