Multilateral War of Attrition with Majority Rule* (preliminary and incomplete)

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Abstract

We analyze a multilateral war of attrition game with majority rule in continuous time. A chair and two competing players decide how to split one unit of surplus. Players have exogenously given demands that are incompatible. At each instance, the players simultaneously choose whether to concede or continue. The chair can concede to either of the two competing players, but the competing players can concede only to the chair. An agreement is reached when at least one player concedes. We characterize the equilibria of this game and establish the necessary and sufficient conditions under which equilibria with delay exist. In contrast to the bilateral case, delay equilibria are less likely and must involve an asymmetry: they exist only when the demands of the competing players are identical and sufficiently large, and only the chair can concede with a strictly positive probability at the start of the game. This has the surprising implication that the chair may be worse off when bargaining with two players under majority rule than with one who has veto power.

1 Introduction

In early 2023, Kevin McCarthy was elected as the speaker of the House after a prolonged election. An article published in The Spectator observes that the election process looked like a "war of attrition" game among the House Republications.[1] War of attrition is a concession

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[1] Charles Lipson, January 4, 2023, "Kevin McCarthy's war of attrition", https://thespectator.com/topic/kevin-mccarthys-war-of-attrition/

game with a large number of applications and is studied widely in economics and political science.[2] Two aspects of the above example distinguish it from the standard war of attrition games studied in the literature. First, it is a multilateral war of attrition with majority rule, and second, one of the players (Kevin McCarthy in this case) must either concede or be conceded to. We study a multilateral war of attrition game with these two features.

Specifically, we analyze a continuous-time game in which three players are deciding how to split one unit of surplus. We assume that one of the players (the chair) must be included in the agreement, but only one of the remaining two players (competing players) is required to reach an agreement. Each player has an exogenously given demand, and the sum of the demands of every potential winning coalition (i.e., the sum of the demands of the chair and each other competing player) exceeds one. Thus, it is not possible to reach an agreement that satisfies the demands of all members of a winning coalition that supports the agreement. The game ends as soon as at least one player concedes. At any moment, the players simultaneously choose whether to concede by accepting the demand of another player or continue, given that the game has not ended thus far. Since any winning coalition must include the chair, the competing players can only concede to the chair, while the chair can concede to either of the two competing players. In addition to the example in our opening paragraph and various other political applications, the model can apply to situations in which a single buyer is negotiating with two sellers.

We first characterize the equilibria in which agreement is reached immediately. We show that there is a continuum of such equilibria (Proposition 1 and Proposition 2). When the demands of the competing players are not equal to each other, or when their demands are sufficiently small, in every immediate-agreement equilibrium, both of the competing players concede at the beginning of the game with certainty, and the chair concedes later. When the demands of the competing players are equal to each other and are sufficiently large, the type of equilibria just described exists, but there is also another type n which the chair concedes at the beginning of the game, and the competing players concede later.

We then turn to characterizing equilibria with delay. We show that when competing players' demands differ, delay equilibria cannot exist (Lemma 2). As such, to characterize the equilibria with delay, we assume the demands of the competing players are identical. We show that if the game does not end immediately, while the total probability of concession by the chair has a constant hazard rate, the hazard rates of concession to each of the competing players can vary over the course of the game (Proposition 3). Similarly, the hazard rates of concession by the competing players to the chair can also vary throughout the game. We show that there exist equilibria in which the competing players (one at a time) have rates of concession arbitrarily close to zero. In such equilibria, the competing players alternate in "holding out", i.e., not

[2] For example, price wars (Fudenberg and Tirole, 1986), bargaining (Ordover and Rubinstein, 1986, Osborne, 1985), fiscal policy negotiations (Alesina and Drazen, 1991), litigation disputes (McAfee, 2009), protests (Cai, 2016), boycotts (Egorov and Harstad, 2017)

conceding to the chair over non-trivial time intervals.

In contrast to the bilateral case (Hendricks, Weiss and Wilson (1988)), delay equilibria are less likely and must involve an asymmetry: they exist only when the demands of the competing players are identical and sufficiently large, only the chair can concede with a strictly positive probability at the start of the game. This has the surprising implication that the chair may be worse off when bargaining with two players under majority rule than with one who has veto power.

For delay equilibria in which all players gradually concede throughout the game, we find that the magnitude of the interest incompatibility, measured by the total demands of two players in the agreement minus the total surplus to be divided, decreases the concession rate for the chair and the aggregate concession rate for competing players. This captures the sense that the agreement is harder to achieve with more intense interest conflict. We also show that these concession rates strictly increase in the chair's impatience; in particular, the concession rate for the chair also strictly increases in both competing players' impatience. Furthermore, the chair obtains the same payoff in the bilateral war of attrition and the three-player war of attrition, whereas competing players obtain strictly lower payoffs in the three-player war of attrition due to the bargaining competition.

The outline of the rest of the paper is as follows. In the next subsection, we review the related literature. In Section 2, we present the model and define equilibrium. In Section 3, we show that a unique pure strategy equilibrium exists. We also show a continuum of mixed strategy immediate agreement equilibria and characterize these equilibria. In Section 4, we establish necessary conditions for the existence of delay equilibria and then characterize the delay equilibria in which all players gradually concede over time. In Section 5, we compare the results to those of a bilateral war of attrition game. In Section 6, we conclude.

1.1 Literature

The war of attrition was introduced by John Maynard Smith in the context of evolutionary games (Smith, 1974; Bishop and Cannings, 1978) and since then has been applied to a wide array of problems in economics, in particular, in bargaining. The first complete characterization of Nash equilibria in the bilateral war of attrition with complete information has appeared in Hendricks, Weiss and Wilson (1988). Our goal is to fully characterize Nash equilibria in the multilateral war of attrition arising in bargaining situations with majority rule and a veto player.

Our paper is related to the literature on reputational bargaining that takes the war of attrition as an essential ingredient. This literature is initiated by Myerson (1991) and Abreu and Gül (2000).[3] Here, we only discuss the most closely related papers. Özyurt (2015) studies a

[3]For a complete overview, we refer the reader to an exceptional survey by Fanning and Wolitzky (2022).

model of bargaining between a buyer and two sellers of an identical good. Similar to our model, there is an asymmetry between the players: the buyer must be included in any agreement while only one of the sellers is required to strike a deal. However, the model restricts which seller can reach an agreement at any given time: the bargaining procedure consists of alternating sequences of bilateral negotiations with the possibility for the buyer to switch between sellers at a cost. By contrast, our bargaining procedure is always multilateral: both competing players can concede to the chair at any time, and the chair can concede to any competing player at any time.

Ma (2022) studies the effect of reputational concerns in a Baron and Ferejohn (1989) model of legislative bargaining with three players and majority rule. This time, any two players can form a winning coalition, and the players differ only in the probability of being a non-compromising type. Ma (2022) constructs a class of equilibria in which obtaining a reputation for being a non-compromising type guarantees that such a player is excluded from the winning coalition. In turn, the threat of exclusion forces the rational players to end the game immediately (and thus quickly reveal the irrational types).

Kambe (2019) studies an incomplete information multilateral war of attrition in which a single exit is required to end the game. Kambe (2019) assumes that the payoffs of the players who remain do depend on the identity of the player who exits. He shows that the equilibrium is unique when each player has a positive probability of being a noncompromising type. A natural application of this model is a provision of a public good, for example, a group's selection of a volunteer. In contrast, we allow the possibility of being excluded from the winning coalition when dividing a surplus to study coalition-building, and hence, in our setting, unlike in Kambe (2019), the payoffs of the players who remain do depend on the identity of the player who yields first. In addition, our setting features asymmetry among players regarding who can yield to whom. Similar to our model, Kambe (2019) shows that some players may wait in equilibrium while other players exit at positive rates. However, the combined exit rates are weakly decreasing instead of being constant.

In an earlier paper, Bulow and Klemperer (1999) consider a generalized multilateral war of attrition game with N prizes and $N + K$ players where K players must exit for the game to end. In our setting, there is only one prize, but there are three players. Thus, their setup can be viewed as bargaining under the unanimity rule, while ours is bargaining under the majority rule with a veto player.

Finally, several researchers have investigated the role of strategic commitments that do not rely on incomplete information.[4] Ellingsen and Miettinen (2008) study strategic commitments that are both observable and revocable. They show that incompatible commitments arise as a robust prediction. Ellingsen and Miettinen (2014) soften this prediction in a dynamic model with commitment decay and demonstrate that the agreements are reached following some

[4]For an excellent review of this literature, see Miettinen (2022).

delay. In a mutlilateral setting, Miettinen and Vanberg (2020) show that commitment causes delay only under the unanimity rule, but delay disappears under any generalized majority rule, even the most demanding one (all-but-one).

2 The Model

Consider the following model of bargaining. Players in the set $N = \{0, 1, 2\}$ are deciding how to split one unit of surplus. Player 0 must receive a share of the surplus, and at most two players can receive positive shares. We call player 0 the *chair*, and call player 1 and 2 *competing players*. Each player $i \in N$ has an exogenously given demand $\alpha_i \in (0, 1)$. These demands satisfy $\alpha_0 + \alpha_i > 1$ for $i = 1, 2$, that is, the demand of player 0 is incompatible with the demand of any other player.

The game is played in continuous time with player i discounting future at rate $r_i > 0$. In each period $t \geq 0$, players simultaneously choose whether to concede or continue. Players 1 and 2 ("he") can only concede to player 0 ("she"), while player 0 chooses to concede to either of the other two players. The game ends as soon as at least one player concedes. There are 12 action profiles, with all but one having at least one player conceding and the game ending. These action profiles can be written as: $(0 \to 1)$, $(0 \leftarrow 1)$, $(0 \leftrightarrow 1)$, $(0 \to 2)$, $(0 \leftarrow 2)$, $(0 \leftrightarrow 2)$, $(1 \to 0 \to 2)$, $(1 \to 0 \leftarrow 2)$, $(1 \leftarrow 0 \leftarrow 2)$, $(1 \leftrightarrow 0 \leftarrow 2)$, $(1 \to 0 \leftrightarrow 2)$, and (none concedes).

Fix $t \geq 0$ and consider a pair of players who can divide the surplus among themselves, i.e., fix $i = 1, 2$ and consider the pair $(0, i)$. Suppose no concession has been made before time t and that $j \neq i$ does not concede at t. If player 0 concedes to i and player i does not concede, then player i receives his claim α_i, and player 0 receives the remainder of surplus $1 - \alpha_i$. If player 0 does not concede to any player and player i concedes, then player 0 gets her claim α_0, and player i gets $1 - \alpha_0$. If player 0 concedes to j and player i does not concede, then player i gets 0, and player 0 gets $1 - \alpha_j$. These are *base* outcomes. If more than one player concedes at t, the outcome is chosen uniformly from the base outcomes determined by concessions.

A pure strategy of player i specifies for each time t whether to concede (and to which player if $i = 0$) or continue given that no player has conceded before time t. Since the game ends after the first concession, all pure strategies of player i with the same earliest concession time are payoff-equivalent. Therefore, we can index each pure strategy by the earliest concession time, i.e., let t_i be a pure strategy such that t_i is the earliest time at which player $i = 1, 2$ concedes, and write $t_i = \infty$ if player i never concedes. Similarly, let (t_0, i) be a pure strategy of player 0 such that t_0 is the earliest time she concedes to player $i = 1, 2$. Again, $(t_0, i) = (\infty, i)$ if player 0 never concedes to i.

For any pure strategy profile $\boldsymbol{t} = ((t_0, \kappa), t_1, t_2) \in (\bar{\mathbb{R}}_+ \times \{1, 2\}) \times \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$, let $u_i(\boldsymbol{t})$ denote the payoff of player $i \in N$. Then,

$$u_0(\boldsymbol{t}) = \begin{cases} \alpha_0 e^{-r_0 \min\{t_1,t_2\}} & \text{if } t_0 > \min\{t_1,t_2\}, \\ (1-\alpha_\kappa)e^{-r_0 t_0} & \text{if } t_0 < \min\{t_1,t_2\}, \\ (\frac{1}{2}\alpha_0 + \frac{1}{2}(1-\alpha_\kappa))e^{-r_0 t_0} & \text{if } t_0 = \min\{t_1,t_2\} < \max\{t_1,t_2\}, \\ (\frac{2}{3}\alpha_0 + \frac{1}{3}(1-\alpha_\kappa))e^{-r_0 t_0} & \text{if } t_0 = t_1 = t_2, \end{cases}$$

and for $i = 1,2$ and $j \neq i$

$$u_i(\boldsymbol{t}) = \begin{cases} \alpha_i e^{-r_i t_0} & \text{if } t_0 < \min\{t_1,t_2\} \text{ and } i = \kappa, \\ (1-\alpha_0)e^{-r_i t_0} & \text{if } t_i < \min\{t_0,t_j\}, \\ (\frac{1}{2}\alpha_i + \frac{1}{2}(1-\alpha_0))e^{-r_i t_0} & \text{if } t_0 = t_i < t_j \text{ and } i = \kappa, \\ \frac{1}{2}\alpha_i e^{-r_i t_0} & \text{if } t_0 = t_j < t_i \text{ and } i = \kappa, \\ \frac{1}{2}(1-\alpha_0)e^{-r_i t_0} & \text{if } t_0 = t_i < t_j \text{ and } i \neq \kappa, \text{ or } t_1 = t_2 < t_0, \\ (\frac{1}{3}\alpha_i + \frac{1}{3}(1-\alpha_0))e^{-r_i t_0} & \text{if } t_0 = t_1 = t_2 \text{ and } i = \kappa, \\ \frac{1}{3}(1-\alpha_0)e^{-r_i} t_0 & \text{if } t_0 = t_1 = t_2 \text{ and } i \neq \kappa, \\ 0 & \text{if } \min\{t_0, t_\kappa\} < t_i \text{ and } i \neq \kappa. \end{cases}$$

A mixed strategy of player $i = 0,1,2$ is a distribution over pure strategies. Denote a mixed strategy of player $i = 1,2$ by $G_i : \bar{\mathbb{R}}_+ \to [0,1]$, where $G_i(t)$ is the probability that player i concedes by time t. The probability that player i never concedes is given by $1 - \lim_{t\to\infty} G_i(t)$. Notice that we do not require $\lim_{t\to\infty} G_i(t) = 1$ and that G_i is weakly increasing and right-continuous by definition.

Similarly, denote a mixed strategy of player 0 by $G_0 = (G_{0,1}, G_{0,2})$ where $G_{0,i} : \bar{\mathbb{R}}_+ \to [0,1]$ and $G_{0,i}(t)$ denotes the probability that player 0 concedes to player i by time t. We assume that $G_{0,i}$ is weakly increasing and right-continuous for each $i = 1,2$, and that $G_{0,1}(t) + G_{0,2}(t) \leq 1$ for all t. Define $\tilde{G}_0(t) = G_{0,1}(t) + G_{0,2}(t)$. Then, $\tilde{G}_0(t)$ denotes the probability that player 0 concedes by time t, and the probability that player 0 never concedes is given by $1 - \lim_{t\to\infty} \tilde{G}_0(t)$.

Let $\mathcal{G}_i$ denote the set of mixed strategies for player $i = 0,1,2$. For any $(G_1, G_2) \in \mathcal{G}_1 \times \mathcal{G}_2$ and $t \geq 0$, let $U_0(t, \kappa, G_1, G_2)$ denote the expected utility of player 0 from conceding to player κ at time t with certainty when players 1 and 2 use mixed strategies G_1 and G_2 respectively. Then,

$$U_0(t,\kappa,G_1,G_2) = \iint u_0((t,\kappa),t_1,t_2)\mathrm{d}G_1(t_1)\mathrm{d}G_2(t_2).$$

Likewise, for $i = 1,2$ and $j \neq i$, for any $(G_0, G_j) \in \mathcal{G}_0 \times \mathcal{G}_j$ and $t \geq 0$, let $U_i(t, G_0, G_j)$ denote the expected utility of player i from conceding at time t with certainty when the players 0 and

j use mixed strategies G_0 and G_j respectively. Then,

$$U_i(t, G_0, G_j) = \iint u_i((t_0, \kappa), t, t_j) \mathrm{d}G_0(t_0, \kappa) \mathrm{d}G_j(t_j).$$

In what follows, for any $f : \mathbb{R}_+ \to \mathbb{R}$, any $G : \bar{\mathbb{R}}_+ \to [0, 1]$ non-decreasing and right-continuous, and any $s < t$, let

$$\int_s^t f(v) \mathrm{d}G(v) = \lim_{\tau \uparrow t} \int_s^\tau f(v) \mathrm{d}G(v).$$

In other words, all integrals omit the mass point (if any) at the upper bound of integration.

We next define Nash equilibrium for this game and describe some of its properties. For ease of exposition, we focus on Nash equilibria, but we also argue that every Nash equilibrium outcome of the game studied here can be supported in a subgame perfect Nash equilibrium.

Definition 1. *A mixed strategy profile $(G_0, G_1, G_2) \in \mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2$ is a Nash equilibrium if*

(i) $\int U_0(t, \kappa, G_1, G_2) \mathrm{d}G_0(t, \kappa) \geq \int U_0(t, \kappa, G_1, G_2) \mathrm{d}\hat{G}_0(t, \kappa)$ *for all* $\hat{G}_0 \in \mathcal{G}_0$, *and*

(ii) $\int U_i(t, G_0, G_j) \mathrm{d}G_i(t) \geq \int U_i(t, G_0, G_j) \mathrm{d}\hat{G}_i(t)$ *for all* $i, j \in \{1, 2\}$, $i \neq j$, *and all* $\hat{G}_i \in \mathcal{G}_i$.

For the rest of this section, fix a mixed strategy profile (G_0, G_1, G_2) and let $q_i(t)$ denote the probability that player $i = 1, 2$ concedes exactly at time t, $q_{0,i}(t)$ denote the probability that player 0 concedes to player i exactly at time t, and $\tilde{q}_0(t)$ denote the probability that player 0 concedes exactly at time t, i.e., $\tilde{q}_0(t) = q_{0,1}(t) + q_{0,2}(t)$. Formally, $q_i(t) = G_i(t) - \lim_{\tau \uparrow t} G_i(\tau)$, and $q_{0,i}(t) = G_{0,i}(t) - \lim_{\tau \uparrow t} G_{0,i}(\tau)$. The probability that player i never concedes is denoted by $q_i(\infty)$, and the probability that player 0 never concedes is $q_0(\infty)$. Clearly, if G_i (respectively $G_{0,i}$) is continuous, then $q_i(t) = 0$ (respectively $q_{0,i}(t) = 0$) for all $t \geq 0$.

For player $i \in \{1, 2\}$, define $T_i^+ = \{t \in [0, \infty] : q_i(t) > 0\}$; and for player 0, define $T_{0,i}^+ = \{t \in [0, \infty] : q_{0,i}(t) > 0\}$. In what follows, we refer to a point $t \in T_i^+$ (or $t \in T_{0,i}^+$) as an atom point in concession time for player i (or for player 0 conceding to player i respectively).

Slightly abusing notation, let $G_i(\mathcal{T})$ denote the probability that player i assigns to a measurable set $\mathcal{T}$ according to a mixed strategy $G_i \in \mathcal{G}_i$. Likewise, let $G_{0,i}(\mathcal{T})$ denote the probability that player 0 assigns to a measurable set $\mathcal{T}$ according to a measure $G_{0,i}$.

Lemma 1. *Let (G_0, G_1, G_2) be a mixed strategy Nash equilibrium and fix player $i \in \{1, 2\}$. For all measurable $\mathcal{T} \subseteq [0, \infty]$, we have:*

(i) if there exists $s \notin \mathcal{T}$ such that $U_i(s, G_0, G_j) > U_i(\tau, G_0, G_j)$ for all $\tau \in \mathcal{T}$, then $G_i(\mathcal{T}) = 0$;

(ii) if there exists $s \notin \mathcal{T}$ or $j \neq i$ such that $U_0(s, j, G_1, G_2) > U_0(\tau, i, G_1, G_2)$ for all $\tau \in \mathcal{T}$, then $G_{0,i}(\mathcal{T}) = 0$.

Lemma 1 implies that, in equilibrium, if a player concedes with a strictly positive probability at time t, then no other concession time can yield her a strictly higher utility. In other words, every atom point in concession time must be a best response to the strategies of other players, analogous to a well-known mixed strategy Nash equilibrium characterization for games in which players have a continuum of actions.[5] In fact, an equilibrium mixed strategy must place zero weight on any measurable set of pure strategies (here, concession times) that are not best responses to the strategies of other players. By Lemma 1, we can derive the following corollary, which will be useful in characterizing equilibria with delayed agreement.

Corollary 1. *Let (G_0, G_1, G_2) be a mixed strategy Nash equilibrium and fix player $i \in \{1, 2\}$. For any time $s, \tau_1, \tau_2 \in \bar{\mathbb{R}}$ such that $\tau_1 < \tau_2$, we have:*

(i) if G_i is strictly increasing over (τ_1, τ_2), then $U_i(t, G_0, G_j) \geq U_i(s, G_0, G_j)$ for $t \in (\tau_1, \tau_2)$ almost everywhere;

(ii) if $G_{0,i}$ is strictly increasing over (τ_1, τ_2), then $U_0(t, i, G_0, G_j) \geq U_0(s, \kappa, G_0, G_{\bar{\kappa}})$ for $t \in (\tau_1, \tau_2)$ almost everywhere, and $j, \kappa, \bar{\kappa} \in \{1, 2\}$, $j \neq i$, $\kappa \neq \bar{\kappa}$.

3 Immediate-Agreement Equilibria

To begin, we show that both competing players concede at time $t = 0$ in every pure-strategy Nash equilibrium. To see why this is true, note first that there is always a pure-strategy equilibrium where the competing players concede immediately, and the chair never concedes. This is an equilibrium since the chair attains the highest utility achievable, i.e., α_0, and thus has no strictly profitable deviation; and neither competing player prefers to continue at $t = 0$ because $\frac{1-\alpha_0}{2} > 0$. Furthermore, the competing players concede immediately in every pure-strategy Nash equilibrium $((t_0, \kappa), t_1, t_2) \in (\bar{\mathbb{R}}_+ \times \{1, 2\}) \times \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$, as we now show. First, since $\alpha_i, \alpha_{0,i} < 1$ for both $i \in \{1, 2\}$, there exists a player who eventually concedes, that is, $\min\{t_0, t_1, t_2\} < \infty$. We denote the earliest concession time by $\underline{t}$. Next, player 0 cannot concede at $\underline{t}$. If $t_0 = \underline{t}$ and $t_j > \underline{t}$ for player $j \neq \kappa$, then j can deviate to $t_j = \underline{t}$ and get a payoff $\frac{1-\alpha_0}{2}$ instead of 0. And if $t_0 = \underline{t} = t_i$ for any $i \in \{1, 2\}$, then player 0 can deviate to $t_0 > \underline{t}$ and get a payoff α_0, which is greater than any payoff player 0 can receive by conceding at $t_0 = \underline{t}$. Further, the game must end immediately, that is, $\underline{t} = 0$. If $\underline{t} > 0$, then each player $i \in \{1, 2\}$ has a profitable deviation to $t_i < \underline{t}$ because it saves the waiting cost and guarantees the highest possible share $1 - \alpha_0$ that player i can achieve by conceding before player 0. Finally, player 1 and player 2 must concede simultaneously, that is, $t_1 = t_2$. If $t_i < t_j$, then player j has a profitable deviation to $t_j = 0$. It follows that in every pure-strategy Nash equilibrium, the competing players concede at $t = 0$, and player 0 concedes either later or never.[6] These results give us the following proposition.

[5] See, for example, Proposition 142.2 in Osborne (2004).

[6] There is a continuum of pure-strategy Nash equilibria that differ only in the action of player 0, namely the concession time t_0 and the player to whom she concedes.

Proposition 1 (Pure-Strategy Nash Equilibrium). *There is a unique pure-strategy Nash equilibrium outcome. In this outcome, only player 1 and player 2 concede at time $t = 0$. The equilibrium payoffs are α_0 for player 0 and $\frac{1}{2}(1-\alpha_0)$ for players 1 and 2.*

Proposition 1 shows that an equilibrium always exists in which the game ends at time $t = 0$. Furthermore, this equilibrium is in pure strategies, and the competing players are the ones who concede immediately. However, this is not the only possible equilibrium outcome in which the game ends immediately because there also may exist equilibria (necessarily in mixed strategies) in which the chair concedes immediately. This type of mixed-strategy equilibrium is characterized in part (ii)(b) of Proposition 2 below.

Fix an equilibrium mixed strategy profile $(G_{0,1}, G_{0,2}, G_1, G_2)$. We further denote the longest possible duration of the game by $\hat{t} \in \bar{\mathbb{R}}_+$, which is given by

$$\hat{t} = \min\{t \geq 0 \ : \max\{\tilde{G}_0(t), G_1(t), G_2(t)\} = 1\}$$

with the convention that $\hat{t} = \infty$ if the set $\{t \geq 0 \ : \max\{\tilde{G}_0(t), G_1(t), G_2(t)\} = 1\}$ is empty.[7] As such, $\hat{t}$ is the earliest time when some player ends the game with probability 1. Therefore, $\hat{t} = 0$ corresponds to the case when the game must end immediately; $\hat{t} \in (0, \infty)$ corresponds to the case when the game must end but can end with a delay; and $\hat{t} = \infty$ corresponds to the case when, with positive probability, the game never ends. In what follows, we call a strategy profile *an immediate-agreement equilibrium* if $\hat{t} = 0$, and we call it *a delay equilibrium* if $\hat{t} > 0$. Formally,

Definition 2. *An equilibrium $(G_0, G_1, G_2) \in \mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2$ is an* **immediate-agreement equilibrium** *if and only if $(1 - G_{0,1}(0) - G_{0,2}(0))(1 - G_1(0))(1 - G_2(0)) = 0$. An equilibrium $(G_0, G_1, G_2) \in \mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2$ is a* **delay equilibrium** *if and only if $(1 - G_{0,1}(0) - G_{0,2}(0))(1 - G_1(0))(1 - G_2(0)) \neq 0$.*

Recall that $q_i(t)$ is the probability that player $i = 1, 2$ concedes exactly at time t, and $\tilde{q}_0(t)$ is the probability that player 0 concedes exactly at time t. It follows that an equilibrium (G_0, G_1, G_2) is a delay equilibrium if and only if, $q_1(0) < 1, q_2(0) < 1$, and $\tilde{q}_0(0) < 1$.

We are ready to characterize the immediate-agreement Nash equilibrium outcomes fully. The following proposition says that any immediate-agreement equilibrium outcome is generically a pure-strategy equilibrium outcome described in Proposition 1. Only when the demands of the two competing players are equal and are sufficiently high, i.e., $\alpha_1 = \alpha_2 \geq 2(1 - \alpha_0)$, it is ever possible for the chair to concede at time $t = 0$ with certainty.

Proposition 2 (Immediate-Agreement Nash Equilibrium).

[7] If there exists some $s \in \mathbb{R}_+$ such that $\max\{\tilde{G}_0(s), G_1(s), G_2(s)\} = 1$, then by the right continuity of cumulative distribution functions, the set $\{t \geq 0 \ : \max\{\tilde{G}_0(t), G_1(t), G_2(t)\} = 1\}$ has a minimum.

(i) If $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2 < 2(1-\alpha_0)$, there is a unique immediate-agreement Nash equilibrium outcome. In this outcome, only player 1 and player 2 concede at time $t = 0$.

(ii) If $\alpha_1 = \alpha_2 = \alpha \geq 2(1-\alpha_0)$, there are two types of immediate-agreement Nash equilibrium outcomes:

(a) Only player 1 and player 2 concede at time $t = 0$.

(b) Only player 0 concedes at time $t = 0$; player 0 concedes to player 1 with probability $q_{0,1}(0) \in [\frac{1-\alpha_0}{\alpha}, 1 - \frac{1-\alpha_0}{\alpha}]$ and to player 2 with complementary probability $q_{0,2}(0) = 1 - q_{0,1}(0)$.

Note that there always exists an immediate-agreement equilibrium in which both competing players concede at the start of the game. By contrast, there need not exist an equilibrium in which the chair concedes at the start of the game. Part (ii) of Proposition 2 gives the necessary and sufficient conditions for such an equilibrium to exist. First, if the demands of the competing players are not equal, there cannot exist an equilibrium in which the chair concedes with certainty at the start. This is because if the chair is ever to concede, she will concede to the competing player with the lower demand. Knowing this, the competing player with the higher demand would concede immediately. Next, if the demands of the competing players are equal and sufficiently low, then again, there is an incentive for the competing players to concede, but this time, it is because the gain from waiting is not worth the risk that the chair might concede to the other player. Finally, if the demands of the competing players are equal and sufficiently high, then there is an equilibrium in which the chair concedes to each competing player with a strictly positive probability. In particular, if $\alpha_1 = \alpha_2 = 2(1-\alpha_0)$, then in any immediate-agreement equilibrium in which the chair concedes at $t = 0$, she must concede to each competing player with equal probability, i.e., $q_{0,1}(0) = q_{0,2}(0) = \frac{1}{2}$. As the conflict of interests increases, $\alpha_1 = \alpha_2 > 2(1-\alpha_0)$, other immediate-agreement equilibria arise in which the chair concedes to the competing players at $t = 0$ with unequal probabilities. Yet the gap in concession probabilities cannot be too wide. Formally, $|q_{0,1}(0) - q_{0,2}(0)| \leq 1 - \frac{2(1-\alpha_0)}{\alpha}$.

Given $\alpha_1 = \alpha_2 = \alpha$, now we explain more explicitly where the condition $\alpha \geq 2(1-\alpha_0)$ comes from. If both competing players are waiting and the chair concedes with certainty, then the expected payoff for competing player i is $\alpha q_{0,i}(0)$. If instead player i unilaterally deviates, his payoff would be $\frac{1}{2}(1-\alpha_0) + \frac{1}{2}\alpha q_{0,i}(0)$. When the competing players do not concede at the beginning of the game, waiting must be sufficiently profitable for each of them compared to conceding. Thus, $\sum_{i\in\{1,2\}} \alpha q_{0,i}(0) \geq \sum_{i\in\{1,2\}}[\frac{1}{2}(1-\alpha_0) + \frac{1}{2}\alpha q_{0,i}(0)]$, i.e., $\alpha \geq 2(1-\alpha_0)$.

Note that unlike in Proposition 1 and part (i) of Proposition 2, the multiplicity of equilibria in part (ii) of Proposition 2 is accompanied with multiple equilibrium outcomes and equilibrium payoffs.

Remark 1. *The equilibrium outcomes described in Proposition 2 can survive the refinement of sequential rationality and arise as a subgame perfect equilibrium outcome. For the first* *type of equilibrium outcome in which two competing players concede immediately, there is a pure-strategy subgame perfect equilibrium wherein the competing players concede with certainty whenever the game has not ended thus far, and the chair never concedes. For the second type of equilibrium outcome in which the chair concedes immediately, there is a mixed-strategy subgame perfect equilibrium in which the chair concedes to the two competing players with the prescribed probabilities whenever the game has not ended thus far, and the competing players never concede.*

4 Delay Equilibria

In this section, we characterize delay equilibria, i.e., a mixed strategy equilibrium (G_0, G_1, G_2) with $q_1(0) < 1, q_2(0) < 1$, and $\tilde{q}_0(0) < 1$.

We start by showing that if $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2 \leq 2(1-\alpha_0)$, then it is impossible to have a delay equilibrium. Intuitively, if $\alpha_i > \alpha_j$, then the chair either does not concede or concedes to player j leaving player i with nothing. In either case, player i has an incentive to concede immediately at the start and the game ends without any delay. Also, if the demands of competing players are equal but sufficiently small, then competing players will not wait and risk being excluded from the agreement. This intuition leads us to the first part of the following result. To see the intuition for the second part, suppose $\alpha = \alpha_1 = \alpha_2 \leq 2(1-\alpha_0)$ and hold the behavior of all players at the start of the game fixed. Delaying has the benefit of being conceded to with some probability. For at least one of the competing players, this probability cannot exceed 1/2, and therefore, the benefit cannot exceed $\alpha/2$. By contrast, conceding right after the start of the game gives a payoff of roughly $1-\alpha_0$ in the continuation when the other competing player is not conceding. When $\alpha/2 \leq 1-\alpha_0$, the payoff from delaying is not sufficiently high compared to conceding right after the start of the game. Thus, we have the following result.

Lemma 2. *If $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2 \leq 2(1-\alpha_0)$, then there does not exist a delay equilibrium.*

Consequently, when $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2 \leq 2(1-\alpha_0)$, the characterization of equilibria is complete by Proposition 2. The rest of this section is devoted to characterizing delay equilibria when $\alpha_1 = \alpha_2 = \alpha > 2(1-\alpha_0)$. In particular, we study the following five scenarios:

(i) Start: What can happen at the start of the game? (Lemma 3).

(ii) Jumps: Do two players concede with strictly positive probability at the same time? Is it ever possible that some player concedes with strictly positive probability before the game ends? (Lemma 4 and Lemma 5).

(iii) Suspense: How will others react if some player has no intention to concede for a period of time? Is it ever possible that some player does not concede for a period of time before the game ends? (Lemma 6).

(iv) End: How will others react if some player concedes with strictly positive probability at the end of the game? Is it ever possible that some player ends the game in finite time? (Lemma 7).

(v) Gradual concession: How will others react to a player who concedes gradually over time? (Lemma 8).

After answering these questions, we characterize delay equilibrium in which both competing players gradually concede over time in Proposition 3.

Remark 2. *Note that in any delay equilibrium, we have $q_{0,1}(0) + q_{0,2}(0) < 1$. Thus there exists a competing player $i \in \{1, 2\}$ who receives a payoff strictly smaller than $\frac{\alpha}{2}$. Consider a delay equilibrium in which $q_j = \tilde{q}_0 = 0$. If player i unilaterally deviates to concede immediately with certainty, then his payoff would be $1-\alpha_0$. Therefore, we must further require $1-\alpha_0 < \frac{\alpha}{2}$, i.e., $\alpha > 2(1-\alpha_0)$, to support the sort of delay equilibrium wherein $q_j = \tilde{q}_0 = 0$.*

4.1 Properties of Best Responses

We start by answering the questions posed above to establish the properties of the best responses of players to the strategies of other players.

4.1.1 Start

Consider the case in which some player concedes with positive probability at the start of the game but does not concede with certainty. The following result states that the chair cannot concede at the start of the game when one of the competing players concedes.

Lemma 3. $q_i(0)[q_{0,1}(0) + q_{0,2}(0)] = 0$*.*

The intuition behind Lemma 3 is as follows. First consider the case when some competing player $i \in \{1, 2\}$ concedes at time 0 with positive probability $q_i(0) > 0$. In this case, player 0 would rather wait for a little longer to concede than concede at time 0. That is because for player 0, the cost of waiting for an infinitesimal amount of time is negligible whereas its benefit discretely jumps when $q_i(0) > 0$. Therefore, if either of the competing players concede at time 0 with strictly positive probability, player 0 does not concede at time 0. The same logic applies when $t > 0$. See Lemma 4.

So what if $\tilde{q}_0(0) > 0$ in equilibrium? A direct implication of Lemma 3 is that: if $\tilde{q}_0(0) > 0$, then $q_1(0) = q_2(0) = 0$. To see this, suppose for the purpose of contradiction that $q_i(0) > 0$ for some $i \in \{1, 2\}$. Then by Lemma 3, we have $\tilde{q}_0(0) = q_{0,1}(0) + q_{0,2}(0) = 0$, a contradiction. Therefore, a competing player and the chair cannot concede simultaneously at the start of the game. Formally, we have $\tilde{q}_0(0)q_i(0) = 0$ for any $i \in \{1, 2\}$.

For the time being, we cannot say much about how the competing player $j \neq i$ would react in equilibrium if competing player i concedes with positive probability (i.e., $q_i(0) > 0$). Player

j's best response depends on player 0's equilibrium strategy. If player 0 concedes to player j shortly after the game starts, player j will not concede at the start (i.e., $q_j(0) = 0$); however, if player 0 only concedes to player j after a long time, player j may weakly prefer to concede immediately (i.e., $q_j(0) \geq 0$). Later in Lemma 12, we will establish that in all delay equilibria, $q_1(0) = q_2(0) = 0$, and $\min\{q_{0,1}(0), q_{0,2}(0)\} \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1} \max\{q_{0,1}(0), q_{0,2}(0)\}$.

4.1.2 Jumps

So far we only considered the start of the game, which is special because it marks the earliest possible concession time. Now we move on to characterizing what happens after the game starts in any delay equilibrium. The following result is analogous to Lemma 3 but describes what happens when player $i \in \{1, 2\}$ concedes with a strictly positive probability at time $t > 0$ rather than $t = 0$. However, Lemma 4 takes into account the possibility that the game might have ended before time t, something that cannot happen at the start of the game.

Lemma 4. *For any $t > 0$, if the game has not ended by player j before time t, i.e., $\lim_{\tau \uparrow t} G_j(\tau) < 1$, then $q_i(t)[q_{0,1}(t) + q_{0,2}(t)] = 0$.*

The message from Lemma 4 is that whenever player $i \in \{1, 2\}$ concedes with strictly positive probability, player 0 will not concede at the same time as long as the game has not ended by then. The intuition behind this result is similar to Lemma 3 since for player 0, the cost of waiting for an infinitesimal time is negligible while the benefit from it is strictly positive. The difference between these two results is that Lemma 4 describes delay equilibrium at $t > 0$ while Lemma 3 is for the start of the game.

We show that if player i concedes with positive probability at time $t > 0$, then the competing player j does not have an incentive to concede at the same time.

Lemma 5. *For any $t > 0$, if $\lim_{\tau \uparrow t} G_{0,1}(\tau) + G_{0,2}(\tau) < 1$, then $q_1(t)q_2(t) = 0$.*

The message from Lemma 5 is that whenever player $i \in \{1, 2\}$ concedes with strictly positive probability, player $j \neq i$ will not concede at the same time as long as the game has not ended by then. This result looks quite similar to Lemma 4 but the intuition behind them is quite different. In Lemma 4, player 0 will not concede at the same time with player i because she has an incentive to wait a little *longer*, while in Lemma 5, player j will not concede at the same time with player i because he has the incentive to concede a little *earlier*.

4.1.3 Suspense

We study how the other players will react if some player does not move for an interval.

Lemma 6. *If the strategies of competing players are constant over some interval before the game ends, then the strategy of player 0 is constant in that interval, and vice versa. Formally, for any t, s satisfying $0 \leq t < s \leq \hat{t}$,* *(i) if $G_1(t) = G_1(s)$, $G_2(t) = G_2(s)$, then $G_{0,\kappa}(t) = G_{0,\kappa}(s)$ for $\kappa = 1, 2$;*

(ii) if $G_{0,i}(t) = G_{0,i}(s)$, then $G_i(t) = G_i(s)$ for $i \in \{1, 2\}$.

The message from Lemma 6 is that whenever players 1 and 2 do not move for some interval, player 0 does not move during the same interval as long as the game has not ended by then. Likewise, whenever player 0 does not move during some interval, then the other two players do not move during that same interval. The force driving Lemma 6 is that any player would rather concede earlier than concede within the suspense period in which (s)he is not conceded to. This is because by conceding earlier, the player saves waiting cost and does not harm his/her chance of being conceded to (since the opponent(s) concession choices are constant over this suspense period).

4.1.4 End

The next result shows that if a competing player concedes with strictly positive probability at time $t > 0$, then the game must end by time t (i.e., $\hat{t} \leq t$).

Lemma 7. *(i) For any $t > 0$ and $i, j \in \{1, 2\}$ such that $j \neq i$, if $q_i(t) > 0$, then $\lim_{\tau \uparrow t} G_{0,1}(\tau) + G_{0,2}(\tau) = 1$ or $\lim_{\tau \uparrow t} G_j(\tau) = 1$.*

(ii) For any $t > 0$, if $\tilde{q}_0(t) > 0$, $\min\{q_{0,1}(t), q_{0,2}(t)\} > \frac{1-\alpha_0}{\alpha}\tilde{q}_0(t)$, then $\lim_{\tau \uparrow t} G_1(\tau) = 1$ or $\lim_{\tau \uparrow t} G_2(\tau) = 1$.

Loosely speaking, the proof is as follows. Suppose player i concedes at time t with strictly positive probability to player 0. Then player 0 has an incentive to concede later. As such, if, to the contrary of the statement of the lemma, the game has not ended by t, then there is a short suspense period right before t during which player 0 does not move. Therefore, by Lemma 6 player i should not have moved for that period as well, which is a contradiction to i jumping at t. The same logic apply to the case when the chair concedes at time t with strictly positive probability to both competing players. We need the additional condition $\frac{\min\{q_{0,1}(t), q_{0,2}(t)\}}{\max\{q_{0,1}(t), q_{0,2}(t)\}} > \frac{1-\alpha_0}{\alpha+\alpha_0-1}$ to ensure that both competing players do not move right before the atom point of the chair; that is, they strictly prefer to concede after the atom point than right before it.

4.1.5 Concede over Time

Consider an interval $[t, s]$ during which at least one player concedes gradually, i.e., without jumps or suspense. In Lemma 8, we characterize the best responses to strictly increasing concession strategies which includes gradual concession as a special case. Note however that, for competing players, as long as the game has not ended, strictly increasing concession strategies are equivalent to gradual concession by Lemma 7.

Let $H(\tau)$ be the probability that at least one of the competing players concedes by time $\tau \in \mathbb{R}_+$:

$$\begin{aligned} H(\tau) &= G_1(\tau)(1 - G_2(\tau)) + G_2(\tau)(1 - G_1(\tau)) + G_1(\tau)G_2(\tau) \\ &= G_1(\tau) + G_2(\tau) - G_1(\tau)G_2(\tau). \end{aligned} \tag{1}$$

We show that when the chair's total probability of concession $\tilde{G}_0$ strictly increases, H must have a constant hazard rate on $[t, s]$ given by

$$\mu = \frac{(1-\alpha)r_0}{\alpha_0 + \alpha - 1} > 0. \tag{2}$$

Similarly, $\tilde{G}_0$ must have a constant hazard rate on $[t, s]$ when both competing players concede gradually. We show that this hazard rate is given by

$$\rho = \frac{\mu + r_1 + r_2}{\frac{\alpha}{1-\alpha_0} - 2} > 0. \tag{3}$$

Since G_i and $G_{0,i}$, $i = 1, 2$ are cumulative distribution functions, they are increasing and therefore they are differentiable almost everywhere. Let g_i denote the derivative of G_i when it exists, and let $g_{0,i}$ denote the derivative of $G_{0,i}$ when it exists. Also, let $g_0(\cdot) = g_{0,1}(\cdot) + g_{0,2}(\cdot)$. For any $\tau \in [t, s]$, let $\lambda_i(\tau)$ denote the hazard rate of the concession strategy G_i of player $i \in \{1, 2\}$. Formally,

$$\lambda_i(\tau) = \frac{g_i(\tau)}{1 - G_i(\tau)} \tag{4}$$

Similarly define $\lambda_{0,i}(\tau)$ as

$$\lambda_{0,i}(\tau) = \frac{g_{0,i}(\tau)}{1 - \tilde{G}_0(\tau)}. \tag{5}$$

Note that the sum of λ_1 and λ_2 is the hazard rate of the competing players' aggregate concession strategy H, and the sum of $\lambda_{0,1}$ and $\lambda_{0,2}$, denoted by λ_0, is the hazard rate of the chair's aggregate concession strategy $\tilde{G}_0$. In Lemma 8, we show that the sum of λ_1 and λ_2 is constant and equal to μ when $\tilde{G}_0$ is strictly increasing. We also show that the sum of $\lambda_{0,1}$ and $\lambda_{0,2}$ is constant and equal to ρ when G_1 and G_2 are strictly increasing.

Lemma 8. *Consider an interval $[t, s]$ with $0 < t < s < \hat{t}$.*

(i) If $\tilde{G}_0$ is strictly increasing on $[t, s]$, then

$$\mu = \lambda_1(\tau) + \lambda_2(\tau) \tag{6}$$

for all $\tau \in [t, s)$ where μ is given by (2).

(ii) If G_i is strictly increasing on $[t, s]$ for some player $i \in \{1, 2\}$, then

$$\lambda_{0,i}(\tau) = \frac{1-\alpha_0}{\alpha+\alpha_0-1}[r_i + \lambda_j(\tau) + \lambda_{0,j}(\tau)] = \frac{1-\alpha_0}{\alpha}(r_i + \lambda_j(\tau) + \lambda_0(\tau)) \tag{7}$$

for $\tau \in (t, s)$ almost everywhere. Additionally,

(a) if $\tilde{q}_0(\tau) > 0$ for some $\tau \in (t, s)$, then $q_{0,i}(\tau) = \frac{1-\alpha_0}{\alpha}\tilde{q}_0(\tau)$;

(b) if $\tilde{q}_0(s) > 0$, then $q_{0,i}(s) \le \frac{1-\alpha_0}{\alpha}\tilde{q}_0(s)$;

(c) if $\tilde{q}_0(t) > 0$, then $q_{0,i}(t) \ge \frac{1-\alpha_0}{\alpha}\tilde{q}_0(t)$.

(iii) If G_1 and G_2 are both strictly increasing on $[t, s]$, then

$$\lambda_i(\tau) = -r_j - \rho + \frac{\alpha}{1-\alpha_0}\lambda_{0,j}(\tau), \quad i = 1, 2, \ j \ne i \tag{8}$$

for $\tau \in [t, s)$ almost everywhere, where ρ is given by (3);

By Lemma 6, $\tilde{G}_0$ must be strictly increasing when G_i is strictly increasing for some $i \in \{1, 2\}$. Together with part (i) of Lemma 8, when G_1 and G_2 are strictly increasing, we must have $\lambda_1(\tau) + \lambda_2(\tau) = \mu$. By adding the two equations in part (iv) of Lemma 8 and using the observation that $\lambda_1(\tau) + \lambda_2(\tau) = \mu$, we obtain $\lambda_{0,1}(\tau) + \lambda_{0,2}(\tau) = \rho$ for all τ, that is, $\tilde{G}_0$ has a constant hazard rate. To summarize, we have the following corollary.

Corollary 2. *Consider an interval $[t, s]$ with $0 < t < s \le \hat{t}$. Both G_1 and G_2 are strictly increasing over the time interval $[t, s]$ only when $\alpha > 2(1 - \alpha_0)$. In this case, $\tilde{G}_0$ has a constant hazard rate ρ over $[t, s)$, where ρ is given by (3).*

4.2 Characterization of Mixed Strategy Equilibria with Delay

In this section, we characterize mixed strategy equilibria with delay. Throughout this section, fix an equilibrium mixed strategy profile (G_0, G_1, G_2) with delay. As before, let $\tilde{G}_0(t) = G_{0,1}(t) + G_{0,2}(t)$ and $H(t) = G_1(t) + G_2(t) - G_1(t)G_2(t)$ for all $t \in \mathbb{R}_+$.

We start by examining the possibility that no one moves in some interval by the end of the game. Since we focus on the delay equilibrium in this section, we must have $\hat{t} > 0$.

The following lemma shows that the aggregate strategies $\tilde{G}_0$ and H prescribe a gradual concession from the beginning of the game, $t = 0$, until its end, $t = \hat{t}$.

Lemma 9. *Aggregate strategies $\tilde{G}_0$ and H are strictly increasing on $[0, \hat{t})$.*

There two immediate implications of this lemma. First, player 0 must concede gradually throughout the game. Second, at any point during the game, at least one of the competing players must be conceding. Therefore, combined with Lemma 7, there are only two possible cases in delay equilibria:

1. Both competing players gradually concede throughout the game, i.e., G_1 and G_2 are both strictly increasing over $[0, \hat{t}]$.
2. One and only one competing player does not concede over some interval(s), i.e., $\exists a, b \in [0, \hat{t}]$ such that $G_i(a) = G_i(b)$ and G_j is strictly increasing over $[a, b]$.

In both cases, by Lemma 7, there is no atom point in any player's strategy before the end of the game, that is, $G_1, G_2, G_{0,1}$ and $G_{0,2}$ are all continuous over $[0, \hat{t})$. We next characterize delay equilibria in these two cases separately.

4.2.1 Both Competing Players Gradually Concede

Since G_1 and G_2 are strictly increasing, Lemma 6 implies that $G_{0,1}$ and $G_{0,2}$ are both strictly increasing over $[0, \hat{t})$. The next result shows that the probability that the game does not end by time t is strictly positive for each $t \in \mathbb{R}_+$, that is, $\hat{t} = \infty$.

Lemma 10 (No End with Jump by a Competing Player)**.** *There does not exist a delay equilibrium in which $G_i(T) = 1$, $q_i(T) > 0$, $\lim_{\tau \uparrow T} G_j(\tau) < 1$, and $\lim_{\tau \uparrow T} \tilde{G}_0(\tau) < 1$ for some $T > 0$, $i \in \{1, 2\}$, and $j \neq i$.*

Proof of Lemma 10. Both parts follow immediately from part (i) of Lemma 7. ■

Lemma 11. *In any delay equilibrium when both competing players gradually concede throughout the game, we have $\hat{t} = \infty$.*

We next reconsider what can happen at the start of the game. By Lemma 3, we have already shown that player 0 and any competing player cannot simultaneously concede at $t = 0$ with positive probability. The next result further establishes that no competing player concedes at the start of the game with positive probability. It also establishes that if player 0 ever concedes at the start of the game, she needs to concede to both competing players, and moreover, she the probability of conceding to player 1 must be "close" to the probability of conceding to player 2.

Lemma 12. *In any delay equilibrium when both competing players gradually concede throughout the game, we have $q_1(0) = q_2(0) = 0$ and*

$$\min\{q_{0,1}(0), q_{0,2}(0)\} \geq \frac{1 - \alpha_0}{\alpha + \alpha_0 - 1} \max\{q_{0,1}(0), q_{0,2}(0)\}.$$

Finally, we are ready to fully characterize the type of delay equilibrium in which both competing players gradually concede by the end of the game.

Proposition 3. *A mixed strategy profile* (G_0, G_1, G_2) *with* $\tilde{q}_0(0) < 1$*,* $q_1(0) < 1$ *and* $q_2(0) < 1$ *is a Nash equilibrium profile in which both competing players concede throughout the game if and only if the following conditions hold:*

(i) $G_{0,1}, G_{0,2}, G_1$ *and* G_2 *are all continuous over* $(0, \infty)$*;*

(ii) $G_1(0) = G_2(0) = 0$*, and* $\min\{G_{0,1}(0), G_{0,2}(0)\} \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1} \max\{G_{0,1}(0), G_{0,2}(0)\}$*;*

(iii) $\lambda_1(t) + \lambda_2(t) = \mu$ *and* $\lambda_1(t), \lambda_2(t) > 0$ *for* $t \geq 0$ *almost everywhere, where* μ *is given by* (2)*;*

(iv) $\lambda_{0,j}(t) = \frac{1-\alpha_0}{\alpha}(\lambda_i(t) + r_j + \rho)$ *for* $t > 0$ *almost everywhere and* $i, j \in \{1, 2\}, j \neq i$*, where* ρ *is given by* (3).

From Proposition 3, it can be seen that there are multiple equilibria in which both competing players concede gradually throughout the game. There are two sources of this indeterminacy. First, at the start of the game, player 0's strategy is not fully pinned down; the only requirement is that it must satisfy $\min\{G_{0,1}(0), G_{0,2}(0)\} \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1} \max\{G_{0,1}(0), G_{0,2}(0)\}$. Second, after the start of the game, there is one degree of freedom in nailing down the hazard rates $\lambda_{0,1}(t), \lambda_{0,2}(t), \lambda_1(t)$, and $\lambda_2(t)$. Once one of these four are fixed, the remaining can be determined by the three equations in parts (iii) and (iv) of Proposition 3. Of course, once the hazard rates and player 0's strategy at the start of the game is determined , the mixed strategy profile $(G_{0,1}, G_{0,2}, G_1, G_2)$ is also fully characterized.

Table 1 compares players' expected payoffs across different equilibria. Different rows correspond to different equilibrium types. The columns correspond to players. Not surprisingly, any delay equilibrium entails welfare loss compared to any immediate equilibrium. When agreement is achieved immediately with certainty, the total payoff of three players is 1, regardless of who concedes at the beginning of the game. By contrast, welfare loss occurs in any equilibrium with delay. In particular, for any delay equilibrium with gradual concession, the total payoff is decreased to $1 - \alpha + \alpha q_{0,1}(0) + (1 - \alpha_0)(1 - \tilde{q}_0(0)) + \alpha q_{0,2}(0) + (1 - \alpha_0)(1 - \tilde{q}_0(0)) = 1 - (\alpha - 2(1 - \alpha_0))(1 - \tilde{q}_0(0)) < 1$.

	Player 0	Player i
Immediate Concession by Competing Players	α_0	$(1 - \alpha_0)/2$
Immediate Concession by the Chair	$1 - \alpha$	$\alpha q_{0,i}(0)$
Gradual Concession	$1 - \alpha$	$\alpha q_{0,i}(0) + (1 - \alpha_0)(1 - \tilde{q}_0(0))$

Table 1: Equilibrium Expected Payoffs

As expected, the chair strictly prefers the equilibria in which one of the competing players concedes immediately to any other equilibria. Somewhat surprisingly, however the chair is indifferent between the gradual concession equilibria and the equilibria in which she concedes immediately to one of the competing players. The reason behind this result can be easily seen from the following two observations. First, in any gradual concession equilibria, the chair plays a strictly mixed strategy over time $[0, \infty)$. Thus, her equilibrium payoff equals her payoff if she plays a pure strategy and concedes at the start of the game, given the competing players' equilibrium strategies. Second, neither competing player concedes at the start of the game in any gradual concession equilibrium. This coincides with the competing players' strategy at the start of the game when the chair makes an immediate concession with certainty. Therefore, the chair's payoff is $1 - \alpha$, both in an immediate-agreement equilibrium in which the chair concedes and in a delay equilibrium in which both competing players concede gradually.

Turning to the competing players, first, they strictly prefer any immediate agreement equilibrium in which the chair concedes to an immediate agreement equilibrium in which the competing players concede. This is not an obvious result since the expected payoff of a competing player in an immediate agreement equilibrium in which the chair concedes could potentially be very low if the chair concedes to that player with a low probability. However, there is a lower bound on the chair's concession probability to each of the two competing players. Specifically, since $q_{0,i} \geq \frac{1-\alpha_0}{\alpha}$ by Proposition 2 for any $i \in \{1, 2\}$, we have $\alpha q_{0,i} \geq 1 - \alpha_0 > \frac{1-\alpha_0}{2}$. Second, as long as $\alpha > 2(1 - \alpha_0)$, for any immediate-agreement equilibrium, there exists a delay equilibrium that yields a strictly higher payoff to one of the competing players. This follows because both in an immediate-agreement equilibrium in which the chair concedes and in a gradual concession equilibrium, the probability that the chair concedes to player $i \in \{1, 2\}$ at the start is in the same range $[\frac{1-\alpha_0}{\alpha}, 1 - \frac{1-\alpha_0}{\alpha}]$. Thus, for any immediate agreement equilibrium in which the chair concedes, there is a gradual concession equilibrium, which makes a competing player better off.

To summarize, equilibria with delay are the most preferable equilibria for the competing players, whereas it is the least preferable equilibria for the chair.

5 Comparison to Bilateral War of Attrition

In this section, we compare the equilibria of the multilateral war of attrition game that we characterized to the equilibria of the bilateral war of attrition game characterized by Hendricks, Weiss and Wilson (1988) (henceforth HWW). The players are symmetric in their voting power in the bilateral war of attrition game. By contrast, in the three-player war of attrition that we analyze, player 0 has veto power and can concede to either of the competing players. Comparison to bilateral war of attrition is not trivial because of multiple equilibria in both cases.

HWW analyze a game with a more general payoff structure than ours. For the comparison, we focus on the particular case of their game between player 0 and player 1 with demands α_0 and α_1. In their analysis, like us, HWW focus on characterizing mixed strategy Nash equilibria for ease of exposition. As in our case, any mixed strategy Nash equilibrium is also subgame perfect when there is delay. Furthermore, they provide necessary and sufficient conditions under which there exists a subgame perfect Nash equilibrium that induces immediate agreement with certainty (their Theorem 4). The case we consider in this section satisfies those conditions, particularly the condition in part (b) of their Theorem 4. In what follows, when comparing our results to theirs, we will compare the mixed strategy Nash equilibria we characterized with the mixed strategy Nash equilibria they characterize. In both cases, these equilibria are also subgame perfect; therefore, we simply refer to them as equilibria.

5.1 Conditions for Equilibrium

We start by comparing the conditions for the existence of different kinds of equilibria. HWW show that there always exists an immediate agreement equilibrium and a delay equilibrium (see HWW, Theorem 1(b) and Theorem 2). In any immediate agreement equilibrium, one and only one player concedes at time 0 with certainty. This player is arbitrary since the players are symmetric in their setting. In the three-player war of attrition game we analyze, an immediate-agreement equilibrium still always exists (see Proposition 2). But by contrast to the bilateral game, an additional restriction on the players' demands is needed for the chair to concede immediately: $\alpha_1 = \alpha_2 \geq 2(1-\alpha_0)$. Additionally, a delay equilibrium exists only when $\alpha_1 = \alpha_2 > 2(1-\alpha_0)$ (see Lemma 2).

5.2 Concession Rate

Next, we compare the equilibrium concession rates when there is a delay. HWW show that a strategy profile (G_0, G_1) with $(1-G_0(0))(1-G_1(0)) > 0$ is an equilibrium if and only if

1. $q_0(0)q_1(0) = 0$;

2. for $i = 0, 1$ and $t > 0$,

$$G_i(t) = 1 - (1 - q_i(0))e^{-\lambda_i t}$$

where λ_i denotes the concession rate of player i, given by

$$\lambda_i = \frac{(1-\alpha_i)r_j}{\alpha_0 + \alpha_1 - 1} \tag{9}$$

with $j \neq i$.

Recall from equation (2) that in the three-player war of attrition, the aggregate concession rate of the competing players is given by $\mu = \frac{(1-\alpha)r_0}{(\alpha_0+\alpha-1)}$. This is identical to the concession rate of the single opponent of the chair in the bilateral war of attrition. Thus, somewhat surprisingly, each competing player concedes at a lower rate in the three-player war of attrition. Turning to the chair's concession rate, recall by equation (3) that is equal to $\rho = \frac{\mu+r_1+r_2}{\frac{\alpha}{1-\alpha_0}-2}$ in the three-player war of attrition game. It is straightforward to show that this is strictly larger than her concession rate in the bilateral war of attrition λ_0. Even more surprisingly, the chair must concede to each competing player at a higher rate in the three-player war of attrition.[8] The reason for the accelerated concession in the three-player war of attrition is that conditional on a delay equilibrium, all players must be indifferent between conceding and waiting. Since the chair concedes to two different players, she must concede at a higher rate to each player compared to the bilateral case to make them indifferent between conceding and waiting. Likewise, since two different players can concede to her, each player must concede at a lower rate than in the bilateral case to leave the chair indifferent between conceding and waiting. Thus, the concession in the three-player war of attrition is accelerated compared to the bilateral case. Taken together with the results in Section 5.1, it is less likely to have a delay equilibrium, and the delays are more likely to be shorter in the three-player war of attrition.

5.3 Payoff

Player 0's expected payoff in the bilateral war of attrition is given by $\alpha_0 q_1(0) + (1 - \alpha_1)(1 - q_1(0))$ in any equilibrium. Now consider increasing competition by adding another player, player 2, with whom player 0 can reach an agreement but player 1 still needs player 0's consent for an agreement. If player 2's demand is different from player 1's demand (i.e., $\alpha_2 \neq \alpha_1$), then by Lemma 2, there is no delay equilibrium, and by Proposition 2 player 0 must receive her demand α_0. Thus, when the new competing player has a different demand than the existing one, player 0 is always better off.

[8] Formally, $\lambda_{0,1}(t) > \lambda_0$ for all $t > 0$ where $\lambda_{0,1}(t)$ is given by Proposition 3 and λ_0 is given by equation (9).

Surprisingly, increasing competition can make the chair worse off. To see, note that by the above arguments, this can happen only when the new competing player has the same demand as the existing competing player, and there is a delay in equilibrium. In this case, in the three-player war of attrition, the expected payoff of player 0 is always given by $1 - \alpha_1$ in any delay equilibrium (see Table 1). By contrast, in the bilateral war of attrition, the expected payoff of player 0 exceeds $1 - \alpha_1$ as long as $q_1(0) > 0$. The reason player 0 can be worse off in a delay equilibrium in the three-player war of attrition is that, by Proposition 3, a delay equilibrium exists in the three-player war of attrition only when player 0 is conceded to with zero probability at the start of the game, whereas in the bilateral war of attrition, player 0 can be conceded to with positive probability at the start of the game.

Also, note that there is inefficiency arising from concession with delay. We end the discussion about the gradual concession equilibrium by the comparative statics of concession rates with respect to the magnitude of conflicts of interest and players' impatience. Recall that the aggregate concession rate of competing players is $\mu = (1-\alpha)r_0/(\alpha_0+\alpha-1)$ and the concession rate of player 0 is $\rho = (\mu + r_1 + r_2)/(\frac{\alpha}{1-\alpha_0} - 2)$. The following implications directly follow from these expressions. First, for a given demand of the competing players α, a larger demand of the chair, α_0, increases the conflicts of interest. In this case, the concession rates μ and ρ are both lower. Similarly, for a given demand of the chair α_0, a larger demand by the competing players α increases the conflicts of interests, and again, the concession rates are lower. Second, μ and ρ are strictly increasing in the chair's impatience r_0; additionally, the concession rate for the chair ρ is strictly increasing in both competing players' impatience r_1 and r_2 whereas μ is constant in r_1 and r_2. Loosely speaking, these comparative statics results imply that agreement is likely to be reached faster when the conflicts of interest are larger and players are more impatient.

Now we can describe equilibria with delay by answering the questions at the start of Section 4 and compare these answers to those in Hendricks et al. (1988) for the bilateral war of attrition.

1. Start:

    (i) If both competing players gradually concede after the game begins, there is an asymmetry in concession choices between player 0 and competing players at the start of the game. Only player 0 can concede with strictly positive probability at the start of the game — when this happens, player 0 must concede to both competing players and the relative concession probability must fall in a moderate range $\frac{\min\{q_{0,1}(0),q_{0,2}(0)\}}{\max\{q_{0,1}(0),q_{0,2}(0)\}} \in [\frac{1-\alpha_0}{\alpha+\alpha_0-1}, 1]$.

(ii) If one competing player does not concede after the game begins, then this competing player cannot concede with strictly positive probability at the start of the game.

(iii) In contrast, for the bilateral war of attrition, either (and at most one) player can concede with strictly positive probability at the start of the game, and the concession probability can be chosen freely in the range $[0, 1)$.

2. Atom points: no atom points for all players ($q_{0,1}(t) = q_{0,2}(t) = q_1(t) = q_2(t) = 0$ for any $t > 0$). This result is the same as the bilateral case.

3. Suspense: suspense can occur for one competing player in equilibrium for three-player war of attrition. In contrast, both players must gradually concede in any delay equilibrium in the bilateral case.

4. End: when both competing players gradually concede throughout the game, the game does not terminate in finite time for any delay equilibrium as in the bilateral case.

6 Concluding Remarks

In this paper, we study a concession game between three players in continuous time in which only two players are needed to reach an agreement on how to split a unit of surplus. In their influential paper on reputational bargaining, Abreu and Gül (2000) study a bilateral concession game and show that players can benefit from building a reputation for being stubborn, i.e., irrationally following a strategy of demanding a high share of surplus and not conceding to any offer below a specified low level. In multilateral bargaining with majority rule, such a reputation has its costs. When one of the players can be excluded from the agreement, players can benefit from building a reputation for being compliant, i.e., irrationally following a strategy of demanding a low share of the surplus. Players prefer to split the surplus with compliant types by conceding to them, so compliant types may become more likely to be included in the winning coalition. As a preliminary for analyzing a reputational multilateral bargaining game with majority rule, we first analyze a multilateral war of attrition game with majority rule and complete information.

References

Abreu, Dilip and Faruk Gül, "Bargaining and Reputation," *Econometrica*, 2000, *68* (1), 85–117.

Baron, David P and John A Ferejohn, "Bargaining in legislatures," *American political science review*, 1989, *83* (4), 1181–1206.

Bishop, DT and Chris Cannings, "A generalized war of attrition," *Journal of theoretical biology*, 1978, *70* (1), 85–124.

Bulow, Jeremy and Paul Klemperer, "The generalized war of attrition," *American Economic Review*, 1999, *89* (1), 175–189.

Ellingsen, Tore and Topi Miettinen, "Commitment and Conflict in Bilateral Bargaining," *American Economic Review*, 2008, *98* (4), 1629–35.

_ and _ , "Tough negotiations: Bilateral bargaining with durable commitments," *Games and Economic Behavior*, 2014, *87*, 353–366.

Fanning, Jack and Alexander Wolitzky, "Reputational Bargaining," in "Bargaining," Springer, 2022, pp. 35–60.

Hendricks, Ken, Andrew Weiss, and Charles Wilson, "The War of Attrition in Continuous Time with Complete Information," *International Economic Review*, 1988, pp. 663–680.

Kambe, Shinsuke, "An N-person war of attrition with the possibility of a noncompromising type," *Theoretical Economics*, 2019, *14* (3), 849–886.

Ma, Zizhen, "Efficiency and Surplus Distribution in Majoritarian Reputational Bargaining," 2022.

Miettinen, Topi, "A Review on Commitment Tactics in Bargaining under Complete Information," in "Bargaining: Current Trends and Future Directions," Palgrave Macmillan, 2022, pp. 9–33.

_ and Christoph Vanberg, "Commitment and Conflict in Multilateral Bargaining," 2020.

Myerson, Roger B, *Game theory: analysis of conflict*, Harvard university press, 1991.

Osborne, Martin J, *An Introduction to Game Theory*, Vol. 3, Oxford university press New York, 2004.

Özyurt, Selçuk, "Searching for a Bargain: Power of Strategic Commitment," *American Economic Journal: Microeconomics*, 2015, *7* (1), 320–53.

Royden, HL and Patrick Fitzpatrick, "Real Analysis (4th Edtion)," *New Jersey: Printice-Hall Inc*, 2010.

Smith, J Maynard, "The theory of games and the evolution of animal conflicts," *Journal of theoretical biology*, 1974, *47* (1), 209–221.

A Appendix

The following technical lemma is used extensively in our analysis.

Lemma A1. *Fix a mixed strategy profile $G = (G_0, G_1, G_2) \in \mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2$. Then, for each $(a, b) \subseteq \mathbb{R}_+$ with $a < b$, there exists $t \in (a, b)$ such that G is continuous at t.*

Proof of Lemma A1. Follows immediately from the monotonicity of each component of G and the fact that monotone functions are continuous almost everywhere.[9] ∎

Lemma A2. *For any $\varepsilon > 0$, any $t \geq 0$, and any $j \in \{1, 2\}$, there exists $\delta > 0$ such that:*

(a) $q_j(t + \delta) = 0$, $q_{0,j}(t + \delta) = 0$;

(b) $G_j(t + \delta) - G_j(t) < \varepsilon$, $G_{0,j}(t + \delta) - G_{0,j}(t) < \varepsilon$;

(c) $1 - e^{-r_0 \delta} < \varepsilon$, $1 - e^{-r_j \delta} < \varepsilon$;

(d) if $t > 0$, *then* $q_j(t - \delta) = 0$ *and* $q_{0,j}(t - \delta) = 0$;

(e) if $t > 0$, *then* $G_j(t) - G_j(t - \delta) - q_j(t) < \varepsilon$ *and* $G_{0,j}(t) - G_{0,j}(t - \delta) - q_{0,j}(t) < \varepsilon$.

Proof of Lemma A2. Fix any $\varepsilon > 0$, any t, any $j \in \{1, 2\}$. To prove (a), (b), and (c), first notice that by the definition of right-continuity there exists $\delta_1 > 0$ such that $G_j(t + \delta) - G_j(t) < \varepsilon$ and $G_{0,j}(t + \delta) - G_{0,j}(t) < \varepsilon$ for all $\delta < \delta_1$. Next, by continuity there exists $\delta_2 > 0$ such that $1 - e^{-r_0 \delta} < \varepsilon$ for all $\delta < \delta_2$. Finally, by Lemma A1 there

[9] See, e.g., Royden and Fitzpatrick (2010, p. 108).

exists $t_\varepsilon \in (t, t+\min\{\delta_1, \delta_2\})$ such that $q_j(t_\varepsilon) = 0$. Letting $\delta = t_\varepsilon - t$, we conclude that (a), (b), and (c) hold.

To prove (d) and (e), suppose $t > 0$. First, by the definitions of $q_j(\cdot)$ and $q_{0,j}(\cdot)$, there exists $\delta_1 > 0$ such that $G_j(t) - G_j(t-\delta) - q_j(t) < \varepsilon$ and $G_{0,j}(t) - G_{0,j}(t-\delta) - q_{0,j}(t) < \varepsilon$ for all $\delta < \delta_1$. Next, by continuity there exists $\delta_2 > 0$ such that $1 - e^{-r_0\delta} < \varepsilon$ for all $\delta < \delta_2$. Finally, by Lemma A1 there exists $t_\varepsilon \in (t - \min\{\delta_1, \delta_2\}, t)$ such that $q_j(t_\varepsilon) = 0$. Letting $\delta = t - t_\varepsilon$, we conclude that (d) and (e) hold. ■

A.1 Expected Payoff in a Delay Equilibrium

For any delay equilibrium, players' expected payoffs depend on their concession time: whether to concede at the start, concede later, or never concede.

Player 0

$$
\begin{aligned}
&U_0(0, \kappa, G_1, G_2) && \text{(A1)}\\
&= (1-\alpha_\kappa)(1-q_1(0))(1-q_2(0)) && \text{[neither player concedes at 0]}\\
&\quad + \Big(\frac{1-\alpha_\kappa}{3} + \frac{2\alpha_0}{3}\Big) q_1(0)q_2(0) && \text{[both concede at 0]}\\
&\quad + \Big(\frac{1-\alpha_\kappa}{2} + \frac{\alpha_0}{2}\Big) q_1(0)(1-q_2(0)) && \text{[player 1 concedes at 0]}\\
&\quad + \Big(\frac{1-\alpha_\kappa}{2} + \frac{\alpha_0}{2}\Big) q_2(0)(1-q_1(0)). && \text{[player 2 concedes at 0]}
\end{aligned}
$$

$$
\begin{aligned}
&U_0(\infty, \kappa, G_1, G_2) && \text{(A2)}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_0(\infty, \kappa, t_1, t_2)dG_1(t_1)dG_2(t_2)\\
&= \alpha_0\Big[1-(1-q_1(0))(1-q_2(0))\Big] && \text{[at least one player concedes at 0]}\\
&\quad + \sum_{v>0} \alpha_0 e^{-r_0 v} q_1(v)q_2(v) && \text{[both players concede at } v>0\text{]}\\
&\quad + \sum_{i=1}^{2}\int_0^{\infty}\int_0^{t_j} \alpha_0 e^{-r_0 t_i} dG_i(t_i)dG_j(t_j) && \text{[}i\text{ concedes before }j\text{ concedes]}\\
&\quad + \sum_{i=1}^{2}\int_0^{\infty} \alpha_0 e^{-r_0 t_i} q_j(\infty)dG_i(t_i) && \text{[}i\text{ concedes and }j\text{ never concedes]}\\
&= \alpha_0\Big[1-(1-q_1(0))(1-q_2(0))\Big]\\
&\quad + \sum_{v>0} \alpha_0 e^{-r_0 v} q_1(v)q_2(v)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{2}\int_0^\infty \alpha_0 e^{-r_0 v}(1-q_j(\infty)-G_j(v))dG_i(v)\\
&+\sum_{i=1}^{2}\int_0^\infty \alpha_0 e^{-r_0 v}q_j(\infty)dG_i(v)\\
=\ &\alpha_0\Big[1-(1-q_1(0))(1-q_2(0))\Big] && \text{[at least one player concedes at 0]}\\
&+\sum_{v>0}\alpha_0 e^{-r_0 v}q_1(v)q_2(v) && \text{[both players concede at } v>0\text{]}\\
&+\sum_{i=1}^{2}\int_0^\infty \alpha_0 e^{-r_0 v}(1-G_j(v))dG_i(v). && \text{[only } i \text{ concedes at } v>0\text{]}
\end{aligned}
$$

To see why the third equation holds, i.e.,

$$
\int_0^\infty\int_0^{t_j}\alpha_0 e^{-r_0 t_i}dG_i(t_i)dG_j(t_j)=\int_0^\infty \alpha_0 e^{-r_0 v}(1-q_j(\infty)-G_j(v))dG_i(v),
$$

we provide two approaches to prove:
Approach 1: *Change the Order of Integration*

$$
\begin{aligned}
\int_0^\infty\int_0^{t_j}\alpha_0 e^{-r_0 t_i}dG_i(t_i)dG_j(t_j)&=\int_0^\infty\int_{t_i}^{\infty}\alpha_0 e^{-r_0 t_i}dG_j(t_j)dG_i(t_i)\\
&=\int_0^\infty \alpha_0 e^{-r_0 t_i}(1-q_j(\infty)-G_j(t_i))dG_i(t_i).
\end{aligned}
$$

Approach 2: *Integral by Parts*

$$
\begin{aligned}
\int_0^\infty\int_0^{t_j}\alpha_0 e^{-r_0 t_i}dG_i(t_i)dG_j(t_j)&=\int_\infty^0\int_0^{t_j}\alpha_0 e^{-r_0 t_i}dG_i(t_i)d(1-G_j(t_j))\\
&=\int_0^{t_j}\alpha_0 e^{-r_0 t_i}dG_i(t_i)(1-G_j(t_j))\Big|_\infty^0-\int_\infty^0(1-G_j(t_j))\alpha_0 e^{-r_0 t_j}dG_i(t_j)\\
&=-\int_0^\infty \alpha_0 e^{-r_0 t_i}dG_i(t_i)q_j(\infty)+\int_0^\infty(1-G_j(v))\alpha_0 e^{-r_0 v}dG_i(v)\\
&=\int_0^\infty \alpha_0 e^{-r_0 v}(1-q_j(\infty)-G_j(v))dG_i(v).
\end{aligned}
$$

For any $t>0$, we have:

$$
\begin{aligned}
&U_0(t,\kappa,G_1,G_2) && \text{(A3)}\\
=\ &\alpha_0\Big[1-(1-q_1(0))(1-q_2(0))\Big] && \text{[at least one player concedes at 0]}\\
&+(1-\alpha_\kappa)e^{-r_0 t}(1-G_1(t))(1-G_2(t)) && \text{[neither concedes by } t\text{]}
\end{aligned}
$$

$$+\sum_{i=1}^{2}\int_0^t \alpha_0 e^{-r_0 v}(1-G_j(v))\mathrm{d}G_i(v) \qquad \text{[only } i \text{ concedes at } v\in(0,t)]$$

$$+\sum_{0<v<t}\alpha_0 e^{-r_0 v}q_1(v)q_2(v) \qquad \text{[both concede at } v\in(0,t)]$$

$$+\sum_{i=1}^{2}\Big(\frac{1-\alpha_\kappa}{2}+\frac{\alpha_0}{2}\Big)e^{-r_0 t}\Big[q_i(t)(1-G_j(t))\Big] \qquad \text{[only } i \text{ concedes at } t]$$

$$+\Big(\frac{1-\alpha_\kappa}{3}+\frac{2\alpha_0}{3}\Big)e^{-r_0 t}q_1(t)q_2(t). \qquad \text{[both concede at } t]$$

Player $i=1,2$

$$U_i(0,G_0,G_j) \tag{A4}$$
$$= (1-\alpha_0)\underbrace{(1-\tilde{q}_0(0))(1-q_j(0))}_{\text{neither 0 nor } j \text{ concedes at 0}}+\frac{1-\alpha_0}{2}\Big[\underbrace{q_{0,j}(0)(1-q_j(0))}_{i\to 0\to j \text{ at } 0}+\underbrace{(1-\tilde{q}_0(0))q_j(0)}_{i\to 0\leftarrow j \text{ at } 0}\Big]$$
$$+\Big(\frac{1-\alpha_0}{2}+\frac{\alpha_i}{2}\Big)\underbrace{q_{0,i}(0)(1-q_j(0))}_{i\leftrightarrow 0\not\leftarrow j \text{ at } 0}+\Big(\frac{1-\alpha_0}{3}+\frac{\alpha_i}{3}\Big)\underbrace{q_{0,i}(0)q_j(0)}_{i\leftrightarrow 0\leftarrow j \text{ at } 0}+\frac{1-\alpha_0}{3}\underbrace{q_{0,j}(0)q_j(0)}_{i\to 0\leftrightarrow j \text{ at } 0}.$$

$$U_i(\infty,G_0,G_j) = \alpha_i\underbrace{q_{0,i}(0)(1-q_j(0))}_{j\not\to 0\to i \text{ at } 0}+\frac{\alpha_i}{2}\underbrace{q_{0,i}(0)q_j(0)}_{j\to 0\to i \text{ at } 0}+\int_0^\infty \alpha_i e^{-r_i v}\underbrace{(1-G_j(v))\mathrm{d}G_{0,i}(v)}_{j\not\to 0\to i \text{ at } v} \tag{A5}$$
$$+\sum_{v>0}\frac{\alpha_i}{2}e^{-r_i v}\underbrace{q_{0,i}(v)q_j(v)}_{j\to 0\to i \text{ at } v}.$$

$$U_i(t,G_0,G_j) \tag{A6}$$
$$= \alpha_i q_{0,i}(0)(1-q_j(0)) \qquad [j\not\to 0\to i \text{ at } 0]$$
$$+\frac{\alpha_i}{2}q_{0,i}(0)q_j(0) \qquad [j\to 0\to i \text{ at } 0]$$
$$+\int_0^t \alpha_i e^{-r_i v}(1-G_j(v))\mathrm{d}G_{0,i}(v) \qquad [j\not\to 0\to i \text{ at } v\in(0,t)]$$
$$+\sum_{0<v<t}\frac{\alpha_i}{2}e^{-r_i v}q_{0,i}(v)q_j(v) \qquad [j\to 0\to i \text{ at } v\in(0,t)]$$
$$+(1-\alpha_0)e^{-r_i t}(1-\tilde{G}_0(t))(1-G_j(t)) \qquad \text{[neither 0 nor } j \text{ concedes by } t]$$
$$+\frac{1-\alpha_0}{2}e^{-r_i t}q_{0,j}(t)(1-G_j(t)) \qquad [i\to 0\to j \text{ at } t]$$

$$+\frac{1-\alpha_0}{2}e^{-r_it}(1-\tilde{G}_0(t))q_j(t) \qquad [i \to 0 \leftarrow j \text{ at } t]$$
$$+\Big(\frac{1-\alpha_0}{2}+\frac{\alpha_i}{2}\Big)e^{-r_it}q_{0,i}(t)(1-G_j(t)) \qquad [i \leftrightarrow 0 \not\leftarrow j \text{ at t}]$$
$$+\Big(\frac{1-\alpha_0}{3}+\frac{\alpha_i}{3}\Big)e^{-r_it}q_{0,i}(t)q_j(t) \qquad [i \leftrightarrow 0 \leftarrow j \text{ at t}]$$
$$+\frac{1-\alpha_0}{3}e^{-r_it}q_{0,j}(t)q_j(t). \qquad [i \to 0 \leftrightarrow j \text{ at t}]$$

A.2 Omitted Proofs

Proof of Lemma 1. We only consider player $i \in \{1,2\}$, the Proof of player 0 is analogous. Suppose that $G_i(\mathcal{T}) > 0$. Then, player i can profitably deviate by shifting weight from $\mathcal{T}$ to s. Let a mixed strategy $\hat{G}_i \in \mathcal{G}_i$ be such that: (1) $\hat{G}_i(B) = G_i(B) + G_i(\mathcal{T})$ for all measurable $B \subseteq [0,\infty]$ containing s, and (2) $\hat{G}_i(B) = G_i(B \setminus \mathcal{T})$ for all measurable $B \subseteq [0,\infty]$ not containing s. We have

$$\begin{aligned}
&\int_0^\infty U_i(t,G_0,G_j)\mathrm{d}\hat{G}_i(t)\\
&= \int_0^\infty U_i(t,G_0,G_j)\mathrm{d}G_i(t) - \int_{\mathcal{T}} U_i(\tau,G_0,G_j)dG_i(\tau) + U_i(s,G_0,G_j)G_i(\mathcal{T})\\
&> \int_0^\infty U_i(t,G_0,G_j)\mathrm{d}G_i(t) - \int_{\mathcal{T}} U_i(s,G_0,G_j)dG_i(\tau) + U_i(s,G_0,G_j)G_i(\mathcal{T})\\
&= \int_0^\infty U_i(t,G_0,G_j)\mathrm{d}G_i(t),
\end{aligned}$$

which implies that G_i is not an equilibrium strategy. ■

Proof of Proposition 2. First, we establish that in any immediate-agreement equilibrium, either $q_1(0) = q_2(0) = 1$ or $q_1(0), q_2(0) < 1$. To see this, notice that if $q_i(0) = 1$ and $q_j(0) < 1$, then an argument in the proof of Proposition 1 shows that player j has a profitable deviation to an immediate concession. Moreover, Proposition 1 implies that there exists a continuum of pure-strategy Nash equilibria, in all of which $q_1(0) = q_2(0) = 1$ and $\tilde{q}_0(0) = q_{0,1}(0) + q_{0,2}(0) = 0$. Therefore, there always exists a continuum of immediate-agreement Nash equilibria in which players 1 and 2 concede at time t $= 0$ with certainty and player 0 concedes later. To complete the characterization of immediate-agreement equilibria, we only need to check if there exist equilibria with $q_1(0), q_2(0) < 1$ and $\tilde{q}_0(0) = 1$.

(i) Suppose first $\alpha_1 \neq \alpha_2$ and assume without loss of generality that $\alpha_1 < \alpha_2$. We claim that there does not exists an equilibrium in which $q_1(0), q_2(0) < 1$. Suppose to the contradiction that there exists such an equilibrium, i.e. $q_1(0), q_2(0) < 1$. Since

$1-\alpha_1 > 1-\alpha_2$, we have $q_{0,1}(0) = 1$. But then player 2 can profitably deviate to conceding at $t = 0$ with certainty since $\frac{1-\alpha_0}{2} > 0$, which is a contradiction.

Suppose next $\alpha_1 = \alpha_2 < 2(1-\alpha_0)$, and again suppose to the contrary of the claim that there exists an equilibrium with $q_1(0), q_2(0) < 1$. Assume without loss of generality that $q_{0,1}(0) \leq q_{0,2}(0)$, and let $\gamma = q_{0,1}(0) \in [0, \frac{1}{2}]$, which implies $q_{0,2}(0) = 1-\gamma$. Notice that for $q_1(0) < 1$ to hold in equilibrium, player 1 must weakly prefer conceding at time $t > 0$ to conceding at $t = 0$, i.e., we must have

$$q_2(0)\Big(\frac{\gamma}{3}\alpha_1 + \frac{1}{3}(1-\alpha_0)\Big) + (1-q_2(0))\Big(\frac{\gamma}{2}\alpha_1 + \frac{1}{2}(1-\alpha_0)\Big) \leq q_2(0)\frac{\gamma}{2}\alpha_1 + (1-q_2(0))\gamma\alpha_1 \tag{A7}$$

where the left hand side is player 1's expected payoff if she concedes at time $t = 0$, and the right hand side is his expected payoff if he concedes at time $t > 0$. However, since $\gamma \in [0, \frac{1}{2}]$ and $\alpha_1 < 2(1-\alpha_0)$, we have $\frac{\gamma}{3}\alpha_1 + \frac{1}{3}(1-\alpha_0) > \frac{\gamma}{2}\alpha_1$ and $\frac{\gamma}{2}\alpha_1 + \frac{1}{2}(1-\alpha_0) > \gamma\alpha_1$, a contradiction to the inequality A7.

(ii) Let $\alpha_1 = \alpha_2 \equiv \alpha \geq 2(1-\alpha_0)$. We will construct an immediate-agreement equilibrium in which $\tilde{q}_0(0) = 1$. Since $0 < 1-\alpha < \alpha_0$, we can find $T \in \mathbb{R}_+$ such that $1-\alpha \geq \alpha_0 e^{-r_0 T}$. Then, for any $t_1, t_2 \geq T$, the following comprises an immediate-agreement Nash equilibrium: $q_{0,1}(0) = q_{0,2}(0) = \frac{1}{2}$, $q_1(t_1) = 1$, $q_2(t_2) = 1$. Player 0 does not have an incentive to deviate since any deviation gives her utility not greater than $\alpha_0 e^{-r_0 T}$ which is in turn less than $1-\alpha$. Consider a deviation by player $i = 1, 2$ to conceding at time $t = 0$. The payoff of player i along the path is $\frac{1}{2}\alpha$, and the payoff from the deviation is $\frac{1}{4}\alpha + \frac{1}{2}(1-\alpha_0) \leq \frac{1}{2}\alpha$, so player i will not deviate either. ■

Proof of Lemma 2: We first prove that there is no delay equilibrium if $\alpha_1 \neq \alpha_2$. Suppose $\alpha_i > \alpha_j$. Let $\hat{t} \equiv \sup\{t \geq 0 : G_1(t) < 1, G_2(t) < 1\}$ with the convention that $\hat{t} = 0$ if the set $\{t \geq 0 : G_1(t) < 1, G_2(t) < 1\}$ is empty. In particular, the equilibrium is immediate-agreement if $\hat{t} = 0$. Consider a delay equilibrium, i.e., $\hat{t} > 0$.

Step 1*: Player 0 does not concede to player i strictly before time $\hat{t}$, that is, $\lim_{t\uparrow\hat{t}_0} G_{0,i}(t) = 0$. Additionally, $q_{0,i}(\hat{t}) > 0$ implies that there exists $k \in \{1, 2\}$ such that $\lim_{t\uparrow\hat{t}} G_k(t) = 1$.*

Since player 0 strictly prefers to concede to player j rather than concede to player i prior to $\hat{t}$, player 0 would never concede to player i with positive probability. Formally, for any $t \in (0, \hat{t})$ we have

$$\begin{aligned} U_0(t, j, G_1, G_2) - U_0(t, i, G_1, G_2) = {} & \frac{\alpha_i - \alpha_j}{2} e^{-r_0 t}\Big[q_1(t)(1-G_2(t)) + q_2(t)(1-G_1(t))\Big] \\ & + \frac{\alpha_i - \alpha_j}{3} e^{-r_0 t} q_1(t) q_2(t) + (\alpha_i - \alpha_j) e^{-r_0 t}(1-G_1(t))(1-G_2(t)) \end{aligned}$$

$$\geq (\alpha_i - \alpha_j)e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) > 0,$$

where the last inequality holds since $G_1(t) < 1, G_2(t) < 1$ by the definition of $\hat{t}$.

For $t = 0$, we have

$$\begin{aligned}
&U_0(0, j, G_1, G_2) - U_0(0, i, G_1, G_2)\\
=&(\alpha_i - \alpha_j)(1 - q_1(0))(1 - q_2(0)) + \frac{\alpha_i - \alpha_j}{2}\Big[q_1(0)(1 - q_2(0)) + q_2(0)(1 - q_1(0))\Big] + \frac{\alpha_i - \alpha_j}{3} q_1(0)q_2(0)\\
\geq&(\alpha_i - \alpha_j)(1 - q_1(0))(1 - q_2(0)) > 0,
\end{aligned}$$

where the last inequality holds since $q_1(0) < 1$ and $q_2(0) < 1$ for any non-degenerate equilibrium. Therefore, by Lemma 1, $\lim_{t\uparrow\hat{t}} G_{0,i}(t) = G_{0,i}\Big([0, \hat{t})\Big) = 0$.

From Lemma 1, we also know that $q_{0,i}(\hat{t}) > 0$ only if $U_0(\hat{t}, j, G_1, G_2) - U_0(\hat{t}, i, G_1, G_2) \leq 0$, that is

$$\begin{aligned}
0 \geq\ & U_0(\hat{t}, j, G_1, G_2) - U_0(\hat{t}, i, G_1, G_2)\\
=\ & \frac{\alpha_i - \alpha_j}{2} e^{-r_0\hat{t}}\Big[q_1(\hat{t})(1 - G_2(\hat{t})) + q_2(\hat{t})(1 - G_1(\hat{t}))\Big]\\
& + \frac{\alpha_i - \alpha_j}{3} e^{-r_0\hat{t}} q_1(\hat{t})q_2(\hat{t}) + (\alpha_i - \alpha_j)e^{-r_0\hat{t}}(1 - G_1(\hat{t}))(1 - G_2(\hat{t})).
\end{aligned}$$

Since $\alpha_i > \alpha_j$, there exists $k \in \{1, 2\}$ such that $G_k(\hat{t}) = 1$ and $q_k(\hat{t}) = 0$, that is, $\lim_{t\uparrow\hat{t}} G_k(t) = 1$.

Step 2: *Player i concedes before time $\hat{t}$ with certainty, that is, $G_i(\hat{t}_0) = 1$.*

If $\hat{t} = \infty$, then $G_i(\hat{t}) = 1$ holds trivially. If $\hat{t} < \infty$, suppose by contradiction that $G_i(\hat{t}) < 1$. By the definition of $\hat{t}$, we have $G_j(\hat{t}) = 1$. In turn, *Step 1* implies that whenever $q_{0,i}(\hat{t}) > 0$ we must also have $q_j(\hat{t}) = 0$, that is, $q_{0,i}(\hat{t})q_j(\hat{t}) = 0$. Since $\lim_{t\uparrow\hat{t}} G_{0,i}(t) = 0$ and $\hat{t} > 0$, we have $q_{0,i}(v) = G_{0,i}(v) = 0$ for all $v \in [0, \hat{t})$. Therefore, we have $U_i(t, G_0, G_j) = 0$ for all $t \in (\hat{t}, \infty]$. And we also have

$$U_i(0, G_0, G_j) = (1 - \alpha_0)(1 - q_j(0)) + \frac{1 - \alpha_0}{3} q_{0,j}(0)q_j(0) > 0$$

since $q_j(0) < 1$ in any non-degenerate equilibrium. It follows that $U_i(0, G_0, G_j) - U_i(t, G_0, G_j) > 0$ for all $t \in (\hat{t}, \infty]$ and thus, by Lemma 1, $G_i\Big((\hat{t}, \infty]\Big) = 0$, i.e., $G_i\Big([0, \hat{t}]\Big) = G_i(\hat{t}) = 1$, which is a contradiction.

Step 3: *Player i concedes at time $\hat{t}$ with zero probability, that is $q_i(\hat{t}) = 0$.*

Suppose, by contradiction, that $q_i(\hat{t}) > 0$. There are two distinct cases: (i) $q_{0,i}(\hat{t}) > 0$; (ii) $q_{0,i}(\hat{t}) = 0$. If $q_{0,i}(\hat{t}) > 0$, we have $G_j(\hat{t}) = 1$ and $q_j(\hat{t}) = 0$, and thus $U_i(\hat{t}, G_0, G_j) =$

$0 < U_i(0, G_0, G_j)$, similar to *Step 2*. But then $q_i(\hat{t}) = 0$ by Lemma 1, a contradiction. And if $q_{0,i}(\hat{t}) = 0$, then we have $G_{0,i}(\hat{t}) = 0$ by *Step 1*. Hence,

$$\begin{aligned} U_i(\hat{t}, G_0, G_j) &= (1-\alpha_0)e^{-r_i\hat{t}}(1 - G_{0,j}(\hat{t}))(1 - G_j(\hat{t})) \\ &\quad + \frac{1-\alpha_0}{2}e^{-r_i\hat{t}}\Big[q_{0,j}(\hat{t})(1 - G_j(\hat{t})) + (1 - G_{0,j}(\hat{t}))q_j(\hat{t})\Big] \\ &\quad + \frac{1-\alpha_0}{3}e^{-r_i\hat{t}}q_{0,j}(\hat{t})q_j(\hat{t}) \\ &\leq (1-\alpha_0)e^{-r_i\hat{t}}\Big(1 - \lim_{s\uparrow\hat{t}} G_{0,j}(s)\Big)\Big(1 - \lim_{s\uparrow\hat{t}} G_j(s)\Big). \end{aligned}$$

On the other hand, we can find $\delta > 0$ sufficiently small such that $q_j(\delta) = 0$, $q_{0,j}(\delta) = 0$, and thus

$$\begin{aligned} U_i(\delta, G_0, G_j) &= (1-\alpha_0)e^{-r_i\delta}(1 - G_{0,j}(\delta))(1 - G_j(\delta)) \\ &> (1-\alpha_0)e^{-r_i\hat{t}}\Big(1 - \lim_{s\uparrow\hat{t}} G_{0,j}(s)\Big)\Big(1 - \lim_{s\uparrow\hat{t}} G_j(s)\Big). \end{aligned}$$

Again, Lemma 1 implies $q_i(\hat{t}_0) = 0$, a contradiction.

Step 4: *Player i must concede at time 0 with certainty, that is, $G_i(0) = 1$, contradicting $\hat{t} > 0$.*

For any $t \in (0, \hat{t}_0)$, and for any $\zeta \in (0, 1)$ there exists a sufficiently small number $\delta \in (0, \zeta t)$ such that $q_j(\delta) = q_{0,j}(\delta) = 0$. Then, for any $\tau \in [\zeta t, t]$ and analogous to *Step 3*, we have

$$U_i(\tau, G_0, G_j) < U_i(\delta, G_0, G_j).$$

Therefore, by Lemma 1, $G_i\Big([\zeta t, t]\Big) = 0$ for any $t \in (0, \hat{t}_0)$, and any $\zeta \in (0, 1)$. Finally, it follows that

$$1 - G_i(0) = G_i\Big((0, \hat{t}_0)\Big) = G_i\Big(\bigcup_{k=3}^{\infty}[\frac{1}{k}\hat{t}_0, \frac{k-1}{k}\hat{t}_0]\Big) = \lim_{k\to\infty} G_i\Big([\frac{1}{k}\hat{t}_0, \frac{k-1}{k}\hat{t}_0]\Big) = 0,$$

which implies that player i concedes at the start of the game with certainty. This implies that $\hat{t} = 0$, which is a contradiction. Therefore, we must have $\alpha_1 = \alpha_2$ in every delay equilibrium.

We next prove that there is no delay equilibrium if $\alpha \equiv \alpha_1 = \alpha_2 \leq 2(1-\alpha_0)$. Suppose, for the purpose of contradiction, that $\alpha \leq 2(1-\alpha_0)$ and there exists a delay equilibrium (G_0, G_1, G_2). By Lemma A2 and equation (A6), there exists $\delta > 0$ sufficiently small such that

$$U_i(\delta, G_0, G_j) = \alpha q_{0,i}(0)(1-q_j(0)) + \frac{\alpha}{2} q_{0,i}(0) q_j(0) + (1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0)) + o(\varepsilon)$$

for any $i, j \in \{1, 2\}$, $j \neq i$ and any arbitrarily small ε. Conditional on the agreement reached with a delay, let $P_i \in [0, 1]$ be the conditional probability that player 0 concedes to player i, let $Q_i \in [0, 1]$ be the conditional probability that player i concedes to player 0, and let U_i^{eqm} denote player i's equilibrium payoff. We have

$$U_i^{eqm} < \alpha q_{0,i}(0)(1-q_j(0)) + \frac{\alpha}{2} q_{0,i}(0) q_j(0) + (1-\tilde{q}_0(0))(1-q_j(0))[\alpha P_i + (1-\alpha_0) Q_i]$$

where $P_1 + P_2 + Q_1 + Q_2 = 1$. Here, the inequality holds strictly since the agreement is delayed with strictly positive probability and payoffs are discounted for a delayed agreement. Therefore,

$$\begin{aligned}\sum_{i=1}^{2} \frac{U_i^{eqm} - \alpha q_{0,i}(0)(1-q_j(0)) - \frac{\alpha}{2} q_{0,i}(0) q_j(0)}{1-q_j(0)} &< (1-\tilde{q}_0(0))[\alpha(P_1+P_2) + (1-\alpha_0)(Q_1+Q_2)] \\ &\leq (1-\tilde{q}_0(0))\alpha\end{aligned}$$

where the second inequality follows from the assumption that $1 - \alpha_0 < \alpha$. Therefore, there exists $\kappa, \bar{\kappa} \in \{1, 2\}$, $\kappa \neq \bar{\kappa}$ such that

$$\frac{U_\kappa^{eqm} - \alpha q_{0,\kappa}(0)(1-q_{\bar{\kappa}}(0)) - \frac{\alpha}{2} q_{0,\kappa}(0) q_{\bar{\kappa}}(0)}{1-q_{\bar{\kappa}}(0)} < \frac{\alpha(1-\tilde{q}_0(0))}{2},$$

that is,

$$U_\kappa^{eqm} < \alpha q_{0,\kappa}(0)(1-q_{\bar{\kappa}}(0)) + \frac{\alpha}{2} q_{0,\kappa}(0) q_{\bar{\kappa}}(0) + \frac{\alpha}{2}(1-\tilde{q}_0(0))(1-q_{\bar{\kappa}}(0)).$$

Since $\alpha \leq 2(1-\alpha_0)$, we have $U_\kappa^{eqm} < U_\kappa(\delta, G_0, G_{\bar{\kappa}})$, a contradiction to Lemma 1.

■

Proof of Lemma 3. By Lemma 1, it is sufficient to show that player 0 strictly prefers conceding to player $\kappa \in \{1, 2\}$ at some time $\delta > 0$ rather than conceding to player κ at time 0.

Fix $\varepsilon > 0$. By Lemma A2, there exists $\delta > 0$ such that for any $j \in \{1, 2\}$:

(i) $q_j(\delta) = 0$;

(ii) $G_j(\delta) - q_j(0) < \varepsilon$;

(iii) $1 - e^{-r_0\delta} < \varepsilon$.

Using the definitions in (A1) and (A3), and substituting $q_1(\delta) = q_2(\delta) = 0$ by (i), we can write:

$$\begin{aligned} U_0(\delta,\kappa,G_1,G_2) - U_0(0,\kappa,G_1,G_2) &= \alpha_0\left[1-(1-q_1(0))(1-q_2(0))\right] \\ &+ (1-\alpha)e^{-r_0\delta}(1-G_1(\delta))(1-G_2(\delta)) \\ &+ \sum_{i=1}^{2}\int_0^{\delta}\alpha_0 e^{-r_0 v}(1-G_i(v))\mathrm{d}G_j(v) + \sum_{0<v<\delta}\alpha_0 e^{-r_0 v}q_1(v)q_2(v) \\ &- (1-\alpha)(1-q_1(0))(1-q_2(0)) - \left(\frac{1-\alpha}{3}+\frac{2\alpha_0}{3}\right)q_1(0)q_2(0) \\ &- \left(\frac{1-\alpha}{2}+\frac{\alpha_0}{2}\right)\left[q_1(0)(1-q_2(0))+q_2(0)(1-q_1(0))\right]. \end{aligned}$$

Note that the expressions in the third line are non-negative, and the expression in the second line can be expressed as $(1-\alpha)(1-q_1(0))(1-q_2(0)) + o(\varepsilon)$ by conditions (ii) and (iii). Combining this with the remaining terms, we obtain that $U_0(\delta,\kappa,G_1,G_2) - U_0(0,\kappa,G_1,G_2)$ is bounded below by

$$\frac{\alpha_0+\alpha-1}{3}q_1(0)q_2(0) + \frac{\alpha_0+\alpha-1}{2}\left[q_1(0)(1-q_2(0))+q_2(0)(1-q_1(0))\right] + o(\varepsilon)$$

which is positive since $\alpha_0+\alpha>1$, $q_i(0)>0$ and ε is arbitrarily small. ■

Proof of Lemma 4. The proof is similar to the proof of Lemma 3. By Lemma 1, it is sufficient to show that player 0 strictly prefers to concede to player $\kappa \in \{1,2\}$ at time $t+\delta$ where $\delta > 0$ rather than concede to player κ at time t.

Fix $\varepsilon > 0$. By Lemma A2, there exists $\delta > 0$ such that for any $i \in \{1,2\}$:

(i) $q_i(t+\delta) = 0$,

(ii) $G_i(t+\delta) - G_i(t) < \varepsilon$,

(iii) $1 - e^{-r_0\delta} < \varepsilon$.

Using the definition in (A3) for $t+\delta$ and t, and substituting $q_1(t+\delta) = q_2(t+\delta) = 0$ by (i), we have:

$$\begin{aligned} &U_0(t+\delta,\kappa,G_1,G_2) - U_0(t,\kappa,G_1,G_2) \\ &= (1-\alpha)e^{-r_0(t+\delta)}(1-G_1(t+\delta))(1-G_2(t+\delta)) - (1-\alpha)e^{-r_0 t}(1-G_1(t))(1-G_2(t)) \\ &\quad + \sum_{i=1}^{2}\int_t^{t+\delta}\alpha_0 e^{-r_0 v}(1-G_i(v))\mathrm{d}G_j(v) + \sum_{t\le v<t+\delta}\alpha_0 e^{-r_0 v}q_1(v)q_2(v) \end{aligned}$$

$$+\sum_{i=1}^{2}\alpha_0 e^{-r_0 t}(1-G_i(t))q_j(v)+\alpha_0 e^{-r_0 v}q_1(v)q_2(v)$$
$$-\sum_{i=1}^{2}\left(\frac{1-\alpha}{2}+\frac{\alpha_0}{2}\right)e^{-r_0 t}(1-G_i(t))q_j(t)-\left(\frac{1-\alpha}{3}+\frac{2\alpha_0}{3}\right)e^{-r_0 t}q_1(t)q_2(t).$$

The two expressions in the second line are non-negative. The first expression in the first line can be written as $(1-\alpha)e^{-r_0 t}(1-G_1(t))(1-G_2(t))+o(\varepsilon)$ by (ii) and (iii). Combining these with the remaining terms, we obtain that $U_0(t+\delta,\kappa,G_1,G_2)-U_0(t,\kappa,G_1,G_2)$ is bounded below by

$$\sum_{i=1}^{2}\frac{\alpha_0+\alpha-1}{2}e^{-r_0 t}q_i(t)(1-G_j(t))+\frac{\alpha_0+\alpha-1}{3}e^{-r_0 t}q_1(t)q_2(t)+o(\varepsilon).$$

For sufficiently small ε, this expression is stricly positive since $\alpha_0+\alpha>1$, $q_i(t)>0$, and $\lim_{\tau\uparrow t}G_j(\tau)<1$, because the latter implies that either $q_j(t)>0$ or $G_j(t)<1$ or both.

■

Proof of Lemma 5. The proof is similar to the proofs of Lemma 3 and Lemma 4. By Lemma 1, it is sufficient to show that player j strictly prefers to concede at an earlier time $t-\delta>0$.

Fix $\varepsilon>0$. By Lemma A2, there exists $\delta>0$ such that:

(i) $1-e^{-r_0\delta}<\varepsilon$,

(ii) $q_i(t-\delta)=q_{0,i}(t-\delta)=0$,

(iii) $G_i(t)-G_i(t-\delta)-q_i(t)<\varepsilon$, and $G_{0,\kappa}(t)-G_{0,\kappa}(t-\delta)-q_{0,\kappa}(t)<\varepsilon$ for any $\kappa\in\{1,2\}$.

First, using the definition in (A6) for $t-\delta$ and t, and substituting $q_i(t-\delta)=q_{0,i}(t-\delta)=0$ from (ii), we have:

$$\begin{aligned}
&U_j(t-\delta,G_0,G_i)-U_j(t,G_0,G_i)\\
&=-\int_{t-\delta}^{t}\alpha e^{-r_j v}(1-G_i(v))\mathrm{d}G_{0,j}(v)-\sum_{t-\delta<v<t}\frac{\alpha_j}{2}e^{-r_j v}q_{0,j}(v)q_i(v)\\
&\quad+(1-\alpha_0)\left[e^{-r_j(t-\delta)}(1-\tilde{G}_0(t-\delta))(1-G_i(t-\delta))-e^{-r_j t}(1-\tilde{G}_0(t))(1-G_i(t))\right]\\
&\quad-\frac{1-\alpha_0}{2}e^{-r_j t}\left[q_{0,i}(t)(1-G_i(t))+(1-\tilde{G}_0(t))q_i(t)\right]\\
&\quad-\left(\frac{1-\alpha_0}{2}+\frac{\alpha}{2}\right)e^{-r_j t}q_{0,j}(t)(1-G_i(t))-\left(\frac{1-\alpha_0}{3}+\frac{\alpha}{3}\right)e^{-r_j t}q_{0,j}(t)q_i(t)-\frac{1-\alpha_0}{3}e^{-r_j t}q_{0,i}(t)q_i(t).
\end{aligned}$$

Next, $e^{-r_j v}(1 - G_i(v)) \leq 1$ for all v implies that

$$\int_{t-\delta}^{t} \alpha e^{-r_j v}(1 - G_i(v))\mathrm{d}G_{0,j}(v) \leq \int_{t-\delta}^{t} \alpha \mathrm{d}G_{0,j}(v) = \alpha(G_{0,j}(t) - q_{0,j}(t) - G_{0,j}(t-\delta)) < \alpha\varepsilon$$

where the last inequality uses (iii). Therefore, for all $i \neq j \in \{1, 2\}$ we have

$$\int_{t-\delta}^{t} \alpha e^{-r_j v}(1 - G_i(v))\mathrm{d}G_{0,j}(v) = o(\varepsilon).$$

Also, $e^{-r_j v} \leq 1$ for all v implies that

$$\sum_{t-\delta<v<t} e^{-r_j v} q_{0,j}(v) q_i(v) \leq \sum_{t-\delta<v<t} q_{0,j}(v) q_i(v) \leq \left(\sum_{t-\delta<v<t} \sqrt{q_{0,j}(v) q_i(v)} \right)^2 \leq \sum_{t-\delta<v<t} q_{0,j}(v) \sum_{t-\delta<v<t} q_i(v) < \varepsilon^2$$

where the third inequality follows from the Cauchy-Schwarz inequality and the fourth inequality uses (iii). Therefore,

$$\sum_{t-\delta<v<t} \frac{\alpha_j}{2} e^{-r_j v} q_{0,j}(v) q_i(v) = \frac{\alpha_j}{2} \sum_{t-\delta<v<t} e^{-r_j v} q_{0,j}(v) q_i(v) = o(\varepsilon).$$

Finally, (i) and (iii) imply that

$$e^{-r_j(t-\delta)}(1 - \tilde{G}_0(t-\delta))(1 - G_i(t-\delta)) = e^{-r_j t}(1 - \tilde{G}_0(t) + \tilde{q}_0(t))(1 - G_i(t) + q_i(t)) + o(\varepsilon).$$

Combining these results, we obtain:

$$\begin{aligned}
&U_j(t-\delta, G_0, G_i) - U_j(t, G_0, G_i) \\
&= o(\varepsilon) + (1-\alpha_0)e^{-r_j t}\left[(1 - \tilde{G}_0(t))q_i(t) + (1 - G_i(t))\tilde{q}_0(t) + \tilde{q}_0(t)q_i(t)\right] \\
&\quad - \frac{1-\alpha_0}{2} e^{-r_j t}\left[q_{0,i}(t)(1 - G_i(t)) + (1 - \tilde{G}_0(t))q_i(t)\right] \\
&\quad - \left(\frac{1-\alpha_0}{2} + \frac{\alpha}{2}\right) e^{-r_j t} q_{0,j}(t)(1 - G_i(t)) - \left(\frac{1-\alpha_0}{3} + \frac{\alpha}{3}\right) e^{-r_j t} q_{0,j}(t) q_i(t) - \frac{1-\alpha_0}{3} e^{-r_j t} q_{0,i}(t) q_i(t).
\end{aligned}$$

Suppose, by contradiction, $q_j(t) > 0$. Then $\lim_{\tau \uparrow t} G_j(\tau) < 1$, and thus by Lemma 4, we have $q_{0,1}(t) = q_{0,2}(t) = 0$ and $\tilde{G}_0(t) = \lim_{\tau \uparrow t} \tilde{G}_0(\tau) < 1$. Therefore,

$$U_j(t-\delta, G_0, G_i) - U_j(t, G_0, G_i) = \frac{1-\alpha_0}{2} e^{-r_j t}(1 - \tilde{G}_0(t))q_i(t) + o(\varepsilon)$$

For sufficiently small ε, this expression is strictly positive since $\alpha_0 < 1$, $\tilde{G}_0(t) < 1$ and $q_i(t) > 0$ (by assumption). This contradicts Lemma 1.

■

Proof of Lemma 6: Fix player $i \in \{1,2\}$ and player $j \in \{1,2\}$ such that $i \neq j$. We prove part (ii). The proof of part (i) is analogous. Fix player $i \in \{1,2\}$ and player $j \in \{1,2\}$ such that $i \neq j$. Lemma A1 implies that for any $\zeta \in (t,s)$ there is an earlier time $\delta \in (t,\zeta)$ such that $q_j(\delta) = 0$ and $\tilde{q}_0(\delta) = 0$.

To begin, we show that for any time $\tau \in (\zeta, s]$, player i strictly prefers to concede at time δ. Fix $\tau \in (\zeta, s]$. Since $G_{0,i}(t) = G_{0,i}(s)$, and $t < \delta < \tau \leq s$, we have $G_{0,i}(\delta) = G_{0,i}(\tau)$ and $q_{0,i}(v) = q_{0,j}(v) = 0$ for any $v \in (t,s]$. Then, (A6) implies

$$\begin{aligned} U_i(\delta, G_0, G_j) - U_i(\tau, G_0, G_j) &= (1-\alpha_0)e^{-r_i\delta}(1-\tilde{G}_0(\delta))(1-G_j(\delta)) \\ &\quad -(1-\alpha_0)e^{-r_i\tau}(1-\tilde{G}_0(\tau))(1-G_j(\tau)) - \frac{1-\alpha_0}{2}e^{-r_i\tau}(1-\tilde{G}_0(\tau))q_j(\tau) \\ &\quad -\frac{1-\alpha_0}{2}e^{-r_i\tau}q_{0,j}(\tau)(1-G_j(\tau)) - \frac{1-\alpha_0}{3}e^{-r_i\tau}q_{0,j}(\tau)q_j(\tau) \\ &\geq (1-\alpha_0)e^{-r_i\delta}(1-\tilde{G}_0(\delta))(1-G_j(\delta)) \\ &\quad -(1-\alpha_0)e^{-r_i\tau}(1-\lim_{w\uparrow\tau}\tilde{G}_0(w))(1-\lim_{w\uparrow\tau}G_j(w)) \\ &\geq (1-\alpha_0)(e^{-r_i\delta} - e^{-r_i\tau})(1-\lim_{w\uparrow\tau}\tilde{G}_0(w))(1-\lim_{w\uparrow\tau}G_j(w)) \end{aligned}$$

where $1-\lim_{w\uparrow\tau}\tilde{G}_0(w) = 1-\tilde{G}_0(\tau)+\tilde{q}_0(\tau) = 1-\tilde{G}_0(\tau)+q_{0,j}(\tau)$, and $1-\lim_{w\uparrow\tau}G_j(w) = 1-G_j(\tau)+q_j(\tau)$. If $(1-\lim_{w\uparrow\tau}\tilde{G}_0(w))(1-\lim_{w\uparrow\tau}G_j(w)) > 0$, then we have $U_i(\delta, G_0, G_j) - U_i(\tau, G_0, G_j) > 0$ as desired. If $(1-\lim_{w\uparrow\tau}\tilde{G}_0(w))(1-\lim_{w\uparrow\tau}G_j(w)) = 0$, which implies that either (i) $\tilde{G}_0(\tau) = 1$ and $\tilde{q}_0(\tau) = 0$, or (ii) $G_j(\tau) = 1$ and $q_j(\tau) = 0$, or both. In any case, it follows that

$$U_i(\delta, G_0, G_j) - U_i(\tau, G_0, G_j) = (1-\alpha_0)e^{-r_i\delta}(1-\tilde{G}_0(\delta))(1-G_j(\delta)) > 0.$$

The last strict inequality holds by the definition of $\hat{t}$ and the fact that $\delta < \hat{t}$.

Next, by part (i) of Lemma 1, for any $\zeta \in (t,s)$ we have $G_i(s) - G_i(\zeta) = G_i((\zeta, s]) = 0$. Finally we have

$$G_i(s) - G_i(t) = G_i((t,s]) = G_i\left(\bigcup_{k=1}^{\infty}\left(t + \frac{1}{k}\left(\frac{s-t}{2}\right), s\right]\right) = \lim_{k\to\infty} G_i\left(\left(t + \frac{1}{k}\left(\frac{s-t}{2}\right), s\right]\right) = 0,$$

which concludes the proof. ■

Proof of Lemma 7: **Proof of Part (i)**: The proof is by contradiction. Suppose $q_i(t) > 0$, $\lim_{\tau\uparrow t} G_{0,1}(\tau) + G_{0,2}(\tau) < 1$, and $\lim_{\tau\uparrow t} G_j(\tau) < 1$. Since $q_i(t) > 0$, Lemma 4 and Lemma 5 imply $q_j(t) = q_{0,1}(t) = q_{0,2}(t) = 0$ and therefore $G_{0,1}(t) + G_{0,1}(t) < 1$ and $G_j(t) < 1$.

We first establish that there exists $\delta > 0$ such that $G_{0,1}(t) = G_{0,1}(t-\delta)$ and $G_{0,2}(t) = G_{0,2}(t-\delta)$.

Fix $\varepsilon > 0$. By Lemma A2, there exists $\delta \in (0, \varepsilon)$ such that:

(i) $G_i(t+\delta) - G_i(t-\delta) = G_i(t+\delta) - G_i(t) + q_i(t) + \lim_{\tau\uparrow t} G_i(\tau) - G_i(t-\delta) < \frac{\varepsilon}{2} + q_i(t) + \frac{\varepsilon}{2} = q_i(t) + \varepsilon$, and similarly, $G_j(t+\delta) - G_j(t-\delta) < q_j(t) + \varepsilon = \varepsilon$;

(ii) $q_i(\tau) \in [0, \frac{\varepsilon}{2})$ for $\tau \in [t-\delta, t+\delta]\backslash\{t\}$, and $q_j(\tau), q_{0,1}(\tau), q_{0,2}(\tau) \in [0, \frac{\varepsilon}{2})$ for $\tau \in [t-\delta, t+\delta]$;

(iii) $1 - e^{-r_0\delta} < \varepsilon$.

Then, for any $\tau \in (t-\delta, t)$ we have for any $\kappa \in \{1, 2\}$:

$$
\begin{aligned}
& U_0(t+\delta, \kappa, G_1, G_2) - U_0(\tau, \kappa, G_1, G_2) \\
= & \int_\tau^{t+\delta} \alpha_0 e^{-r_0 v}(1 - G_2(v))\mathrm{d}G_1(v) + \int_\tau^{t+\delta} \alpha_0 e^{-r_0 v}(1 - G_1(v))\mathrm{d}G_2(v) \\
& +(1-\alpha)e^{-r_0(t+\delta)}(1 - G_1(t+\delta))(1 - G_2(t+\delta)) - (1-\alpha)e^{-r_0\tau}(1 - G_1(\tau))(1 - G_2(\tau)) + o(\varepsilon) \\
= & \alpha_0 e^{-r_0 t}(1 - G_j(t))q_i(t) \\
& +(1-\alpha)e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) - (1-\alpha)e^{-r_0 t}(1 - G_i(t) + q_i(t))(1 - G_j(t)) + o(\varepsilon) \\
= & (\alpha_0 + \alpha - 1)e^{-r_0 t}(1 - G_j(t))q_i(t) + o(\varepsilon) \\
> & 0,
\end{aligned}
$$

where the last inequality holds when ε is sufficiently small, $q_i(t) > 0$, and $G_j(t) < 1$. Then, Lemma 1 and $q_{0,\kappa}(t) = 0$ imply $G_{0,\kappa}(t) = G_{0,\kappa}(t-\delta)$.

We are now ready to show a contradiction. Since $G_{0,\kappa}(t-\delta) = G_{0,\kappa}(t)$ for $\kappa \in \{1, 2\}$, $\lim_{\tau\uparrow t} G_{0,1}(\tau) + G_{0,2}(\tau) < 1$, and $\lim_{\tau\uparrow t} G_j < 1$, Lemma 6 implies that $G_i(t-\delta) = G_i(t)$, which contradicts $q_i(t) > 0$.

Proof of Part (ii): Suppose $q_{0,1}(t), q_{0,2}(t) > 0$, $\frac{\min\{q_{0,1}(t), q_{0,2}(t)\}}{\max\{q_{0,1}(t), q_{0,2}(t)\}} > \frac{1-\alpha_0}{\alpha+\alpha_0-1}$, $\lim_{\tau\uparrow t} G_1(\tau) < 1$, $\lim_{\tau\uparrow t} G_2(\tau) < 1$. Since $q_{0,i}(t) > 0$, Lemma 4 imply $q_1(t) = q_2(t) = 0$ and therefore $G_1(t) < 1$ and $G_2(t) < 1$.

We first establish that there exists $\delta > 0$ such that $G_1(t) = G_1(t-\delta)$ and $G_2(t) = G_2(t-\delta)$.

Fix $\varepsilon > 0$. By Lemma A2, there exists $\delta \in (0, \varepsilon)$ such that:

(i) $G_{0,i}(t+\delta) - G_{0,i}(t-\delta) < q_{0,i}(t) + \varepsilon$, $\tilde{G}_0(t+\delta) - \tilde{G}_0(t-\delta) < \tilde{q}_0(t) + \varepsilon$, and $G_j(t+\delta) - G_j(t-\delta) < \varepsilon$;

(ii) $q_{0,i}(\tau) \in [0, \frac{\varepsilon}{2})$ for $\tau \in [t-\delta, t+\delta]\backslash\{t\}$, and $q_1(\tau), q_2(\tau) \in [0, \frac{\varepsilon}{2})$ for $\tau \in [t-\delta, t+\delta]$;

(iii) $1 - e^{-r_i\delta} < \varepsilon$.

Then, for any $\tau \in (t-\delta, t)$ we have for any $i, j \in \{1,2\}$, $j \neq i$:

$$\begin{aligned}
& U_i(t+\delta, G_0, G_j) - U_i(\tau, G_0, G_j) \\
= & \int_\tau^{t+\delta} \alpha e^{-r_i v}(1-G_j(v))\mathrm{d}G_{0,i}(v) + (1-\alpha_0)e^{-r_i(t+\delta)}(1-\tilde{G}_0(t+\delta))(1-G_j(t+\delta)) \\
& -(1-\alpha_0)e^{-r_i\tau}(1-\tilde{G}_0(\tau))(1-G_j(\tau)) + o(\varepsilon) \\
= & \alpha e^{-r_i t}(1-G_j(t))q_{0,i}(t) + (1-\alpha_0)e^{-r_i(t)}(1-\tilde{G}_0(t))(1-G_j(t)) \\
& -(1-\alpha_0)e^{-r_i t}(1-\tilde{G}_0(t)+\tilde{q}_0(t))(1-G_j(t)) + o(\varepsilon) \\
= & [\alpha q_{0,i}(t) - (1-\alpha_0)\tilde{q}_0(t)]e^{-r_i t}(1-G_j(t)) + o(\varepsilon) \\
> & \; 0,
\end{aligned}$$

where the last inequality holds when ε is sufficiently small, $\frac{\min\{q_{0,1}(t), q_{0,2}(t)\}}{\max\{q_{0,1}(t), q_{0,2}(t)\}} > \frac{1-\alpha_0}{\alpha+\alpha_0-1}$, and $G_j(t) < 1$. Then, Lemma 1 and $q_i(t) = 0$ implies $G_i(t-\delta) = G_i(t)$.

We are now ready to show a contradiction. Since $G_i(t-\delta) = G_i(t)$ for $i \in \{1,2\}$, $\lim_{\tau\uparrow t} G_1(\tau) < 1$, and $\lim_{\tau\uparrow t} G_2(\tau) < 1$, Lemma 6 implies that $G_{0,i}(t-\delta) = G_{0,i}(t)$, which contradicts $q_{0,i}(t) > 0$.

■

Proof of Lemma 8: **Proof of Part (i)** By hypothesis, no one ends the game by time s (i.e., $\lim_{\tau\uparrow s} G_1(\tau) < 1$, $\lim_{\tau\uparrow s} G_2(\tau) < 1$, and $\lim_{\tau\uparrow s} \tilde{G}_0(\tau) < 1$). Therefore, part (ii) of Lemma 7 implies that G_1 and G_2 are continuous on $(0, s)$. Since $\tilde{G}_0$ is strictly increasing over the interval $[t, s]$, the chair is indifferent between conceding at time τ and conceding time $\tau + \mathrm{d}\tau$ with $t \leq \tau < \tau + \mathrm{d}\tau < s$. She is also indifferent between conceding to player 1 and conceding to player 2 at any of those times since both of the competing players have the same demand. Therefore, for all $i, j \in \{1,2\}$ we have

$$\begin{aligned}
0 = & \; U_0(\tau + \mathrm{d}\tau, i, G_1, G_2) - U_0(\tau, j, G_1, G_2) \\
= & \int_\tau^{\tau+\mathrm{d}\tau} \alpha_0 e^{-r_0 v}(1-G_2(v))\mathrm{d}G_1(v) + \int_\tau^{\tau+\mathrm{d}\tau} \alpha_0 e^{-r_0 v}(1-G_1(v))\mathrm{d}G_2(v) \\
& +(1-\alpha)\Big[e^{-r_0(\tau+\mathrm{d}\tau)}(1-G_1(\tau+\mathrm{d}\tau))(1-G_2(\tau+\mathrm{d}\tau)) - e^{-r_0\tau}(1-G_1(\tau))(1-G_2(\tau))\Big].
\end{aligned}$$

For sufficiently small $\mathrm{d}\tau > 0$, the right hand side is equal to

$$\begin{aligned}
& \alpha_0 e^{-r_0\tau}[(1-G_2(\tau))g_1(\tau) + (1-G_1(\tau))g_2(\tau)]\mathrm{d}\tau \\
& -(1-\alpha)e^{-r_0\tau}\Big[r_0(1-G_1(\tau))(1-G_2(\tau)) + g_1(\tau)(1-G_2(\tau)) + g_2(\tau)(1-G_1(\tau))\Big]\mathrm{d}\tau.
\end{aligned}$$

It follows that for almost all $\tau \in [t, s)$ we have

$$(1-G_1(\tau))g_2(\tau)+(1-G_2(\tau))g_1(\tau) = \frac{(1-\alpha)r_0}{\alpha_0+\alpha-1}(1-G_1(\tau))(1-G_2(\tau)) = \mu(1-G_1(\tau))(1-G_2(\tau)).$$

Dividing both sides by $(1-G_1(\tau))(1-G_2(\tau))$, we find

$$\lambda_1(\tau) + \lambda_2(\tau) = \mu.$$

Proof of Part (ii): Since $\lim_{\tau\uparrow s} G_j(\tau) < 1$ and $\lim_{\tau\uparrow s} \tilde{G}_0(\tau) < 1$, Lemma 7 implies that $q_i(\tau) = 0$ for any $i \in \{1, 2\}$ and $\tau \in [t, s]$. Provided that $\tilde{G}_0$ is increasing, we have $\tilde{q}_i(\tau) = 0$ for $\tau \in [t, s]$ almost everywhere. Since G_i is strictly increasing on $[t, s]$, we have $G_i(\tau) < G_i(s) \le 1$ for all $t \le \tau < s$ and player i must attain her maximal utility almost everywhere on $[t, s)$. Therefore, player i must be indifferent between conceding at times τ and $\tau + \mathrm{d}\tau$ such that $t \le \tau < \tau + \mathrm{d}\tau < s$ and $\tilde{q}_0(\tau) = \tilde{q}_0(\tau + \mathrm{d}\tau) = 0$, that is, we must have:

$$\begin{aligned}
0 &= U_i(\tau + \mathrm{d}\tau, G_0, G_j) - U_i(\tau, G_0, G_j) \\
&= \int_\tau^{\tau+\mathrm{d}\tau} \alpha e^{-r_i v}(1 - G_j(v))\mathrm{d}G_{0,i}(v) \\
&\quad +(1-\alpha_0)\Big[e^{-r_i(\tau+\mathrm{d}\tau)}(1-\tilde{G}_0(\tau+\mathrm{d}\tau))(1-G_j(\tau+\mathrm{d}\tau)) - e^{-r_i\tau}(1-\tilde{G}_0(\tau))(1-G_j(\tau))\Big]
\end{aligned}$$

For sufficiently small $\mathrm{d}\tau > 0$, we have:

$$\begin{aligned}
&0 = \alpha e^{-r_i\tau}(1-G_j(\tau))g_{0,i}(\tau)\mathrm{d}\tau \\
&\quad -(1-\alpha_0)e^{-r_i\tau}\Big[r_i(1-\tilde{G}_0(\tau))(1-G_j(\tau)) + \tilde{g}_0(\tau)(1-G_j(\tau)) + g_j(\tau)(1-\tilde{G}_0(\tau))\Big]\mathrm{d}\tau.
\end{aligned}$$

It follows that

$$\alpha(1-G_j(\tau))g_{0,i}(\tau) = (1-\alpha_0)\Big[r_i(1-\tilde{G}_0(\tau))(1-G_j(\tau)) + \tilde{g}_0(\tau)(1-G_j(\tau)) + g_j(\tau)(1-\tilde{G}_0(\tau))\Big].$$

$$\iff \quad \frac{g_j(\tau)}{1-G_j(\tau)} = -r_i - \frac{\tilde{g}_0(\tau)}{1-\tilde{G}_0(\tau)} + \frac{\alpha}{1-\alpha_0}\frac{g_{0,i}(\tau)}{1-\tilde{G}_0(\tau)}.$$

for almost all $\tau \in [t, s)$.

Proof of Part (ii) (a): if there exists $\hat{\tau} \in (t, s)$ such that $\tilde{q}_0(\hat{\tau}) > 0$, player i must be indifferent between conceding at times $\hat{\tau} - \mathrm{d}\tau$ and $\hat{\tau} + \mathrm{d}\tau$ such that $t \le \hat{\tau} - \mathrm{d}\tau < \hat{\tau} + \mathrm{d}\tau < s$ and $\tilde{q}_0(\hat{\tau} - \mathrm{d}\tau) = \tilde{q}_0(\hat{\tau} + \mathrm{d}\tau) = 0$, that is, we must have:

$$0 = U_i(\hat{\tau} + \mathrm{d}\tau, G_0, G_j) - U_i(\hat{\tau} - \mathrm{d}\tau, G_0, G_j)$$

$$
\begin{aligned}
= & \int_{\hat{\tau}-\mathrm{d}\tau}^{\hat{\tau}+\mathrm{d}\tau} \alpha e^{-r_i v}(1-G_j(v))\mathrm{d}G_{0,i}(v) \\
& +(1-\alpha_0)\Big[e^{-r_i(\hat{\tau}+\mathrm{d}\tau)}(1-\tilde{G}_0(\hat{\tau}+\mathrm{d}\tau))(1-G_j(\hat{\tau}+\mathrm{d}\tau)) - e^{-r_i(\hat{\tau}-\mathrm{d}\tau)}(1-\tilde{G}_0(\hat{\tau}-\mathrm{d}\tau))(1-G_j(\hat{\tau}-\mathrm{d}\tau))\Big]
\end{aligned}
$$

For sufficiently small $\mathrm{d}\tau > 0$, we have:

$$
0 = \alpha e^{-r_i\hat{\tau}}(1-G_j(\hat{\tau}))q_{0,i}(\hat{\tau}) - (1-\alpha_0)e^{-r_i\hat{\tau}}\tilde{q}_0(\hat{\tau})(1-G_j(\hat{\tau}))
$$

Therefore, $q_{0,i}(\hat{\tau}) = \frac{1-\alpha_0}{\alpha}\tilde{q}_0(\hat{\tau}) = \frac{1-\alpha_0}{\alpha+\alpha_0-1}q_{0,j}(\hat{\tau})$.

Proof of Part (ii) (b): if $\tilde{q}_0(s) > 0$, for $\mathrm{d}\tau \in (0, s-t)$ such that $\tilde{q}_0(s-\mathrm{d}\tau) = 0$,

$$
\begin{aligned}
0 \geq & \; U_i(s, G_0, G_j) - U_i(s-\mathrm{d}\tau, G_0, G_j) \\
= & \int_{s-\mathrm{d}\tau}^{s} \alpha e^{-r_i v}(1-G_j(v))\mathrm{d}G_{0,i}(v) \\
& + \frac{1-\alpha_0}{2}e^{-r_i s}q_{0,j}(s)(1-G_j(s)) + (\frac{1-\alpha_0}{2}+\frac{\alpha}{2})e^{-r_i s}q_{0,i}(s)(1-G_j(s)) \\
& +(1-\alpha_0)\Big[e^{-r_i(s)}(1-\tilde{G}_0(s))(1-G_j(s)) - e^{-r_i(s-\mathrm{d}\tau)}(1-\tilde{G}_0(s-\mathrm{d}\tau))(1-G_j(s-\mathrm{d}\tau))\Big]
\end{aligned}
$$

For sufficiently small $\mathrm{d}\tau > 0$, we have:

$$
0 \geq \frac{1-\alpha_0}{2}e^{-r_i s}q_{0,j}(s)(1-G_j(s)) + (\frac{1-\alpha_0}{2}+\frac{\alpha}{2})e^{-r_i s}q_{0,i}(s)(1-G_j(s)) - (1-\alpha_0)e^{-r_i s}\tilde{q}_0(s)(1-G_j(s)).
$$

Therefore, $q_{0,i}(\hat{\tau}) \leq \frac{1-\alpha_0}{\alpha}\tilde{q}_0(s)$, i.e., $q_{0,i}(\hat{\tau}) \leq \frac{1-\alpha_0}{\alpha+\alpha_0-1}q_{0,j}(s)$.

Proof of Part (ii) (c): Similarly to Part (ii) (b), if $\tilde{q}_0(t) > 0$, we have $q_{0,i}(\hat{\tau}) \geq \frac{1-\alpha_0}{\alpha}\tilde{q}_0(t)$, i.e., $q_{0,i}(\hat{\tau}) \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1}q_{0,j}(t)$.

Proof of Part (iii): Since G_1 and G_2 are both strictly increasing over $[t, s]$, by (7) we have

$$
\begin{aligned}
\frac{g_1(\tau)}{1-G_1(\tau)} + \frac{\tilde{g}_0(\tau)}{1-\tilde{G}_0(\tau)} &= -r_2 + \frac{\alpha g_{0,2}(\tau)}{(1-\alpha_0)(1-\tilde{G}_0(\tau))}, \\
\frac{g_2(\tau)}{1-G_2(\tau)} + \frac{\tilde{g}_0(\tau)}{1-\tilde{G}_0(\tau)} &= -r_1 + \frac{\alpha g_{0,1}(\tau)}{(1-\alpha_0)(1-\tilde{G}_0(\tau))}.
\end{aligned}
$$

Adding these equations, we obtain

$$
\frac{g_1(\tau)}{1-G_1(\tau)} + \frac{g_2(\tau)}{1-G_2(\tau)} + 2\frac{\tilde{g}_0(\tau)}{1-\tilde{G}_0(\tau)} = -r_1 - r_2 + \frac{\alpha\tilde{g}_0(\tau)}{(1-\alpha_0)(1-\tilde{G}_0(\tau))}.
$$

Integrating both sides yields

$$
\ln(1-G_1(\tau)) + \ln(1-G_2(\tau)) = (r_1+r_2)\tau + C_2 + (\frac{\alpha}{1-\alpha_0} - 2)\ln(1-\tilde{G}_0(\tau)).
$$

Taking the exponentials of both sides, we obtain

$$(1-G_1(\tau))(1-G_2(\tau)) = C_3 e^{(r_1+r_2)\tau}(1-\tilde{G}_0(\tau))^{\frac{\alpha}{1-\alpha_0}-2}.$$

Using the initial condition at t, we have

$$\Big[\frac{1-\tilde{G}_0(\tau)}{1-\tilde{G}_0(t)}\Big]^{\frac{\alpha}{1-\alpha_0}-2} = \frac{(1-G_1(\tau))(1-G_2(\tau))}{(1-G_1(t))(1-G_2(t))}e^{-(r_1+r_2)(\tau-t)}.$$

By part (i)

$$\frac{1-\tilde{G}_0(\tau)}{1-\tilde{G}_0(t)} = e^{-\frac{\lambda+r_1+r_2}{\frac{\alpha}{1-\alpha_0}-2}(\tau-t)} = e^{-\rho(\tau-t)}.$$

It follows that $\frac{\tilde{g}_0(\tau)}{1-\tilde{G}_0(\tau)} = \rho$. Using (7) again, we obtain the desired result. ■

Proof of Lemma 9: We only prove that $\tilde{G}_0$ must be strictly increasing on $[0,\hat{t})$. The Proof of H is analogous.

The proof is by contradiction. Suppose $\tilde{G}_0(t) = \tilde{G}_0(s)$ for some $t, s \in [0,\hat{t})$ with $t < s$. The definition of $\hat{t}$ implies $\tilde{G}_0(t) = \tilde{G}_0(s) < 1$. This in turn implies by Lemma 6 that $H(t) = H(s)$. Using the definition of $\hat{t}$ one more time, we must have $H(t) = H(s) < 1$. Define $s^* = \sup\{s' > 0 \ : \ \tilde{G}_0(s') = \tilde{G}_0(t) < 1, \ H(s') = H(t) < 1\}$. We next show that $s^* \leq \hat{t}$. If not, then take $\tilde{t} \in (\hat{t}, s^*)$. Since $t < \tilde{t} < s^*$, the definition of s^* implies $\tilde{G}_0(\tilde{t}) = \tilde{G}_0(t) < 1$ and $H(\tilde{t}) = H(t) < 1$. But since $\tilde{t} > \hat{t}$ we also have $\max\{\tilde{G}_0(\tilde{t}), \ H(\tilde{t})\} = 1$ by the definition of $\hat{t}$ which is a contradiction.

Note that $\tilde{G}_0$ and H cannot have an atom point at s^* by Lemma 7 and the fact that $\lim_{s'\uparrow s^*}\tilde{G}_0(s') = \tilde{G}_0(t) < 1$ and $\lim_{s'\uparrow s^*} H(s') = H(t) < 1$. Therefore, we have $\tilde{G}_0(s^*) = \tilde{G}_0(t) < 1$ and $H(s^*) = H(t) < 1$, and $s^* < \infty$. We will use these facts to show that there exists $\delta > 0$ such that $\tilde{G}_0$ and H are constant on $[t, s^*+\delta)$, which is a contradiction of the definition of s^*.

It suffices to show that $\tilde{G}_0$ is constant on $[t, s^*+\delta)$, because Lemma 6 implies that if $\tilde{G}_0$ is constant, then H is constant as well. Since $\tilde{G}_0$ is constant over $[t, s^*]$, it is turn sufficent to show that $\tilde{G}_0[s^*, s^*+\delta) = 0$. By Lemma 1, we only need to show that player 0 strictly prefers to concede at time $t \notin [s^*, s^*+\delta)$ rather than conceding at any time $\tau \in [s^*, s^*+\delta)$. Fix $\varepsilon > 0$. By Lemma A2, there exists $\delta > 0$ such that $H(s^*+\delta) - H(s^*) = H(s^*+\delta) - H(t) < \varepsilon$. For any $\tau \in [s^*, s^*+\delta)$ and κ, we can write

$$\begin{aligned} U_0(\tau,\kappa,G_1,G_2) - U_0(t,\kappa,G_1,G_2) \quad &= \quad (1-\alpha)e^{-r_0\tau}(1-H(\tau))) - (1-\alpha)e^{-r_0 t}(1-H(t)) \\ &\quad + \int_t^\tau \alpha_0 e^{-r_0 v}\mathrm{d}H(v) + o(\varepsilon) \end{aligned}$$

$$= \quad (1-\alpha)(e^{-r_0\tau} - e^{-r_0 t})(1-H(t)) + o(\varepsilon) < 0,$$

as desired. ■

Proof of Lemma 11: To establish a contradiction, suppose that $\hat{t} < \infty$. First, this implies that $\tilde{q}_0(\hat{t}) > 0$, or $q_i(\hat{t}) > 0$ for some player $i \in \{1,2\}$. To see why, suppose $\tilde{q}_0(\hat{t}) = q_1(\hat{t}) = q_2(\hat{t}) = 0$. Then, by the definition of $\hat{t}$, either $\lim_{t\uparrow\hat{t}} \tilde{G}_0(t) = \tilde{G}_0(\hat{t}) = 1$ or there must exist a player $i \in \{1,2\}$ such that $\lim_{t\uparrow\hat{t}} G_i(t) = G_i(\hat{t}) = 1$. There are two cases, each leading to a contradiction.

Suppose first $\lim_{t\uparrow\hat{t}} \tilde{G}_0(t) = 1$. By the definition of $\hat{t}$, $G_1(t) < 1$, $G_2(t) < 1$, $\tilde{G}_0(t) < 1$ for all $t < \hat{t}$. Then, since G_1 and G_2 are strictly increasing, by Corollary 2, $\tilde{G}_0(t) = 1 - (1-\tilde{G}_0(0))e^{-\rho t}$. Taking the limit as t converges to $\hat{t}$, we establish contradiction as $\lim_{t\uparrow\hat{t}} \tilde{G}_0(t) = 1 - (1-\tilde{G}_0(0))e^{-\rho\hat{t}} < 1$.

Suppose now $\lim_{t\uparrow\hat{t}} G_i(t) = 1$ for some $i = 1,2$. This implies $\lim_{t\uparrow\hat{t}} H(t) = 1$. By the definition of $\hat{t}$, $G_1(t) < 1$, $G_2(t) < 1$, $\tilde{G}_0(t) < 1$ for all $t < \hat{t}$. Then, since $\tilde{G}_0$ is strictly increasing over $[0,\hat{t})$, part (i) of Lemma 8 implies $H(t) = 1-(1-H(0))e^{-\mu t} < 1$ for any $t \in [0,\hat{t})$. Taking the limit as t converges to $\hat{t}$ and recalling that $H(0) < 1$, we have $\lim_{t\uparrow\hat{t}} H(t) = 1-(1-H(0))e^{-\mu\hat{t}} < 1$, which is a contradiction.

Thus, we have shown that either $\tilde{q}_0(\hat{t}) > 0$ or $q_i(\hat{t}) > 0$ for some $i \in \{1,2\}$. Then Lemma 7 implies that $\lim_{t\uparrow\hat{t}} G_j(t) = 1$ for some $j \in \{1,2\}$ or $\lim_{t\uparrow\hat{t}} \tilde{G}_0(t) = 1$. But we have already shown that this leads to a contradiction. It follows that $\hat{t} = \infty$. ■

Proof of Lemma 12: Fix competing player $i = 1,2$, and let $j \neq i$. Since G_i is strictly increasing, there exists $t > 0$ such that player i weakly prefers conceding at time t to conceding at time 0 by Lemma 1, that is $U_i(t, G_0, G_j) - U_i(0, G_0, G_j) \geq 0$. Note that by Lemma 3, we have $q_j(0)\tilde{q}_0(0) = 0$. In addition, Lemma 7, Lemma 9, and Lemma 11 imply that $\tilde{q}_0(\tau) = q_j(\tau) = 0$ for any $\tau > 0$. Finally, $1 - \tilde{G}_0(t) = (1-\tilde{q}_0(0))e^{-\rho t}$ by Lemma 8. Together with A4 and A6 these imply that $U_i(t, G_0, G_j) - U_i(0, G_0, G_j)$ is equal to

$$\begin{aligned} &\alpha q_{0,i}(0) \int_0^t \alpha e^{-r_i v}(1-G_j(v))dG_{0,i}(v) + (1-\alpha_0)e^{-r_i t}(1-\tilde{q}_0(0))e^{-\rho t}(1-G_j(t)) \\ &\quad -(1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0)) - \frac{1-\alpha_0}{2}[q_{0,j}(0)+q_j(0)] - \frac{1-\alpha_0+\alpha}{2} q_{0,i}(0). \quad \text{(A8)} \end{aligned}$$

By equation (8), the definition of $\lambda_{0,i}$, and Corollary 2), we have

$$G_{0,i}(t) = q_{0,i}(0) + \frac{(1-\alpha_0)(1-\tilde{q}_0(0))}{\alpha} \int_0^t (\lambda_j(\tau) + \rho + r_i)e^{-\rho\tau} \mathrm{d}\tau.$$

By the definition of λ_j, we have

$$1 - G_j(t) = (1 - q_j(0))e^{-\Lambda_j(t)}$$

where $\Lambda_j(t) = \int_0^t \lambda_j(\tau)\mathrm{d}\tau$. Using these facts, (A8) becomes

$$\begin{aligned}\alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0))(1-e^{-(r_i+\rho)t-\Lambda_j(t)})\\ +(1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0))e^{-(r_i+\rho)t-\Lambda_j(t)} - (1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0))\\ -\frac{1-\alpha_0}{2}[q_{0,j}(0)+q_j(0)] - \frac{1-\alpha_0+\alpha}{2}q_{0,i}(0)\end{aligned}$$

It follows (after some algebra) that the condition $U_i(t, G_0, G_j) - U_i(0, G_0, G_j) \geq 0$ is equivalent to the condition

$$\frac{\alpha}{2}q_{0,i}(0) - \frac{1-\alpha_0}{2}[\tilde{q}_0(0) + q_j(0)] \geq 0.$$

For this to be possible, we must have $q_j(0) = 0$. Otherwise, $q_j(0) > 0$ implies $\tilde{q}_0(0) = 0$ by Lemma 3, and then $U_i(t, G_0, G_j) - U_i(0, G_0, G_j) = -\frac{1-\alpha_0}{2}q_j(0) < 0$, which is a contradiction. Therefore, $q_j(0) = 0$ and $\frac{\alpha}{2}q_{0,i}(0) \geq \frac{1-\alpha_0}{2}\tilde{q}_0(0)$. Since i was arbitrary, we have $q_1(0) = q_2(0) = 0$, and $q_{0,i}(0) \in [\frac{1-\alpha_0}{\alpha+\alpha_0-1}q_{0,j}(0), \frac{\alpha+\alpha_0-1}{1-\alpha_0}q_{0,j}(0)]$ as desired. ■

Proof of Proposition 3. Necessity: (i) By Lemma 7, there are no atom points in $G_{0,1}, G_{0,2}, G_1$ and G_2 over $(0, \hat{t})$ after the start of the game and before the end of the game. By Lemma 11, when both competing players gradually concede throughout the game we must have $\hat{t} = \infty$. Therefore, $G_{0,1}, G_{0,2}, G_1$ and G_2 must be continuous over $(0, \infty)$ in an equilibrium in which both competing players concede throughout the game. (ii) Immediately follows from Lemma 12. (iii) The fact that $\lambda_1(t) + \lambda_2(t) = \mu$ immediately follows from part (i) of Lemma 8. Since both competing players concede throughout the game, we must have $\lambda_1(t), \lambda_2(t) > 0$ for $t \geq 0$ almost everywhere. (iv) Immediately follows from part (iii) of Lemma 8.

Sufficiency: By condition (i), G_1 and G_2 are continuous over $(0, \infty)$, and thus λ_1 and λ_2 are well defined over $(0, \infty)$. Additionally, by (iii), $\lambda_i(t) = \frac{G_i'(t)}{1-G_i(t)} > 0$ for $i = 1, 2$. Therefore, $G_1' > 0$ and $G_2' > 0$, which implies that G_1 and G_2 are strictly increasing on $\bar{\mathbb{R}}_+$, i.e. both players gradually concede throughout the game.

We now establish that no player has strictly profitable deviation from the proposed strategy profile $(G_{0,1}, G_{0,2}, G_1, G_2)$. First consider player 0. By (A1) and condition (ii), we have

$$U_0(0, \kappa, G_1, G_2) = 1 - \alpha.$$

By (A2) and conditions (i) and (ii), we have

$$U_0(\infty, \kappa, G_1, G_2) = \sum_{i=1}^{2} \int_0^\infty \alpha_0 e^{-r_0 v}(1-G_j(v))dG_i(v) = \alpha_0 \int_0^\infty -e^{-r_0 v} d(1-G_1(v))(1-G_2(v))$$

where the second inequality follows from the product rule for derivatives. Condition (iii) gives us $(1 - G_1(v))(1 - G_2(v)) = e^{-\mu v}$. Therefore,

$$U_0(\infty, \kappa, G_1, G_2) = \alpha_0 \int_0^\infty e^{-r_0 v} \mu e^{-\mu v} dv = \alpha_0 \mu \left. \frac{e^{-(r_0+\mu)v}}{r_0 + \mu} \right|_\infty^0 = \frac{\alpha_0 \mu}{r_0 + \mu} = 1 - \alpha$$

where the last equality follows from the definition of μ given in (2).

By (A3) and conditions (i) and (ii), for any $t > 0$ we have

$$U_0(t, \kappa, G_1, G_2) = (1 - \alpha)e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) + \sum_{i=1}^{2} \int_0^t \alpha_0 e^{-r_0 v}(1 - G_j(v))\mathrm{d}G_i(v)$$

Similar to above, using the product rule for derivatives, the right hand side is equal to

$$(1 - \alpha)e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) + \alpha_0 \int_0^t -e^{-r_0 v} d(1 - G_1(v))(1 - G_2(v)).$$

Using the fact that $(1 - G_1(t))(1 - G_2(t)) = e^{-\mu t}$, this is equal to

$$(1 - \alpha)e^{-(r_0+\mu)t} + \alpha_0 \int_0^t \mu e^{-(r_0+\mu)v} dv = (1 - \alpha)e^{-(r_0+\mu)t} + (1 - \alpha)(1 - e^{-(r_0+\mu)t}).$$

Thus, we conclude that

$$U_0(t, \kappa, G_1, G_2) = 1 - \alpha.$$

for all $t \geq 0$. Therefore, given that players 1 and 2 gradually concede using the proposed mixed strategy profile (G_1, G_2), player 0 is indifferent about the concession time: no matter when she concedes, her expected payoff is always $1 - \alpha$.

Next we consider the competing players $i = 1, 2$. Similar to derivations above, by (A4) and condition (ii), we have

$$U_i(0, G_0, G_j) = (1 - \alpha_0)(1 - \tilde{q}_0(0)) + \frac{1 - \alpha_0}{2} q_{0,j}(0) + \Big(\frac{1 - \alpha_0 + \alpha}{2}\Big) q_{0,i}(0).$$

By (A5) and conditions (i) and (ii), we have

$$\begin{aligned}
U_i(\infty, G_0, G_j) &= \alpha q_{0,i}(0) + \int_0^\infty \alpha e^{-r_i v}(1-G_j(v))\mathrm{d}G_{0,i}(v) \\
&= \alpha q_{0,i}(0) + \int_0^\infty \alpha e^{-r_i v} e^{-\Lambda_j(v)}\mathrm{d}G_{0,i}(v) \qquad \text{(A9)}
\end{aligned}$$

where $\Lambda_j(v) \equiv \int_0^v \lambda_j(\tau)\mathrm{d}\tau$ and the second equation follows from the definition of λ_j which implies $1 - G_j(v) = (1 - q_j(0))e^{-\Lambda_j(v)} = e^{-\Lambda_j(v)}$ by condition (ii) for any $v \in (0,\infty)$. By the definition of $\lambda_{0,i}$, we have $G'_{0,i}(v) = \lambda_{0,i}(v)[1 - \tilde{G}_0(v)]$. By Corollary 2, the right hand side is equal to $\lambda_{0,i}(v)(1-\tilde{q}_0(0))e^{-\rho v}$. Thus using condition (iv), we obtain $G'_{0,i}(v) = \frac{1-\alpha_0}{\alpha}(\lambda_j(v) + r_i + \rho)(1-\tilde{q}_0(0))e^{-\rho v}$. Plugging in (A9), we obtain

$$\begin{aligned}
U_i(\infty, G_0, G_j) &= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0)) \int_0^\infty e^{-r_i v} e^{-\Lambda_j(v)}(\lambda_j(v) + r_i + \rho)e^{-\rho v}\mathrm{d}v \\
&= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0))\ e^{-(r_i+\rho)v - \Lambda_j(v)}\big|_\infty^0 \\
&= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0)) \\
&\geq U_i(0, G_0, G_j)
\end{aligned}$$

where the inequality follows from condition (ii).

Finally, by (A6) and conditions (i) and (ii), for any $t > 0$, we have

$$\begin{aligned}
U_i(t, G_0, G_j) &= \alpha q_{0,i}(0) + \int_0^t \alpha e^{-r_i v}(1-G_j(v))dG_{0,i}(v) + (1-\alpha_0)e^{-r_i t}(1-\tilde{G}_0(t))(1-G_j(t)) \\
&= \alpha q_{0,i}(0) + \int_0^t \alpha e^{-r_i v} e^{-\Lambda_j(v)} G'_{0,i}(v)dv + (1-\alpha_0)e^{-r_i t}(1-\tilde{q}_0(t))e^{-\rho t}e^{-\Lambda_j(t)} \\
&= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0))(1-e^{-(r_i+\rho)t-\Lambda_j(t)}) + (1-\alpha_0)(1-\tilde{q}_0(0))e^{-(r_i+\rho)t-\Lambda_j(t)} \\
&= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0)) \\
&= U_i(\infty, G_0, G_j) \\
&\geq U_i(0, G_0, G_j)
\end{aligned}$$

where the second equation follows from $1 - G_j(v) = e^{-\Lambda_j(v)}$, and the third equation follows from $G'_{0,i}(v) = \frac{1-\alpha_0}{\alpha}(\lambda_j(v) + r_i + \rho)(1-\tilde{q}_0(0))e^{-\rho v}$. Therefore, player i weakly prefers to concede after the start of the game, and is indifferent about the concession time after the start of the game. Thus, any (mixed) strategy that assigns zero probability mass at $t = 0$, as in the proposed strategy profile, is a best response. ■