# Multilateral War of Attrition with Majority Rule

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# Motivation

- Reputational bargaining pioneered by Abreu & Gül (2000)
  - players can benefit from building a reputation for being stubborn
  - build on Hendricks & Weiss & Wilson (1988)
- Limitation: bilateral
- Ultimate goal: extend Abreu & Gül (2000) to <u>multilateral reputational bargaining</u>
  - companion paper in progress: in multilateral bargaining with majority rule, players may benefit from a reputation for being compliant so as to be included
  - intemediate step: **this paper** extends bilateral war of attrition, characterized by Hendricks & Weiss & Wilson (1988), to <u>multilateral war of attrition</u>

# Motivation

- A simple and relevant extension to multilateral bargaining
  - agreement requires approval by a majority
  - a particular party, the chair, must approve the aggreement
  - e.g., committee decisions under majority rule, with a president having the veto
- The demands of all parties are exogenous and incompatible
  - competiting players decide whether & when to concede to the chair
  - the chair decides whether & when & to whom to concede

# Observations

Forces at work in bilateral war of attrition

1. Being conceded to by others is better than conceding to others
2. Earlier agreement is better than later agreement (holding fixed the agreement)

Additional force for competing players in multilateral war of attrition

- **Competition motive**: being included in the agreement is better than excluded
  - *"Conceding is better than the other two players reaching the agreement without me"*

# Questions

How would introducing a competitor and a chance of being excluded from the agreement affect the bargaining?

- Under what conditions the agreement is reached with a delay?
- How does the bargaining outcome vary with three players' demands, impatience?

# Model

Three players decide how to split one unit of surplus with majority rule

- **Time horizon**: $t \in [0, \infty]$
- **Players**: $N = \{0, 1, 2\}$
  - majority rule: an agreement is reached by two players
  - player 0 (the chair/ "she"): must be part of the agreement
  - player $i = 1, 2$ (competing players/ "he"): one and only one is part of the agreement
- **Exogenous demand**: $\alpha_i \in (0, 1)$ for $i \in N$
  - incompatible demands: $\alpha_0 + \alpha_i > 1$ for $i = 1, 2$

# Model

- **Actions**: the three players simultaneously choose their actions
  - player $i$: whether to concede or continue
  - player 0: whether to concede or continue; to whom to concede
  - the game ends as soon as at least one player concedes
- **Payoffs**
  - if player $i$ concedes to player $j$, with player $k$ being excluded from the agreement:

$$(u_i, u_j, u_k) = (1 - \alpha_j, \alpha_j, 0)$$

  - if more than one player concedes simultaneously: the outcome is chosen uniformly
  - discount rate: $r_i > 0$ for $i \in N$

# Model: Strategies

- Pure strategy: indexed by the earliest concession time for player $i \in N$
  - player $i$: $t_i \in [0, \infty]$
  - player 0: $(t_0, \kappa) \in [0, \infty] \times \{1, 2\}$ where $\kappa$ denotes to whom the chair concedes to
  - $t_i = \infty$: never concede

- Mixed strategy
  - $G_i : [0, \infty] \to [0, 1]$ where $G_i(t)$ is the probability that player $i$ concedes by time $t$
  - $G_0 = (G_{0,1}, G_{0,2}) : [0, \infty] \to [0, 1]^2$
    - $G_{0,\kappa}(t)$ is the probability that player 0 concedes to player $\kappa$ by time $t$
    - $G_{0,1}(t) + G_{0,2}(t) \leq 1$ for all $t \in [0, \infty]$

# Model: Equilibrium

Nash equilibrium strategy profile $(G_1, G_2, (G_{0,1}, G_{0,2}))$

We distinguish two types of equilibria

- **Immediate-agreement equilibrium**: the game ends at time $t = 0$ with certainty
- **Delay equilibrium**: the game ends later than time $t = 0$ with positive probability

# Characterization of Immediate-Agreement Equilibrium

**Proposition (Immediate-Agreement Equilibrium)**

There is a continuum of immediate-agreement Nash equilibria.

(i) If $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2 < 2(1 - \alpha_0)$, then in every immediate-agreement equilibrium, player 1 and player 2 concede at time $t = 0$ with certainty and player 0 concedes later.

(ii) If $\alpha_1 = \alpha_2 \geq 2(1 - \alpha_0)$, then in any immediate-agreement equilibrium, either player 1 and player 2 concede at time $t = 0$ with certainty and player 0 concedes later, or player 0 concedes at time $t = 0$ with certainty and player 1 and player 2 concede later.

# If some competing player concedes immediately with certainty...

- The chair waits to be conceded to at the start of the game
- The other competing player is surely excluded if he does not concede immediately $\Rightarrow$ the other competing player also concedes immediately
- This equilibrium always exists, and it is the unique pure-strategy equilibrium outcome

# If the chair concedes immediately with certainty...

- Neither of competing players concedes immediately
- The chair must concede to both players with positive probability
  $\Rightarrow$ the demands of competing players must be equal
- The wait is better for competing players than the immediate concession
  $\Rightarrow$ the demands of the competing players must be sufficiently high
- This equilibrium exists if and only if $\alpha_1 = \alpha_2 > 2(1 - \alpha_0)$

# Characterization of Delay Equilibrium

**Lemma (Necessary Condition for Delay Equilibrium)**

If $\alpha_1 \neq \alpha_2$, then there does not exists a delay equilibrium.

- In what follows, assume $\alpha_1 = \alpha_2 = \alpha$ to characterize delay equilibria

# Characterization of Delay Equilibrium

Only two possible cases

1. All players gradually concede throughout the game
   - $G_1$, $G_2$, $G_{0,1}$ and $G_{0,2}$ are all strictly increasing over $(0, \infty]$ with no atom point
2. One and only one competing player does not concede over some interval(s)
   - the other competing player concedes to the chair with a constant hazard rate

# Sketch of Proof

1. Atom points do not overlap
   - concede later: $q_i(t)[q_{0,1}(t) + q_{0,2}(t)] = 0$ for $t \geq 0, i = 1, 2$
   - concede sooner: $q_1(t)q_2(t) = 0$ for $t > 0$
2. Suspense over a period
   - concede sooner: if my opponent(s) concede with zero probability, then me too
3. Game ends before any atom point $t > 0$
   - concede later $\Rightarrow$ a short suspense before the atom point $\Rightarrow$ contradiction
4. No suspense period of three players (otherwise, concede sooner)

# Characterization of Delay Equilibrium — Gradual Concession

**Proposition (Delay Equilibrium with Gradual Concession)**

Let $\lambda_i(t)$ be the hazard rate at time $t \in \mathbb{R}_+$ for player $i \in \{1,2\}$, and let $\lambda_{0,i}(t) = G'_{0,i}(t)/[1 - G_{0,1}(t) - G_{0,2}(t)]$ for any $t$. The mixed strategy profile $(G_1, G_2, (G_{0,1}, G_{0,2}))$ is a Nash equilibrium profile in which both competing players concede throughout the game if and only if the following conditions hold:

(i) $G_1, G_2, G_{0,1}$ and $G_{0,2}$ are continuous over $(0, \infty]$.

(ii) $G_1(0) = G_2(0) = 0$.

(iii) $\frac{\min\{G_{0,1}(0), G_{0,2}(0)\}}{\max\{G_{0,1}(0), G_{0,2}(0)\}} \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1}$.

(iv) $\lambda_1(t) + \lambda_2(t) = \mu \equiv \frac{(1-\alpha)r_0}{\alpha_0+\alpha-1}$.

(v) $\lambda_{0,j}(t) = \frac{1-\alpha_0}{\alpha}(\lambda_i(t) + r_j + \rho)$ for $j \neq i$, where $\rho \equiv \frac{\mu+r_1+r_2}{\frac{\alpha}{1-\alpha_0}-2}$.

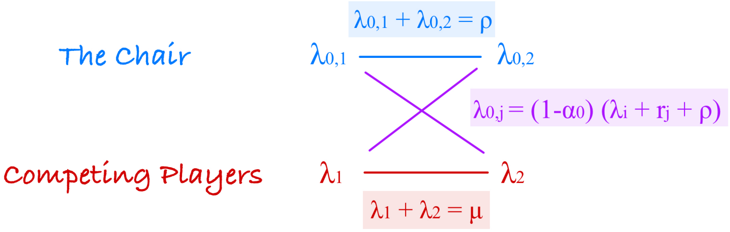
# Multiplicity of Gradual Concession Equilibria

Two sources of indeterminacy

1. At the start of the game, player 0's strategy is not fully pinned down
   - as long as conceding to both competing players with "balanced" probabilities:

$$\frac{\min\{G_{0,1}(0), G_{0,2}(0)\}}{\max\{G_{0,1}(0), G_{0,2}(0)\}} \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1}$$

2. For any $t > 0$, there is only one degree of freedom in $(G_1, G_2, (G_{0,1}, G_{0,2}))(t)$



# Compared to Bilateral War of Attrition

Introduce a competitor $\Rightarrow$ asymmetric voting power & a chance of exclusion

- Conditions to support equilibria:

| **Equilibrium Type** | **Two Players** | **Three Players (majority)** |
|---|---|---|
| *Immediate Concession by Player i* | Always | Always |
| *Immediate Concession by Player* 0 | Always | $\alpha_1 = \alpha_2 \geq 2(1 - \alpha_0)$ |
| *Delay Equilibrium* | Always | $\alpha_1 = \alpha_2 > 2(1 - \alpha_0)$ |

  - Immediate concession by the chair or concession with delay is harder

- The chair's welfare:
  - weakly better off if $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2 < 2(1 - \alpha_0)$
  - weakly worse off if $\alpha_1 = \alpha_2 > 2(1 - \alpha_0)$

- Concession rate: accelerated

# Comparative Statics

Recall that for any gradual concession,

- the aggregate concession rate of competing players: $\mu = (1-\alpha)r_0/(\alpha_0+\alpha-1)$
- the concession rate of player 0 is $\rho = (\mu + r_1 + r_2)/(\frac{\alpha}{1-\alpha_0} - 2)$

A delayed agreement is reached faster when

- players' demands are less incompatible; or
- players become more impatient

# Main Results: Equilibrium Characterization

We characterize three-player war of attrition game with majority rule

- **Immediate-agreement equilibrium**
  - always exists: competing players concede at the start and the chair concedes later
  - $\alpha_1 = \alpha_2 > 2(1 - \alpha_0)$: the chair concedes at the start and the competing players concede later
- **Delay equilibrium**

# Main Results: Equilibrium Characterization

We characterize three-player war of attrition game with majority rule

- **Immediate-agreement equilibrium**
- **Delay equilibrium**
    - exists if and only if $\alpha_1 = \alpha_2 > 2(1 - \alpha_0)$
    - $\lambda_1(t) + \lambda_2(t) = \mu$ throughout the game
    - there exist equilibria in which the competing players alternate in holding out
    - if no competing players holds out over time, then $\lambda_{0,1}(t) + \lambda_{0,2}(t) = \rho$ for all $t > 0$

# Bibliography

**Abreu & Gül (2000)**, Ellingsen & Miettinen (2008), **Hendricks & Weiss & Wilson (1988)**, Ma (2022), Miettinen & Vanberg (2020), Miettinen (2022), Osborne & Rubinstein (1994), Osborne (2004), Özyurt (2015), Royden & Fitzpatrik (2010)