# PARAMETERS PREDICTION ON DIVIDEND YIELD AND S&P REAL RETURN MODEL

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## 1. Algorithm I

Given 2-dimensional model

$$\begin{cases} dX_t = k(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t \\ dR_t = \mu X_t dt + a\sqrt{X_t}dZ_t \end{cases}$$

where $d\langle Z_t, W_t\rangle = \rho dt$ and

$$\begin{cases} Y_{1,t} = X_t e^{Q_1 \times B_{1,n}} \\ Y_{2,t} = R_t + Q_2 B_{2,n} \end{cases}$$

where $Q = \begin{Bmatrix} Q_1^2 & 0 \\ 0 & Q_2^2 \end{Bmatrix}$ and $B = \begin{Bmatrix} B_{1,n} \\ B_{2,n} \end{Bmatrix}$, $B_{1,n}$, $B_{2,n}$ are independent Brownian motions.

By Ito formula, we formed the dynamic of $\ln(X_t)$.

$$d\ln(X_t) = (\frac{k\theta}{X_t} - k - \frac{1}{2}\sigma^2)dt + \frac{\sigma}{\sqrt{X_t}}dW_t$$

Then SDE should be

$$\ln X_{t+h} = \ln X_t + \int_t^{t+h} (\frac{k\theta}{X_s} - k - \frac{1}{2}\sigma^2)ds + \int_t^{t+h} \frac{\sigma}{\sqrt{X_s}}dW_s$$

After discretising the SDE by using Forward Euler, formulas of $X_n$ and $R_n$ are formed below.

$$\begin{cases} X_n = X_{n-1}e^{\frac{2k\theta - 2kX_{n-1} - \sigma^2}{2X_{n-1}}} e^{\frac{\sigma}{\sqrt{X_{n-1}}}\Delta W_{1,n}} \\ R_n = \mu X_{n-1} + a\sqrt{X_{n-1}}(\rho\Delta W_{1,n} + \sqrt{1-\rho^2}\Delta W_{2,n}) \end{cases}$$

We take $V_n = \ln(X_n)$ and the model equation reads:

$$x_n = \begin{pmatrix} V_n \\ R_n \end{pmatrix} = \begin{pmatrix} V_{n-1} + \frac{2k\theta - \sigma^2}{2e^{V_{n-1}}} - k + \frac{\sigma}{\sqrt{e^{V_{n-1}}}}\Delta W_{1,n} \\ \mu e^{V_{n-1}} + a\sqrt{e^{V_{n-1}}}(\rho\Delta W_{1,n} + \sqrt{1-\rho^2}\Delta W_{2,n}) \end{pmatrix}$$

where $\Delta W_{i,n} = W_{i,n+1} - W_{i,n}$, $i = 1, 2$ and $\Delta W_{1,n}, \Delta W_{2,n}$are independent. The observation equation reads

$$y_n = \begin{pmatrix} Y_{1,n} \\ Y_{2,n} \end{pmatrix} = \begin{pmatrix} V_n + Q_1 B_{1,n} \\ R_n + Q_2 B_{2,n} \end{pmatrix}.$$

We are using maximum loglikelihood to estimate the paramaters. Let $\mathcal{F}_n$ denote all the measurements available until and including time $t_n$. Then we can write the likelihood function for the set of observations $Y = y_1, y_2, \dots, y_N$ as

$$L(Y) = p(y_1)\prod_{i=1}^{N} p(y_i|\mathcal{F}_{i-1})$$

We let $\nu_n$ represents information which could not have been derived from data up to time $t_{n-1}$ and are called *innovations*. According to previous equations, $\nu_n$ is formed below.

$$\begin{aligned}\nu_n &= y_n - E(y_n|\mathcal{F}_{n-1}) = y_n - E(x_n|\mathcal{F}_{n-1}) \\ &= y_n - \begin{Bmatrix} V_{n-1} + \frac{2k\theta-\sigma^2}{2e^{V_{n-1}}} - k \\ \mu e^{V_{n-1}} \end{Bmatrix} \end{aligned} \tag{1.1}$$

Then the covariance matrix of $\nu_n$ can be calculated by

$$\begin{aligned}\Sigma_n &= \mathrm{var}(\nu_n|\mathcal{F}_{n-1}) = \mathrm{cov}(x_n|\mathcal{F}_{n-1}) + Q^{1/2}(Q^{1/2})^\top \\ &= \begin{Bmatrix} \mathrm{var}(V_n|\mathcal{F}_{n-1}) & \mathrm{var}(V_n, R_n|\mathcal{F}_{n-1}) \\ \mathrm{cov}(V_n, R_n|\mathcal{F}_{n-1}) & \mathrm{cov}(R_n|\mathcal{F}_{n-1}) \end{Bmatrix} + Q^{1/2}(Q^{1/2})^\top \\ &= \begin{Bmatrix} \frac{\sigma^2}{e^{V_{n-1}}} & a\sigma\rho \\ a\sigma\rho & a^2 e^{V_{n-1}} \end{Bmatrix} + Q^{1/2}(Q^{1/2})^\top \\ &= \begin{Bmatrix} \frac{\sigma^2}{e^{V_{n-1}}} + Q_1^2 & a\sigma\rho \\ a\sigma\rho & a^2 e^{V_{n-1}} + Q_2^2 \end{Bmatrix} \end{aligned} \tag{1.2}$$

It is usually simpler to work with logarithm of likelihood, which is given by

$$\log L(Y) = \sum_{i=1}^{N} \log p(y_i|\mathcal{F}_{i-1}) = -\frac{1}{2}\sum_{i=1}^{N}(\log|\Sigma_i| + \nu_i^T \Sigma_i^{-1}\nu_i)$$

when the constant terms are ignored. The above function can then be maximized to find the parameter vectors $k$,$\theta$,$\sigma$,$\mu$, $a$, $\rho$ and matrices $Q$ using an off-the-shelf nonlinear solver such as *fminsearch* in MATLAB.

1.1. **Numerical results.** Finally we got our result in the following table:

| $Q_1$ | $Q_2$ | $k$ | $\theta$ | $\sigma$ | $\mu$ | $a$ | $\rho$ |
|---|---|---|---|---|---|---|---|
| 2.0714 | 2.0451 | 0.3003 | 0.1907 | 0.9197 | 1.6309 | 0.0310 | -0.8857 |

## 2. Algorithm II

Given 2-dimensional model

$$\begin{cases} dX_t = k(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t \\ dR_t = \mu X_t dt + a\sqrt{X_t}dZ_t \end{cases}$$

where $d\langle Z_t, W_t\rangle = \rho dt$ and

$$\begin{cases} Y_{1,t} = X_t + Q_1 B_{1,n} \\ Y_{2,t} = R_t + Q_2 B_{2,n} \end{cases}$$

where $Q = \begin{Bmatrix} Q_1^2 & 0 \\ 0 & Q_2^2 \end{Bmatrix}$ and $B = \begin{Bmatrix} B_{1,n} \\ B_{2,n} \end{Bmatrix}$, $B_{1,n}$, $B_{2,n}$ are independent Brownian motions. Rather than using Forward Euler, we use Backward Euler to solve SDE:

$$\begin{cases} X_{t+h} = X_t + \int_t^{t+h} k(\theta - X_s)ds + \int_t^{t+h} \sigma\sqrt{X_s}dW_s \\ R_{t+h} = R_t + \int_t^{t+h} \mu X_s ds + \int_t^{t+h} a\sqrt{X_s}dZ_s \end{cases}$$

We have

$$x_n = \begin{pmatrix} X_n \\ R_n \end{pmatrix} = \begin{pmatrix} \frac{1}{1+k}X_{n-1} + \frac{k\theta}{1+k} + \frac{\sigma}{1+k}\sqrt{X_{n-1}}\Delta W_{1,n} \\ \mu X_n + a\sqrt{X_{n-1}}(\rho\Delta W_{1,n} + \sqrt{1-\rho^2}\Delta W_{2,n}) \end{pmatrix}$$

where $\Delta W_{i,n} = W_{i,n+1} - W_{i,n},\ i = 1, 2$ and $\Delta W_{1,n}, \Delta W_{2,n}$are independent. The observation equation reads

$$y_n = \begin{pmatrix} Y_{1,n} \\ Y_{2,n} \end{pmatrix} = \begin{pmatrix} X_n + Q_1 B_{1,n} \\ R_n + Q_2 B_{2,n} \end{pmatrix}.$$

We are using maximum loglikelihood to estimate the paramaters. Let $\mathcal{F}_n$ denote all the measurements available until and including time $t_n$. Then we can write the likelihood function for the set of observations $Y = y_1, y_2, \ldots, y_N$ as

$$L(Y) = p(y_1)\prod_{i=1}^{N} p(y_i|\mathcal{F}_{i-1})$$

We let $\nu_n$ represents information which could not have been derived from data up to time $t_{n-1}$ and are called *innovations*. According to previous equations, $\nu_n$ is formed below.

$$\begin{aligned} \nu_n &= y_n - E(y_n|\mathcal{F}_{n-1}) = y_n - E(x_n|\mathcal{F}_{n-1}) \\ &= y_n - \begin{Bmatrix} \frac{1}{1+k}X_{n-1} + \frac{k\theta}{1+k} \\ \mu(\frac{1}{1+k}X_{n-1} + \frac{k\theta}{1+k}) \end{Bmatrix} \end{aligned} \tag{2.1}$$

Then the covariance matrix of $\nu_n$ can be calculated by

$$\begin{aligned} \Sigma_n &= \mathrm{var}(\nu_n|\mathcal{F}_{n-1}) = \mathrm{cov}(x_n|\mathcal{F}_{n-1}) + Q^{1/2}(Q^{1/2})^\top \\ &= \end{aligned} \tag{2.2}$$

It is usually simpler to work with logarithm of likelihood, which is given by

$$\log L(Y) = \sum_{i=1}^{N} \log p(y_i|\mathcal{F}_{i-1}) = -\frac{1}{2}\sum_{i=1}^{N}(\log|\Sigma_i| + \nu_i^T\Sigma_i^{-1}\nu_i)$$

when the constant terms are ignored. The above function can then be maximized to find the parameter vectors $k$,$\theta$,$\sigma$,$\mu$, $a$, $\rho$ and matrices $Q$ using an off-the-shelf nonlinear solver such as *fminsearch* in MATLAB.

## 3. Algorithm III

In this section, the only difference is that for $X_n$, the scheme is positivity-preserving. All the computation will be similar to what we had in the section above.

Given 2-dimensional model

$$\begin{cases} dX_t = k(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t \\ dR_t = \mu X_t dt + a\sqrt{X_t}dZ_t \end{cases}$$

where $d\langle Z_t, W_t\rangle = \rho dt$ and

$$\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} X_t \\ R_t \end{pmatrix} + \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} B_{1,t} \\ B_{2,t} \end{pmatrix} \tag{3.1}$$

Here $B_{1,t},\ B_{2,t}$ are independent Brownian motions.

$$\begin{cases} X_n = \left|X_{n-1} + k(\theta - X_{n-1}) + \sigma\sqrt{X_{n-1}}\Delta W_{1,n}\right| (\text{ positivity preserving}) \\ R_n = \mu X_{n-1} + a\sqrt{X_{n-1}}(\rho\Delta W_{1,n} + \sqrt{1-\rho^2}\Delta W_{2,n}) \end{cases}$$

We are using maximum loglikelihood to estimate the parameters. Let $\mathcal{F}_n$ denote all the measurements available until and including time $t_n$. Then we can write the likelihood function for the set of observations $Y = \{y_1, y_2, \ldots, y_N\}$ as

$$L(Y) = p(y_1)\prod_{i=1}^{N} p(y_i|\mathcal{F}_{i-1}).$$

We let $\nu_n$ represents information which could not have been derived from data up to time $t_{n-1}$ and are called *innovations*. According to previous equations, $\nu_n$ is calculated below.

$$\nu_n = y_n - E(y_n|\mathcal{F}_{n-1}) = y_n - E(x_n|\mathcal{F}_{n-1}) = y_n - ?? \tag{3.2}$$

Then the covariance matrix of $\nu_n$ can be calculated by

$$\begin{aligned} \Sigma_n &= var(\nu_n|\mathcal{F}_{n-1}) = cov(x_n|\mathcal{F}_{n-1}) + QQ^T \\ &= \begin{Bmatrix} var(V_n|\mathcal{F}_{n-1}) & cov(V_n, R_n|\mathcal{F}_{n-1}) \\ cov(V_n, R_n|\mathcal{F}_{n-1}) & var(R_n|\mathcal{F}_{n-1}) \end{Bmatrix} + QQ^T \\ &= ?? + QQ^T \end{aligned} \tag{3.3}$$

It is usually simpler to work with logarithm of likelihood, which is given by

$$\ln L(Y) = \sum_{i=1}^{N} \ln p(y_i|\mathcal{F}_{i-1}) = -\frac{1}{2}\sum_{i=1}^{N}(\ln|\Sigma_i| + \nu_i^T\Sigma_i^{-1}\nu_i)$$

when the constant terms are ignored. The above function can then be maximized to find the parameter vectors $k$,$\theta$,$\sigma$,$\mu$, $a$, $\rho$ and matrices $Q$ using an off-the-shelf nonlinear solver such as *fminsearch* in MATLAB.

Finally we got our result in the following table:

| $Q_1$ | $Q_2$ | $k$ | $\theta$ | $\sigma$ | $\mu$ | $a$ | $\rho$ |
|---|---|---|---|---|---|---|---|
| | | | | | | | |

## Appendix A. Inverse of matrices

**Theorem A.1.** *Suppose $A$ is an invertible square matrix and $uv$ are column vectors. Suppose furthermore that $1 + v^TA^{-1}u \neq 0$. Then the ShermanMorrison formula states that*

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}.$$

**Theorem A.2.** *Assume that we have some $n \times n$ matrix, $A$. $U$ is a $n \times k$ matrix and $V$ is a $k \times n$ matrix, $B = A + UV$. Then,*

$$B^{-1} = A^{-1} - A^{-1}U\big(I_k + VA^{-1}U\big)^{-1}VA^{-1}.$$

**Theorem A.3.** *The Woodbury matrix identity*

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1},$$