From Markov Chains to 1D Markov Random Fields

Discrete state space MC

Suppose $(X_n; n \in \mathbb{N})$ is a discrete state Markov Chain with stationary transition probabilities $p_{ij} = P(X_n = j | X_{n-1} = i)$. We can apply Bayes' rule in the probability space determined by $\{X_{n-1} = i\}$:

$$\begin{aligned} P(X_n = j | X_{n-1} = i, X_{n+1} = k) &= \frac{P(X_n = j | X_{n-1} = i) \cdot P(X_{n+1} = k | X_n = j, X_{n-1} = i)}{P(X_{n+1} = k | X_{n-1} = i)} \\ &= \frac{P(X_n = j | X_{n-1} = i) \cdot P(X_{n+1} = k | X_n = j)}{\sum_{\ell \in \mathbb{N}} P(X_n = \ell | X_{n-1} = i) \cdot P(X_{n+1} = k | X_n = \ell)} \\ &= \frac{p_{ij} \cdot p_{jk}}{\sum_{\ell \in \mathbb{N}} p_{i\ell} \cdot p_{\ell k}} \end{aligned}$$

A two-state Markov Chain

Suppose now that the state space is binary, with transition probabilities $\alpha = P(X_n = 1 | X_{n-1} = -1)$ and $\beta = P(X_n = -1 | X_{n-1} = 1)$.

Then

$$\begin{aligned} P(X_n = 1 | X_{n-1} = 1, X_{n+1} = 1) &= \frac{(1-\beta)^2}{(1-\beta)^2 + \alpha\beta} \\ P(X_n = -1 | X_{n-1} = 1, X_{n+1} = 1) &= \frac{\alpha\beta}{(1-\beta)^2 + \alpha\beta} \\ P(X_n = 1 | X_{n-1} = -1, X_{n+1} = 1) &= \frac{\alpha(1-\beta)}{\alpha(1-\beta) + \alpha(1-\alpha)} \\ P(X_n = -1 | X_{n-1} = -1, X_{n+1} = 1) &= \frac{\alpha(1-\alpha)}{\alpha(1-\beta) + \alpha(1-\alpha)} \\ P(X_n = 1 | X_{n-1} = 1, X_{n+1} = -1) &= \frac{(1-\beta)\beta}{(1-\beta)\beta + \beta(1-\alpha)} \\ P(X_n = -1 | X_{n-1} = 1, X_{n+1} = -1) &= \frac{\beta(1-\alpha)}{(1-\beta)\beta + \beta(1-\alpha)} \\ P(X_n = 1 | X_{n-1} = -1, X_{n+1} = -1) &= \frac{\alpha\beta}{\alpha\beta + (1-\alpha)^2} \\ P(X_n = -1 | X_{n-1} = -1, X_{n+1} = -1) &= \frac{(1-\alpha)^2}{\alpha\beta + (1-\alpha)^2} \end{aligned}$$

We know that in a two-state Gibbs Random Field with an Ising model

$$\begin{aligned} & P(X_n = x_n | X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}) \\ = & \frac{\exp\left(-\tau(x_n x_{n-1} + x_n x_{n+1})\right) \cdot \exp\left(-\gamma x_n\right)}{\exp\left(-\tau(x_{n-1} + x_{n+1})\right) \exp\left(-\gamma x_n\right) + \exp\left(+\tau(x_{n-1} + x_{n+1})\right) \exp\left(\gamma x_n\right)} \end{aligned}$$

So, for instance, we can state

$$\frac{\alpha(1-\beta)}{\alpha(1-\beta)+\alpha(1-\alpha)} = P(X_n = 1|X_{n-1} = -1, X_{n+1} = 1) = \frac{e^{-\gamma}}{e^{-\gamma}+e^{\gamma}}$$

and

$$\frac{(1-\beta)^2}{(1-\beta)^2+\alpha\beta} = P(X_n = 1|X_{n-1} = 1, X_{n+1} = 1) = \frac{e^{-2\tau}e^{-\gamma}}{e^{-2\tau}e^{-\gamma}+e^{2\tau}e^{\gamma}}$$

By taking inverses on both sides, the former equation can be reworked into

$$\frac{1-\alpha}{1-\beta} = e^{2\gamma} \Leftrightarrow \gamma = \log\sqrt{\frac{1-\alpha}{1-\beta}}$$

The latter equation becomes

$$\frac{\alpha\beta}{(1-\beta)^2} = e^{2\gamma}e^{4\tau}$$

Merging the two expressions

$$\frac{\alpha\beta}{(1-\alpha)(1-\beta)} = e^{4\tau} \Leftrightarrow \tau = \frac{1}{4}\log\left(\frac{\alpha\beta}{(1-\alpha)(1-\beta)}\right)$$

The other equations would lead to the same expressions.

Alternatively, we can compute in this Ising model the one-side Markov transition probabilities. Denote $\mathcal{C}_n$ the cliques that contain site n, then

$$\begin{aligned}
& P(X_{n+1} = x_{n+1}|X_n = x_n) \\
= & \frac{P(\boldsymbol{X}_{(n,n+1)} = \boldsymbol{x}_{(n,n+1)})}{P(X_n = x_n)} \\
= & \frac{\sum_{\boldsymbol{y}_{I\setminus\{n,n+1\}}} \prod_{C\in\mathcal{C}_{n+1}} f_C(x_n x_{n+1} \boldsymbol{y}_{C\cap\partial(n+1)}) \prod_{C\in\mathcal{C}_n\setminus\mathcal{C}_{n+1}} f_C(x_n \boldsymbol{y}_{C\cap\partial n}) \prod_{C\in\mathcal{C}\setminus\mathcal{C}_n\setminus\mathcal{C}_{n+1}} f_C(\boldsymbol{y}_C)}{\sum_{\boldsymbol{y}_{I\setminus\{n+1\}}} \prod_{C\in\mathcal{C}_{n+1}} f_C(x_{n+1} \boldsymbol{y}_{C\cap\partial(n+1)}) \prod_{C\in\mathcal{C}_n\setminus\mathcal{C}_{n+1}} f_C(\boldsymbol{y}_C) \prod_{C\in\mathcal{C}\setminus\mathcal{C}_n\setminus\mathcal{C}_{n+1}} f_C(\boldsymbol{y}_C)} \\
= & \frac{\sum_{\boldsymbol{y}_{I\setminus\{n,n+1\}}} \prod_{C\in\mathcal{C}\setminus\mathcal{C}_n\setminus\mathcal{C}_{n+1}} f_C(\boldsymbol{y}_C) \prod_{C\in\mathcal{C}_n} f_C(x_n x_{n+1} \boldsymbol{y}_{C\cap\partial(n+1)}) \prod_{C\in\mathcal{C}_n\setminus\mathcal{C}_{n+1}} f_C(x_n \boldsymbol{y}_{C\cap\partial n})}{\sum_{\boldsymbol{y}_{I\setminus\{n,n+1\}}} \prod_{C\in\mathcal{C}\setminus\mathcal{C}_n\setminus\mathcal{C}_{n+1}} f_C(\boldsymbol{y}_C) \sum_{y_n} \prod_{C\in\mathcal{C}_{n+1}} f_C(x_{n+1} \boldsymbol{y}_{C\cap\partial(n+1)}) \prod_{C\in\mathcal{C}_n\setminus\mathcal{C}_{n+1}} f_C(\boldsymbol{y}_C)}
\end{aligned}$$

Denoting $a = 1-\alpha$, $b = 1-\beta$, $G = e^{2\gamma}$ and $T = e^{4\tau}$, the above system can be written as $\left\{\begin{array}{rcl} \frac{(1-a)(1-b)}{ab} & = & T \\ \frac{a}{b} & = & G \end{array}\right.$

Solving the system for a and b yields $\left\{\begin{array}{rcl} a & = & \frac{(G+1)\pm\sqrt{(G-1)^2+4GT}}{2(G-GT)} \\ b & = & \frac{(G+1)\pm\sqrt{(G-1)^2+4GT}}{2(1-T)} \end{array}\right.$

If $T \geq 1$, then $\sqrt{(G-1)^2+4GT} \geq G+1$, and $G - GT \leq 0$, along with $1 - T \leq 0$ so we must take the solution with the minus-sign (otherwise, we would find $a, b < 0$, which is impossible, as a and b are probabilities). If $0 \leq T \leq 1$, then the plus-sign solution satisfies $a = \frac{(G+1)+\sqrt{(G-1)^2+4GT}}{2G(1-T)} \geq \frac{(G+1)+(G-1)}{2G(1-T)} \geq 1$ which is impossible, since a is a probability.

Hence, we have $\begin{cases} \alpha &= 1 - \frac{(e^{2\gamma}+1)-\sqrt{(e^{2\gamma}-1)^2+4e^{2\gamma+4\tau}}}{2e^{2\gamma}(1-e^{4\tau})} \\ \beta &= 1 - \frac{(e^{2\gamma}+1)-\sqrt{(e^{2\gamma}-1)^2+4e^{2\gamma+4\tau}}}{2(1-e^{4\tau})} \end{cases}$

A two-state Markov process with transition matrix $\boldsymbol{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$ has as stationary distribution

$$\begin{aligned} P(X_n = -1) &= \frac{\beta}{\alpha+\beta} \\ P(X_n = 1) &= \frac{\alpha}{\alpha+\beta} \end{aligned}$$