3 Matrix Algebra

3.1 Vector and matrix

- A brief review of vector and matrix

$\boxed{\text{Remark 3.1}}$ $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\underset{\sim}{A}_{n\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

表示一个向量或矩阵.

* Rank(A)
* Square matrix $m=n$. $A=A'$. symmetric

Covariance matrix $\underset{\sim}{A} = Cov(\underset{\sim}{x}) = \begin{pmatrix} Cov(x_1,x_1) & \cdots & Cov(x_1,x_n) \\ \vdots & & \vdots \\ Cov(x_n,x_1) & \cdots & Cov(x_n,x_n) \end{pmatrix}$

$A \geq 0$. eigenvalue ≥ 0

* for $n\times n$ matrix.
  If $rank(A) = n$. (full rank)
  — unique A^{-1} s.t. $AA^{-1} = A^{-1}A = I_{n\times n} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

  — $\underset{n\times n}{A}\ \underset{n\times 1}{x} = \underset{n\times 1}{c} \Rightarrow x = A^{-1}c$ $(AB)^{-1} = B^{-1}A^{-1}$

* Partition of matrix. (样本扩大时,原来的$A_m \to A_{11}$)

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ 将原本的A_{11}^{-1}保存下来

以A_{11}为标准 ☆ $A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B^{-1} \\ -B^{-1}A_{21}A_{11}^{-1} & B^{-1} \end{pmatrix}$

where $B = A_{22} - A_{21}A_{11}^{-1}A_{12}$

逆唯一

以A_{22}为标准 ☆ $A^{-1} = \begin{pmatrix} D^{-1} & -D^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}D^{-1}A_{12}A_{22}^{-1} \end{pmatrix}$

where $D = A_{11} - A_{12}A_{22}^{-1}A_{21}$

☆ $B^{-1} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$
$= A_{22}^{-1} + A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1}$

☆ special cases: $\underset{(n+1)\times(n+1)}{A} = \begin{pmatrix} A_{11} & a_{12} \\ a_{12}' & a_{22} \end{pmatrix}$ A_{11} $n\times n$
仅增加一列观测

$b = a_{22} - a_{12}'A_{11}^{-1}a_{12}$

$A^{-1} = \frac{1}{b}\begin{pmatrix} bA_{11}^{-1} + A_{11}^{-1}a_{12}a_{21}'A_{11}^{-1} & -A_{11}^{-1}a_{12} \\ -a_{12}'A_{11}^{-1} & 1 \end{pmatrix}$

* $(\underset{n\times n}{A} + \underset{n\times 1}{c}\ \underset{1\times n}{c'})^{-1} = A^{-1} - \frac{A^{-1}cc'A^{-1}}{1 + c'A^{-1}c}$

Pf: $A_{22} = A$, $A_{21} = c$, $A_{11} = -1$, $A_{12} = c_{12}'$

or $\begin{pmatrix} -1 & c' \\ c & A_{n\times n} \end{pmatrix}$ + (*) ⇒ (**)

- Full Rank Factorization

Theorem:

$\mathbf{A}_{p\times q}$ of rank r can always be factorized as

$$\mathbf{A} = \mathbf{K}_{p\times r}\mathbf{L}_{r\times q}$$

where $\mathbf{K}$ and $\mathbf{L}$ have full column and full row rank respectively.

Proof:

?

There exist nonsingular matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\Rightarrow \qquad \mathbf{A} = \mathbf{P}^{-1}\begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}^{-1}$$

Partition $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$ as

$$\mathbf{P}^{-1} = [\mathbf{K}_{p\times r} \quad \mathbf{W}_{p\times(p-r)}]$$

$$\mathbf{Q}^{-1} = \begin{bmatrix} \mathbf{L}_{r\times q} \\ \mathbf{Z}_{(q-r)\times q} \end{bmatrix}$$

$$\begin{aligned} \mathbf{A} &= [\mathbf{K} \quad \mathbf{W}]\begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{L} \\ \mathbf{Z} \end{bmatrix} \\ &= [\mathbf{K} \quad \mathbf{0}]\begin{bmatrix} \mathbf{L} \\ \mathbf{Z} \end{bmatrix} \\ &= \mathbf{KL} \end{aligned}$$

幂等的

- Idempotent Matrices ($\mathbf{A}^2 = \mathbf{A}$)

  - All idempotent matrices (except $\mathbf{I}$) are singular

    Proof: Since $\mathbf{A}^2 = \mathbf{A}$ and if $\mathbf{A}$ is nonsingular,

$$\mathbf{A} = \mathbf{A}^{-1}\mathbf{A}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

  - $r(\mathbf{A}) = tr\mathbf{A}$

    Proof:

    Consider the full rank factorization, let

$$\mathbf{A} = \mathbf{BC} \quad \text{and} \quad \mathbf{A}^2 = \mathbf{BCBC} = \mathbf{BC}$$

    But $\mathbf{B}$ has a left inverse $\mathbf{U}$ and $\mathbf{C}$ has a right inverse $\mathbf{R}$

    列满 →左逆　　行满→右逆

$$\begin{aligned} \mathbf{UBCBCR} &= \mathbf{UBCR} \\ \Rightarrow \qquad \mathbf{CB} &= \mathbf{I}_{r\times r} \end{aligned}$$

    So,

$$\begin{aligned} tr(\mathbf{A}) &= tr(\mathbf{BC}) \\ &= tr(\mathbf{CB}) \\ &= tr(\mathbf{I}_{r\times r}) \\ &= r \\ &= r(\mathbf{A}) \end{aligned}$$

- Eigenvalues of idempotent matrices are either 0 or 1.

Proof:

Let $\lambda, \mathbf{x}$ be a pair of eigenvalue and eigenvector

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\Rightarrow \mathbf{A}^2\mathbf{x} = \mathbf{A}(\mathbf{A}\mathbf{x}) = \mathbf{A}(\lambda\mathbf{x}) = \lambda\mathbf{A}\mathbf{x} = \lambda^2\mathbf{x}$$

But

$$\mathbf{A}^2\mathbf{x} = \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\lambda^2\mathbf{x} = \lambda\mathbf{x} \Rightarrow \lambda(\lambda - 1)\mathbf{x} = 0$$

Since

$$\mathbf{x} \neq 0 \Rightarrow \lambda = 0 \;\; or \;\; 1$$

- Theorem:

For symmetric matrix $\mathbf{A}$, if all eigenvalues are 1 or 0, $\mathbf{A}$ is idempotent

Proof:

For $\mathbf{A}$ symmetric, there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P'AP} = \mathbf{D}$$

where $\mathbf{D}$ is a diagonal matrix with eigenvalues of $\mathbf{A}$ on the diagonal

So, $\quad \mathbf{P'APP'AP} = \mathbf{D}^2$

But, $\quad \mathbf{P'APP'AP} = \mathbf{P'AAP}$

However if all eigenvalues are 1 or 0

$\Rightarrow \quad \mathbf{D} = \mathbf{D}^2$

$\Rightarrow \quad \mathbf{P'AP} = \mathbf{P'AAP}$

$\Rightarrow \quad \mathbf{A} = \mathbf{AA}$

$\Rightarrow \quad \mathbf{A}$ is idempotent consider $y = X\beta + \varepsilon$

$$X_{n\times p} = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{np} \end{pmatrix}$$

$p >> n$, $\underset{p\times 1}{\hat{\beta}} = (\underset{p\times n}{X^T}\ \underset{n\times p}{X})^{-1} X^T y$

用来找有效的variables，

若奇异，逆无意义，可用g-逆替代

3.2 Generalized Inverse

Definition: Let $\mathbf{A}$ be $m \times n$ and the generalized inverse $\mathbf{A}^-$ satisfies

$$\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$$

- g-inverse may not be unique

Example 3.1

$\underset{\sim}{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, then $\underset{\sim}{x_1}^{-1} = (1, 0, 0, 0)$ is one of the g-inverse

$$\underset{\sim}{x}\underset{\sim}{x}^{-1}x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\left.\begin{array}{l} x_2^- = (0, 0, \frac{1}{3}, 0) \\ x_3^- = (0, 0, \frac{1}{3}, 0) \\ x_4^{-1} = (0, 0, 0, \frac{1}{4}) \end{array}\right\} \text{are also g-inverse of } \underset{\sim}{x}$$

$\Rightarrow$ x^{-1} is usually not unique

Theorem: Suppose $\boldsymbol{A}$ is $n \times p$ f rank r and $\boldsymbol{A}$ is partitioned by

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}$$

where $\boldsymbol{A}_{11}$ is $r \times r$ of rank r (full rank). Then a generalized inverse of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{-} = \begin{bmatrix} \boldsymbol{A}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Proof.

$$AA^{-}A = \begin{bmatrix} I_r & 0 \\ A_{21}A_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

$$A_{21}A_{11}^{-1}A_{12} \overset{?}{=} A_{22}$$

let $B = \begin{bmatrix} I_r & 0 \\ -A_{21}A_{11}^{-1} & I_{n-r} \end{bmatrix}$ — nonsingular

note that $\boxed{\text{rank}(BA) = \text{rank}(A) = r}$

$$BA = \begin{pmatrix} A_{11} & A_{12} \\ 0 & \underbrace{A_{22} - A_{21}A_{11}^{-1}A_{12}}_{\text{可以写成线性组合}} \end{pmatrix} \Rightarrow \begin{pmatrix} A_{12} \\ A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix} = \begin{pmatrix} A_{11} \\ 0 \end{pmatrix} Q = \begin{pmatrix} A_{11}Q \\ 0 \end{pmatrix}$$

已经rank=r

$$\Rightarrow A_{22} - A_{21}A_{11}^{-1}A_{12} = 0$$

$$\Rightarrow A_{21}A_{11}^{-1}A_{12} = A_{22}$$

Corollary Suppose $\boldsymbol{A}$ is $n \times p$ f rank r, and $\boldsymbol{A}_{22}$ is $r \times r$ of rank r (full rank). Then a generalized inverse of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{-} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}_{22}^{-1} \end{bmatrix}.$$

Example 3.2

$$\underset{\sim}{A} = \begin{pmatrix} \overset{\underset{\sim}{A}_{11}}{\boxed{\begin{matrix} 2 & 2 \\ 1 & 0 \end{matrix}}} & \begin{matrix} 3 \\ 1 \end{matrix} \\ \begin{matrix} 3 & 2 \end{matrix} & 4 \end{pmatrix} \qquad \operatorname{rank}(\underset{\sim}{A}) = 2.$$

$$\underset{\sim}{A}_1^{-} = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{A}_2^{-} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22}^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

- Let $\mathbf{X}$ be $m \times n$, $\cancel{r(x)}$ $r(\underset{\sim}{X})$ $= k > 0$ $\Rightarrow X^-$ must be $n\times m$.
  - **(i)** $r(\mathbf{X}^-) \geq k$ $\rightarrow k = r(\underset{\sim}{X}) = r(\underset{\sim}{X}\underset{\sim}{X}^-\underset{\sim}{X}) \leq r(\underset{\sim}{X}^-)$ ✓
  - **(ii)** $\mathbf{X}^-\mathbf{X}$ and $\mathbf{X}\mathbf{X}^-$ are idempotent ✓ $\underset{\sim}{X}^-\underset{\sim}{X}\underset{\sim}{X}^-\underset{\sim}{X} = \underset{\sim}{X}^-\underset{\sim}{X}$, $\underset{\sim}{X}\underset{\sim}{X}^-\underset{\sim}{X}\underset{\sim}{X}^- = \underset{\sim}{X}\underset{\sim}{X}^-$
  - **(iii)** $r(\mathbf{X}^-\mathbf{X}) = r(\mathbf{X}\mathbf{X}^-) = k$ $k = \text{rank}(XX^-X) \leq r(X^-X) = r(XX^-) \leq \text{rank}(X) = k$ ✓
  - **(iv)** $\mathbf{X}^-\mathbf{X} = \mathbf{I}$ if and only if $r(\mathbf{X}) = n$ ✓
  - **(v)** $\mathbf{X}\mathbf{X}^- = \mathbf{I}$ if and only if $r(\mathbf{X}) = m$ ✓
  - **(vi)** $tr(\mathbf{X}^-\mathbf{X}) = tr(\mathbf{X}\mathbf{X}^-) = k = r(\mathbf{X})$ ✓ $r(X^-X) = tr(X^-X) = tr(XX^-) = r(XX^-)$
  - **(vii)** If $\mathbf{X}^-$ is any g-inverse of $\mathbf{X}$,then $(\mathbf{X}^-)'$ is a g-inverse of $\mathbf{X}$'
    $XX^-X = X$
    $\Rightarrow X'(X^-)'X' = X'$
- Let $\mathbf{K} = \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$, $\mathbf{K}$ is invariant for any g-inverse of $\mathbf{X}'\mathbf{X}$

Remark 3.2

- For $\mathbf{K} = \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$
  - **(i)** $\mathbf{K} = \mathbf{K}'$, $\mathbf{K} = \mathbf{K}^2$ (So, Symmetric Idempotent)
  - **(ii)** rank($\mathbf{K}$) = rank($\mathbf{X}$) = r (rank($\mathbf{K}$) = tr($\mathbf{K}$) = rank($\mathbf{X}$)) ✓
  - **(iii)** $\mathbf{K}\mathbf{X} = \mathbf{X}$; $\mathbf{X}'\mathbf{K} = \mathbf{X}'$
  - **(iv)** $(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ is a g-inverse of $\mathbf{X}$ for any g-inverse of $\mathbf{X}'\mathbf{X}$
  - **(v)** $\mathbf{X}(\mathbf{X}'\mathbf{X})^-$ is a g-inverse of $\mathbf{X}'$ for any g-inverse of $\mathbf{X}'\mathbf{X}$

Pf: (i). $\underset{\sim}{K}' = (X(X'X)^-X')' = X((X'X)^-)'X' = X(X'X)^-X' = K$

$\underset{\sim}{K}^2 = X(X'X)^-X'X(X'X)^-X' =$

(ii) $k = r(X'X) \leq r(K) \leq r(X) = k$ $\Rightarrow r(k) = tr(k) = r$

(iii). $\underset{\sim}{K}\underset{\sim}{X} = \underset{\sim}{X}$, $\underset{\sim}{X}'\underset{\sim}{K} = \underset{\sim}{X}'$

Want to : $X(X'X)^-X'X = X$

Note : $X'X(X'X)^-X'X = X'X$

$L'K'X(X'X)^-X'KL = L'K'KL$

$K'X(X'X)^-X'K = K'K$ $\rightarrow K'X(X'X)^-L'K'K = K'K$

$K'KL(X'X)^-X'K = K'K$ $\rightarrow K'X(X'X)^-L' = I$

$\Rightarrow L(X'X)^-X'K = I$ $\rightarrow L'K'X(X'X)^-L'K' = L'K'$

$\Rightarrow X(X'X)^-X'X = X$ $\rightarrow X'X(X'X)^-X' = X'$

(iv). (v). By (iii) obviously.

$X_{p\times q} = K_{p\times r} \cdot L_{r\times q}$

Proof: $X'X(X'X)^{-}X'X = X'X$

$$\Rightarrow \underset{R'}{\uparrow} L'K'X(X'X)^{-}X'KL \underset{R}{\uparrow} = \underset{R'}{\uparrow} L'K'KL \underset{R}{\uparrow}$$

$$\Rightarrow K'X(X'X)^{-}X'K = K'K$$

$$\Rightarrow K'KL(X'X)^{-}L'K'K = K'K$$

$$\Rightarrow L(X'X)^{-}L' = (K'K)^{-1}$$

$$\Rightarrow KL(X'X)^{-1}L'K' = K(K'K)^{-1}K'$$

$$\Rightarrow X(X'X)^{-1}X' = \underline{K(K'K)^{-1}K'} \leftarrow$$ then prove this is invariant.

$$K(K'K)^{-1}K' = K^{*}S(S'K^{*\prime}K^{*}S)^{-1}S'K^{*\prime}$$

$$= K^{*}(K^{*\prime}K^{*})^{-1}K^{*}$$

Full rank factorization.

$r = \text{rank}(X)$

$$X = K_{p\times r}L_{r\times q}$$

$$= K^{*}_{p\times r}L^{*}_{r\times q}$$

$K = K^{*}\cdot S_{r\times r}$. $\quad S \sim$ nonsingular

$LR = I_r$

$\uparrow$ right inverse

$\Rightarrow (K'K)$ is nonsingular

Moore - Penrose Inverse

Definition: Let $\mathbf{A}$ be an $m \times n$ matrix.
If a matrix $\mathbf{A}^+$ exists that satisfies

$$\left.\begin{array}{ll} (1) & \mathbf{A}\mathbf{A}^+ \textit{ is symmetric} \\ (2) & \mathbf{A}^+\mathbf{A} \textit{ is symmetric} \\ (3) & \mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A} \\ (4) & \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+ \end{array}\right\} \quad (*)$$

$\mathbf{A}^+$ is defined as a Moore - Penrose inverse of $\mathbf{A}$.

Theorem: Each matrix $(\mathbf{A})$ has an $\mathbf{A}^+$

Proof:

If $\mathbf{A} = 0$, $\mathbf{A}^+ = 0$
If $\mathbf{A} \neq 0$, $\mathbf{A}$ can be factored by full rank factorization

$$\mathbf{A} = \mathbf{A_L}\mathbf{A_R} = \mathbf{B}\mathbf{C}$$

where $\mathbf{B}$ is $m \times r$ of rank r and $\mathbf{C}$ is $r \times n$ of rank r

Hence, $\mathbf{B}$'$\mathbf{B}$ and $\mathbf{C}\mathbf{C}$' are both n.s.
Define

$$\mathbf{A}^+ \;=\; \mathbf{C}'(\mathbf{C}\mathbf{C}')^{-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'$$

and it can be shown that $\mathbf{A}^+$ satisfies the 4 conditions (*) Q.E.D.

- The Moore - Penrose inverse is unique
- $(\mathbf{A}')^+ = (\mathbf{A}^+)'$
- $(\mathbf{A}^+)^+ = \mathbf{A}$
- $r(\mathbf{A}^+) = r(\mathbf{A})$
- If $\mathbf{A} = \mathbf{A}$', then $\mathbf{A}^+ = (\mathbf{A}^+)'$
- If $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1} = \mathbf{A}^+$
- If $\mathbf{A}$ is symmetric idempotent, $\mathbf{A}^+ = \mathbf{A}$
- If $r(\mathbf{A}_{m\times n}) = m$,then $\mathbf{A}^+ = \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}$,$\mathbf{A}\mathbf{A}^+ = \mathbf{I}$
  If $r(\mathbf{A}_{m\times n}) = n$,then $\mathbf{A}^+ = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$,$\mathbf{A}^+\mathbf{A} = \mathbf{I}$
- The matrices $\mathbf{A}\mathbf{A}^+, \mathbf{A}^+\mathbf{A}, \mathbf{I}-\mathbf{A}\mathbf{A}^+$ and $\mathbf{I}-\mathbf{A}^+\mathbf{A}$ are all symmetric idempotent

§3.3 Vector and matric calculations

(I). $u = f(\underset{\sim}{x})$ $\quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$

$$\frac{\partial u}{\partial \underset{\sim}{x}} = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_p} \end{pmatrix}$$

— Let $u = \underset{\sim}{a}'\underset{\sim}{x} = \underset{\sim}{x}'\underset{\sim}{a}$. $\frac{\partial u}{\partial \underset{\sim}{x}} = \frac{\partial(\underset{\sim}{a}'\underset{\sim}{x})}{\partial \underset{\sim}{x}} = \underset{\sim}{a}$ obviously.

— Let $u = x'Ax$. where A—symmetric. $\frac{\partial u}{\partial \underset{\sim}{x}} = \frac{\partial(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x})}{\partial \underset{\sim}{x}} = 2\underset{\sim}{A}\underset{\sim}{x}$

$A = (a_{ij})_{n\times n}$

$\Rightarrow \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} = \sum_{i\neq j}(a_{ij} + \overset{\overset{2a_{ij}x_ix_j}{\shortparallel}}{a_{ij}})x_ix_j + \sum_{k=1}^{n} a_{kk}x_k^2$

$\Rightarrow \frac{\partial u}{\partial \underset{\sim}{x}} = 2\begin{pmatrix} \\ a_{mm}x_1 + \sum_{\substack{k=1 \\ \neq m}}^{n} a_{mk}x_k \\ \\ \end{pmatrix} - m$

$\Rightarrow \underset{\sim}{A}\underset{\sim}{x} = \begin{pmatrix} \\ \sum_{k=1}^{n} a_{mk}x_k \\ \\ \end{pmatrix} - m$

$\Rightarrow \frac{\partial u}{\partial \underset{\sim}{x}} = 2\underset{\sim}{A}\underset{\sim}{x}$

(II). $u = f(\underset{\sim}{X})$ $\quad \underset{\sim}{X} = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pp} \end{pmatrix}_{p\times p}$

$$\frac{\partial u}{\partial \underset{\sim}{X}} = \begin{pmatrix} \frac{\partial u}{\partial x_{11}} & \cdots & \frac{\partial u}{\partial x_{1p}} \\ \vdots & & \vdots \\ \frac{\partial u}{\partial x_{p1}} & \cdots & \frac{\partial u}{\partial x_{pp}} \end{pmatrix}$$

— let $u = tr(\underset{\sim}{X}\underset{\sim}{A})$ $\quad X_{p\times p}$—Symmetric postive definite.

$$\frac{\partial u}{\partial \underset{\sim}{X}} = \frac{\partial tr(\underset{\sim}{X}\underset{\sim}{A})}{\partial \underset{\sim}{X}} = \underset{\sim}{A} + \underset{\sim}{A}' - diag(A)$$

Hint: $tr(\underset{\sim}{X}\underset{\sim}{A}) = \sum_i\sum_j x_{ij}a_{ji}$

$x_{ij} = x_{ji}$ if $i \neq j$

$(\underset{\sim}{X}\underset{\sim}{A})_{(i,j)} = \sum_{k=1}^{n} x_{ik}a_{kj}$

$tr(\underset{\sim}{X}\underset{\sim}{A}) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_{ij}a_{ji}\right) = \sum_i\sum_j x_{ij}a_{ji}$

the coefficients of x_{ij} is $\begin{cases} (a_{ij} + a_{ji}), & i \neq j. \\ a_{ii}, & i = j \end{cases}$

— $u = \ln|\underset{\sim}{X}|$

$$\frac{\partial u}{\partial \underset{\sim}{X}} = \frac{\partial \ln|\underset{\sim}{X}|}{\partial \underset{\sim}{X}} = 2X^{-1} - diag(\underset{\sim}{X}^{-1})$$

(III) $\underset{\sim}{A}_{n\times n} = (a_{ij})$ $\quad a_{ij} = a_{ij}(x)$ — function of x (Scalar)

— $\underset{\sim}{A}$ is nonsingular of order n. with $\frac{dA}{dx}$

$$\Rightarrow \frac{\partial A^{-1}}{\partial x} = -\underset{\sim}{A}^{-1}\frac{\partial A}{\partial x}\underset{\sim}{A}^{-1}$$

Hint: $\underset{\sim}{A}^{-1}\underset{\sim}{A} = \underset{\sim}{I}$

$$\frac{\partial A^{-1}}{\partial x}\underset{\sim}{A} + \underset{\sim}{A}^{-1}\frac{\partial A}{\partial x} = \underset{\sim}{0}$$

— $\underset{\sim}{A}$ is $n\times n$ positive definite matrix

$$\frac{\partial \ln|A|}{\partial x} = tr\left(\underset{\sim}{A}^{-1}\frac{\partial A}{\partial x}\right)$$