Revision.

CH3. vector, matrix.

— Full rank factorization $\quad r = \text{rank}(A)$

$$\underset{\sim}{A}_{p\times q} = \underset{\sim}{K}_{p\times r} \underset{\sim}{L}_{r\times q}$$

· left inverse: $\underset{r\times p}{U}\underset{p\times r}{K} = I_r$

· right inverse: $\underset{r\times q}{L}\underset{q\times r}{R} = I_r$

— Idempotent Matrix $(A^2 = A)$

* $\text{rank}(A) = tr(A)$

* eigenvalues of A: $\underbrace{1 \cdots 1}_{r}\ 0 \cdots 0$

— Generalized Inverse.

* definition: $\underset{\sim}{A}\underset{\sim}{A}^-\underset{\sim}{A} = \underset{\sim}{A}$

· full rank
· $\text{rank}(A_{11}) = \text{rank}(A)$

* how to calculate: e.g. $\underset{\sim}{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

* some properties.

* Morre-Penrose Inverse $\begin{cases} \cdot \text{ special case of } A^- \\ \cdot \text{ unique} \end{cases}$

CH3: — Vector and matrix calculus

* $u = f(\underset{\sim}{x})$. $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$. $\frac{\partial u}{\partial \underset{\sim}{x}}$ and

$$\frac{\partial \underset{\sim}{a}'\underset{\sim}{x}}{\partial \underset{\sim}{x}} = \underset{\sim}{a}. \quad \frac{\partial \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}}{\partial \underset{\sim}{x}} = 2\underset{\sim}{A}\underset{\sim}{x}$$

* $u = f(\underset{\sim}{X})$. $\underset{\sim}{X} = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pp} \end{pmatrix}$ — symmetric positive definite

$$\frac{\partial tr(\underset{\sim}{X}\underset{\sim}{A})}{\partial \underset{\sim}{X}} = \underset{\sim}{A} + \underset{\sim}{A}' - \text{diag}(\underset{\sim}{A})$$

Condition is not sure.

$$\frac{\partial \ln|X|}{\partial \underset{\sim}{X}} = 2\underset{\sim}{X}^{-1} - \text{diag}(\underset{\sim}{X}^{-1})$$

* $\underset{\sim}{A}_{n\times n} = (a_{ij}(x))_{n\times n}$. $\frac{\partial \underset{\sim}{A}}{\partial x} = \left(\frac{\partial a_{ij}(x)}{\partial x}\right)$

$$\frac{\partial \underset{\sim}{A}^{-1}}{\partial x} = -\underset{\sim}{A}^{-1}\frac{\partial \underset{\sim}{A}}{\partial x}\underset{\sim}{A}^{-1}$$

$$\frac{\partial \ln|\underset{\sim}{A}|}{\partial x} = tr\left(\underset{\sim}{A}^{-1}\frac{\partial \underset{\sim}{A}}{\partial x}\right)$$

CH4. Random Vector. Matrix. $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix}$ $X_1, \cdots, X_p$ are r.v.s

— mean vector and covariance matrix.

* $\underset{p\times 1}{E(\underset{\sim}{X})} = \underset{\sim}{\mu}_{p\times 1}$ $\quad \underset{\sim}{\Sigma}_{p\times p} = Cov(\underset{\sim}{X}) = (Cov(X_i, X_j))_{p\times p}$

* Sample mean and sample covariance matrix.

$\hat{\underset{\sim}{\mu}} = \bar{\underset{\sim}{X}} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{X}_i$

$\hat{\underset{\sim}{\Sigma}} = \underset{\sim}{S} = \frac{1}{n}\sum_{i=1}^{n} (\underset{\sim}{X}_i - \bar{\underset{\sim}{X}})(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}})'$

observations. $\underset{\sim}{X}_i = \begin{pmatrix} X_{i1} \\ \vdots \\ X_{ip} \end{pmatrix}$ $i = 1, \cdots, n$

* $Cov(\underset{\sim}{A}\underset{\sim}{X}, \underset{\sim}{B}\underset{\sim}{Y}) = \underset{\sim}{A}\, Cov(\underset{\sim}{X}, \underset{\sim}{Y})\underset{\sim}{B}'$

$Cov(\underset{\sim}{A}\underset{\sim}{X}) = \underset{\sim}{A}\, Cov(\underset{\sim}{X})\underset{\sim}{A}'$

* generalized variance $|\underset{\sim}{\Sigma}|$

* Correlation matrix

* mgf of $\underset{\sim}{Y}$, $M_{\underset{\sim}{Y}}(t) = E(e^{\underset{\sim}{t}'\underset{\sim}{Y}})$

* If $\underset{\sim}{Y}_1, \cdots, \underset{\sim}{Y}_m$ are mutually independent.

$\Rightarrow g_1(\underset{\sim}{Y}_1), g_2(\underset{\sim}{Y}_2), \cdots, g_m(\underset{\sim}{Y}_m)$ are mutually independent

CH5. Multivariate normal

— Density, mgf

— Some important properties. $\underset{\sim}{Y} \sim N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

* $\underset{\sim}{B}\underset{\sim}{Y} + \underset{\sim}{c} \sim N(\underset{\sim}{B}\underset{\sim}{\mu} + \underset{\sim}{c}, \underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}')$

* $\underset{\sim}{Y} = \begin{pmatrix} \underset{\sim}{Y}_1 \\ \underset{\sim}{Y}_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$

* marginal distribution of $\underset{\sim}{Y}_1$ and $\underset{\sim}{Y}_2$

* Conditional distribution $\underset{\sim}{Y}_1 | \underset{\sim}{Y}_2$

* Independence of $\underset{\sim}{Y}_1$ and $\underset{\sim}{Y}_2$ $\Leftrightarrow$ $Cov(\underset{\sim}{Y}_1, \underset{\sim}{Y}_2) = 0$.

— Partial correlation.

* definition.

* meaning.

CH6. Quadratic form.

$\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ $\quad \underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X}$, A — symmetric.

* mgf of $\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X}$

* $E(\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X})$, $Var(\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X})$

— Non-central χ^2, F, t distribution.

- $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, I_n)$. $U = \sum X_i^2 \sim \chi^2(n, \lambda)$.

$$\lambda = \frac{1}{2}\sum_i \mu_i^2$$

- non-central F.

$U_1 \sim \chi^2(p_1, \lambda)$. $U_2 \sim \chi^2(p_2)$. U_1 and U_2 are independent

$$W = \frac{U_1/p_1}{U_2/p_2} \sim F(p_1, p_2, \lambda)$$

- Non-central t.

$z \sim N(\mu, 1)$ $\qquad U \sim \chi^2(n)$. $\quad z$ and u are independent

$$t = \frac{z}{\sqrt{u/n}} \sim t(n, \lambda)$$

Thm 6.1. $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

$q = \underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X} \sim \chi^2(r, \lambda)$, $\lambda = \frac{\underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu}}{2}$, $r = rank(\underset{\sim}{A})$

$\Leftrightarrow \underset{\sim}{A}\underset{\sim}{\Sigma}$ is idempotent

Thm 6.2. $\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X}$ and $\underset{\sim}{B}\underset{\sim}{X}$ are independent.