Algebra

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1 Group

1.1 Groups

Definition 1.1 (Group). *A group $(G, *)$ is a set G together with a law of composition ($G * G \to G$) which is associative and has an identity element, and such that every element of G has an inverse. That is, $(G, *)$ satisfies*

1. *(Closure)* $\forall a, b \in G, a * b \in G$.
2. *(Identity)* $\exists e$, *s.t.* $\forall g \in G, g * e = e * g = g$.
3. *(Inverse)* $\forall g \in G$, $\exists g^{-1} in G$, *s.t.* $g * g^{-1} = e$.
4. *(Associativity)* $\forall a, b, c \in G, (a * b) * c = a * (b * c)$.

Remark 1.1. *Identity e is unique.*

Proof. Suppose to the contrary, $\exists e' \neq e$, s.t. $g * e' = e' * g = g$. Then, $e * g = g \Rightarrow e * e' = e'$; $g * e' = g \Rightarrow e * e' = e$. Therefore, $e * e' = e' = e$. Contradiction. □

Remark 1.2. $g^{-1} * g = e$.

Proof. Suppose $h * g = e$, we want to show $h = g^{-1}$.
$h = h * e = h * (g * g^{-1}) = (h * g) * g^{-1} = e * g^{-1} = g^{-1}$. □

Remark 1.3. $\forall g$, *its inverse* g^{-1} *is unique.*

Proof. Suppose $\exists h \neq g^{-1}$, s.t. $g * h = e$.
$h = e * h = (g^{-1} * g) * h = g^{-1} * (g * h) = g^{-1} * e = g^{-1}$. Contradiction. □

Remark 1.4. *If h is the inverse of g, i.e., $h = g^{-1}$, then g is the inverse of h, i.e., $g = h^{-1}$.*

Proof. $h^{-1} = e * h^{-1} = (g * h) * h^{-1} = g * (h * h^{-1}) = g * e = g$. □

Remark 1.5 (Cancellation Law)**.** *Let $\forall a, b, c \in G$. If $a * b = a * c$, then $b = c$. If $b * a = c * a$, then $b = c$.*

Proof. $b = e * b = (a^{-1} * a) * b = a^{-1} * (a * b) = a^{-1} * (a * c) = (a^{-1} * a) * c = e * c = c$. □

Definition 1.2 (Abelian Group). *An abelian group $(G, *)$ is a group whose law of composition is commutative. That is, $\forall a, b \in G$, $a * b = b * a$.*

Example 1.1. *Abelian Groups:*

1. $\mathbb{Z}^+$*: integers, with addition.*

2. $\mathbb{R}^+$*: real numbers, with addition.*

3. $\mathbb{R}^\times$*: nonzero real numbers, with multiplication.*

4. $\mathbb{C}^+$,$\mathbb{C}^\times$*: complex numbers, with addition or multiplication (nonzero).*

Example 1.2. *Non-Abelian Groups:*

1. *(General Linear Group)*
   $GL_n(\mathbb{R})$*(or* $GL_n(\mathbb{C})$*)* $= \{n \times n$ *real (or complex) matraces* A *with* $\det A \neq 0\}$.

2. *(Symmetric Group)* $S_n =$ *group of permutations of* $\{1, ..., n\}$*. The order of the group is* $n!$.

1.2 Subgroups

Definition 1.3 (Subgroup). *A subset* H *of a group* G *is called a subgroup if it has the following properties:*

1. *Closure: If* $a \in H$ *and* $b \in H$*, then* $a * b \in H$.

2. *Identity:* $e \in H$.

3. *Inverses: If* $a \in H$*, then* $a^{-1} \in H$.

Remark 1.6. *Associativity is automatic.*

Remark 1.7. *Every group has two obvious subgroups:*

1. *the whole group* $(G, *)$*;*

2. $(e, *)$.

Definition 1.4 (Proper Subgroup). *A subgroup is said to be a proper subgroup if it is not* $(G, *)$ *or* $(e, *)$.

Example 1.3. *Proper Subgroups:*

1. *The set* T *of invertible upper triangular* 2×2 *matrices*

   $$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} (a, d \neq 0)$$

   is a subgroup of the general linear group $GL_n(\mathbb{R})$.

2. *The set of complex numbers of absolute value* 1 *– the set of points on the unit circle in the complex plane – is a subgroup of* $\mathbb{C}^\times$.

3. *The subset* $b\mathbb{Z} = \{n \in \mathbb{Z} \mid n = bk$ *for some* $k \in \mathbb{Z}\}$ *is a subgroup of* $\mathbb{Z}^+$*. Moreover, every subgroup* H *of* $\mathbb{Z}^+$ *is of the type* $H = b\mathbb{Z}$ *for some integer* b.

Proposition 1.1. *Let a, b be integers, not both 0, and let d be the positive integer which generates the subgroup $a\mathbb{Z} + b\mathbb{Z}$. Then,*

1. *d can be written in the form $d = ar + bs$ for some integer r and s.*
2. *d divides a and b.*
3. *If integer e divides a and b, it also divides d.*

Definition 1.5 (Order of a Group). *The order of any group G is the number of its elements.*

$$|G| = \text{ number of elements of } G.$$

Definition 1.6 (Order of an Element). *An element of a group is said to have order m (possibly infinity) if the cyclic subgroup it generates has order m. That is, m is the smallest positive integer with the property $x^m = 1$ or, if the order is infinite, that $x^m \neq 1$ for all $m \neq 0$.*

Example 1.4. *For $GL_n(\mathbb{R})$*

1. $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ *is an element of order* 6.
2. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *has infinite order, since* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

Example 1.5. *Non-cyclic Groups:*

1. *The Klein four group $V = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$ is the simplest group which is not cyclic.*
2. *The quaternion group H is another example of a small subgroup of $GL_n(\mathbb{C})$ which is not cyclic. It consists of the eight matrices $H = \{\pm 1, \pm i, \pm j, \pm k\}$, where*

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

1.3 Isomorphisms

Definition 1.7 (Isomorphic). *Let G and G' be two groups. We want to say that they are isomorphic if all properties of the group structure of G hold for G' as well, and conversely. Denote $G \approx G'$.*

Definition 1.8 (Isomorphism). *An isomorphism φ from $(G, *)$ to $(G', *')$ is a bijective map which is compatible with the laws of composition. That is $\varphi(a * b) = \varphi(a) *' \varphi(b)$.*

Example 1.6. *Isomophisms:*

1. *Let $C = \{\ldots, a^{-2}, a^{-1}, 1, a, a^2, \ldots\}$ be an infinite cyclic group. Then the map $\varphi : \mathbb{Z}^+ \to C$ defined by $\varphi(n) = a^n$ is an isomophism.*
2. *Let $G = \{1, x, x^2, \ldots, x^{n-1}\}$ and $G' = \{1, y, y^2, \ldots, y^{n-1}\}$ be two cyclic groups, generated by elements x, y of the same order. Then the map which sends x^i to y^i is an isomorphism: Two cyclic groups of the same order are isomorphic.*

Definition 1.9 (Isomorphism Class). *The groups isomorphic to a given group G form what is called the isomorphism class of G, and any two groups in an isomorphism class are isomorphic.*

Definition 1.10 (Automophism). *A map $\varphi : G \to G$ is called an automophism of G.*

Remark 1.8. *The identity map is an automorphism.*

Remark 1.9. *The most important example of automorphism is conjugation: Let $b \in G$ be a fixed element. Then conjugation by b is the map φ from G to itself defined by*

$$\varphi(x) = bxb^{-1}.$$

This is an automophism. Any noncommutative group has some nontrivial conjugations, and so it has nontrivial automorphisms.

Definition 1.11 (Conjugate). *Two elements a, a' of a group G are called conjugate if $a' = bab^{-1}$ for some $b \in G$.*

Remark 1.10. *The conjugate behaves in much the same way as the element a itself; for example, it has the same order in the group. This follows from the fact that it is the image of a by an automorphism.*

1.4 Homomorphisms

Definition 1.12 (Homomorphism). *Let G, G' be groups. A homomorphism $\varphi : G \to G'$ is any map satisfying $\varphi(a * b) = \varphi(a) *' \varphi(b)$.*

Remark 1.11. *Note that φ does not need to be bijective.*

Example 1.7. *Homomorphisms:*

1. *the determinant function* $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$.
2. *the sign of a permutation* $\text{sign} : S_n \to \{\pm 1\}$.
3. *the map $\varphi : \mathbb{Z}^+ \to G$ defined by $\varphi(n) = a^n$, where a is a fixed element of G.*
4. *the inclusion map: $i : H \to G$ of a subgroup H into a group G, defined by $i(x) = x$.*

Remark 1.12. *A group homomorphism $\varphi : G \to G'$ carries the identity to the identity, and inverses to inverses.*

Proof. $\varphi(e) = \varphi(e * e) = \varphi(e) *' \varphi(e)$; $\varphi(a^{-1}) *' \varphi(a) = \varphi(a^{-1} * a) = \varphi e = e'$. □

Definition 1.13 (Image). *The image of a homomorphism $\varphi : G \to G'$ is the image of the map*

$$\operatorname{im} \varphi = \{x \in G' \mid x = \varphi(a) \text{ for some } a \in G\}.$$

Definition 1.14 (Kernel). *The kernel of φ is the set of elements of G which are mapped to the identity in G' :*

$$\ker \varphi = \{a \in G \mid \varphi(a) = e'\}.$$

Remark 1.13. *The kernel is a subgroup of G.*

Example 1.8. *Kernels:*

1. *The kernel of determinant homomorphism is called the special linear group*

$$SL_n(\mathbb{R}) = \{\textit{real } n \times n \textit{ matrices } A \mid \det A = 1\}.$$

2. *The kernel of the sign homomorphism is called the alternating group*

$$A_n = \{\textit{even permutations}\}.$$

Definition 1.15 (Normal Subgroup). *A subgroup N of a group G is called a normal subgroup if it has the following property: For every $a \in N$ and every $b \in G$, the conjugate $bab^{-1} \in N$.*

Remark 1.14. *The kernel of a homomorphism is a normal subgroup.*

Proof. $\varphi(bab^{-1}) = \varphi(b)\varphi(a)\varphi(b^{-1}) = \varphi(b)e\varphi(b)^{-1} = e'$. □

Remark 1.15. *Any subgroup of an abelian group G is normal.*

Definition 1.16 (Center of a Group). *The center of a group G, sometimes denoted by Z or by $Z(G)$, is the set of elements which commute with every element of G:*

$$Z = \{z \in G \mid zx = xz \textit{ for all } x \in G\}.$$

Remark 1.16. *The center of any group is a normal subgroup of the group.*

Example 1.9. *The center of $GL_n(\mathbb{R})$ is the group of scalar matrices, that is, those of the form $c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.*

1.5 Equivalence Relations and Partitions

Definition 1.17 (Equivalence Relation). *An equivalence relation on S is a relation which holds between certain elements of S. Denoted as $a \sim b$. An equivalence relation is required to be:*

1. *transitive: If $a \sim b$ and $b \sim c$, then $a \sim c$.*
2. *symmetric: If $a \sim b$, then $b \sim a$.*
3. *reflexive: $a \sim a$ for all $a \in S$.*

Definition 1.18 (Equivalence Class). *The equivalence class of a is*

$$C_a = \{b \in S \mid a \sim b\}.$$

Remark 1.17. *If C_a and C_b have an element d in common. Then $C_a = C_b$.*

Proposition 1.2. *Let $\varphi : G \to G'$ be a group homomorphism with kernel N, and let a, b be elements of G. Then $\varphi(a) = \varphi(b)$ if and only if $b = an$ for some element $n \in N$, or equivalently, if $a^{-1}b \in N$.*

Proof. Suppose $\varphi(a) = \varphi(b)$. Then $\varphi(a)^{-1}\varphi(b) = e'$. Since φ is a homomorphism, $\varphi(a^{-1}b) = e'$. By definition of the kernel, $a^{-1}b \in N$. Conversely, if $b = an$ for some $n \in N$. Then $\varphi(b) = \varphi(a)\varphi(n) = \varphi(a)e' = \varphi(a)$. $\square$

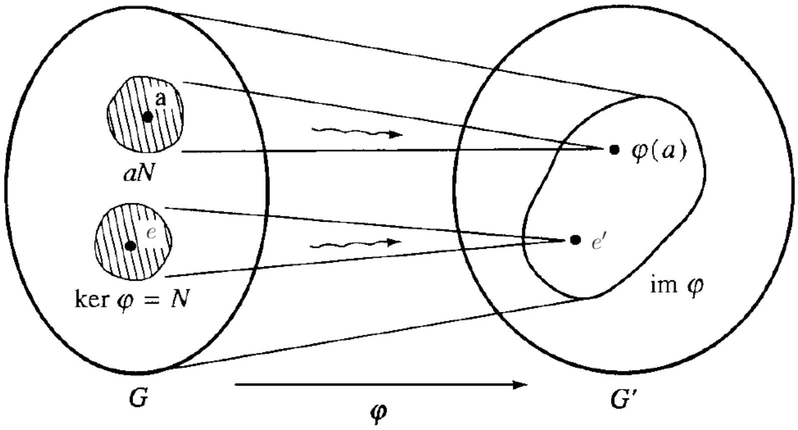

Figure 1: A schematic diagram of a group homomorphism.

Definition 1.19 (Coset). *The set of elements of the form an is denoted by aN and is called a coset of N in G:*

$$aN = \{g \in G \mid g = an \text{ for some } n \in N\}.$$

1.6 Cosets

One can define cosets for any subgroup H of a group G, not only for the kernel of a homomorphism.

Definition 1.20 (Left Coset). *A left coset is a subset of the form*

$$aH = \{ah \mid h \in H\}.$$

Remark 1.18. *The cosets are equivalence classes.*

Remark 1.19. *The left cosets of a subgroup partition the group.*

Remark 1.20. *Each coset aH has the same number of elements as H does.*

Definition 1.21 (Index). *The number of left eosets of a subgroup is called the index of H in G and is denoted by*

$$[G : H].$$

Theorem 1.1 (Lagrange's Theorem). *Let G be a finite group, and let H be a subgroup of G. The order of H divides the order of G.*

Corollary 1.1. *Let $\varphi : G \to G'$ be a homomorphism of finite groups. Then*

$$|G| = |\ker\varphi| \cdot |\operatorname{im}\varphi|.$$

Definition 1.22 (Right Coset). *A right coset is a subset of the form*

$$Ha = \{ha \mid h \in H\}.$$

Remark 1.21. *Right cosets need not be the same as left cosets.*

Proposition 1.3. *A subgroup H of a group G is normal if and only if every left coset is also a right coset. If H is normal, then $aH = Ha$ for every $a \in G$.*

Proof. Suppose that H is normal. For any $h \in H$, and any $a \in G$,

$$ah = (aha^{-1})a \in Ha.$$

This shows that $aH \subset Ha$. Similarly, $Ha \subset aH$. So, $aH = Ha$. Conversely, suppose that H is not normal. Then $\exists h \in H$ and $a \in G$ s.t. $aha^{-1} \notin H$. Then $ah \in aH$ and $ah \notin Ha$. So, $aH \neq Ha$. However, $a \in aH \cap Ha$. Therefore, aH can't be in some other right coset. This shows that the partition into left cosets is not the same as the partition into right cosets. □

1.7 Restriction of a Homomorphism to a Subgroup

Proposition 1.4. *The intersection $K \cap H$ of two subgroups is a subgroup of H. If K is a normal subgroup of G, then $K \cap H$ is a normal subgroup of H.*

Remark 1.22. *If $|H|$ and $|K|$ have no common factor, then $K \cap H = \{1\}$.*

Remark 1.23. *Suppose that a homomorphism $\varphi : G \to G'$ is given and that H is a subgroup of G. Then we may restrict φ to H:*

$$\varphi|_H : H \to G'.$$

The kernel of $\varphi|_H$ is

$$\ker \varphi|_H = (\ker \varphi) \cap H.$$

Proposition 1.5. *Let $\varphi : G \to G'$ be a homomorphism, and let H' be a subgroup of G'. Denote the inverse image $\varphi^{-1}(H') = \{x \in G \mid \varphi(x) \in H'\}$ by $\tilde{H}$.*

1. *$\tilde{H}$ is a subgroup of G.*

2. *If H' is a normal subgroup of G', then $\tilde{H}$ is a normal subgroup of G.*

3. *$\tilde{H}$ contains $\ker \varphi$.*

4. *The restriction of φ to $\tilde{H}$ defines a homomorphism $\tilde{H} \to H'$, whose kernel is $\ker \varphi$.*

Proof. We verify the conditions for a subgroup:

1. Closure: Suppose $x, y \in \tilde{H}$, then $\varphi(x)$ and $\varphi(y)$ are in H'. Since H' is a subgroup, $\varphi(x)\varphi(y) \in H'$. Since φ is a homomorphism, $\varphi(x)\varphi(y) = \varphi(xy) \in H'$. So, $xy \in \tilde{H}$.

2. Identity: $e \in \tilde{H}$ since $\varphi(e) = e' \in H'$.

3. Inverses: Suppose $x \in \tilde{H}$, so that $\varphi(x) \in H'$, then $\varphi(x)^{-1} \in H'$. Since φ is a homomorphism, $\varphi(x)^{-1} = \varphi(x^{-1})$. Thus, $x^{-1} \in \tilde{H}$.

Suppose that H' is a normal subgroup, and let $x \in \tilde{H}$ and $g \in G$. Then $\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1}$, and $\varphi(x) \in H'$. Therefore, $\varphi(gxg^{-1}) \in H'$, and this shows that $gxg^{-1} \in \tilde{H}$.
Next, $\tilde{H}$ contains $\ker \varphi$ because if $x \in \ker \varphi$ then $\varphi(x) = e'$, and $e' \in H'$. So $x \in \varphi^{-1}(H')$. □

Example 1.10. *Consider the determinant homomorphism:* $\det : GL_n(\mathbb{R}) \to \mathbb{R}^{\times}$. *The set P of positive real numbers is a subgroup of $\mathbb{R}$, and its inverse image is the set of invertible $n \times n$ matrices with positive determinant, which is a normal subgroup of $GL_n(\mathbb{R})$.*

1.8 Products of Groups

Definition 1.23 ($G \times G'$)**.** *We define multiplication of pairs by the rule*

$$(a, a'), (b, b') \mapsto (ab, a'b'),$$

for $a, b \in G$ *and* $a', b' \in G'$. *The pair* (e, e') *is an identity. And* $(a, a')^{-1} = (a^{-1}, a'^{-1})$. *The group thus obtained is called the product of* G *and* G' *and is denoted by* $G \times G'$.

Remark 1.24. *The order of* $G \times G'$ *is the product of the orders of* G *and* G'.

Remark 1.25. *The product group is related to the two factors* G, G' *in a simple way, which can be summed up in terms of some homomorphisms:*

$$\begin{tikzcd}
G \arrow[dr, "i"] & & G \\
& G \times G' \arrow[ur, "p"] \arrow[dr, "p'"] & \\
G' \arrow[ur, "i'"] & & G'
\end{tikzcd}$$

defined by

$$i(x) = (x, 1), i'(x') = (1, x');$$
$$p(x, x') = x, p'(x, x') = x'.$$

The maps i, i' *are injective and the maps* p, p' *are surjective.*

$$\ker p = 1 \times G'$$

and

$$\ker p' = G \times 1.$$

These maps are called projections.

Proposition 1.6 (The mapping property of products)**.** *Let* H *be any group. The homomorphisms* $\Phi : H \to G \times G'$ *are in bijective correspondence with pairs* (φ, φ') *of homomorphisms*

$$\varphi : H \to G, \varphi' : H \to G'.$$

The kernel of Φ *is the intersection* $(\ker \varphi) \cap (\ker \varphi')$.

Proof. Given a pair (φ, φ') of homomorphisms, we define the corresponding homomorphism

$$\Phi : H \to G \times G'$$

by the rule $\Phi(h) = (\varphi(h), \varphi'(h))$. Conversely, given Φ, we obtain φ and φ' by composition with the projections, as

$$\varphi = p\Phi, \varphi' = p'\Phi.$$

Obviously, $\Phi(h) = (e, e')$ if and only if $\varphi(h) = e$ and $\varphi'(h) = e'$, which shows that $\ker \Phi = (\ker \varphi) \cap (\ker \varphi')$. □

Example 1.11. *A cyclic group C_6 of order 6 is isomorphic to the product $C_2 \times C_3$ of cyclic groups of orders 2 and 3.*

Proposition 1.7. *Let r, s be integers with no common factor. A cyclic group of order rs is isomorphic to the product of a cyclic group of order r and a cyclic group of order s.*

Definition 1.24. *Let A and B be subsets of a group G. Then we denote the set of products of elements of A and B by*

$$AB = \{x \in G \mid x = ab \text{ for some } a \in A \text{ and } b \in B\}.$$

Proposition 1.8. *Let H and K be subgroups of a group G.*

1. *If $H \cap K = e$, the product map $p : H \times K \to G$ defined by $p(h, k) = hk$ is injective. Its image is the subset HK.*

2. *If either H or K is a normal subgroup of G, then the product sets HK and KH are equal, and HK is a subgroup of G.*

3. *If H and K are normal, $H \cap K = e$, and $HK = G$, then G is isomorphic to the product group $H \times K$.*

Proof.

1. Let $(h_1, k_1), (h_2, k_2)$ be elements of $H \times K$ such that $h_1k_1 = h_2k_2$. multiplying both sides of this equation on the left by h_1^{-1} and on the right by k_2^{-1}, we find $k_1k_2^{-1} = h_1^{-1}h_2$. Since $H \cap K = e$, $k_1k_2^{-1} = h_1^{-1}h_2 = e$, hence $h_1 = h_2$ and $k_1 = k_2$. This shows that p is injective.

2. Suppose that H is a normal subgroup of G, and let $h \in H$ and $k \in K$. Note that $kh = (khk^{-1})k$. Since H is normal, $khk^{-1} \in H$. Therefore $kh \in HK$, which shows that $KH \subset HK$. The proof of the other inclusion is similar. The fact that HK is a subgroup now follows easily. For closure under multiplication, note that in a product $(hk)(h'k') = h(kh')k'$, the middle term kh' is in $KH = HK$, say $kh' = h''k''$. Then $hkh'k' = (hh'')(k''k') \in HK$. Closure under inverses is similar: $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$. And of course, $e = ee \in HK$. Thus HK is a subgroup. The proof is similar in the case that K is normal.

3. Assume that both subgroups are normal and that $H \cap K = e$. Consider the product $(hkh^{-1})k^{-1} = h(kh^{-1}k^{-1})$. Since K is a normal subgroup, the left side is in K. Since H is normal, the right side is in H. Thus this product is the intersection $H \cap K$, i.e., $hkh^{-1}k^{-1} = e$. Therefore $hk = kh$. This being known, the fact that p is a homomorphism follows directly: In the group $H \times K$, the product rule is $(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2)$, and this element corresponds to $h_1h_2k_1k_2 \in G$, while in G the products h_1k_1 and h_2k_2 multiply as $h_1k_1h_2k_2$. Since $h_2k_1 = k_1h_2$, the products are equal. Part (1) shows that p is injective, and the assumption that $HK = G$ shows that p is surjective.

□

Remark 1.26. *It is important to note that the product map $p : H \times K \to G$ will not be a group homomorphism unless the two subgroups commute with each other.*

1.9 Quotient Groups

Lemma 1.1. *Let N be a normal subgroup of a group G. Then the product of two cosets aN, bN is again a coset, in fact*

$$(aN)(bN) = abN.$$

Proof. Note that $Nb = bN$, and since N is a subgroup $NN = N$.

$$(aN)(bN) = a(Nb)N = a(bN)N = abNN = abN.$$

$\square$

Theorem 1.2. *With the law of composition, $\bar{G} = G/N$ is a group, and the map $\pi : G \to \bar{G} = G/N$ sending $a \mapsto \bar{a} = aN$ is a homomorphism with kernel N.*

Corollary 1.2. *Every normal subgroup of a group G is the kernel of a homomorphism.*

Theorem 1.3 (First Isomorphism Theorem)**.** *Let $\varphi : G \to G'$ be a surjective group homomorphism, and let $N = \ker \varphi$. Then G/N is isomorphic to G' by the map $\bar{\varphi}$ which sends the cosets $\bar{a} = aN$ to $\varphi(a)$:*

$$\bar{\varphi}(\bar{a}) = \varphi(a).$$

Example 1.12.

1. *The absolute value map $\mathbb{C}^\times \to \mathbb{R}^\times$ maps the nonzero complex numbers to the positive real numbers, and its kernel is the unit circle U. So the quotient group $\mathbb{C}^\times/U$ is isomorphic to the multiplicative group of positive real numbers.*
2. *The determinant is a surjective homomorphism $GL_n(\mathbb{R}) \to \mathbb{R}^\times$, whose kernel is the special linear group $SL_n(\mathbb{R})$. So the quotient group $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^\times$.*