Statistical linear models.

Assignment 2

牛圣杰 11910901

1. We want to prove

$$(\underset{\sim}{A}+\underset{\sim}{P}\underset{\sim}{B}\underset{\sim}{Q})^{-1}=\underset{\sim}{A}^{-1}-\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B}(\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B})^{-1}\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}$$

Since $\underset{\sim}{A}+\underset{\sim}{P}\underset{\sim}{B}\underset{\sim}{Q}$ is nonsingular $\Rightarrow$ we only need to prove that $(\underset{\sim}{A}+\underset{\sim}{P}\underset{\sim}{B}\underset{\sim}{Q})(\underset{\sim}{A}+\underset{\sim}{P}\underset{\sim}{B}\underset{\sim}{Q})^{-1}=\underset{\sim}{I}$

then we have

$$(\underset{\sim}{A}+\underset{\sim}{P}\underset{\sim}{B}\underset{\sim}{Q})(\underset{\sim}{A}+\underset{\sim}{P}\underset{\sim}{B}\underset{\sim}{Q})^{-1}=(\underset{\sim}{A}+\underset{\sim}{P}\underset{\sim}{B}\underset{\sim}{Q})[\underset{\sim}{A}^{-1}-\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B}(\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B})^{-1}\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}]$$

$$=\underset{\sim}{I}-\underset{\sim}{P}\underset{\sim}{B}(\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B})^{-1}\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}+\underset{\sim}{P}\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}-\underset{\sim}{P}\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B}(\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B})^{-1}\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}$$

$$=\underset{\sim}{I}+\underset{\sim}{P}\left[-\underset{\sim}{B}(\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B})^{-1}+\underset{\sim}{I}-\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B}(\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B})^{-1}\right]\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}$$

$$=\underset{\sim}{I}-\underset{\sim}{P}\left[-(\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B})(\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}\underset{\sim}{P}\underset{\sim}{B})^{-1}+\underset{\sim}{I}\right]\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}$$

$$=\underset{\sim}{I}-\underset{\sim}{P}[-\underset{\sim}{I}+\underset{\sim}{I}]\underset{\sim}{B}\underset{\sim}{Q}\underset{\sim}{A}^{-1}$$

$$=\underset{\sim}{I}\quad\square$$

2. (a) $\underset{\sim}{A}=\begin{pmatrix}4&2&2\\2&2&0\\2&0&2\end{pmatrix}$ let $\underset{\sim}{A}_{22}=\begin{pmatrix}2&0\\0&2\end{pmatrix}$

note that $\text{rank}(\underset{\sim}{A})=2$ and $\text{rank}(\underset{\sim}{A}_{22})=2$

$$\Rightarrow \underset{\sim}{A}_{22}^{-1}=\frac{\begin{pmatrix}2&0\\0&2\end{pmatrix}}{4}=\begin{pmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{pmatrix}$$

$\Rightarrow$ a symmetric generalized inverse for A is $\begin{pmatrix}0&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{2}\end{pmatrix}$

(b) $\underset{\sim}{A}=\begin{pmatrix}4&2&2\\2&2&0\\2&0&2\end{pmatrix}$ We can pick row 1, 3 and column 2, 3

$\Rightarrow$ then we have $\underset{\sim}{C}=\begin{pmatrix}2&2\\0&2\end{pmatrix}$

note that $\text{rank}(\underset{\sim}{A})=2$. and $\text{rank}(\underset{\sim}{C})=2$.

$$\Rightarrow C^{-1}=\frac{\begin{bmatrix}2&-2\\0&2\end{bmatrix}}{4}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix},\quad (C^{-1})'=\begin{bmatrix}\frac{1}{2}&0\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix}$$

then replace the elements of $\underset{\sim}{C}$ by the elements of $(C^{-1})'$ and replace other all elements in $\underset{\sim}{A}$ by 0's.

$\Rightarrow$ a nonsymmetric generalized inverse for $\underset{\sim}{A}$ is $\begin{pmatrix}0&\frac{1}{2}&0\\0&0&0\\0&-\frac{1}{2}&\frac{1}{2}\end{pmatrix}$

3. From the problem, we want to prove that

$$\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{X} = X$$

Note that $\underset{\sim}{X}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{X} = \underset{\sim}{X}'\underset{\sim}{X}$ for any generalized inverse of $X'X$

According to full rank factorization: $\text{rank}(\underset{\sim}{X}) = r$.

$$\Rightarrow \underset{\sim}{X} = \underset{\sim}{K}_{p\times r} L_{r\times q} = \underset{\sim}{K}^*_{p\times r} L^*_{r\times q} \quad \text{where } K^*_{p\times r} = \left[A_{r\times r} \,\vdots\, 0_{(p-r)\times r} \right]$$, A is full rank square matrix.

then $K_{p\times r} = K^*_{p\times r} \cdot S_{r\times r}$, where $S_{r\times r}$ is nonsingular.

$$K'K = (K^*S)'(K^*S) = S'K^{*\prime}K^*S = S'A'AS$$, which is invertible.

and $\underset{\sim}{L}\underset{\sim}{R} = I_r$. $\underset{\sim}{R}$ is the right-inverse of $\underset{\sim}{L}$

then we have $\underset{\sim}{L}'\underset{\sim}{K}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{K}\underset{\sim}{L} = \underset{\sim}{L}'\underset{\sim}{K}'\underset{\sim}{K}\underset{\sim}{L}$

$$\Rightarrow \underset{\sim}{R}'\underset{\sim}{L}'\underset{\sim}{K}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{K}\underset{\sim}{L}\underset{\sim}{R} = \underset{\sim}{R}'\underset{\sim}{L}'\underset{\sim}{K}'\underset{\sim}{K}\underset{\sim}{L}\underset{\sim}{R}$$

$$\Rightarrow \underset{\sim}{K}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{K} = \underset{\sim}{K}'\underset{\sim}{K}$$

$$\Rightarrow \underset{\sim}{K}'\underset{\sim}{K}\underset{\sim}{L}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{K} = \underset{\sim}{K}'\underset{\sim}{K}.$$

$$\Rightarrow (\underset{\sim}{K}'\underset{\sim}{K})^{-1}\underset{\sim}{K}'\underset{\sim}{K}\underset{\sim}{L}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{K} = (\underset{\sim}{K}'\underset{\sim}{K})^{-1}\underset{\sim}{K}'\underset{\sim}{K}$$

$$\Rightarrow \underset{\sim}{L}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{K} = \underset{\sim}{I}$$

$$\Rightarrow \underset{\sim}{K}\underset{\sim}{L}(\underset{\sim}{X}'\underset{\sim}{X})\underset{\sim}{X}'\underset{\sim}{K}\underset{\sim}{L} = \underset{\sim}{K}\cdot\underset{\sim}{I}\cdot\underset{\sim}{L}$$

$$\Rightarrow \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})\underset{\sim}{X}'\underset{\sim}{X} = \underset{\sim}{X} \qquad \square$$