6 Quadratic Forms

6.1 Quadratic Form $\mathbf{x}'\mathbf{A}\mathbf{x}$

Let $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and assume $\mathbf{A}$ symmetric, then m.g.f. of $\mathbf{x}'\mathbf{A}\mathbf{x}$ is

$$M_{\mathbf{x}'\mathbf{A}\mathbf{x}}(t) \ = \ | \ \mathbf{I} - 2t\mathbf{A}\boldsymbol{\Sigma} \ |^{-\frac{1}{2}} \cdot \mathbf{e}^{\{-\frac{1}{2}\boldsymbol{\mu}'[\mathbf{I}-(\mathbf{I}-2\mathbf{t}\mathbf{A}\boldsymbol{\Sigma})^{-1}]\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\}} \tag{6.1}$$

Proof $\longrightarrow$ Remark 6.1

- $E(\mathbf{x}'\mathbf{A}\mathbf{x}) = \text{tr}(\boldsymbol{A\Sigma}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$ — Ch 4
- $\text{Var}(\mathbf{x}'\mathbf{A}\mathbf{x}) = 2\text{tr}[(\boldsymbol{A\Sigma})^2] + 4\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\mu}$

Proof — Remark 6.2

6.2 Non-Central χ^2, F and t distributions

I. Non-Central χ^2

- Let $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_n)$, then $\mathbf{x}'\mathbf{x} \sim \chi^2_{(n)}$
- Let $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{I}_n)$, then

$$u = \mathbf{x}'\mathbf{x} \sim \chi^2_{(n,\ \lambda)}$$

where λ = non-centered parameter $= \frac{1}{2}\boldsymbol{\mu}'\boldsymbol{\mu}$

- Density is

$$f(u) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{u^{\frac{1}{2}n+k-1} e^{-\frac{1}{2}u}}{2^{\frac{1}{2}n+k}\Gamma(\frac{1}{2}n + k)} \qquad \mu > 0, \lambda \geq 0$$

Note: Define $\lambda^k = 1$ when $\lambda = 0, k = 0$, density function of $u \sim \chi^2_{(n,0)}$ is

$$f(u) = \frac{u^{\frac{1}{2}n-1} e^{-\frac{1}{2}u}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}$$

- m.g.f of $u \sim \chi^2_{(n,\lambda)}$ is

$$(1 - 2t)^{-\frac{n}{2}} e^{-\lambda[1-(1-2t)^{-1}]}$$

Note: for $\lambda = 0$, $\Rightarrow M_u(t) = (1 - 2t)^{-\frac{n}{2}}$ which is m.g.f of $\chi^2_{(n)}$

- $\mathrm{E}(u) = n + 2\lambda$ and $\mathrm{Var}(u) = 2n + 8\lambda$;
- If $u_i \sim \chi^2_{(n_i,\lambda_i)}$ independently for $i = 1, \ldots, k$, then

$$\sum_{i=1}^{k} u_i \sim \chi^2_{(\sum_{i=1}^{k} n_i, \sum_{i=1}^{k} \lambda_i)}.$$

II. Non-Central F

Let $u_1 \sim \chi^2_{(p_1,\lambda)}$, $u_2 \sim \chi^2_{(p_2,0)}$, and Let u_1 be independent of u_2, then

$$w = \frac{u_1/p_1}{u_2/p_2} \;\sim\; F_{(p_1,p_2,\lambda)}$$

and

$$\mathrm{E}(w) = \frac{p_2}{p_2 - 2}\left(1 + \frac{2\lambda}{p_1}\right)$$

III. Non-Central t

Let $z \sim N(\mu, 1), u \sim \chi^2_{(n)}$, z is independent of u, then

$$t = \frac{z}{\sqrt{u/n}} \;\sim\; \text{non-centered } t \text{ distribution}$$

Theorem 6.1 Let $\mathbf{x}_{p\times 1} \sim N(\boldsymbol{\mu}, \mathbf{V})$, then $q = \mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi^2_{(r,\ \lambda)}$ where r denoting the rank of $\mathbf{A}$ and $\lambda = \frac{\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}}{2}$ if and only if $\mathbf{AV}$ is idempotent. (let $\mathbf{A}$ be symmetric)

$$\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim \chi^2_{(r,\lambda)} \iff (\underset{\sim}{A}\underset{\sim}{V})^2 = \underset{\sim}{A}\underset{\sim}{V}$$

$$r = \text{rank}(\underset{\sim}{A})$$

$$\lambda = \frac{\underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu}}{2} \qquad \underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$$

Remark 6.3 if $\underset{\sim}{A}\underset{\sim}{V}$ is idempotent

$$\implies \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim \chi^2_{(r,\lambda)}$$

Corollaries

- If $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ is χ^2_r if and only if $\mathbf{A}$ is idempotent of rank r.
- If $\mathbf{x} \sim N(\mathbf{0}, \mathbf{V})$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ is χ^2_r if and only if $\mathbf{AV}$ is idempotent of rank r.
- If $\mathbf{x}$ is $N(\boldsymbol{\mu}, \sigma^2\mathbf{I})$, then $\frac{\mathbf{x}'\mathbf{x}}{\sigma^2}$ is $\chi^2_{(n, \frac{1}{2}\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{\sigma^2})}$
- If $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{I})$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ is $\chi^2_{(r, \frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu})}$ if and only if $\mathbf{A}$ is idempotent of rank r.

6.3 Independence

Theorem 6.2 When $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \Sigma)$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ and $\mathbf{B}\mathbf{x}$ are distributed independently if and only if $\mathbf{B}\Sigma\mathbf{A} = 0$

If $\underset{\sim}{A}$ is symmetric and idempotent

$\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ and $\underset{\sim}{B}\underset{\sim}{x}$ are independent

$\iff \underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{A} = \underset{\sim}{0}$

<u>Remark 6.4</u> proof for the above special case.

Theorem 6.3 Let $\mathbf{x} \sim N(\boldsymbol{\mu}, \Sigma)$, $\mathbf{x}'\mathbf{A}\mathbf{x}$ and $\mathbf{x}'\mathbf{B}\mathbf{x}$ are distributed independently if and only if $\mathbf{A}\Sigma\mathbf{B} = 0$ (or equivalently, $\mathbf{B}\Sigma\mathbf{A} = 0$)

Additional results

Let the $n \times 1$ vector $\mathbf{x} = (x_1, ..., x_n)' \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $q_1 = \mathbf{x}'\mathbf{A}_1\mathbf{x}$, $q_2 = \mathbf{x}'\mathbf{A}_2\mathbf{x}$ and $\mathbf{T} = \mathbf{B}\mathbf{x}$ where $\mathbf{B}$ is $r \times n$ and $\mathbf{A}_1, \mathbf{A}_2$ are symmetric.

1. $\mathrm{E}(q_1) = \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\mathbf{A}_1\boldsymbol{\mu}$.
2. $\mathrm{Var}(q_1) = 2\ \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\Sigma}) + 4\ \boldsymbol{\mu}'\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\mu}$.
3. $\mathrm{Cov}(q_1, q_2) = 2\ \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_2\boldsymbol{\Sigma}) + 4\ \boldsymbol{\mu}'\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_2\boldsymbol{\mu}$.
4. $\mathrm{Cov}(\mathbf{x}, q_1) = 2\ \boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\mu}$.
5. $\mathrm{Cov}(\mathbf{T}, q_1) = 2\ \mathbf{B}\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\mu}$.

Examples

1. Let the $n \times 1$ vector $\mathbf{Y} = (Y_1, ..., Y_n)' \sim N(\alpha \mathbf{1}, \sigma^2 \mathbf{I})$. Define $U = \Sigma_{i=1}^{n}(Y_i - \overline{Y})^2/\sigma^2$ and $V = n(\overline{Y} - \alpha)^2/\sigma^2$. Find the distributions of U and V and show that these two random variables are independent.

$\underset{\sim}{\Sigma}$

$$U = \underset{\sim}{Y}' (\underset{\sim}{B}/\sigma^2) \underset{\sim}{Y} \sim \chi^2_{(n-1)}$$

$$\underset{\sim}{B} = I - \frac{1}{n} \underset{\sim}{J} \qquad \underset{\sim}{J} = 1 1'$$

Thm 6.1 $\quad \underset{\sim}{A}\underset{\sim}{\Sigma} = (\underset{\sim}{B}/\sigma^2) \cdot (\sigma^2 I) = \underset{\sim}{B}$

idempotent

2. Let the $n \times 1$ vector $\mathbf{Y} = (Y_1, ..., Y_n)' \sim N(\mu \mathbf{1}, \sigma^2 \mathbf{I})$. Let

$$\begin{aligned} \overline{Y} &= \frac{\Sigma_{i=1}^n Y_i}{n} \\ Q_1 &= n\overline{Y}^2 \\ Q_2 &= \sum_{i=1}^n (Y_i - \overline{Y})^2 \end{aligned}$$

(a) Prove that $\overline{Y}$ and Q_2 are independent.

(b) Prove that Q_1 and Q_2 are independent.

(c) Find the distributions of Q_1 and Q_2.

special case of
Ex6.1

$\underset{\sim}{\Sigma} = \sigma^2 \underset{\sim}{I}$

3. Suppose $\mathbf{y}$ is $N_3(\boldsymbol{\mu}, \sigma^2\mathbf{I})$ and let $\boldsymbol{\mu}' = [3, \quad -2, \quad 1]$ and

$$\mathbf{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \underset{\sim}{I} - \frac{1}{3}\underset{\sim}{J}$$

$\underset{\sim}{J} = \underset{\sim}{1}\underset{\sim}{1}'$

$$\mathbf{B} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

(a) Find the distribution of $\mathbf{y}'\mathbf{A}\mathbf{y}/\sigma^2$. $\sim \chi^2_{(3-1)} = \chi^2_2$

(b) Are $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{B}\mathbf{y}$ independent?

(c) Are $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $y_1 + y_2 + y_3$ independent?

(b) Need to check if $\underset{\sim}{B}\underset{\sim}{I}\underset{\sim}{A} = \underset{\sim}{B}\underset{\sim}{A} = 0$?

$$\underset{\sim}{B}\underset{\sim}{A} = \frac{1}{9}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & -3 \end{pmatrix} \neq 0$$

$\Rightarrow \underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y}$ and $\underset{\sim}{B}\underset{\sim}{Y}$ are not independent!

(c) $y_1 + y_2 + y_3 = (1 \; 1 \; 1)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$= \underset{\sim}{1}'\underset{\sim}{Y}$ $\qquad \underset{\sim}{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\underset{\sim}{1}'\underset{\sim}{I}\underset{\sim}{B} = \underset{\sim}{1}'\underset{\sim}{B} = 0$

$\Rightarrow \underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y}$ and $y_1+y_2+y_3$ are independent!

4. Suppose $\mathbf{y}$ is $N_n(\mu\mathbf{1}, \mathbf{\Sigma})$ where

$$\mathbf{\Sigma} = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ . & . & & . \\ . & . & & . \\ \rho & \rho & \cdots & 1 \end{pmatrix}. = \sigma^2\left((1-\rho)\underset{\sim}{I} + \rho \underset{\sim}{J}\right)$$

Derive the distribution of

$$u = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma^2(1-\rho)}. = \frac{1}{\sigma^2(1-\rho)} \underset{\sim}{y}' \underset{\sim}{B} \underset{\sim}{y}$$

$\underset{\sim}{B} = (I - \frac{1}{n} J)$

$= \underset{\sim}{y}' \underset{\sim}{A} \underset{\sim}{y}$ $\quad \underset{\sim}{A} = \frac{\underset{\sim}{B}}{\sigma^2(1-\rho)}$

$\underset{\sim}{A}\underset{\sim}{\Sigma}$ is idempotent ?

$\underset{\sim}{A}\underset{\sim}{\Sigma} = \cdots = \underset{\sim}{B}$

Th6.1 $\Longrightarrow$ $u \sim \chi^2_{(n-1, \lambda)} \approx \chi^2_{(n-1)}$

$n-1 = \text{rank}(A) = \text{rank}(B) = tr(B)$

$\lambda = \frac{1}{2}\underset{\sim}{\mu}' \underset{\sim}{A} \underset{\sim}{\mu}$

$= \frac{1}{2}(\mu \underset{\sim}{1})'\left(\frac{\underset{\sim}{B}}{\sigma^2(1-\rho)}\right)(\mu \underset{\sim}{1})$

$= \frac{\mu^2}{2\sigma^2(1-\rho)} \mathbf{1}' \underset{\sim}{B} \underset{\sim}{1} = \cdots = 0$