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Performance evaluation of UKF-based nonlinear filtering☆

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Abstract

The performance of the modified unscented Kalman filter (UKF) for nonlinear stochastic discrete-time system with linear measurement equation is investigated. It is proved that under certain conditions, the estimation error of the UKF remains bounded. Furthermore, it is shown that the design of noise covariance matrix plays an important role in improving the stability of the algorithm. Error behavior of the UKF is then derived in terms of mean square error (MSE), and the Cramér–Rao lower bound (CRLB) is introduced as a performance measure. The modified UKF is found to approach the CRLB if the difference between the real noise covariance matrix and the selected one is small enough. These results are verified by using Monte Carlo simulations on two example systems.

Keywords: Nonlinear systems; Stochastic systems; Extended Kalman filter; Unscented Kalman filter; Stability

1. Introduction

Discrete-time filtering for nonlinear dynamic system is an important research area and has attracted considerable interest. A large number of suboptimal approaches have been developed to solve the nonlinear filtering problem. These include extended Kalman filtering (EKF) (Gobbo, Napolitano, Famouri, & Innocenti, 2001; Goodwin & Sin, 1984; La Scala, Bitmead, & Quinn, 1996; Ljung, 1979), Gaussian sum filter (Alspach & Sorenson, 1972), grid-based methods (Kramer & Sorenson, 1988) and particle filters (Arulampalam, Maskell, Gordon, & Clapp, 2002; Gordon, Salmond, & Smith, 1993). Among these methods, EKF is the most widely used filtering strategy. It is obtained by first-order linearization of nonlinear models so that the traditional linear Kalman filter can be applied. However, it has two well known drawbacks (Hall & Llinas, 2001): (1) the first-order linearization can introduce large errors in mean and covariance of the state vector, and (2) the derivation of Jacobian matrices is nontrivial in many applications.

Recently, a relatively new nonlinear filtering algorithm named unscented Kalman filter (UKF) is proposed as an improvement to EKF (Julier, Uhlmann, & Durrant-Whyte, 1995). This method is based on the unscented transform (UT) technique, a mechanism for propagating mean and covariance through a nonlinear transformation. The state vector is represented by a minimal set of carefully chosen sample points, called sigma points, which approximate the posterior mean and covariance of the Gaussian random variable with a second order accuracy. In contrast, the linearization technique used in the EKF can only achieve first order accuracy. Further, it is not necessary to compute the Jacobian matrices in the UKF. The UKF is widely used in practice, ranging from multi-sensor fusion (Ristic, Farina, Benvenuti, & Arulampalam, 2003), target tracking (Julier et al., 1995), position determination (Julier, 2002), to training of neural networks (Wan & van der Merwe, 2000). Julier compared the performance of the UKF and the EKF for an example system and showed that the UKF performs better than the EKF (Julier, Uhlmann, & Durrant-Whyte, 2000). Similar comparison can be seen in (Ristic et al., 2003).

Despite its superior practical usefulness, the UKF has not been analyzed in a rigorous mathematical way. Some results

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obtained in the study of the EKF can be used to treat the stability properties of the UKF. Using the second method of Lyapunov, Reif and Unbehauen (1999) proved that the estimation error of the EKF applied to deterministic nonlinear systems is exponentially stable if it is initialized closely enough to the actual states. Moreover, a rigorous treatment of the discrete-time EKF for stochastic systems with similar conditions is given by Reif, Gunther, Yaz, and Unbehauen (1999). However, in the implementation of EKF, initial estimation errors may be hard to reduce. Boutayeb, Rafaralahy, and Darouach (1997) developed a new formulation of the first-order linearization technique and proved that if the error introduced in the linearization is negligible, the local asymptotic stability is ensured. In a recent paper (Boutayeb & Aubry, 1999), the sufficient conditions established in Boutayeb et al., 1997 is relaxed and the stability analysis of the EKF is reduced to solve a linear matrix inequality (LMI) problem where noise covariance matrix plays a central role to improve the stability. However, the impact of the covariance matrix on the mean square error (MSE) was neglected in these works.

Motivated by the encouraging results developed in Boutayeb et al., 1997 and (Boutayeb & Aubry, 1999), influence of the noise covariance matrix on the UKF behavior is analyzed in this paper. In order to improve stability, slight modifications of the standard UKF were performed by introducing an extra positive definite matrix in the noise covariance matrix. However, setting high value of the extra additive matrix leads to a large MSE, so the design of the matrix can be seen as a tradeoff between stability and accuracy. In addition, the Cramér–Rao lower bound (CRLB) is introduced as a benchmark to evaluate the performance of the modified algorithm. The CRLB provides a lower bound on the MSE and no matter which method is selected, one cannot get a result better than the CRLB (Caffery & Stuber, 2000; Leung & Zhu, 2001). It is shown that if the extra positive definite matrix is selected properly, the performance of the UKF used for nonlinear systems with linear measurement equation may be improved significantly even for big initial estimation error. Finally, the modified method is used as an estimator for two example systems.

2. The unscented Kalman filter

The considered nonlinear discrete-time system is represented by

$$x_k = f(x_{k-1}) + w_k, \tag{1a}$$

$$y_k = H_k x_k + v_k, \tag{1b}$$

where $k \in N_0$ is discrete time, N_0 denotes the set of natural numbers including zero. $x_k \in R^L$ the state, and $y_k \in R^M$ the measurement. The nonlinear mapping $f(\cdot)$ is assumed to be continuously differentiable with respect to x_k. Moreover, w_k and v_k are uncorrelated zero-mean Gaussian white sequence and their covariances are

$$E[w_k w_j^{\mathrm{T}}] = Q_k \cdot \delta_{kj}, \quad E[v_k v_j^{\mathrm{T}}] = R_k \cdot \delta_{kj}, \quad E[w_k v_j^{\mathrm{T}}] = 0. \tag{2}$$

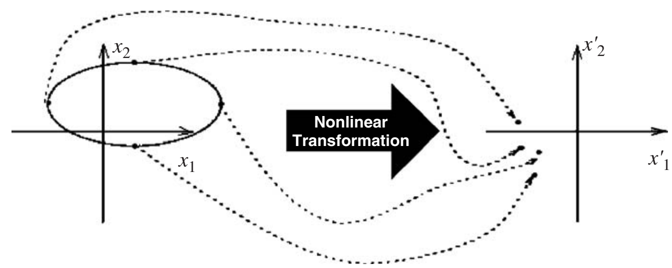

Fig. 1. The unscented transformation.

Instead of linearization required by the EKF, a new approximate method called UT is used in the UKF (Julier & Uhlmann, 2004). A set of weighted sigma points is deterministically chosen so that the sample mean and sample covariance of these points match those of a priori distribution. The nonlinear function is applied to each of these points in turn to yield transformed samples, and the predicted mean and covariance are calculated from the transformed samples as shown in Fig. 1. The expected performance of the UKF is superior to that of the EKF in that the UT algorithm introduces errors in estimating the mean and covariance at the third and higher order terms in the Taylor series.

The procedure for implementing the UKF can be summarized as follows (Wan & Merwe, 2000):

Step 1: The *n*-dimensional random variable x_{k-1} with mean $\hat{x}_{k-1}$ and covariance $\hat{P}_{k-1}$ is approximated by sigma points selected using the following equations:

$$\begin{cases} \chi_{i,k-1} = \hat{x}_{k-1}, & i = 0, \\ \chi_{i,k-1} = \hat{x}_{k-1} + (a\sqrt{L\hat{P}_{k-1}})_i, & i = 1, \ldots, L, \\ \chi_{i,k-1} = \hat{x}_{k-1} - (a\sqrt{L\hat{P}_{k-1}})_{i-L}, & i = L+1, \ldots, 2L, \end{cases} \tag{3}$$

where $a \in R$ is a tuning parameter denoting the spread of the sigma points around $\hat{x}_{k-1}$ and $(\sqrt{L\hat{P}_{k-1}})_i$ is the ith column of the matrix square root of $L\hat{P}_{k-1}$. a is often set to a small positive value.

Step 2: *Prediction*. Each point is instantiated through the process model to yield a set of transformed samples

$$\chi_{i,k|k-1} = f(\chi_{i,k-1}). \tag{4}$$

The predicted mean and covariance are computed as

$$\hat{x}_{k|k-1} = \sum_{i=0}^{2L} \omega_i \chi_{i,k|k-1}, \tag{5}$$

$$\hat{P}_{k|k-1} = \sum_{i=0}^{2L} \omega_i (\chi_{i,k|k-1} - \hat{x}_{k|k-1})(\chi_{i,k|k-1} - \hat{x}_{k|k-1})^{\mathrm{T}} + Q_k, \tag{6}$$

where

$$\begin{cases} \omega_i = 1 - \dfrac{1}{a^2}, & i = 0, \\ \omega_i = \dfrac{1}{2La^2}, & i = 1, \ldots, 2L. \end{cases}$$

Step 3: *Update*. As the measurement equation is linear, measurement update can be performed with the same equations as the classical Kalman filter.

$$\hat{y}_k = H_k \hat{x}_{k|k-1}, \tag{7}$$

$$\hat{P}_{yy} = H_k \hat{P}_{k|k-1} H_k^{\mathrm{T}} + R_k, \tag{8}$$

$$\hat{P}_{xy} = \hat{P}_{k|k-1} H_k^{\mathrm{T}}, \tag{9}$$

$$\hat{x}_k = \hat{x}_{k|k-1} + \hat{P}_{xy} \hat{P}_{yy}^{-1} (y_k - \hat{y}_k), \tag{10}$$

$$\hat{P}_k = \hat{P}_{k|k-1} - \hat{P}_{xy} \hat{P}_{yy}^{-1} \hat{P}_{xy}^{\mathrm{T}}. \tag{11}$$

Step 4: Repeat steps 1 to 3 for the next sample.

Clearly, the implementation of the UKF is extremely convenient, for it does not need to evaluate the Jacobian matrices, which is necessary in the EKF.

3. Stability of the UKF

3.1. Instrumental diagonal matrix and extra positive definite matrix

In this section, a simple approach to present error of the UKF is given. First instrumental time-varying matrices are introduced to give a formulation for the UT technique. Define the estimation error and prediction error by

$$\tilde{x}_k = x_k - \hat{x}_k, \tag{12}$$

$$\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}. \tag{13}$$

Expanding x_k by a Taylor series about $\hat{x}_{k-1}$ gives,

$$x_k = f(\hat{x}_{k-1}) + \nabla f(\hat{x}_{k-1})\tilde{x}_{k-1} + \tfrac{1}{2}\nabla^2 f(\hat{x}_{k-1})\tilde{x}_{k-1}^2 + \cdots + w_k, \tag{14}$$

where

$$\nabla^i f(\hat{x})\tilde{x}^i = \left(\sum_{j=1}^{L} \tilde{x}_j \frac{\partial}{\partial x_j} \right)^i f(x) \Bigg|_{x=\hat{x}_{k-1}},$$

x_j denotes the jth component of x. Expanding $\hat{x}_{k|k-1}$ given in (5) by a Taylor series yields,

$$\begin{aligned}
\hat{x}_{k|k-1} &= \left(1 - \frac{1}{a^2}\right) f(\hat{x}_{k-1}) \\
&\quad + \frac{1}{2La^2} \sum_{i=1}^{L} f[\hat{x}_{k-1} + (a\sqrt{LP_{k-1}})_i] \\
&\quad + \frac{1}{2La^2} \sum_{i=L+1}^{2L} f[\hat{x}_{k-1} - (a\sqrt{LP_{k-1}})_{i-L}] \\
&= f(\hat{x}_{k-1}) + \frac{1}{2}\nabla^2 f(\hat{x}_{k-1}) P_{k-1} + \cdots .
\end{aligned} \tag{15}$$

Substituting (14) and (15) into (13) gives an approximate equality

$$\tilde{x}_{k|k-1} \approx F_k \tilde{x}_{k-1} + w_k, \tag{16}$$

where

$$F_k = \left(\frac{\partial f(x)}{\partial x} \bigg|_{x=\hat{x}_{k-1}} \right).$$

Note that only the first term is shown in (16). It is evident that there always exist residuals of state error prediction $\tilde{x}_{k|k-1}$. In order to take these residuals into account and obtain an exact equality, an unknown instrumental diagonal matrix $\beta_k = diag(\beta_{1,k}, \beta_{2,k}, \ldots, \beta_{M,k})$ is introduced, so that

$$\tilde{x}_{k|k-1} = \beta_k F_k \tilde{x}_{k-1} + w_k. \tag{17}$$

The residual of the measurement is defined by

$$\tilde{y}_k = y_k - \hat{y}_k. \tag{18}$$

Substituting (2) and (7) into (18) gives,

$$\tilde{y}_k = H_k \tilde{x}_{k|k-1} + v_k. \tag{19}$$

In the modified form of the algorithm, the predicted covariance matrix is calculated with enlarged Q_k, i.e.,

$$\hat{P}_{k|k-1} = \sum_{i=0}^{2L} \omega_i (\chi_{i,k|k-1} - \hat{x}_{k|k-1})(\chi_{i,k|k-1} - \hat{x}_{k|k-1})^{\mathrm{T}} + Q_k + \Delta Q_k, \tag{20}$$

where ΔQ_k is an extra positive definite matrix introduced in the calculated covariance matrix as a slight modification of the UKF so that the stability will be improved. The role of ΔQ_k to ensure stability is shown in Section 3.2. In contrast the real prediction error covariance matrix is

$$\begin{aligned}
P_{k|k-1} &= E[\tilde{x}_{k|k-1}\tilde{x}_{k|k-1}^{\mathrm{T}}] \\
&= E[(\beta_k F_k \tilde{x}_{k-1} + w_k)(\beta_k F_k \tilde{x}_{k-1} + w_k)^{\mathrm{T}}] \\
&= \beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \Delta P_{k|k-1} + Q_k,
\end{aligned} \tag{21}$$

where $\Delta P_{k|k-1}$ is the difference between $\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k$ and $E(\beta_k F_k \tilde{x}_{k-1} \tilde{x}_{k-1}^{\mathrm{T}} F_k^{\mathrm{T}} \beta_k)$. Let $\delta P_{k|k-1}$ be the difference between the real covariance matrix $P_{k|k-1}$ and the sample one $\sum_{i=0}^{2L} \omega_i (\chi_{i,k|k-1} - \hat{x}_{k|k-1})(\chi_{i,k|k-1} - \hat{x}_{k|k-1})^{\mathrm{T}} + Q_k$, the calculated covariance matrix shown in (20) becomes

$$\hat{P}_{k|k-1} = P_{k|k-1} + \delta P_{k|k-1} + \Delta Q_k = \beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k, \tag{22}$$

where

$$\hat{Q}_k = \Delta P_{k|k-1} + Q_k + \delta P_{k|k-1} + \Delta Q_k. \tag{23}$$

3.2. Stochastic boundedness of estimation error

For analysis of the error dynamics (12) some standard results about the boundedness of stochastic processes (Agniel & Jury, 1971; Tarn & Rasis, 1976) are recalled.

Lemma 1. *Assume that ξ_k is the stochastic process and there is a stochastic process $V(\xi_k)$ as well as real numbers*

$\upsilon_{\min}, \upsilon_{\max} > 0$, $\mu > 0$ and $0 < \lambda \leqslant 1$ *such that* $\forall k$

$$\upsilon_{\min}\|\xi_k\|^2 \leqslant V(\xi_k) \leqslant \upsilon_{\max}\|\xi_k\|^2, \tag{24}$$

$$E[V(\xi_k)|\xi_{k-1}] - V(\xi_{k-1}) \leqslant \mu - \lambda V(\xi_{k-1}) \tag{25}$$

are fulfilled. Then the stochastic process is bounded in mean square, *i.e.*,

$$E\{\|\xi_k\|^2\} \leqslant \frac{\upsilon_{\max}}{\upsilon_{\min}} E\{\|\xi_0\|^2\}(1-\lambda)^k + \frac{\mu}{\upsilon_{\min}} \sum_{i=1}^{k-1}(1-\lambda)^i. \tag{26}$$

Before establishing sufficient conditions to ensure stability of the UKF, another two lemmas are given.

Lemma 2. *Assume that matrices* $A \in R^{m\times n}$, $B \in R^{m\times n}$ *and* $C \in R^{n\times n}$, *if* $A > 0$ *and* $C > 0$, *then*

$$A^{-1} > B(B^{\mathrm{T}}AB + C)^{-1}B^{\mathrm{T}}. \tag{27}$$

The proof is put in Appendix A.

Lemma 3. *Assume that matrices* $A \in R^{n\times n}$, $C \in R^{n\times n}$, *if* $A > 0$ *and* $C > 0$, *then*

$$A^{-1} > (A + C)^{-1}. \tag{28}$$

Lemma 3 can be seen as a special case of Lemma 2 when $B = I$. With these lemmas and formulations shown in (17) and (22), it is able to state a main result of this paper.

Theorem 1. *Consider a nonlinear stochastic system given by* (1) *and the modified* UKF *as stated by* (3)–(5), (20) *and* (7)–(11). *Let the following assumptions hold*:

(1) There are real numbers $f_{\min}, h_{\min}, \beta_{\min} \neq 0$, $f_{\max}, h_{\max}, \beta_{\max} \neq 0$ such that the following bounds on various matrices are fulfilled for every $k \geqslant 0$:

$$f_{\min}^2 I \leqslant F_k F_k^{\mathrm{T}} \leqslant f_{\max}^2 I, \tag{29}$$

$$h_{\min}^2 I \leqslant H_k H_k^{\mathrm{T}} \leqslant h_{\max}^2 I, \tag{30}$$

$$\beta_{\min}^2 I \leqslant \beta_k \beta_k^{\mathrm{T}} \leqslant \beta_{\max}^2 I. \tag{31}$$

(2) There are real numbers $q_{\max}, \hat{q}_{\max}, \hat{q}_{\min}, r_{\min}, p_{\max}, p_{\min} > 0$ such that the following bounds are fulfilled:

$$Q_k \leqslant q_{\max} I, \tag{32}$$

$$\hat{q}_{\max} I \geqslant \hat{Q}_k \geqslant \hat{q}_{\min} I, \tag{33}$$

$$R_k \geqslant r_{\min} I, \tag{34}$$

$$p_{\min} I \leqslant \hat{P}_k \leqslant p_{\max} I. \tag{35}$$

Then the estimation error $\tilde{x}_k$ is bounded in mean square.

Proof. Choose

$$V_k(\tilde{x}_k) = \tilde{x}_k^{\mathrm{T}} \hat{P}_k^{-1} \tilde{x}_k. \tag{36}$$

From (35) it gives

$$\frac{1}{p_{\max}}\|\tilde{x}_k\|^2 \leqslant V(\tilde{x}_k) \leqslant \frac{1}{p_{\min}}\|\tilde{x}_k\|^2. \tag{37}$$

To satisfy the requirement for the application of Lemma 1, it needs an upper bound on $E[V(\xi_k)|\xi_{k-1}] - V(\xi_{k-1})$ as in (25). Substituting (8) and (9) into (11) gives

$$\hat{P}_k = \hat{P}_{k|k-1} - \hat{P}_{xy}\hat{P}_{yy}^{-1}\hat{P}_{xy}^{\mathrm{T}} = (I - K_k H_k)\hat{P}_{k|k-1}, \tag{38}$$

where

$$K_k = \hat{P}_{k|k-1}H_k^{\mathrm{T}}(H_k \hat{P}_{k|k-1} H_k^{\mathrm{T}} + R_k)^{-1}. \tag{39}$$

Using (10), (12), (18) as well as (39) yields

$$\tilde{x}_k = x_k - (\hat{x}_{k|k-1} + \hat{P}_{xy}\hat{P}_{yy}^{-1}\tilde{y}_k) = \tilde{x}_{k|k-1} - K_k\tilde{y}_k. \tag{40}$$

From (36) and (40) it can be obtained

$$\begin{aligned} V_k(\tilde{x}_k) &= (\tilde{x}_{k|k-1} - K_k\tilde{y}_k)^{\mathrm{T}}\hat{P}_k^{-1}(\tilde{x}_{k|k-1} - K_k\tilde{y}_k) \\ &= \tilde{x}_{k|k-1}^{\mathrm{T}}\hat{P}_k^{-1}\tilde{x}_{k|k-1} - [H_k\tilde{x}_{k|k-1} + v_k]^{\mathrm{T}}K_k^{\mathrm{T}}\hat{P}_k^{-1}\tilde{x}_{k|k-1} \\ &\quad - \tilde{x}_{k|k-1}^{\mathrm{T}}\hat{P}_k^{-1}K_k[H_k\tilde{x}_{k|k-1} + v_k] + [H_k\tilde{x}_{k|k-1} \\ &\quad + v_k]^{\mathrm{T}}K_k^{\mathrm{T}}\hat{P}_k^{-1}K_k[H_k\tilde{x}_{k|k-1} + v_k]. \end{aligned} \tag{41}$$

Rearranging (39) yield

$$K_k = (I - K_k H_k)\hat{P}_{k|k-1}H_k^{\mathrm{T}}R_k^{-1} = \hat{P}_k H_k^{\mathrm{T}} R_k^{-1}. \tag{42}$$

On the other hand, applying the matrix inversion lemma on (38) gives

$$\hat{P}_k^{-1} = \hat{P}_{k|k-1}^{-1} + H_k^{\mathrm{T}}R_k^{-1}H_k. \tag{43}$$

Inserting (42), (43) and (17) into (41), and taking the conditional expectation yields

$$\begin{aligned} &E[V_k(\tilde{x}_k)|\tilde{x}_{k-1}] \\ &\quad = E\{(\beta_k F_k\tilde{x}_{k-1} + w_k)^{\mathrm{T}}\hat{P}_{k|k-1}^{-1}(\beta_k F_k\tilde{x}_{k-1} + w_k) \\ &\qquad - [H_k(\beta_k F_k\tilde{x}_{k-1} + w_k)]^{\mathrm{T}}(R_k^{-1} - R_k^{-1}H_k\hat{P}_k H_k^{\mathrm{T}}R_k^{-1}) \\ &\qquad \times [H_k(\beta_k F_k\tilde{x}_{k-1} + w_k)] \\ &\qquad + v_k^{\mathrm{T}}R_k^{-1}H_k\hat{P}_k H_k^{\mathrm{T}}R_k^{-1}v_k|\tilde{x}_{k-1}\}. \end{aligned} \tag{44}$$

Then we consider the term $R_k^{-1} - R_k^{-1}H_k\hat{P}_k H_k^{\mathrm{T}}R_k^{-1}$ on the right side of (44), with (42) and (39) it can be verified that

$$\begin{aligned} &R_k^{-1} - R_k^{-1}H_k\hat{P}_k H_k^{\mathrm{T}}R_k^{-1} \\ &\quad = R_k^{-1}[1 - H_k\hat{P}_{k|k-1}H_k^{\mathrm{T}}(H_k\hat{P}_{k|k-1}H_k^{\mathrm{T}} + R_k)^{-1}] \\ &\quad = (H_k\hat{P}_{k|k-1}H_k^{\mathrm{T}} + R_k)^{-1} > 0. \end{aligned} \tag{45}$$

Substituting (45) and (22) into (44), and applying Lemma 3, the function becomes

$$\begin{aligned} &E[V_k(\tilde{x}_k)|\tilde{x}_{k-1}] \\ &\quad \leqslant E\{(\beta_k F_k\tilde{x}_{k-1})^{\mathrm{T}}(\beta_k F_k\hat{P}_{k-1}F_k^{\mathrm{T}}\beta_k)^{-1}(\beta_k F_k\tilde{x}_{k-1}) \\ &\qquad + w_k^{\mathrm{T}}(\beta_k F_k\hat{P}_{k-1}F_k^{\mathrm{T}}\beta_k + \hat{Q}_k)^{-1}w_k \\ &\qquad -(H_k\beta_k F_k\tilde{x}_{k-1})^{\mathrm{T}}[H_k(\beta_k F_k\hat{P}_{k-1}F_k^{\mathrm{T}}\beta_k + \hat{Q}_k)H_k^{\mathrm{T}} + R_k]^{-1} \\ &\qquad \times (H_k\beta_k F_k\tilde{x}_{k-1}) - (H_k w_k)^{\mathrm{T}}[H_k(\beta_k F_k\hat{P}_{k-1}F_k^{\mathrm{T}}\beta_k \\ &\qquad + \hat{Q}_k)H_k^{\mathrm{T}} + R_k]^{-1}(H_k w_k) \\ &\qquad + v_k^{\mathrm{T}}R_k^{-1}H_k\hat{P}_k H_k^{\mathrm{T}}R_k^{-1}v_k|\tilde{x}_{k-1}\}. \end{aligned} \tag{46}$$

Inequality (29) and (31) implies that $(\beta_k F_k)^{-1}$ exists. Therefore, it may establish that

$$E\{(\beta_k F_k \tilde{x}_{k-1})^{\mathrm{T}} (\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k)^{-1} (\beta_k F_k \tilde{x}_{k-1}) | \tilde{x}_{k-1}\} = \tilde{x}_{k-1}^{\mathrm{T}} \hat{P}_{k-1} \tilde{x}_{k-1} = V_{k-1}(\tilde{x}_{k-1}). \tag{47}$$

Subtracting (47) from both sides of (46) gives

$$\begin{aligned} &E[V_k(\tilde{x}_k)|\tilde{x}_{k-1}] - V_{k-1}(\tilde{x}_{k-1}) \\ &\quad\leqslant E\{w_k^{\mathrm{T}} (\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k)^{-1} w_k - (H_k w_k)^{\mathrm{T}} \\ &\qquad\times [H_k(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k) H_k^{\mathrm{T}} + R_k]^{-1} \cdot (H_k w_k) \\ &\qquad+ v_k^{\mathrm{T}} R_k^{-1} H_k \hat{P}_k H_k^{\mathrm{T}} R_k^{-1} v_k | \tilde{x}_{k-1}\} - (H_k \beta_k F_k \tilde{x}_{k-1})^{\mathrm{T}} \\ &\qquad\times [H_k(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k) H_k^{\mathrm{T}} + R_k]^{-1} \\ &\qquad\times (H_k \beta_k F_k \tilde{x}_{k-1}). \end{aligned} \tag{48}$$

Now let us focus on the last term in (48), according to Lemma 2

$$\hat{P}_{k-1}^{-1} > (H_k \beta_k F_k)^{\mathrm{T}} [(H_k \beta_k F_k) \hat{P}_{k-1} (H_k \beta_k F_k)^{\mathrm{T}} + H_k \hat{Q}_k H_k^{\mathrm{T}} + R_k]^{-1} (H_k \beta_k F_k). \tag{49}$$

Pre- and post-multiplying both sides of (49) by $\tilde{x}_{k-1}^{\mathrm{T}}$, and $\tilde{x}_{k-1}$, respectively, we have $\forall \tilde{x}_{k-1} \neq 0$

$$\tilde{x}_{k-1}^{\mathrm{T}} \hat{P}_{k-1}^{-1} \tilde{x}_{k-1} > (H_k \beta_k F_k \tilde{x}_{k-1})^{\mathrm{T}} [(H_k \beta_k F_k) \hat{P}_{k-1} (H_k \beta_k F_k)^{\mathrm{T}} + H_k \hat{Q}_k H_k^{\mathrm{T}} + R_k]^{-1} (H_k \beta_k F_k \tilde{x}_{k-1}). \tag{50}$$

Choosing

$$\begin{aligned} \lambda_k = \{&(H_k \beta_k F_k \tilde{x}_{k-1})^{\mathrm{T}} [(H_k \beta_k F_k) \hat{P}_{k-1} (H_k \beta_k F_k)^{\mathrm{T}} \\ &+ H_k \hat{Q}_k H_k^{\mathrm{T}} + R_k]^{-1} (H_k \beta_k F_k \tilde{x}_{k-1})\} \\ &/(\tilde{x}_{k-1}^{\mathrm{T}} \hat{P}_{k-1} \tilde{x}_{k-1}). \end{aligned} \tag{51}$$

From (50) we have $\lambda_k < 1$. Under assumption (29)–(31), and (33)–(35)

$$\begin{aligned} \lambda_k &\geqslant p_{\min} (h_{\min} \beta_{\min} f_{\min})^2 [p_{\max} (h_{\max} \beta_{\max} f_{\max})^2 \\ &\quad+ \hat{q}_{\max} h_{\max}^2 + r_{\max}]^{-1} \\ &\triangleq \lambda_{\min} > 0. \end{aligned} \tag{52}$$

Thus by using (50) and (52) it is easy to show that

$$\begin{aligned} &-(H_k \beta_k F_k \tilde{x}_{k-1})^{\mathrm{T}} [(H_k \beta_k F_k) \hat{P}_{k-1} (H_k \beta_k F_k)^{\mathrm{T}} \\ &\quad+ H_k \hat{Q}_k H_k^{\mathrm{T}} + R_k]^{-1} (H_k \beta_k F_k \tilde{x}_{k-1}) \\ &\qquad\leqslant -\lambda_{\min} V_{k-1}(\tilde{x}_{k-1}). \end{aligned} \tag{53}$$

Now the other items in (48) are considered. Setting

$$\begin{aligned} \mu_k = E\{&w_k^{\mathrm{T}} [(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k)^{-1} \\ &- H_k^{\mathrm{T}} (H_k(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k) H_k^{\mathrm{T}} + R_k)^{-1} H_k] w_k \\ &+ v_k^{\mathrm{T}} R_k^{-1} H_k \hat{P}_k H_k^{\mathrm{T}} R_k^{-1} v_k | \tilde{x}_{k-1}\}. \end{aligned} \tag{54}$$

Since both sides of (54) are scalars, taking trace will not change its value. Applying the well-known matrix identity

$$tr(AB) = tr(BA), \tag{55}$$

where A, B are such matrices that the above matrix multiplication and the trace operation make sense, it gives

$$\begin{aligned} \mu_k &= E\{tr[((\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k)^{-1} \\ &\quad- H_k^{\mathrm{T}} (H_k(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k) H_k^{\mathrm{T}} + R_k)^{-1} H_k)(w_k w_k^{\mathrm{T}})] \\ &\quad+ tr[R_k^{-1} H_k \hat{P}_k H_k^{\mathrm{T}} R_k^{-1} v_k v_k^{\mathrm{T}}] | \tilde{x}_{k-1}\} \\ &= tr\{[(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k)^{-1} - H_k^{\mathrm{T}} (H_k(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k \\ &\quad+ \hat{Q}_k) H_k^{\mathrm{T}} + R_k)^{-1} H_k] Q_k\} + tr(R_k^{-1} H_k \hat{P}_k H_k^{\mathrm{T}}). \end{aligned} \tag{56}$$

Applying Lemma 2 we obtain

$$(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k)^{-1} - H_k^{\mathrm{T}} [H_k(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k) H_k^{\mathrm{T}} + R_k]^{-1} H_k > 0. \tag{57}$$

Now we have $\mu_k > 0$. From (28) and (30)–(35)

$$\begin{aligned} \mu_k &\leqslant tr[(\beta_k F_k \hat{P}_{k-1} F_k^{\mathrm{T}} \beta_k + \hat{Q}_k)^{-1} Q_k] + tr(R_k^{-1} H_k \hat{P}_k H_k^{\mathrm{T}}) \\ &\leqslant \frac{q_{\max}}{\hat{q}_{\min}} \cdot L + \frac{h_{\max}^2 p_{\max}}{r_{\min}} \cdot M \\ &\triangleq \mu_{\max}. \end{aligned} \tag{58}$$

Therefore, it is able to apply Lemma 1 with (48), (53) and (58). Consequently, the inequality

$$E[V_k(\tilde{x}_k)] - V_{k-1}(\tilde{x}_{k-1}) \leqslant \mu_{\max} - \lambda_{\min} V_{k-1}(\tilde{x}_{k-1}) \tag{59}$$

is fulfilled to guarantee the boundedness of $\tilde{x}_k$

$$E\{\|\tilde{x}_k\|^2\} \leqslant \frac{p_{\max}}{p_{\min}} E\{\|\tilde{x}_0\|^2\} (1 - \lambda_{\min})^k + \frac{\mu_{\max}}{p_{\min}} \sum_{i=1}^{k-1} (1 - \lambda_{\min})^i. \quad \square \tag{60}$$

Remark 1. (1) β_k is an unknown instrumental diagonal matrix introduced to evaluate the error introduced by the UT. And stability of the algorithm do not depend on the magnitude of β_k. According to (52), although different β_k may change the value of $\lambda_{\min}$, the last item in (48) will remain negative and the relationship shown in (59) will not be changed so long as the matrix $\hat{Q}_k$ is positive definite. In other words, if $\hat{Q}_k \geqslant \hat{q}_{\min} I$ is fulfilled, an upper bound on $E[V(\xi_k)|\xi_{k-1}] - V(\xi_{k-1})$ can be obtained and the estimation error will remain bounded even for bad approximation to the nonlinear model.

(2) To ensure the stability of UKF, the matrices $\hat{Q}_k$ need to be positive define. From (23), as $\Delta P_{k|k-1}$ and $\delta P_{k|k-1}$ may be not positive definite matrices, extra additive matrix ΔQ_t should be introduced as a modification to the UKF so that $\hat{Q}_k \geqslant \hat{q}_{\min} I$ will be satisfied. Obviously, if $\Delta \hat{Q}_k$ is sufficiently large, condition (33) can always be fulfilled. This means that the modified UKF can tolerate high order error introduced during the UT by enlarging the noise covariance matrix. On the other hand, the precision of the algorithm also relates to the value of $\hat{Q}_k$ and the relation is shown in Section 4.2.

(3) Condition (35) is closely related to observability property of the linearized system and related discussion can be seen in (Reif et al. (1999)).

4. Error performance of the UKF

Cramér–Rao lower bound (CRLB) that provides a lower bound on the MSE is widely used to assess the performance of an estimator. In this section, CRLB is applied to evaluate the effectiveness of the modified UKF.

4.1. Cramér–Rao lower bound

Let $X_0^k=\{x_0, x_1, \ldots, x_k\}$ represent state vector series, $Y_0^k=\{y_0, y_1, \ldots, y_k\}$ noisy observations, $p(Y_0^k, X_0^k)$ the joint probability density of the pair (Y_0^k, X_0^k), and $\hat{x}_k$ a estimate of x_k. $\hat{x}_k$ is a function of Y_0^k. The CRLB on the estimation error has the form (Tichavsky, Muravchik, & Nehorai, 1998)

$$P_k \triangleq E\{[x_k-\hat{x}_k][x_k-\hat{x}_k]^{\mathrm{T}}\} \geqslant J_k^{-1}, \tag{61}$$

where J_k is the Fisher information matrix

$$J_k = E\left[-\frac{\partial^2 \ln p(Y_0^k, X_0^k)}{\partial x_k^2}\right]. \tag{62}$$

An efficient method for computing J_k recursively is given in Farina, Ristic, and Benvenuti (2002).

Proposition 1. *The sequence $\{J_k\}$ of posterior information matrices for estimating state vectors $\{x_k\}$ obeys the recursion*

$$J_k = D_k^{22} - D_k^{21}(J_{k-1}+D_k^{11})^{-1}D_k^{12}, \tag{63}$$

where

$$D_k^{11} = E\left\{-\frac{\partial^2 \ln p(x_k|x_{k-1})}{\partial x_{k-1}^2}\right\},$$

$$D_k^{12} = E\left\{-\frac{\partial^2 \ln p(x_k|x_{k-1})}{\partial x_k \partial x_{k-1}}\right\},$$

$$D_k^{21} = E\left\{-\frac{\partial^2 \ln p(x_k|x_{k-1})}{\partial x_{k-1} \partial x_k}\right\},$$

$$D_k^{22} = E\left\{-\frac{\partial^2 \ln p(x_k|x_{k-1})}{\partial x_k^2}\right\} + E\left\{-\frac{\partial^2 \ln p(y_k|x_k)}{\partial x_k^2}\right\}.$$

From Proposition 1, it follows that

Proposition 2. *Assume that the nonlinear filtering is applied to system* (1), *the CRLB is given by*

$$P_k \geqslant J_k^{-1}, \tag{64}$$

where

$$J_k = Q_k^{-1} + H_k^{\mathrm{T}} R_k^{-1} H_k - Q_k^{-1} F_k (J_{k-1} + F_k^{\mathrm{T}} Q_k^{-1} F_k)^{-1} F_k^{\mathrm{T}} Q_k^{-1}.$$

The proof of Proposition 2 can be found in Appendix B.

4.2. The mean square error of the UKF

In this subsection, the MSE for the UKF is derived and compared to the CRLB.

Theorem 2. *Assuming that the modified UKF for* (1) *is expressed as* (3)–(5), (20) *and* (7)–(11), *the MSE of the UKF-based estimation method is given by*

$$P_k = [\hat{Q}_k^{-1} + H_k^{\mathrm{T}} R_k^{-1} H_k - \hat{Q}_k^{-1} \beta_k F_k (\hat{P}_{k-1}^{-1} + F_k^{\mathrm{T}} \beta_k \hat{Q}_k^{-1} \beta_k F_k)^{-1} F_k^{\mathrm{T}} \beta_k \hat{Q}_k^{-1}]^{-1} + \Delta P_k, \tag{65}$$

where

$$\Delta P_k = (I - K_k H_k)(P_{k|k-1} - \hat{P}_{k|k-1})(I - K_k H_k)^{\mathrm{T}}. \tag{66}$$

The proof of Theorem 2 can be found in Appendix C.

Remark 2. Compared to the CRLB shown in (64), it is obvious that the MSE of the UKF (65) depends on the instrumental matrix β_k and the calculated covariance matrix $\hat{Q}_k$, while the CRLB is related to Q_k. If the error introduced by the UT is negligible, and $\hat{Q}_k = Q_k$, or equivalently, $\beta_k = I$, $\Delta P_k = 0$, (17) and (22) can be reduced to

$$\tilde{x}_{k|k-1} = F_k \tilde{x}_{k-1} + w_k, \tag{67}$$

$$\hat{P}_{k|k-1} = P_{k|k-1}. \tag{68}$$

Then the MSE of the UKF is equal to the CRLB. Thus, the UKF reaches the optimal performance if high order error is negligible and the difference between $\hat{Q}_k$ and Q_k is small enough. Clearly, setting high value of ΔQ_k may enlarge the difference between $\hat{Q}_k$ and $Q_k$and make the MSE of the modified UKF deviate from CRLB. Therefore, when the matrix $\hat{Q}_k$ is enlarged, stability of the modified UKF will be improved, but the precision may be decreased.

5. Numerical simulations

The results in the preceding two sections clarify the upper-bound and lower-bound of the UKF by (60) and (64), respectively. It is shown that the estimation error of the modified UKF is bounded if the extra positive definite matrix ΔQ_k is set properly and the nonlinear system to be observed satisfies appropriate conditions given in Theorem 1. These conditions exclude the requirement of small initial estimation error.

In order to show the efficiency of the modified UKF, it is applied to two example systems in comparison with the standard UKF and the EKF. Error performances of the filters are evaluated with Monte Carlo simulations.

Example 1. The first numerical example considered in this section is a fifth-order nonlinear model given by (1) with

(Boutayeb & Aubry, 1999)

$$f(x_k,u_k) = \begin{bmatrix} x_{1,k-1}+h\left(-\gamma x_{1,k-1}+\dfrac{K}{T_r}x_{3,k-1}+Kpx_{5,k-1}x_{4,k-1} \right. \\ \left. +\dfrac{1}{\sigma L_x}u_{1,k}\right) \\ x_{2,k-1}+h\left(-\gamma x_{2,k-1}-Kpx_{5,k-1}x_{3,k-1}+\dfrac{K}{T_r}x_{4,k-1} \right. \\ \left. +\dfrac{1}{\sigma L_s}u_{2,k}\right) \\ x_{3,k-1}+h\left(\dfrac{M}{T_r}x_{2,k-1}-\dfrac{1}{T_r}x_{3,k-1}-px_{5,k-1}x_{4,k-1}\right) \\ x_{4,k-1}+h\left(\dfrac{M}{T_r}x_{2,k-1}+px_{5,k-1}x_{3,k-1}-\dfrac{1}{T_r}x_{4,k-1}\right) \\ x_{5,k-1}+h\left(\dfrac{pM}{JL_r}(x_{3,k-1}x_{2,k-1}-x_{4,k-1}x_{1,k-1})-\dfrac{T_r}{J}\right) \end{bmatrix}, \tag{69}$$

$$H_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{70}$$

The covariance matrices of w_t and v_t are

$$Q_k = 0.01^2 I_5, \quad R_k = 0.1^2 I_2.$$

The time constant T_r and the parameters σ, K, and γ are defined as follows:

$$T_r = \frac{L_r}{R_r}, \quad \sigma = 1-\frac{M^2}{L_s L_r}, \quad K = \frac{M}{\sigma L_s L_r},$$
$$\gamma = \frac{R_s}{\sigma L_s} + \frac{R_r M^2}{\sigma L_s L_r^2}.$$

The numerical values are set to

$$R_s=0.18, \quad R_r=0.15, \quad M=0.068, \quad L_s=0.0699,$$

$$L_r=0.0699, \quad J=0.0586, \quad T_l=10, \quad p=1, \quad h=0.0001.$$

The input signals are

$$u_{1,k} = 350\cos(0.003k), \quad u_{2,k} = 300\sin(0.003k).$$

The initial conditions for the system and the filters are

$$x_{1,0} = x_{2,0} = x_{3,0} = x_{4,0} = x_{5,0} = 0,$$

$$\hat{x}_{1,0} = 200, \quad \hat{x}_{2,0} = 200, \quad \hat{x}_{3,0} = 50,$$
$$\hat{x}_{4,0} = 50, \quad \hat{x}_{5,0} = 350.$$

And the initial covariance matrix is chosen as

$$\hat{P}_0 = 100^2 I_5.$$

The standard UKF is performed according to Eqs. (3)–(11), and in the modified UKF algorithm, Eq. (20) is used instead of (6). To fulfill the assumption of Theorem 1 shown in (33), the extra additive matrix is designed as $\Delta Q_k = 0.005^2 I_5$ by experiment. The discrete-time EKF equations can be found in literatures (see, e.g., Goodwin & Sin, 1984). The simulation result of the modified UKF is depicted in Fig. 2. The trajectory of the state $x_{5,k}$ and the estimated state $\hat{x}_{5,k}$, that cannot be obtained directly from the measurements, are plotted versus k.

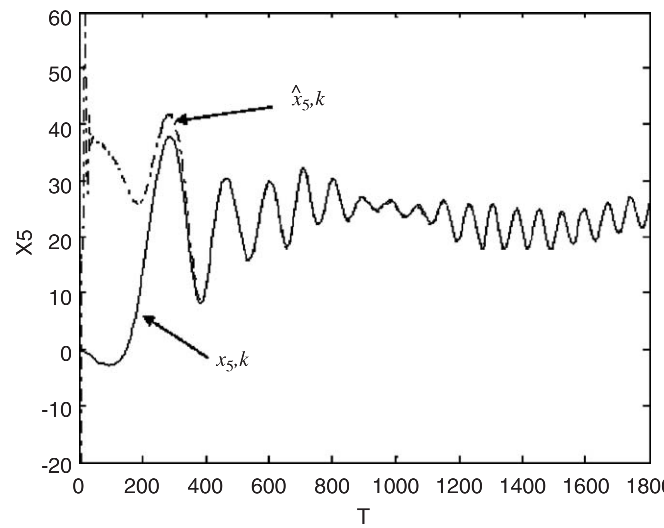

Fig. 2. State $x_{5,k}$ and its estimate $\hat{x}_{5,k}$ for example system 1.

Then Monte Carlo simulations are carried out for 10 times, yielding 10 realizations of state and measurement trajectories. The standard UKF, the modified one, as well as the EKF are used to estimate the state vector respectively, and each of the filters generates 10 trajectories of state estimates. The MSEs are calculated by averaging the square error between $x_{5,k}$ and $\hat{x}_{5,k}$ over 10 trails. The error performance of the three filters is compared to the theoretically derived CRLB, which is calculated using (64) and the initial information matrix for computation of the CRLB is set to $J_0 = \hat{P}_0^{-1}$.

In order to illustrate the effect of ΔQ_k on error performance, another simulation is performed with extra additive matrix $\Delta Q_k = 0.5^2 I_5$. The standard deviations of the estimation error for various filters and the square root of the diagonal element of CRLB are shown in Fig. 3.

As can be seen from Figs. 2 and 3, with a relevant choice of instrumental matrix ΔQ_k, the error of the modified UKF remains bounded while the standard UKF fails to converge. It indicates that, when the standard UKF is used, or $\Delta Q_k = 0_5$, condition (33) may be violated and the estimate error is divergent. By introducing sufficient large additive matrix $\Delta Q_k = 0.05^2 I_5$, the condition (33) is satisfied, and the stability of UKF is ensured. On the other hand, the modified UKF with $\Delta Q_k = 0.05^2 I_5$ follows the theoretical CRLB bound closely, while the standard deviation of error using the modified UKF with the choice of $\Delta Q_k = 0.5^2 I_5$ is much worse than the theoretical bound. It confirms that setting high values of ΔQ_k leads to large estimation error. In order to improve the performance of the modified UKF, the extra positive definition matrix ΔQ_k should be set properly.

The error curve of the EKF cannot be found in Fig. 3 because it jumps out of the scope of the figure immediately after the simulation starts. It shows that the EKF is not stable in this situation.

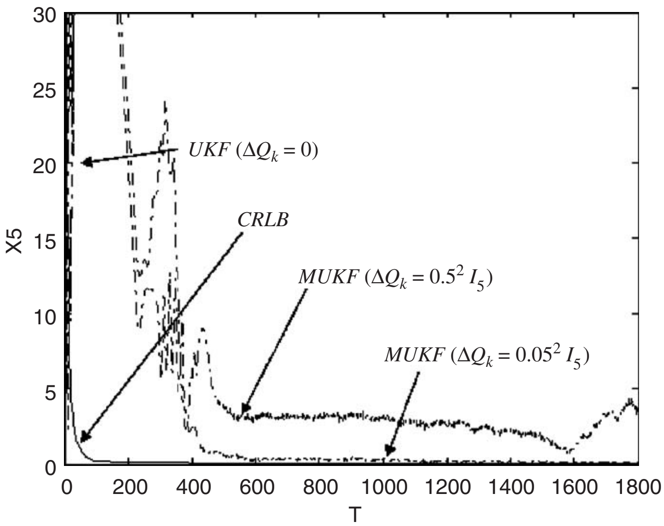

Fig. 3. Square root of CRLB corresponding to $\tilde{x}_{5,k}$ and MSE of the filters (MUKF denotes the modified UKF).

Example 2. The second numerical example concerns another nonlinear stochastic system (Reif et al., 1999)

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} x_{1,k-1}+\tau x_{2,k-1} \\ x_{2,k-1}+\tau[-x_{1,k-1}+(x_{1,k-1}^2+x_{2,k-1}^2-1)x_{2,k-1}] \end{bmatrix} + w_k, \tag{71}$$

$$y_k = x_{1,k} + v_k, \tag{72}$$

where $\tau = 0.001$. Covariance matrices of w_t and v_t are

$Q_k = 0.003^2 I_2, \quad R_k = 0.001^2.$

The initial conditions are taken as

$x_{1,0} = 0.8, \quad x_{2,0} = 0.2, \quad \hat{x}_{1,0} = 2.3, \quad \hat{x}_{2,0} = 2.2.$

And the initial covariance matrix is chosen as

$\hat{P}_0 = I_2.$

To fulfill the assumption of Theorem 1 shown in (33), in the modified UKF, the extra matrix is designed as $\Delta Q_k = \mathrm{diag}([0.015^2\ 0.02^2])$ by experiment. The state $x_{2,k}$ and estimate $\hat{x}_{2,k}$ are shown in Fig. 4.

The same Monte Carlo simulation procedure is applied to this nonlinear model. 10 trajectories are generated and each of the nonlinear filters is performed for 10 times. The error standard deviations of the UKF, the EKF, the modified UKF with different choices of ΔQ_k, and the CRLB are plot versus k in Fig. 5.

Again, the estimation error of standard UKF diverges, as shown in Fig. 5. Meanwhile, the error curve of the modified UKF is close to the CRLB. However, the deviation between the standard deviation of the modified UKF and the CRLB becomes significant when the extra additive matrix is enlarged to $\Delta Q_k = \mathrm{diag}([0.018^2\ 0.4^2])$. The plots confirm that with a sufficiently large matrix ΔQ_k, the modified UKF will be robust. However, setting very large value of ΔQ_k will make the MSE deviates from the CRLB.

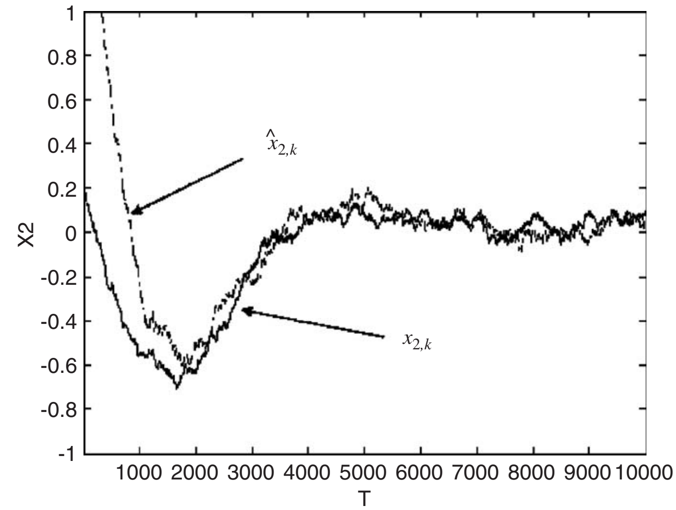

Fig. 4. State $x_{2,k}$ and its estimate $\hat{x}_{2,k}$ for example system 2.

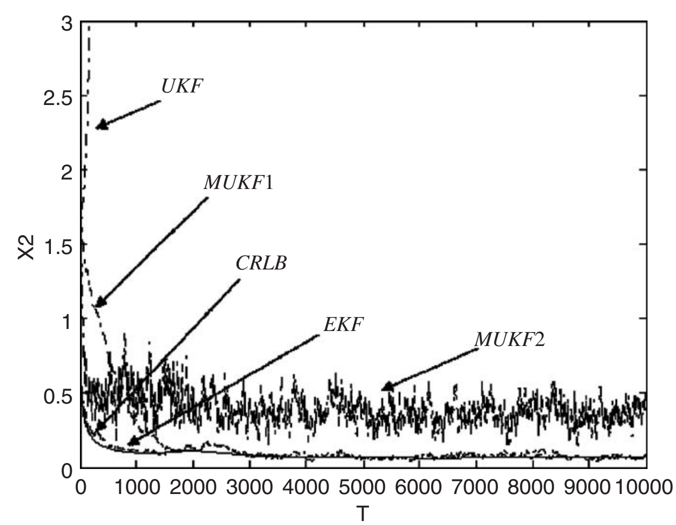

Fig. 5. Square root of CRLB corresponding to $\tilde{x}_{2,k}$ and MSE of the filters (MUKF1 denotes the modified UKF with $\Delta Q_k = \mathrm{diag}([0.015^2\ 0.02^2])$, and MUKF2 denotes the modified UKF with $\Delta Q_k = \mathrm{diag}([0.018^2\ 0.4^2])$).

In addition, the EKF appears to be an efficient estimator for this application as its error standard deviation practically equal to the square root of the CRLB.

6. Conclusion

The error behavior of the modified UKF is analyzed for estimation problems of nonlinear stochastic discrete-time systems with linear measurement equations. According to some standard results about the boundedness of stochastic processes, it is pointed out that, with appropriate choice of the extra positive definite matrix ΔQ_k, stability of the modified UKF may be ensured without the requirement of small initial estimation error. Furthermore, the CRLB is introduced as the performance measure of the error behavior and it is shown that the MSE performance of the modified UKF approaches the CRLB when the calculated covariance matrix $\hat{Q}_k$ is close to the real noise covariance matrix Q_k. However, the deviation becomes significant when the extra positive definite matrix ΔQ_k is set too large. Consequently, design of ΔQ_k can be seen as a tradeoff between stability and accuracy requirement. High performances of the modified UKF are shown through numerical examples under the worst initial conditions.

Further work is necessary to analyze the performance of the UKF with nonlinear measurement equation. Moreover, an adaptive scheme to adjust the matrices ΔQ_k in response to the changing environment is under investigation.

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Appendix A

Eq. (27) is easy to be verified according to the matrix inversion lemma (see, e.g., Lewis, 1986, Appendix A2, p. 347).

Appendix B

This appendix shows the proof of Proposition 2.

Proof. As the noise is assumed to be Gaussian, the conditional density functions can be expressed as

$$p(x_k|x_{k-1}) = \frac{1}{\sqrt{2\pi|Q_k|}} \mathrm{e}^{\{-1/2[x_k - f(x_{k-1})]^{\mathrm{T}} Q_k^{-1} [x_k - f(x_{k-1})]\}}, \tag{73}$$

$$p(y_k|x_k) = \frac{1}{\sqrt{2\pi|R_k|}} \mathrm{e}^{\{-1/2[y_k - H_k x_k]^{\mathrm{T}} R_k^{-1} [y_k - H_k x_k]\}}. \tag{74}$$

According to Proposition 1

$$\begin{aligned} D_k^{11} &= -F_k^{\mathrm{T}} Q_k^{-1} F_k, \\ D_k^{12} &= -F_k^{\mathrm{T}} Q_k^{-1}, \\ D_k^{21} &= (D_k^{12})^{\mathrm{T}} = -Q_k^{-1} F_k, \\ D_k^{22} &= Q_k^{-1} + H_k^{\mathrm{T}} R_k^{-1} H_k. \end{aligned} \tag{75}$$

Thus we obtain the desired formula (64) from (61), (63). □

Appendix C

This appendix shows the proof of Theorem 2.

Proof. Substituting (10) into (61), and using (12), (13), (40), (66) we have

$$\begin{aligned} P_k &= E[\tilde{x}_k \tilde{x}_k^{\mathrm{T}}] = E[(\tilde{x}_{k|k-1} - K_k \tilde{y}_k)(\tilde{x}_{k|k-1} - K_k \tilde{y}_k)^{\mathrm{T}}] \\ &= \hat{P}_{k|k-1} - \hat{P}_{k|k-1} H_k^{\mathrm{T}} K_k^{\mathrm{T}} - K_k H_k \hat{P}_{k|k-1} \\ &\quad + K_k (H_k \hat{P}_{k|k-1} H_k^{\mathrm{T}} + R_k) K_k^{\mathrm{T}} + \Delta P_k. \end{aligned} \tag{76}$$

Substituting (39) into (76) and applying the matrix inversion lemma.

$$\begin{aligned} P_k &= \hat{P}_{k|k-1} - \hat{P}_{k|k-1} H_k^{\mathrm{T}} \\ &\quad \times (H_k \hat{P}_{k|k-1} H_k^{\mathrm{T}} + R_k)^{-1} H_k \hat{P}_{k|k-1} + \Delta P_k \\ &= (\hat{P}_{k|k-1}^{-1} + H_k^{\mathrm{T}} R_k^{-1} H_k)^{-1} + \Delta P_k. \end{aligned} \tag{77}$$

Substituting (22) into (77) and using the matrix inversion lemma again, we can find that the MSE of the UKF-based estimation method is given by (65). □

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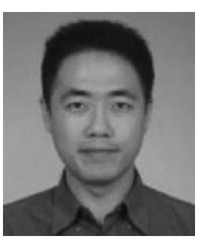

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