6 Quadratic Forms

6.1 Quadratic Form $\mathbf{x}'\mathbf{A}\mathbf{x}$

Let $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and assume $\mathbf{A}$ symmetric, then m.g.f. of $\mathbf{x}'\mathbf{A}\mathbf{x}$ is

$$M_{\mathbf{x}'\mathbf{A}\mathbf{x}}(t) = \mid \mathbf{I} - 2t\mathbf{A}\boldsymbol{\Sigma} \mid^{-\frac{1}{2}} \cdot \mathbf{e}^{\{-\frac{1}{2}\boldsymbol{\mu}'[\mathbf{I}-(\mathbf{I}-2\mathbf{t}\mathbf{A}\boldsymbol{\Sigma})^{-1}]\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\}} \tag{6.1}$$

Proof: $\longrightarrow$ Remark 6.1

Remark 6.1

Proof: $M_{\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}}(t) = E_{\underset{\sim}{x}}(e^{t\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}})$

$$= \int_{R^p} e^{t\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}} \frac{1}{(\sqrt{2\pi})^p|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})} d\underset{\sim}{x}$$

$$= c\int_{R^p} e^{-\frac{h}{2}} d\underset{\sim}{x} \qquad c = \frac{1}{(\sqrt{2\pi})^p|\Sigma|^{\frac{1}{2}}}$$

$$h = (\underset{\sim}{x}-\underset{\sim}{\mu})'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}) + t\,\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$$

$$= \underset{\sim}{x}'(I-2t\underset{\sim}{A}\underset{\sim}{\Sigma})\underset{\sim}{\Sigma}^{-1}\underset{\sim}{x} - 2\underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{x} + \underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}$$

$$= (\underset{\sim}{x}-\underset{\sim}{\theta})'\underset{\sim}{V}^{-1}(\underset{\sim}{x}-\underset{\sim}{\theta}) + \underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu} - \underset{\sim}{\theta}'\underset{\sim}{V}^{-1}\underset{\sim}{\theta}$$

$$\boxed{\begin{array}{l} V^{-1} = (I-2tA\Sigma)\Sigma^{-1} \\ \theta' = \mu'(I-2tA\Sigma)^{-1} \end{array}}$$

- $E(\mathbf{x}'\mathbf{A}\mathbf{x}) = \text{tr}(\boldsymbol{A\Sigma}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$ — ch4.
- $\text{Var}(\mathbf{x}'\mathbf{A}\mathbf{x}) = 2\text{tr}[(\boldsymbol{A\Sigma})^2] + 4\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\mu}$ — Remark 6.2

Remark 6.2

$$M_{X'AX}(t) = |C(t)|^{-\frac{1}{2}} e^{-\frac{1}{2}\mu'(I - C(t)^{-1})\Sigma^{-1}\mu}$$

Note: $C(t) = I - 2tA\Sigma$

$C(0) = I$, $C^{-1}(0) = I$

$$\frac{\partial C^{-1}}{\partial t} = -C^{-1}\frac{\partial C}{\partial t}C^{-1} = 2C^{-1}A\Sigma C^{-1}$$

Let $K(t) = \ln M_{X'AX}$

$\Rightarrow \text{Var}(X'AX) = \left.\frac{\partial^2 K(t)}{\partial t^2}\right|_{t=0}$

$K(t) = -\frac{1}{2}\ln|C(t)| - \frac{1}{2}\mu'(I - C^{-1}(t))\Sigma^{-1}\mu$

$\Rightarrow \frac{\partial K(t)}{\partial t} = -\frac{1}{2}\text{tr}\left(C^{-1}(t)\frac{\partial C}{\partial t}\right) + \mu' C^{-1} A\Sigma C^{-1}\Sigma^{-1}\mu$

$\frac{\partial^2 K(t)}{\partial t^2} = \cdots$ $\qquad C = C(t)$

$= 2\text{tr}(C^{-1}A\Sigma\, C^{-1}A\Sigma) + 2\mu'[C^{-1}A\Sigma C^{-1}]A\Sigma C^{-1}\Sigma^{-1}\mu + 2\mu' C^{-1}A\Sigma[C^{-1}A\Sigma C^{-1}]\Sigma^{-1}\mu$

$$\left.\frac{\partial^2 K(t)}{\partial t^2}\right|_{t=0} = 2\text{tr}(A\Sigma)^2 + 4\mu' A\Sigma A\mu$$

6.2 Non-Central χ^2, F and t distributions

I. Non-Central χ^2

$X_i \overset{iid}{\sim} N(0,1)$

- Let $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_n)$, then $\mathbf{x}'\mathbf{x} \sim \chi^2_{(n)}$

$X_i \overset{ind}{\sim} N(\mu_i, 1)$

- Let $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{I}_n)$, then

degree of freedom

$$u = \mathbf{x}'\mathbf{x} \sim \chi^2_{(n,\ \lambda)}$$

where $\lambda =$ non-centered parameter $= \frac{1}{2}\boldsymbol{\mu}'\boldsymbol{\mu}$ $= \lambda = \frac{1}{2}(\mu_1^2 + \cdots + \mu_n^2)$

- Density is

$$f(u) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{u^{\frac{1}{2}n+k-1} e^{-\frac{1}{2}u}}{2^{\frac{1}{2}n+k}\Gamma(\frac{1}{2}n+k)} \qquad \mu > 0, \lambda \geq 0$$

Note: Define $\lambda^k = 1$ when $\lambda = 0, k = 0$, density function of $u \sim \chi^2_{(n,0)}$ is

$$f(u) = \frac{u^{\frac{1}{2}n-1} e^{-\frac{1}{2}u}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}$$

- m.g.f of $u \sim \chi^2_{(n,\lambda)}$ is $\quad Mgf_{cu}(t) = mgf_u(ct) = (1-2ct)^{-\frac{n}{2}} e^{-\lambda[1-(1-2ct)^{-1}]}$

$=$

$$(1-2t)^{-\frac{n}{2}} e^{-\lambda[1-(1-2t)^{-1}]}$$

Note: for $\lambda = 0$, $\Rightarrow M_u(t) = (1-2t)^{-\frac{n}{2}}$ which is m.g.f of $\chi^2_{(n)}$

- $\mathrm{E}(u) = n + 2\lambda$ and $\mathrm{Var}(u) = 2n + 8\lambda$;
- If $u_i \sim \chi^2_{(n_i,\lambda_i)}$ independently for $i = 1, \ldots, k$, then

$$\sum_{i=1}^{k} u_i \sim \chi^2_{(\sum_{i=1}^k n_i, \sum_{i=1}^k \lambda_i)}.$$

II. Non-Central F

Let $u_1 \sim \chi^2_{(p_1,\lambda)}$, $u_2 \sim \chi^2_{(p_2,0)}$, and Let u_1 be independent of u_2, then

$$w = \frac{u_1/p_1}{u_2/p_2} \ \sim \ F_{(p_1,p_2,\lambda)}$$

and

$$\mathrm{E}(w) = \frac{p_2}{p_2-2}\left(1+\frac{2\lambda}{p_1}\right)$$

III. Non-Central t

Let $z \sim N(\mu, 1), u \sim \chi^2_{(n)}$, z is independent of u, then

$$t = \frac{z}{\sqrt{u/n}} \ \sim \ \text{non-centered } t \text{ distribution}$$

$\sim t_{n,\lambda} \quad (\lambda = \frac{\mu^2}{2})$

Theorem 6.1 Let $\mathbf{x}_{p\times 1} \sim N(\boldsymbol{\mu}, \mathbf{V})$, then $q = \mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi^2_{(r,\ \lambda)}$ where r denoting the rank of $\mathbf{A}$ and $\lambda = \frac{\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}}{2}$ if and only if $\mathbf{AV}$ is idempotent. (let $\mathbf{A}$ be symmetric)

$$\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim \chi^2(r, \lambda) \iff (\underset{\sim}{A}\underset{\sim}{V})^2 = \underset{\sim}{A}\underset{\sim}{V}$$

$r = \text{rank}(\underset{\sim}{A})$

$\lambda = \dfrac{\underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu}}{2} \quad \underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$

Remark 6.3

If $\underset{\sim}{A}\underset{\sim}{V}$ is idempotent. $\Rightarrow \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim \chi^2(r, \lambda)$

Proof: mgf of $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ is

$$M_{\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}}(t) = |\underset{\sim}{I} - 2t\underset{\sim}{A}\underset{\sim}{V}|^{-\frac{1}{2}} e^{-\frac{1}{2}\underset{\sim}{\mu}'[\underset{\sim}{I} - (\underset{\sim}{I} - 2t\underset{\sim}{A}\underset{\sim}{V})^{-1}]\underset{\sim}{V}^{-1}\underset{\sim}{\mu}}$$

let λ_i be the eigenvalue of $\underset{\sim}{A}\underset{\sim}{V}$. then $1 - 2t\lambda_i$ is the eigenvalue of $(\underset{\sim}{I} - 2t\underset{\sim}{A}\underset{\sim}{V})$

thus $|I - 2t\underset{\sim}{A}\underset{\sim}{V}| = \prod_{i=1}^{p}(1 - 2t\lambda_i)$

$\boxed{(I - A)^{-1} = I + A + A^2 + \cdots \text{ if all the eigenvalue are in } (-1, 1)}$

$(I - 2t\underset{\sim}{A}\underset{\sim}{V})^{-1} = I + \sum_{k=1}^{\infty}(2t)^k(\underset{\sim}{A}\underset{\sim}{V})^k$ —— $-1 < 2t\lambda_i < 1$

$(\underset{\sim}{A}\underset{\sim}{V})^k = \underset{\sim}{A}\underset{\sim}{V}$

λ_i: $\underbrace{1, 1, \cdots, 1}_{r}, 0, 0, \cdots, 0$

Corollaries

- If $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ is χ^2_r if and only if $\mathbf{A}$ is idempotent of rank r.
- If $\mathbf{x} \sim N(\mathbf{0}, \mathbf{V})$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ is χ^2_r if and only if $\mathbf{AV}$ is idempotent of rank r.
- If $\mathbf{x}$ is $N(\boldsymbol{\mu}, \sigma^2\mathbf{I})$, then $\frac{\mathbf{x}'\mathbf{x}}{\sigma^2}$ is $\chi^2_{\left(n, \frac{1}{2}\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{\sigma^2}\right)}$
- If $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{I})$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ is $\chi^2_{\left(r, \frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}\right)}$ if and only if $\mathbf{A}$ is idempotent of rank r.

6.3 Independence

Theorem 6.2 When $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \Sigma)$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ and $\mathbf{B}\mathbf{x}$ are distributed independently if and only if $\mathbf{B}\Sigma\mathbf{A} = 0$

If A is symmetric and idempotent, $x'Ax$ and Bx are independent.

$\Leftrightarrow B\Sigma A = 0$

Remark 6.4

Proof for the alone special case.

If $B\Sigma A = 0$

$Cov(BX, AX) = B\Sigma A' = B\Sigma A = 0$

$\Rightarrow BX$ and AX are independent (normal distribution)

$\Rightarrow BX$ and $X'AX$ are independent

If A^{-1} exist

BX and AX are independent

$\Rightarrow BX$ and $A^{\frac{1}{2}}AX = A^{\frac{1}{2}}X$ are independent

$\Rightarrow BX$ and $(A^{\frac{1}{2}}X)'(A^{\frac{1}{2}}X)$ are independent

If A^{-1} do not exist

also true...

If BX and $X'AX$ are independent

$\Rightarrow 0 = Cov(BX, X'AX)$

$= B\,Cov(X, X'AX)$

$= B\,E[(X-\mu)(X'AX - E(X'AX))]$

$E(X'AX) = tr(A\Sigma) + \mu' A \mu$

$= B\,E[(X-\mu)(\underline{X'AX - \mu'A\mu} - tr(A\Sigma))]$

$X'AX - \mu'A\mu$

$= (X-\mu)'A(X-\mu) + 2(X-\mu)'A\mu$

$= B\{E[(X-\mu)(X-\mu)'A(X-\mu)] + 2E[(X-\mu)(X-\mu)'A\mu] - E[(X-\mu)tr(A\Sigma)]\}$

$\overset{\shortparallel}{0}$ (third central moment of normal) $\qquad \overset{\shortparallel}{0}$

$= 2B\Sigma A\mu$ for any μ

$\Rightarrow B\Sigma A = 0$

tip: $Ax = 0$ for any x

$\Rightarrow A = 0$

Theorem 6.3 Let $\mathbf{x} \sim N(\boldsymbol{\mu}, \Sigma)$, $\mathbf{x}'\mathbf{A}\mathbf{x}$ and $\mathbf{x}'\mathbf{B}\mathbf{x}$ are distributed independently if and only if $\mathbf{A}\Sigma\mathbf{B} = 0$ (or equivalently, $\mathbf{B}\Sigma\mathbf{A} = 0$)

Additional results

Let the $n \times 1$ vector $\mathbf{x} = (x_1, ..., x_n)' \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $q_1 = \mathbf{x}'\mathbf{A}_1\mathbf{x}$, $q_2 = \mathbf{x}'\mathbf{A}_2\mathbf{x}$ and $\mathbf{T} = \mathbf{B}\mathbf{x}$ where $\mathbf{B}$ is $r \times n$ and $\mathbf{A}_1, \mathbf{A}_2$ are symmetric.

1. $\mathrm{E}(q_1) = \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\mathbf{A}_1\boldsymbol{\mu}$.
2. $\mathrm{Var}(q_1) = 2\ \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\Sigma}) + 4\ \boldsymbol{\mu}'\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\mu}$. $= \mathrm{Cov}(\underset{\sim}{x}'A_1\underset{\sim}{x}, \underset{\sim}{x}'A_1\underset{\sim}{x})$
3. $\mathrm{Cov}(q_1, q_2) = 2\ \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_2\boldsymbol{\Sigma}) + 4\ \boldsymbol{\mu}'\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_2\boldsymbol{\mu}$. $= \mathrm{Cov}(\underset{\sim}{x}'A_1\underset{\sim}{x}, \underset{\sim}{x}'A_2\underset{\sim}{x})$
4. $\mathrm{Cov}(\mathbf{x}, q_1) = 2\ \boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\mu}$.
5. $\mathrm{Cov}(\mathbf{T}, q_1) = 2\ \mathbf{B}\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\mu}$.

   $\underset{\sim}{B}\underset{\sim}{x}$

Examples important

1. Let the $n \times 1$ vector $\mathbf{Y} = (Y_1, ..., Y_n)' \sim N(\underset{\sim}{\alpha \mathbf{1}}, \underset{\sim}{\sigma^2 \mathbf{I}})$. Define $U = \Sigma_{i=1}^{n}(Y_i - \overline{Y})^2/\sigma^2$ and $V = n(\overline{Y} - \alpha)^2/\sigma^2$. Find the distributions of U and V and show that these two random variables are independent.

($\underset{\sim}{\mu} = \alpha\mathbf{1}$, $\underset{\sim}{\Sigma} = \sigma^2\mathbf{I}$)

1°. distribution of U.

$$\overline{Y} = \frac{y_1+y_2+\cdots+y_n}{n} = \frac{1}{n}\underset{\sim}{Y}'\underset{\sim}{1} = \frac{1}{n}\underset{\sim}{1}'\underset{\sim}{Y}$$

$$\begin{pmatrix} y_1-\bar{y} \\ \vdots \\ y_n-\bar{y} \end{pmatrix} = \underset{\sim}{Y} - \underset{\sim}{1}\overline{Y} = (\underset{\sim}{Y} - \tfrac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'\underset{\sim}{Y}) = (\underline{\underset{\sim}{I} - \tfrac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'})\underset{\sim}{Y} = \underset{\sim}{B}\underset{\sim}{Y}$$

$\underset{\sim}{B}$ want to prove it's idempotent

Note: $\underset{\sim}{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}$, $\underset{\sim}{J} = \underset{\sim}{1}\underset{\sim}{1}' = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{n\times n}$

$$B^2 = (I - \tfrac{1}{n}\underset{\sim}{1}\underset{\sim}{1}')^2 = \underset{\sim}{I} - \tfrac{2}{n}\underset{\sim}{1}\underset{\sim}{1}' + J^2/n^2 = I - \tfrac{2}{n}\underset{\sim}{1}\underset{\sim}{1}' + \frac{n\underset{\sim}{1}\underset{\sim}{1}'}{n^2} = I - \tfrac{1}{n}\underset{\sim}{1}\underset{\sim}{1}' = B$$

$\Rightarrow$ B is idempotent.

$$B = \begin{pmatrix} 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \ddots & \vdots \\ \vdots & & \\ -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{pmatrix}$$

$\text{rank}(B) = \text{tr}(B) = n(1-\frac{1}{n}) = n-1$

$$\begin{aligned} U &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \frac{1}{\sigma^2}(\underset{\sim}{Y} - \underset{\sim}{1}\overline{Y})'(\underset{\sim}{Y} - \underset{\sim}{1}\overline{Y}) \\ &= \frac{1}{\sigma^2}\underset{\sim}{Y}'\underset{\sim}{B}'\underset{\sim}{B}\underset{\sim}{Y} \\ &= \frac{1}{\sigma^2}\underset{\sim}{Y}'\underset{\sim}{B}\underset{\sim}{Y} \\ &= \underset{\sim}{Y}'(\tfrac{B}{\sigma^2})\underset{\sim}{Y} \sim \chi^2_{n-1,\lambda} \end{aligned}$$

$$\begin{aligned} \lambda &= \tfrac{1}{2}\underset{\sim}{\mu}'\, B/\sigma^2\, \underset{\sim}{\mu} \\ &= \frac{\alpha^2}{2\sigma^2}\underset{\sim}{1}'(I - \tfrac{1}{n}\underset{\sim}{1}\underset{\sim}{1}')\underset{\sim}{1} \\ &= 0 \end{aligned}$$

$\Rightarrow U \sim \chi^2_{(n-1)}$

2°. distribution of $\overline{Y}$

note that $\overline{Y} \sim N(\alpha, \frac{\sigma^2}{n}) \Rightarrow \dfrac{\overline{Y}-\alpha}{\sigma/\sqrt{n}} \sim N(0,1)$

$\Rightarrow \dfrac{\sqrt{n}(\overline{Y}-\alpha)}{\sigma} \sim N(0,1)$

$\Rightarrow V = \dfrac{n(\overline{Y}-\alpha)^2}{\sigma^2} \sim \chi^2_1$

3°. Show $U \perp\!\!\!\perp V$.

At first. we want to prove $U \perp\!\!\!\perp \bar{Y}$.

$$\bar{Y} = \frac{1}{n} \cdot \underset{\sim}{1}' \underset{\sim}{Y}. \Rightarrow \underset{\sim}{1}' \underset{\sim}{B} = \underset{\sim}{1}' \left(\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \right)$$

$$= \underset{\sim}{1}' - \frac{1}{n} \underbrace{\underset{\sim}{1}' \underset{\sim}{1}} \underset{\sim}{1}' = 0$$

$\overset{\text{Thm.2}}{\Longrightarrow}$ $U = \frac{1}{\sigma^2} \underset{\sim}{Y}' \underset{\sim}{B} \underset{\sim}{Y}$ and $\bar{Y}$ are independent

$\Rightarrow$ U and $\bar{Y} - \alpha$ are independent

$\Rightarrow$ U and V are independent

2. Let the $n \times 1$ vector $\mathbf{Y} = (Y_1, ..., Y_n)' \sim N(\mu\mathbf{1}, \sigma^2\mathbf{I})$. Let

$$\begin{aligned} \overline{Y} &= \frac{\sum_{i=1}^n Y_i}{n} \\ Q_1 &= n\overline{Y}^2 \\ Q_2 &= \sum_{i=1}^n (Y_i - \overline{Y})^2 \end{aligned}$$

(a) Prove that $\overline{Y}$ and Q_2 are independent.

(b) Prove that Q_1 and Q_2 are independent.

(c) Find the distributions of Q_1 and Q_2.

(a). Note that $\bar{Y} = \frac{1}{n}\underset{\sim}{1}'\underset{\sim}{Y} \triangleq \underset{\sim}{B}\underset{\sim}{Y}$

$\Rightarrow \begin{pmatrix} y_1-\bar{y} \\ \vdots \\ y_n-\bar{y} \end{pmatrix} = (\underset{\sim}{Y} - \frac{1}{n}\underset{\sim}{1}\bar{Y}) = (\underset{\sim}{Y} - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'\underset{\sim}{Y}) = (\underset{\sim}{I} - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}')\underset{\sim}{Y} \triangleq A\underset{\sim}{Y}$

$\Rightarrow Q_2 = (\underset{\sim}{A}\underset{\sim}{Y})'A\underset{\sim}{Y} = \underset{\sim}{Y}'A'A\underset{\sim}{Y} = \underset{\sim}{Y}'A\underset{\sim}{Y}$

$\Rightarrow \underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{A} = \underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{A} = \frac{\sigma^2}{n}\underset{\sim}{1}'\underset{\sim}{I}(\underset{\sim}{I} - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}')$

$= \frac{\sigma^2}{n}(\underset{\sim}{1}' - \frac{1}{n}\underset{\sim}{1}'\underset{\sim}{1}\underset{\sim}{1}')$

$= 0$

$\Rightarrow \bar{Y} \perp\!\!\!\perp Q_2$

$Q_1 = n(\frac{1}{n}\underset{\sim}{1}'\underset{\sim}{Y})(\frac{1}{n}\underset{\sim}{1}'\underset{\sim}{Y})$

$= \frac{1}{n}\underset{\sim}{Y}'\underset{\sim}{1}\underset{\sim}{1}'\underset{\sim}{Y}$

$= \underset{\sim}{Y}'\frac{\underset{\sim}{1}\underset{\sim}{1}'}{n}\underset{\sim}{Y}$

(b). Since $Q_2 \perp\!\!\!\perp \bar{Y}$.

$Q_1 = f(\bar{Y})$

$\Rightarrow Q_1 \perp\!\!\!\perp Q_2$

(c). Note that $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$

$\Rightarrow \frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma} \sim N(0,1)$

$\Rightarrow \frac{\sqrt{n}\bar{Y} - \sqrt{n}\mu}{\sigma} \sim N(0,1)$

$\Rightarrow \frac{\sqrt{n}\bar{Y}}{\sigma} \sim N(\frac{\sqrt{n}\mu}{\sigma}, 1)$

$\Rightarrow \frac{n\bar{Y}^2}{\sigma^2} \sim N(1, \frac{n\mu^2}{2\sigma^2})$

$\Downarrow$?

$n\bar{Y}^2$

$Q_2 = \sum_{i=1}^n (Y_i - \bar{Y})^2$

$= \underset{\sim}{Y}'A\underset{\sim}{Y}$

$\Rightarrow \underset{\sim}{A}\underset{\sim}{\Sigma} = \sigma^2(\underset{\sim}{I} - \frac{1}{n}\underset{\sim}{J})$ Thm 6.1 not apply

$(\underset{\sim}{A}\underset{\sim}{\Sigma})^2 = \sigma^4\underset{\sim}{A} \neq \underset{\sim}{A}\underset{\sim}{\Sigma} = \sigma^2\underset{\sim}{A}$

$\frac{Q_2}{\sigma^2} \sim \chi^2_{(n-1)} \sim \text{Gamma}(\frac{n-1}{2}, \frac{1}{2})$

$\Rightarrow Q_2 \sim \text{Gamma}(\frac{n-1}{2}, \frac{1}{2\sigma^2})$

3. Suppose $\mathbf{y}$ is $N_3(\boldsymbol{\mu}, \sigma^2\mathbf{I})$ and let $\boldsymbol{\mu}' = [3, \quad -2, \quad 1]$ and

$$\mathbf{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \underset{\sim}{I} - \frac{1}{3}\underset{\sim}{J}$$

$$\mathbf{B} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

(a) Find the distribution of $\mathbf{y}'\mathbf{A}\mathbf{y}/\sigma^2$. $\sim \chi^2_{(3-1)} = \chi^2_2$

(b) Are $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{B}\mathbf{y}$ independent?

(c) Are $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $y_1 + y_2 + y_3$ independent?

(b). Thm 6.2

Want to check if $\underset{\sim}{B}\underset{\sim}{I}\underset{\sim}{A} = \underset{\sim}{B}\underset{\sim}{A} \overset{?}{=} 0$

$$BA = \frac{1}{9}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & -3 \end{pmatrix} \neq 0$$

$\Rightarrow$ y'Ay and By are not independent.

(c). $y_1 + y_2 + y_3 = (1 \ 1 \ 1)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 1'\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad 1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$1' I B = 1' B = 0$

$\Rightarrow \underset{\sim}{y'Ay}$ and $y_1 + y_2 + y_3$ are independent.

4. Suppose $\mathbf{y}$ is $N_n(\mu\mathbf{1}, \mathbf{\Sigma})$ where

$$\mathbf{\Sigma} = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \rho & \rho & \cdots & 1 \end{pmatrix} := \sigma^2 \left((1-\rho)\underset{\sim}{I} + \rho J \right)$$

Derive the distribution of

$$\frac{\Sigma_{i=1}^n (y_i - \bar{y})^2}{\sigma^2(1-\rho)} = \frac{1}{\sigma^2(1-\rho)} \underset{\sim}{y}' \underset{\sim}{B} \underset{\sim}{y} \quad \text{where } \underset{\sim}{B} = (\underset{\sim}{I} - \tfrac{1}{n}\underset{\sim}{J})$$

$$= \underset{\sim}{y}' \underset{\sim}{A} \underset{\sim}{y} \qquad \left(\underset{\sim}{A} = \frac{1}{\sigma^2(1-\rho)} \underset{\sim}{B} \right)$$

Step 1: Want to prove $\underset{\sim}{A}\underset{\sim}{\Sigma}$ is idempotent.

$$\Rightarrow \underset{\sim}{A}\underset{\sim}{\Sigma} = \frac{1}{\sigma^2(1-\rho)} \underset{\sim}{B} \cdot \sigma^2 \left((1-\rho)\underset{\sim}{I} + \rho \underset{\sim}{J} \right)$$

$$= \frac{1}{(1-\rho)} \left((1-\rho) BI + \rho \underset{\sim}{B} \underset{\sim}{J} \right)$$

$$= \underset{\sim}{B} + \frac{\rho}{1-\rho} \underset{\sim}{B}\underset{\sim}{J} = \underset{\sim}{B} \left[1 + \frac{\rho}{1-\rho} \right] \underset{\sim}{J} = \frac{1}{1-\rho} \underset{\sim}{B} \underset{\sim}{1} \underset{\sim}{1}'.$$

Step 1: $A\Sigma$ is idempotent ?

$$\underset{\sim}{A}\underset{\sim}{\Sigma} = \cdots = \underset{\sim}{B}$$

$$\overset{\text{Thm } b.1}{\Longrightarrow} \quad u \sim \chi^2_{(n-1, \lambda)} \quad \underline{\text{degree of freedom}}$$

need to prove

$$\lambda = \frac{1}{2} \underset{\sim}{\mu}' \underset{\sim}{A} \underset{\sim}{\mu} = \frac{1}{2} (\mu \underset{\sim}{1})' \frac{B}{\sigma^2(1-\rho)} \cdot (\mu \underset{\sim}{1}) = \cdots = 0$$

$$J^2 = nJ$$