Statistic Linear Model

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Assigment 3

1. Note that A is nonsingular of order n with derivative $\partial A / \partial x$.

then we have $\dfrac{\partial A^{-1}}{\partial x} = -A^{-1}\dfrac{\partial A}{\partial x}A^{-1}$

$$\Rightarrow \frac{\partial H}{\partial x} = \frac{\partial\left[B'(BAB')^{-1}B\right]}{\partial x} = B'\frac{\partial (BAB')^{-1}}{\partial x}B \quad \cdots (*)$$

Since BAB' is nonsingular of order n

$$\Rightarrow \frac{\partial (BAB')^{-1}}{\partial x} = -(BAB')^{-1}\frac{\partial (BAB')}{\partial x}(BAB')^{-1} = -(BAB')^{-1}B\frac{\partial (A)}{\partial x}B'(BAB')^{-1}$$

$$(*) = -B'(BAB')^{-1}B\frac{\partial (A)}{\partial x}B'(BAB')^{-1}B$$

$$= -H\frac{\partial (A)}{\partial x}H \quad \square$$

2. (a). $\operatorname{Cov}(X_2, 2X_1 - X_3) = 2\operatorname{Cov}(X_2, X_1) - \operatorname{Cov}(X_2, X_3)$

$$= 2\Sigma_{21} - \Sigma_{23}$$

$$= 0 - 0 = 0$$

Note that both X_2 and $2X_1 - X_3$ are normal distribution.

Hence, $\operatorname{Cov}(X_2, 2X_1 - X_3) = 0 \rightarrow X_2$ and $2X_1 - X_3$ are independent.

(b). For $Z_1 = 2X_1 - 5X_3$, $Z_2 = X_1 + X_2$

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

denote $A = \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix}$

$$\Rightarrow A\mu = \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \overset{d}{=} \mu^*$$

$$A\Sigma A' = \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 0 & -3 \\ 0 & 9 & 0 \\ -3 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -5 & 0 \end{pmatrix} = \begin{pmatrix} 25 & 0 & -16 \\ 5 & 9 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -5 & 0 \end{pmatrix} = \begin{pmatrix} 130 & 25 \\ 25 & 14 \end{pmatrix} \overset{d}{=} \Sigma^*$$

$$\Rightarrow \begin{pmatrix} 2X_1 - 5X_3 \\ X_1 + X_2 \end{pmatrix} \sim N(\mu^*, \Sigma^*) \quad \text{where } \mu^* = \begin{pmatrix} 6 \\ 1 \end{pmatrix},\ \Sigma^* = \begin{pmatrix} 130 & 25 \\ 25 & 14 \end{pmatrix}$$

(c) Suppose $Y_1 = X_3$, $Y_2 = (X_1, X_2)'$

$$\Rightarrow Y_1 \mid Y_2 = y_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_2 - \mu_2),\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

Since $\underset{\sim}{X} \sim N_3(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$. $\underset{\sim}{\mu}' = (3,-2,0)$. $\underset{\sim}{\Sigma} = \begin{pmatrix} 5 & 0 & -3 \\ 0 & 9 & 0 \\ -3 & 0 & 2 \end{pmatrix}$

$\Rightarrow$ By Suppose. $\rightarrow$ $\Sigma_{11} = 2$, $\underset{\sim}{\Sigma}_{22} = \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix}$ $\underset{\sim}{\Sigma}_{21} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ $\underset{\sim}{\Sigma}_{12} = (-3, 0)$, i.e. $\underset{\sim}{\Sigma} = \begin{array}{cc} \overset{d}{\Sigma_{22}} & \\ \left(\begin{array}{cc|c} 5 & 0 & -3 \\ 0 & 9 & 0 \\ \hline -3 & 0 & 2 \end{array}\right) & \begin{array}{l} \overset{d}{\Sigma_{21}} \\ \\ \overset{d}{\Sigma_{11}} \end{array} \\ \overset{d}{\Sigma_{12}} & \end{array}$

$\Rightarrow \mu_1 = 0, \quad \underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-1} (\underset{\sim}{y}_2 - \underset{\sim}{\mu}_2) = (-3, 0) \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \begin{pmatrix} 1-3 \\ -2+2 \end{pmatrix}$

$= \left(-\frac{3}{5}, 0\right) \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \frac{6}{5}$

$\Sigma_{11} - \underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-1} \underset{\sim}{\Sigma}_{21} = 2 - (-3, 0) \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \begin{pmatrix} -3 \\ 0 \end{pmatrix} = 2 - \left(-\frac{3}{5}, 0\right) \begin{pmatrix} -3 \\ 0 \end{pmatrix} = 2 - \frac{9}{5} = \frac{1}{5}$

$\Rightarrow X_3 \big| x_1 = 1, x_2 = -2 \sim N\left(\frac{6}{5}, \frac{1}{5}\right)$

3. Note that $\bar{Y} = \frac{y_1 + y_2 + \cdots + y_n}{n} = \frac{1}{n} \underset{\sim}{Y}' \underset{\sim}{1} = \frac{1}{n} \underset{\sim}{1}' \underset{\sim}{Y}$

$\rightarrow \begin{pmatrix} y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix} = \underset{\sim}{Y} - \underset{\sim}{1} \bar{Y} = (\underset{\sim}{Y} - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \underset{\sim}{Y}) = (\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}') \underset{\sim}{Y} = \underset{\sim}{B} \underset{\sim}{Y}$ where $B \overset{d}{=} (\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}')$

$\rightarrow$ then $\sum_{i=1}^{n} (y_i - \bar{y})^2 = \begin{pmatrix} y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix}' \begin{pmatrix} y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix} = \underset{\sim}{Y}' \underset{\sim}{B}' \underset{\sim}{B} \underset{\sim}{Y} \cdots (*)$

note that $B'B = (\underset{\sim}{I}' - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}')(\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}') = I - \frac{2}{n} \underset{\sim}{1} \underset{\sim}{1}' + \frac{1}{n^2} \cdot n \underset{\sim}{1} \underset{\sim}{1}' = I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' = B$

$\Rightarrow (*) = \underset{\sim}{Y}' \underset{\sim}{B} \underset{\sim}{Y}$

$\rightarrow$ For Q3. we have $n = 3$. $\rightarrow B = I - \frac{1}{3} (\underset{\sim}{1} \underset{\sim}{1}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

and $\underset{\sim}{U} = \underset{\sim}{Y}' \underset{\sim}{B} \underset{\sim}{Y}$, note that B is symmetric

$\rightarrow E(\underset{\sim}{U}) = E(\underset{\sim}{Y}' \underset{\sim}{B} \underset{\sim}{Y}) = tr(\underset{\sim}{B} \underset{\sim}{\Sigma}) + \underset{\sim}{\mu}' \underset{\sim}{B} \underset{\sim}{\mu}$

$tr(\underset{\sim}{B} \underset{\sim}{\Sigma}) = tr \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} = \frac{5}{3} + \frac{1}{3} + \frac{6}{3} = 4$

$\underset{\sim}{\mu}' \underset{\sim}{B} \underset{\sim}{\mu} = (2, 3, 4) \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = (-1, 0, 1) \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 2$

Thus, $E(U) = tr(\underset{\sim}{B} \underset{\sim}{\Sigma}) + \underset{\sim}{\mu}' \underset{\sim}{B} \underset{\sim}{\mu}$

$= 4 + 2$

$= 6$

4. Note that $U = \sum_{i<j} (Y_i - Y_j)^2 = \frac{1}{2} \sum_{i \neq j} (Y_i - Y_j)^2$

$\Rightarrow E(U) = E\left(\frac{1}{2} \sum_{i \neq j} (Y_i - Y_j)^2\right)$

$= \frac{1}{2} \sum_{i \neq j} E(Y_i^2 - 2 Y_i Y_j + Y_j^2)$

$= \frac{1}{2} \sum_{i \neq j} \left(E(Y_i^2) + E(Y_j^2) - 2 E(Y_i Y_j) \right)$

Note that $Cov(Y_i, Y_j) = E(Y_i - \mu)(Y_j - \mu) = E(Y_i Y_j - \mu Y_i - \mu Y_j + \mu^2) = E(Y_i Y_j) - \mu^2$

when $i \neq j$. by covariance matrix of $\underset{\sim}{X}$ is $\sigma^2 \underset{\sim}{I}$ $\Rightarrow Cov(X_i, X_j) = E(Y_i Y_j) - \mu^2 = 0$

$\Rightarrow E(Y_i Y_j) = \mu^2$. and $E(Y_i^2) = Var(Y_i) + E(Y_i)^2 = \sigma^2 + \mu^2$

$$\Rightarrow E(U) = \frac{1}{2} \sum_{i \neq j} \left(E(Y_i^2) + E(Y_j^2) - 2\mu^2 \right)$$

$$= \frac{1}{2} \cdot (n^2 - n) (\sigma^2 + \mu^2 + \sigma^2 + \mu^2 - 2\mu^2)$$

$$= (n^2 - n) \sigma^2 .$$

We want to find k s.t. kU is an unbiased estimator of σ^2.

i.e. $E(kU) = \sigma^2 = k(n^2 - n)\sigma^2$

$$\Rightarrow k = \frac{1}{n^2 - n}$$

Since $\varepsilon \sim N(0, \sigma^2)$ for all errors, $\Rightarrow \begin{cases} Y_{1i} \overset{iid}{\sim} N(\theta, \sigma^2) & \Rightarrow \sum_{i=1}^{m} Y_{1i} \sim N(m\theta, m\sigma^2) \\ Y_{2i} \overset{iid}{\sim} N(\theta + \phi, \sigma^2) & \Rightarrow \sum_{i=1}^{m} Y_{2i} \sim N(m(\theta + \phi), m\sigma^2) \\ Y_{3i} \overset{iid}{\sim} N(\theta - 2\phi, \sigma^2) & \Rightarrow \sum_{i=1}^{n} Y_{3i} \sim N(n(\theta - 2\phi), n\sigma^2) \end{cases}$

$$\Rightarrow \begin{cases} \hat{\theta} \sim N\left[\frac{m(2\theta + \phi) + n(\theta - 2\phi)}{5n}, \frac{(2m + n)\sigma^2}{25n^2} \right] = N\left(\theta, \frac{\sigma^2}{5n}\right) \\ \hat{\phi} \sim N\left[\frac{m(\theta + \phi) - 2n(\theta - 2\phi)}{6n}, \frac{m\sigma^2 + 4n\sigma^2}{36n^2} \right] = N\left(\phi, \frac{\sigma^2}{6n}\right) \end{cases}$$

Consider the $Cov(\hat{\theta}, \hat{\phi}) = E(\hat{\theta} - E(\hat{\theta}))(\hat{\phi} - E(\hat{\phi})) = E(\hat{\theta}\hat{\phi}) - E(\hat{\phi})E(\hat{\theta})$

By the property of independence,

$$E(\hat{\theta}\hat{\phi}) = \frac{1}{30n^2} E\left(\sum_{i=1}^{m} Y_{1i} + \sum_{i=1}^{m} Y_{2i} + \sum_{i=1}^{n} Y_{3i}\right)\left(\sum_{i=1}^{m} Y_{2i} - 2\sum_{i=1}^{n} Y_{3i}\right)$$

$$= \frac{1}{30n^2} E\left(\left(\sum_{i=1}^{m} Y_{2i}\right)\left(\sum_{i=1}^{m} Y_{2i}\right) - 2\left(\sum_{i=1}^{n} Y_{3i}\right)\left(\sum_{i=1}^{n} Y_{3i}\right) \right)$$

$$= \frac{1}{30n^2} E\left[m^2(\theta + \phi)^2 + m\sigma^2 - 2\left(n^2(\theta - 2\phi)^2 + n\sigma^2 \right) \right]$$

$$= \frac{1}{30n^2} E\left[4n^2(\theta + \phi)^2 + 2n\sigma^2 - 2\left(n^2(\theta^2 - 4\theta\phi + 4\phi^2) + n\sigma^2 \right) \right]$$

$$= \frac{1}{30n^2} E\left[4n^2(\theta + 2\theta\phi + \phi^2) - 2n^2(\theta^2 - 4\theta\phi + 4\phi^2) \right.$$