## 5 Multivariate Normal Distribution

$\underset{\sim}{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$

- Density (Let $\boldsymbol{Y}_{p\times 1} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$)

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = |\boldsymbol{\Sigma}|^{-\frac{1}{2}}(2\pi)^{-\frac{p}{2}}e^{-\frac{1}{2}\{(\boldsymbol{y}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\}}$$

- MGF: $M_{\boldsymbol{Y}}(\mathbf{t}) = \mathbf{e}^{\mathbf{t}'\boldsymbol{\mu}+\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}}$ ← Remark 5.1

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right)$$

$\underline{Y_1 \sim} N(\mu_1, \sigma_{11})$ $\qquad \sigma_{11} = \sigma_1^2$

$cov(Y_1, Y_2) = \sigma_{12}$,

$corr(Y_1, Y_2) = \dfrac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$

$Y_2 | Y_1 \sim N( \quad , \quad )$

- Let $\boldsymbol{B}$ be a constant matrix and $\boldsymbol{C}$ be a constant vector

$$\boldsymbol{BY} + \boldsymbol{C} \;\sim\; N(\boldsymbol{B\mu} + \boldsymbol{C}, \boldsymbol{B\Sigma B}')$$

Remark 5.2

- Marginal Distribution, Condition Distribution and independence
  Let

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix} \;\sim\; N\left[\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}\right]$$

  then

  **(i)** $\boldsymbol{Y}_1 \;\sim\; N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$

  **(ii)** $\boldsymbol{Y}_1 | \boldsymbol{Y}_2 = \boldsymbol{y}_2 \;\sim\; N(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{y}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})$

  **(iii)** $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are independent iff $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$

Remark 5.3

- Partial Correlation

Let $\boldsymbol{v} \sim N_q(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{x} \end{pmatrix}; \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_y \\ \boldsymbol{\mu}_x \end{pmatrix}; \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{yy} & \boldsymbol{\Sigma}_{yx} \\ \boldsymbol{\Sigma}_{xy} & \boldsymbol{\Sigma}_{xx} \end{pmatrix}$$

where $\boldsymbol{y} = (y_1, y_2, ..., y_{r-1})'$ and $\boldsymbol{x} = (x_r, ..., x_q)'$. Let $\rho_{ij.r...q}$ be the partial correlation between $y_i$ and $y_j$, $1 \leq i < j \leq r-1$, in the conditional distribution of $\boldsymbol{y}$ given $\boldsymbol{x}$. By the definition of correlation, we have

Remark 5.4

$$\rho_{ij.r...q} = \frac{\sigma_{ij.r...q}}{\sqrt{\sigma_{ii.r...q}\sigma_{jj.r...q}}}.$$

Matrix of partial correlations

$$\boldsymbol{\Omega}_{y.x} = \boldsymbol{D}_{y.x}^{-1} \boldsymbol{\Sigma}_{y.x} \boldsymbol{D}_{y.x}^{-1}$$

where $\boldsymbol{\Sigma}_{y.x} = \boldsymbol{\Sigma}_{yy} - \boldsymbol{\Sigma}_{yx}\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xy}$ and $\boldsymbol{D}_{y.x} = [diag(\boldsymbol{\Sigma}_{y.x})]^{1/2}$.

**Example** 5.1

**Example** 5.2