Bias in estimating expected ratio of two gamma variables

Problem

Suppose $X \sim \Gamma(\lambda_x, \alpha_x)$ and $Y \sim \Gamma(\lambda_y, \alpha_y)$. That is,

$$\begin{aligned} f_X(x) &= \frac{1}{\Gamma(\alpha_x)} \cdot x^{\alpha_x - 1} e^{-\lambda_x x} \lambda_x^{\alpha_x} \\ f_Y(y) &= \frac{1}{\Gamma(\alpha_y)} \cdot y^{\alpha_y - 1} e^{-\lambda_y y} \lambda_y^{\alpha_y} \end{aligned}$$

We also take X and Y independent.

We want to estimate $\theta = E(X)/E(Y)$.

We define as estimator $T = \overline{X}/\overline{Y}$.

The exact value of the expected ratio of two Gamma r.v.'s

If $X \sim \Gamma(\lambda_x, \alpha_x)$ and $Y \sim \Gamma(\lambda_y, \alpha_y)$, then we can compute $\mu = E(X/Y)$ exactly. Indeed, because of independency

$$\mu = E(X) \cdot E(1/Y)$$

and

$$\begin{aligned} E\left(\frac{1}{Y}\right) &= \int_0^\infty \frac{1}{y} f_Y(y)\ dy \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha_y)} \cdot y^{\alpha_y - 2} e^{-\lambda_y y} \lambda_y^{\alpha_y}\ dy \\ &= \frac{\lambda_y}{\alpha_y - 1} \cdot \int_0^\infty \frac{1}{\Gamma(\alpha_y - 1)} \cdot y^{\alpha_y - 2} e^{-\lambda_y y} \lambda_y^{\alpha_y - 1}\ dy \\ &= \frac{\lambda_y}{\alpha_y - 1} \cdot 1 \end{aligned}$$

The last integral must be equal to one, as the integrand is the density of a $\Gamma(\lambda_y, \alpha_y - 1)$ variable.

Hence

$$\mu = \frac{\alpha_x}{\lambda_x} \cdot \frac{\lambda_y}{\alpha_y - 1}$$

Note that $\mu \neq \theta$.

The distributions of the sample means

If $X \sim \Gamma(\lambda_x, \alpha_x)$, then $n_x\overline{X} \sim \Gamma(\lambda_x, n_x\alpha_x)$ and $\overline{X} \sim \Gamma(n_x\lambda_x, n_x\alpha_x)$

We thus have

$$E(T) = \frac{\alpha_x}{\lambda_x} \cdot \frac{n_y\lambda_y}{n_y\alpha_y - 1} = \frac{\alpha_x}{\lambda_x} \cdot \frac{\lambda_y}{\alpha_y - 1/n_y}$$

The bias is than

$$\begin{aligned} E(T) - \theta &= \frac{\alpha_x}{\lambda_x} \cdot \frac{\lambda_y}{\alpha_y - 1/n_y} - \frac{\alpha_x}{\lambda_x} \cdot \frac{\lambda_y}{\alpha_y} = \frac{\alpha_x}{\lambda_x} \cdot \frac{\lambda_y}{\alpha_y} \cdot \left(\frac{1}{1 - 1/(n_y \alpha_y)} - 1 \right) \\ &= \theta \cdot \frac{1}{n_y \alpha_y - 1} \end{aligned}$$