3 Matrix Algebra

3.1 Vector and matrix

- A brief review of vector and matrix

Remark 3.1

- Full Rank Factorization

$r = \text{Rank}(A)$

<u>Theorem</u>:

$\mathbf{A}_{p\times q}$ of rank r can always be factorized as

$$\underset{p\times q}{\mathbf{A}} = \mathbf{K}_{p\times r}\mathbf{L}_{r\times q}$$

where $\mathbf{K}$ and $\mathbf{L}$ have full column and full row rank respectively.

<u>Proof</u>:

There exist nonsingular matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\underset{p\times p}{\mathbf{P}}\mathbf{A}\underset{q\times q}{\mathbf{Q}} = \underset{p\times q}{\begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}$$

$$\Rightarrow \qquad \mathbf{A} = \mathbf{P}^{-1}\begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}^{-1}$$

similarly, there is a right inverse of $\underset{\sim}{L}$,

$\underset{r\times q}{\underset{\sim}{L}} \; \underset{q\times r}{\underset{\sim}{R}} = I_{r\times r}$

Partition $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$ as

Let $\underset{p\times p}{\underset{\sim}{P}} = \begin{pmatrix} \underset{r\times p}{U} \\ \underset{(p-r)\times p}{V} \end{pmatrix}$ ....

$$\mathbf{P}^{-1} = \underline{[\mathbf{K}_{p\times r} \quad \mathbf{W}_{p\times(p-r)}]}$$

$\Rightarrow \underset{\sim}{P}\,\underset{\sim}{P}^{-1} = \begin{pmatrix} \underset{\sim}{U}\underset{\sim}{K} & \underset{\sim}{U}\underset{\sim}{W} \\ \underset{\sim}{V}\underset{\sim}{K} & \underset{\sim}{V}\underset{\sim}{W} \end{pmatrix} = I_{p\times p}$

$$\mathbf{Q}^{-1} = \begin{bmatrix} \mathbf{L}_{r\times q} \\ \mathbf{Z}_{(q-r)\times q} \end{bmatrix}$$

$\underset{r\times p}{\underset{\sim}{U}}\;\underset{p\times r}{\underset{\sim}{K}} = I_{r\times r}$

$\underset{\sim}{U}_{r\times p}$ is the left inverse of $\underset{\sim}{K}_{p\times r}$

$$\begin{aligned} \mathbf{A} &= [\mathbf{K} \quad \mathbf{W}]\begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{L} \\ \mathbf{Z} \end{bmatrix} \\ &= [\mathbf{K} \quad \mathbf{0}]\begin{bmatrix} \mathbf{L} \\ \mathbf{Z} \end{bmatrix} \\ &= \mathbf{KL} \end{aligned}$$

- Idempotent Matrices ($\mathbf{A}^2 = \mathbf{A}$)
  - All idempotent matrices (except $\mathbf{I}$) are singular

    Proof: Since $\mathbf{A}^2 = \mathbf{A}$ and if $\mathbf{A}$ is nonsingular,

$$\mathbf{A} = \mathbf{A}^{-1}\mathbf{A}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

  - $r(\mathbf{A}) = tr\mathbf{A}$

    Proof:

    Consider the full rank factorization, let

$$\mathbf{A} = \mathbf{BC} \quad \text{and} \quad \mathbf{A}^2 = \mathbf{BCBC} = \mathbf{BC}$$

    But $\mathbf{B}$ has a left inverse $\mathbf{U}$ and $\mathbf{C}$ has a right inverse $\mathbf{R}$

$$\begin{aligned} \mathbf{UBCBCR} &= \mathbf{UBCR} \\ \Rightarrow \qquad \mathbf{CB} &= \mathbf{I}_{r\times r} \end{aligned}$$

    So,

$$\begin{aligned} tr(\mathbf{A}) &= tr(\mathbf{BC}) \\ &= tr(\mathbf{CB}) \\ &= tr(\mathbf{I}_{r\times r}) \\ &= r \\ &= r(\mathbf{A}) \end{aligned}$$

- Eigenvalues of idempotent matrices are either 0 or 1.

Proof:

Let $\lambda, \mathbf{x}$ be a pair of eigenvalue and eigenvector

$$\mathbf{A}\mathbf{x} \;=\; \lambda\mathbf{x}$$

$$\Rightarrow \mathbf{A}^2\mathbf{x} \;=\; \mathbf{A}(\mathbf{A}\mathbf{x}) \;=\; \mathbf{A}(\lambda\mathbf{x}) \;=\; \lambda\mathbf{A}\mathbf{x} \;=\; \lambda^2\mathbf{x}$$

But

$$\mathbf{A}^2\mathbf{x} \;=\; \mathbf{A}\mathbf{x} \;=\; \lambda\mathbf{x}$$

$$\lambda^2\mathbf{x} \;=\; \lambda\mathbf{x} \;\Rightarrow\; \lambda(\lambda-1)\mathbf{x} \;=\; 0$$

Since

$$\mathbf{x} \neq 0 \;\Rightarrow\; \lambda = 0 \;\; or \;\; 1$$

- Theorem:

For symmetric matrix $\mathbf{A}$, if all eigenvalues are 1 or 0, $\mathbf{A}$ is idempotent

Proof:

For $\mathbf{A}$ symmetric, there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{D}$$

where $\mathbf{D}$ is a diagonal matrix with eigenvalues of $\mathbf{A}$ on the diagonal

So, $\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{D}^2$

But, $\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{P}'\mathbf{A}\mathbf{A}\mathbf{P}$

However if all eigenvalues are 1 or 0

$\Rightarrow$ $\mathbf{D} = \mathbf{D}^2$

$\Rightarrow$ $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{P}'\mathbf{A}\mathbf{A}\mathbf{P}$

$\Rightarrow$ $\mathbf{A} = \mathbf{A}\mathbf{A}$

$\Rightarrow$ $\mathbf{A}$ is idempotent

3.2 Generalized Inverse

Definition: Let $\mathbf{A}$ be $m \times n$ and the generalized inverse $\mathbf{A}^-$ satisfies

$$\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$$

- g-inverse may not be unique

Example 3.1

$$\underset{\sim}{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

then $x_1^- = (1, 0, 0, 0)$ is $\underset{\sim}{x}$'s g-inverse.

$$\underset{\sim}{x}\,\underset{\sim}{x}_1^-\,\underset{\sim}{x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\left.\begin{aligned} \underset{\sim}{x}_2^- &= (0, \tfrac{1}{2}, 0, 0) \\ \underset{\sim}{x}_3^- &= (0, 0, \tfrac{1}{3}, 0) \\ \underset{\sim}{x}_4^- &= (0, 0, 0, \tfrac{1}{4}) \end{aligned}\right\} \text{ are also g-inverse of } \underset{\sim}{x}$$

$\Rightarrow$ $\underset{\sim}{x}^-$ is usually not unique.

<u>Theorem</u>: Suppose $\boldsymbol{A}$ is $n \times p$ f rank r and $\boldsymbol{A}$ is partitioned by

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}$$

where $\boldsymbol{A}_{11}$ is $r \times r$ of rank r (full rank). Then a generalized inverse of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{-} = \begin{bmatrix} \boldsymbol{A}_{11}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

Proof.

$$A A^{-} A = \underbrace{\begin{bmatrix} I_r & 0 \\ A_{21} A_{11}^{-1} & 0 \end{bmatrix}}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \underbrace{A_{21} A_{11}^{-1} A_{12}}_{\overset{?}{=} A_{22}} \end{bmatrix}$$

Let

$$B = \begin{pmatrix} I_r & 0 \\ -A_{21} A_{11}^{-1} & I_{n-r} \end{pmatrix} \text{ — non singular}$$

$$\boxed{\operatorname{rank}(BA) = \operatorname{rank}(A) = r}$$

$$BA = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A_{12} \\ A_{22} - A_{21} A_{11}^{-1} A_{12} \end{pmatrix} = \begin{pmatrix} A_{11} \\ 0 \end{pmatrix} Q = \begin{pmatrix} A_{11} Q \\ 0 \end{pmatrix}$$

Corollary Suppose $\boldsymbol{A}$ is $n \times p$ f rank r, and $\boldsymbol{A}_{22}$ is $r \times r$ of rank r (full rank). Then a generalized inverse of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{-} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_{22}^{-1} \end{bmatrix}.$$

$$\Rightarrow A_{22} - A_{21} A_{11}^{-1} A_{12} = 0$$

$$\Rightarrow A_{21} A_{11}^{-1} A_{12} = A_{22}$$

Ex 3.2

Example 3.2

- Let $\mathbf{X}$ be $m \times n$, $r(x) = k > 0$
  - **(i)** $r(\mathbf{X}^{-}) \geq k$
  - **(ii)** $\mathbf{X}^{-}\mathbf{X}$ and $\mathbf{X}\mathbf{X}^{-}$ are idempotent
  - **(iii)** $r(\mathbf{X}^{-}\mathbf{X}) = r(\mathbf{X}\mathbf{X}^{-}) = k$
  - **(iv)** $\mathbf{X}^{-}\mathbf{X} = \mathbf{I}$ if and only if $r(\mathbf{X}) = n$
  - **(v)** $\mathbf{X}\mathbf{X}^{-} = \mathbf{I}$ if and only if $r(\mathbf{X}) = m$
  - **(vi)** $tr(\mathbf{X}^{-}\mathbf{X}) = tr(\mathbf{X}\mathbf{X}^{-}) = k = r(\mathbf{X})$
  - **(vii)** If $\mathbf{X}^{-}$ is any g-inverse of $\mathbf{X}$,then $(\mathbf{X}^{-})'$ is a g-inverse of $\mathbf{X}$'
- Let $\mathbf{K} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$, $\mathbf{K}$ is invariant for any g-inverse of $\mathbf{X}'\mathbf{X}$
- For $\mathbf{K} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$
  - **(i)** $\mathbf{K} = \mathbf{K}'$, $\mathbf{K} = \mathbf{K}^2$ (So, Symmetric Idempotent)
  - **(ii)** rank($\mathbf{K}$) = rank($\mathbf{X}$) = r (rank($\mathbf{K}$) = tr($\mathbf{K}$) = rank($\mathbf{X}$))
  - **(iii)** $\mathbf{K}\mathbf{X} = \mathbf{X}$; $\mathbf{X}'\mathbf{K} = \mathbf{X}'$
  - **(iv)** $(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$ is a g-inverse of $\mathbf{X}$ for any g-inverse of $\mathbf{X}'\mathbf{X}$
  - **(v)** $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}$ is a g-inverse of $\mathbf{X}'$ for any g-inverse of $\mathbf{X}'\mathbf{X}$

Remark 3.2

Moore - Penrose Inverse

Definition: Let $\mathbf{A}$ be an $m \times n$ matrix.
If a matrix $\mathbf{A}^+$ exists that satisfies

$$\left.\begin{array}{ll} (1) & \mathbf{A}\mathbf{A}^+ \textit{ is symmetric} \\ (2) & \mathbf{A}^+\mathbf{A} \textit{ is symmetric} \\ (3) & \mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A} \\ (4) & \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+ \end{array}\right\} \quad (*)$$

$\mathbf{A}^+$ is defined as a Moore - Penrose inverse of $\mathbf{A}$.

Theorem: Each matrix ($\mathbf{A}$) has an $\mathbf{A}^+$

Proof:

If $\mathbf{A} = 0$, $\mathbf{A}^+ = 0$
If $\mathbf{A} \neq 0$, $\mathbf{A}$ can be factored by full rank factorization

$$\mathbf{A} = \mathbf{A_L}\mathbf{A_R} = \mathbf{BC}$$

where $\mathbf{B}$ is $m \times r$ of rank r and $\mathbf{C}$ is $r \times n$ of rank r

Hence, $\mathbf{B}$'$\mathbf{B}$ and $\mathbf{CC}$' are both n.s.
Define

$$\mathbf{A}^+ \;=\; \mathbf{C}'(\mathbf{CC}')^{-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'$$

and it can be shown that $\mathbf{A}^+$ satisfies the 4 conditions (*) Q.E.D.

- The Moore - Penrose inverse is unique
- $(\mathbf{A}')^+ = (\mathbf{A}^+)'$
- $(\mathbf{A}^+)^+ = \mathbf{A}$
- $r(\mathbf{A}^+) = r(\mathbf{A})$
- If $\mathbf{A} = \mathbf{A}$', then $\mathbf{A}^+ = (\mathbf{A}^+)'$
- If $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1} = \mathbf{A}^+$
- If $\mathbf{A}$ is symmetric idempotent, $\mathbf{A}^+ = \mathbf{A}$
- If $r(\mathbf{A}_{m\times n}) = m$,then $\mathbf{A}^+ = \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}$,$\mathbf{A}\mathbf{A}^+ = \mathbf{I}$
  If $r(\mathbf{A}_{m\times n}) = n$,then $\mathbf{A}^+ = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$,$\mathbf{A}^+\mathbf{A} = \mathbf{I}$
- The matrices $\mathbf{A}\mathbf{A}^+, \mathbf{A}^+\mathbf{A}, \mathbf{I}-\mathbf{A}\mathbf{A}^+$ and $\mathbf{I}-\mathbf{A}^+\mathbf{A}$ are all symmetric idempotent

3.3 Vector a