Department of Statistics and Data Science at SUSTech

MAT7035: Computational Statistics

Tutorial 3: The SR method and The conditional sampling (CS) method

G. The SR method

G.1 The basic idea

(a) Univariate case: If a random variable X is very difficult to generate, but we have $X = \psi(Y_1, \ldots, Y_n)$ for some known function ψ and it is easy to generate $\{Y_j\}_{j=1}^n$, then we first generate $\{Y_j\}_{j=1}^n$ and second set $X = \psi(Y_1, \ldots, Y_n)$.

(b) Multivariate case: Suppose that we have the following one-to-one mapping

$$X_i = g_i(Y_1, \ldots, Y_n), \quad i = 1, \ldots, d$$

for a set of known functions $\{g_i\}_{i=1}^d$ and it is easy to generate $\{Y_j\}_{j=1}^n$, then we first generate $\{Y_j\}_{j=1}^n$ and second set $X_i = g_i(Y_1, \ldots, Y_n)$ for $i = 1, \ldots, d$.

G.2 Remarks

(a) The inversion method is a special case of the SR.

(b) The SR method is also known as the transformation method.

Example T3.1 (Standard Cauchy distribution). Let $X_1, X_2 \stackrel{\text{iid}}{\sim} N(0,1)$ and define $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$. (i) Find the joint density of $(Y_1, Y_2)^\top$ and the marginal density of Y_2; (ii) State the SR method to generate a sample from the standard Cauchy distribution.

Solution: (i) From $y_1 = x_1 + x_2$ and $y_2 = x_1/x_2$, we have $x_1 = y_1y_2/(1+y_2)$ and $x_2 = y_1/(1+y_2)$. The Jacobian determinant is

$$J(x_1, x_2 \to y_1, y_2) = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \det \begin{pmatrix} \dfrac{y_2}{1+y_2} & \dfrac{y_1}{(1+y_2)^2} \\ \dfrac{1}{1+y_2} & -\dfrac{y_1}{(1+y_2)^2} \end{pmatrix} = -\frac{y_1}{(1+y_2)^2}$$

so that the joint density of $(Y_1, Y_2)^\top$ is

$$\begin{aligned} f_{(Y_1,Y_2)}(y_1, y_2) &= f_{(X_1,X_2)}(x_1, x_2) \times |J(x_1, x_2 \to y_1, y_2)| \\ &= \frac{1}{2\pi} \exp\left[-\frac{1}{2}\left\{\frac{(y_1y_2)^2}{(1+y_2)^2} + \frac{y_1^2}{(1+y_2)^2}\right\}\right] \times \frac{|y_1|}{(1+y_2)^2} \\ &= \frac{1}{2\pi}\frac{|y_1|}{(1+y_2)^2} \exp\left[-\frac{1}{2}\left\{\frac{(1+y_2^2)y_1^2}{(1+y_2)^2}\right\}\right]. \end{aligned}$$

The marginal density of Y_2 is given by

$$\begin{aligned} f_{Y_2}(y_2) &= \int_{-\infty}^{\infty} f_{(Y_1,Y_2)}(y_1, y_2)\,\mathrm{d}y_1 \\ &= \frac{1}{2\pi}\frac{1}{(1+y_2)^2}\int_{-\infty}^{\infty} |y_1| \exp\left[-\frac{1}{2}\left\{\frac{(1+y_2^2)y_1^2}{(1+y_2)^2}\right\}\right] \mathrm{d}y_1. \end{aligned}$$

Let

$$u = \frac{1}{2}\frac{(1+y_2^2)y_1^2}{(1+y_2)^2}, \quad \text{then} \quad u \geqslant 0 \quad \text{and} \quad \mathrm{d}u = \frac{(1+y_2^2)y_1}{(1+y_2)^2}\,\mathrm{d}y_1,$$

so

$$f_{Y_2}(y_2) = \frac{1}{2\pi(1+y_2)^2} \cdot 2\int_0^{\infty} \mathrm{e}^{-u}\frac{(1+y_2)^2}{(1+y_2^2)}\,\mathrm{d}u = \frac{1}{\pi(1+y_2^2)},$$

which is the standard Cauchy density.

(ii) The SR method for drawing $X \sim \text{Cauchy}(0,1)$ is as follows:

Step 1: Draw $X_1 = x_1, X_2 = x_2 \overset{\text{iid}}{\sim} N(0,1)$;

Step 2: Return $y_2 = x_1/x_2$. ‖

Example T3.2 (Standard bivariate logistic distribution). Let U, V, W be i.i.d. random variables following the standard Gumbel–maximum distribution [see Exercise 1.5(f)] with pdf

$$f_U(u) = \mathrm{e}^{-u} \exp(-\mathrm{e}^{-u}), \quad u \in \mathbb{R}.$$

Define $X = V - U$, $Y = W - U$ and $Z = U$. (i) Find the joint distribution of $(X, Y, Z)^\top$; (ii) Find the joint distribution of $(X, Y)^\top$; (iii) State the SR method to generate a sample from the standard bivariate logistic distribution.

Solution: (i) From $x = v - u$, $y = w - u$ and $z = u$, we have $u = z$, $v = x + z$ and $w = y + z$. The Jacobian determinant is

$$J(u, v, w \to x, y, z) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1$$

so that the joint density of $(X, Y, Z)^\top$ is

$$\begin{aligned} f_{(X,Y,Z)}(x, y, z) &= f_{(U,V,W)}(u, v, w) \times |J(u, v, w \to x, y, z)| = f_U(u) f_V(v) f_W(w) \\ &= \mathrm{e}^{-u-v-w} \exp(-\mathrm{e}^{-u} - \mathrm{e}^{-v} - \mathrm{e}^{-w}) \\ &= \mathrm{e}^{-x-y-3z} \exp(-\beta\, \mathrm{e}^{-z}), \quad x, y, z \in \mathbb{R}. \end{aligned}$$

where $\beta \mathrel{\hat=} 1 + \mathrm{e}^{-x} + \mathrm{e}^{-y} > 0$.

(ii) The joint density of $(X, Y)^\top$ is given by

$$\begin{aligned} f_{(X,Y)}(x, y) &= \int_{-\infty}^{\infty} f_{(X,Y,Z)}(x, y, z)\, \mathrm{d}z \\ &= \mathrm{e}^{-x-y} \int_{-\infty}^{\infty} \mathrm{e}^{-3z} \exp(-\beta\, \mathrm{e}^{-z})\, \mathrm{d}z \qquad [\text{Let } \mathrm{e}^{-z} = s] \\ &= \mathrm{e}^{-x-y} \int_{\infty}^{0} s^3\, \mathrm{e}^{-\beta s} \cdot (-1) s^{-1}\, \mathrm{d}s = \mathrm{e}^{-x-y} \int_0^{\infty} s^2\, \mathrm{e}^{-\beta s}\, \mathrm{d}s \\ &= \mathrm{e}^{-x-y} \cdot \frac{\Gamma(3)}{\beta^3} = \frac{2\, \mathrm{e}^{-x}\, \mathrm{e}^{-y}}{(1 + \mathrm{e}^{-x} + \mathrm{e}^{-y})^3}, \end{aligned}$$

which is the standard bivariate logistic density.

(iii) The SR method for drawing X from the standard bivariate logistic distribution is as follows:

Step 1: Draw $U = u, V = v, W = w \overset{\text{iid}}{\sim} f_U(u) = \mathrm{e}^{-u} \exp(-\mathrm{e}^{-u})$, and the corresponding generating method is given by Exercise 1.5(f);

Step 2: Return $x = v - u$ and $y = w - u$. ‖

Example T3.3 (Multivariate Poisson distribution). Assume that $\{Y_i\}_{i=0}^m \overset{\text{ind}}{\sim} \text{Poisson}\,(\lambda_i)$. Define $X_i = Y_0 + Y_i$, $i = 1, \ldots, m$. Then, the discrete random vector $\mathbf{x} = (X_1, \ldots, X_m)^\top$ is said to follow an m-dimensional Poisson distribution, denoted by $\mathbf{x} \sim \text{MP}\,(\lambda_0, \lambda_1, \ldots, \lambda_m)$. (i) Find the joint distribution of $\mathbf{x}$; (ii) State the SR method to generate a sample from the multivariate Poisson distribution.

Solution: (i) Let $\{x_i\}_{i=1}^m$ be the realizations of $\{X_i\}_{i=1}^m$. The joint pmf of $\mathbf{x}$ is

$$
\begin{aligned}
\Pr(\mathbf{x} = \boldsymbol{x}) &= \Pr(X_1 = x_1, \ldots, X_m = x_m) \\
&= \Pr(Y_0 + Y_1 = x_1, \ldots, Y_0 + Y_m = x_m) \\
&= \sum_k \Pr(Y_0 = k) \cdot \Pr(Y_1 = x_1 - k, \ldots, Y_m = x_m - k | Y_0 = k) \\
&= \sum_{k=0}^{\min(\boldsymbol{x})} \frac{\lambda_0^k \, \mathrm{e}^{-\lambda_0}}{k!} \prod_{i=1}^m \frac{\lambda_i^{x_i - k} \, \mathrm{e}^{-\lambda_i}}{(x_i - k)!},
\end{aligned}
$$

where $\boldsymbol{x} = (x_1, \ldots, x_m)^\top$ and $\min(\boldsymbol{x}) \,\hat{=}\, \min(x_1, \ldots, x_m)$.

(ii) The SR method for drawing $\mathbf{x} \sim \text{MP}\,(\lambda_0, \lambda_1, \ldots, \lambda_m)$ is as follows:

Step 1: Independently draw $Y_i = y_i \sim \text{Poisson}\,(\lambda_i)$ for $i = 0, 1, \ldots, m$;

Step 2: Return $x_i = y_0 + y_i$ for $i = 1, \ldots, m$. ‖

H. The CS method

H.1 The idea

- Let $X = (X_1, \ldots, X_d)^\top$ and its joint density can be factorized as

$$f(x) = f_1(x_1) \prod_{i=2}^{d} f_i(x_i | x_1, x_2, \ldots, x_{i-1}).$$

- The beauty of the conditional sampling method is that it reduces the problem of generating a d-dimensional random vector into d univariate generation problems.

H.2 The algorithm

Step 1: Draw $X_1 = x_1$ from $f_1(x_1)$;

Step 2: Draw $X_2 = x_2$ from $f_2(x_2|x_1)$;

Step 3: Draw $X_3 = x_3$ from $f_3(x_3|x_1, x_2)$;

$\vdots$

Step d: Draw $X_d = x_d$ from $f_d(x_d|x_1, x_2, \ldots, x_{d-1})$.

Example T3.4 (Type I bivariate Pareto distribution). A random variable X is said to have a Type I Pareto distribution, denoted by $X \sim \text{Pareto}^{(\text{I})}(\sigma, a)$ with $\sigma, a > 0$, if its cdf and pdf are given by

$$F_X(x) = 1 - \left(\frac{x}{\sigma}\right)^{-a} \quad \text{and} \quad f_X(x) = \frac{a\sigma^a}{x^{a+1}} I(x \geqslant \sigma), \tag{T3.1}$$

where a is called the Pareto index parameter. The bivariate distribution with joint pdf

$$f_{(X_1,X_2)}(x_1, x_2) = \frac{(a+1)a(\theta_1\theta_2)^{a+1}}{(\theta_2 x_1 + \theta_1 x_2 - \theta_1\theta_2)^{a+2}}, \quad x_1 \geqslant \theta_1 > 0,\ x_2 \geqslant \theta_2 > 0,\ a > 0, \tag{T3.2}$$

is called the Type I bivariate Pareto distribution. Show that (i) $X_i \sim \text{Pareto}^{(\text{I})}(\theta_i, a)$ for $i = 1, 2$; (ii) $(\theta_1 X_2 + \theta_2(x_1 - \theta_1))|(X_1 = x_1) \sim \text{Pareto}^{(\text{I})}(\theta_2 x_1, a+1)$; (iii) State the CS method to generate a sample from the Type I bivariate Pareto distribution.

Solution: (i) The marginal density of X_1 is given by

$$
\begin{aligned}
f_{X_1}(x_1) &= \int_{\theta_2}^{\infty} f_{(X_1,X_2)}(x_1, x_2)\,\mathrm{d}x_2 \\
&= \int_{\theta_2}^{\infty} \frac{(a+1)a(\theta_1\theta_2)^{a+1}}{(\theta_2 x_1 + \theta_1 x_2 - \theta_1\theta_2)^{a+2}}\,\mathrm{d}x_2 \qquad [\text{Let } \theta_2 x_1 + \theta_1 x_2 - \theta_1\theta_2 = y] \\
&= \int_{\theta_2 x_1}^{\infty} \frac{(a+1)a(\theta_1\theta_2)^{a+1}}{y^{a+2}} \cdot \theta_1^{-1}\,\mathrm{d}y \\
&= (a+1)a\theta_1^a\theta_2^{a+1} \cdot \frac{-1}{(a+1)y^{a+1}}\bigg|_{\theta_2 x_1}^{\infty} = \frac{a\theta_1^a}{x_1^{a+1}} I(x_1 \geqslant \theta_1),
\end{aligned}
$$

indicating that $X_1 \sim \text{Pareto}^{(\text{I})}(\theta_1, a)$. Similarly, we have $X_2 \sim \text{Pareto}^{(\text{I})}(\theta_2, a)$.

(ii) The conditional pdf of $X_2|(X_1 = x_1)$ is given by

$$
\begin{aligned}
f_{(X_2|X_1)}(x_2|x_1) &= \frac{f_{(X_1,X_2)}(x_1, x_2)}{f_{X_1}(x_1)} \\
&= \frac{(a+1)a(\theta_1\theta_2)^{a+1}}{(\theta_2 x_1 + \theta_1 x_2 - \theta_1\theta_2)^{a+2}} \cdot \frac{x_1^{a+1}}{a\theta_1^a} \\
&= \frac{(a+1)\theta_1(\theta_2 x_1)^{a+1}}{(\theta_2 x_1 + \theta_1 x_2 - \theta_1\theta_2)^{a+2}}, \quad x_2 \geqslant \theta_2 > 0.
\end{aligned}
$$

Let $Y = \theta_1 X_2 + \theta_2(x_1 - \theta_1)$, then the conditional pdf of $Y|(X_1 = x_1)$ is given by

$$
f_{(Y|X_1)}(y|x_1) = f_{(X_2|X_1)}(x_2|x_1) \cdot \left|\frac{\mathrm{d}x_2}{\mathrm{d}y}\right| = \frac{(a+1)(\theta_2 x_1)^{a+1}}{y^{a+2}} I(y \geqslant \theta_2 x_1),
$$

indicating that $Y|(X_1 = x_1) \sim \text{Pareto}^{(\text{I})}(\theta_2 x_1, a+1)$.

(iii) The CS method to generate a sample from the Type I bivariate Pareto distribution is as follows:

Step 1: Draw $X_1 = x_1 \sim \text{Pareto}^{(\text{I})}(\theta_1, a)$, and the corresponding generating method is given by Exercise 1.5(d);

Step 2: Draw $Y = y \sim \text{Pareto}^{(\text{I})}(\theta_2 x_1, a+1)$, and return $x = [y - \theta_2(x_1 - \theta_1)]/\theta_1$. $\|$