SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

<u>MA215 Probability Theory</u>

1. The covariance between X and Y, denoted by $\mathrm{C}ov(X,Y)$, is defined by

$$\mathrm{C}ov(X,Y) = \mathrm{E[(X - E(X))(Y - E(Y))]}.$$

Show that

$$\mathrm{C}ov(X,Y) = \mathrm{E[XY] - E[X]E[Y]}.$$

2. Let X be a discrete random variable with p.m.f as

$$\mathrm{P}\{X=0\} = \mathrm{P}\{X=1\} = \mathrm{P}\{X=-1\} = \frac{1}{3}.$$

Define

$$Y = \begin{cases} 0 & \text{if} \quad X \neq 0, \\ 1 & \text{if} \quad X = 0. \end{cases}$$

(i) Show that $\mathrm{C}ov(X,Y) = 0$.

(ii) Write down the joint p.m.f of X and Y, and show that X and Y are not independent.

3. Show that the following conclusions are true:

(i) $\mathrm{C}ov(X,Y) = \mathrm{C}ov(Y,X)$;

(ii) $\mathrm{C}ov(X,X) = \mathrm{V}ar(X)$;

(iii) $\mathrm{C}ov(aX,Y) = a\mathrm{C}ov(X,Y)$, where a is a constant;

(iv) $\mathrm{C}ov(\sum_{i=1}^{m} X_i, \sum_{j=1}^{n} Y_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{C}ov(X_i, Y_j)$;

(v) If X is a random variable and C is a constant, then $\mathrm{C}ov(X,C) = 0$.

(vi) Show that the following statements are true:

$$\mathrm{V}ar(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{V}ar(X_i) + \sum\sum_{1\le i\neq j\le n} \mathrm{C}ov(X_i, X_j),$$

or, equivalently,

$$\mathrm{V}ar(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{V}ar(X_i) + 2\sum\sum_{1\le i< j\le n} \mathrm{C}ov(X_i, X_j).$$

Further show that if $X_1, \ldots, X_n$ are pairwise independent (i.e. X_i and X_j are independent for $1 \le i \neq j \le n$), then we have

$$\mathrm{V}ar(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{V}ar(X_i).$$

4. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common (mean)expectation μ and common variance σ^2. Let $\overline{X}$ and S^2 be defined as follows.

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

$$S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2.$$

The two random variables $\bar{X}$ and $\frac{S^2}{n-1}$ are called the sample mean and sample variance, respectively. Find

(i) $\mathrm{E}[\overline{\mathrm{X}}]$;

(ii) $\mathrm{V}ar(\overline{X})$;

(iii) $\mathrm{E}[\frac{\mathrm{S}^2}{\mathrm{n}-1}]$.

5. Let I_A and I_B be the indicator variables for the events A and B. That is,

$$I_A(\omega) = \begin{cases} 1 & \omega \in A, \\ 0 & \omega \notin A. \end{cases}$$

$$I_B(\omega) = \begin{cases} 1 & \omega \in B, \\ 0 & \omega \notin B. \end{cases}$$

Show that

(i)

$$\begin{aligned} \mathrm{E}[\mathrm{I_A}] &= \mathrm{P(A)}; \\ \mathrm{E}[\mathrm{I_B}] &= \mathrm{P(B)}; \\ \mathrm{E}[\mathrm{I_A I_B}] &= \mathrm{P(AB)}. \end{aligned}$$

(ii)

$$Cov(I_A, I_B) = P(AB) - P(A)P(B).$$

6. Let$X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common variance σ^2. Show that for any fixed $i (1 \le i \le n)$,

$$Cov(X_i - \overline{X}, \overline{X}) = 0,$$

where $\overline{X}$ is the sample mean (i.e. $\overline{X} = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$).

1. 设X, Y是互相独立的随机变量，且有$E(X)=3$, $E(Y)=1$, $D(X)=4$, $D(Y)=9$. 令$Z=5X-2Y+15$, 求 $E(Z)$ 和 $D(Z)$.

2. 设随机变量X_1, X_2, X_3, X_4互相独立，且有$E(X_i)=2i$, $D(X_i)=5-i$, 其中$i=1,2,3,4$. 令$Z=2X_1-X_2+3X_3-\frac{1}{2}X_4$, 求 $E(Z)$ 和 $D(Z)$.

设随机变量X的概率密度为$f(x)=\frac{1}{2}e^{-|x|}, -\infty<x<+\infty$,

(1)求出$E(X), D(X)$.

(2)X与$|X|$是否独立？说明理由.

(3)X与$|X|$是否相关？说明理由.

1. 设随机变量(X, Y)的概率密度为

$$f(x,y)=\begin{cases} \frac{1}{8}(x+y), & 0\leqslant x\leqslant 2,\ 0\leqslant y\leqslant 2, \\ 0, & \text{其他}. \end{cases}$$

求 $E(X)$, $E(Y)$, $Cov(X,Y)$, ρ_{XY}, $D(X+Y)$.

2. 设随机变量X和Y独立同分布于$N(\mu,\sigma^2)$. 令$Z=\alpha X+\beta Y$, $W=\alpha X-\beta Y$, 求$Cov(Z,W)$, ρ_{ZW}.