5 Multivariate Normal Distribution

- Density (Let $\boldsymbol{Y}_{p\times 1} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$) $\quad \underset{\sim}{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = |\boldsymbol{\Sigma}|^{-\frac{1}{2}}(2\pi)^{-\frac{p}{2}}e^{-\frac{1}{2}\{(\boldsymbol{y}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\}}$$

- MGF: $M_{\boldsymbol{Y}}(\mathbf{t}) = \mathbf{e}^{\mathbf{t}'\boldsymbol{\mu}+\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}}$ – Remark 5.1

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right)$$

$Y_1 \sim N(\mu_1, \sigma_{11}) \quad \sigma_{11} = \sigma_1^2$

$Y_2 \sim N(\mu_2, \sigma_{22}) . \quad \sigma_{22} = \sigma_2^2 \qquad \Sigma$: Covariance matrix

Remark 5.1

$$M_{Y}(t) = E_{Y}(e^{t'Y})$$

$$= |\Sigma|^{-\frac{1}{2}} (2\pi)^{-\frac{p}{2}} \int_{R^p} e^{t'y} e^{-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)} dy$$

$$= |\Sigma|^{-\frac{1}{2}} (2\pi)^{-\frac{p}{2}} \int e^{-\frac{1}{2}[(y-\mu)'\Sigma^{-1}(y-\mu) - 2t'y]} dy$$

1°. If A n.s. $\rightarrow A'$. n.s.
$(A')^{-1} = (A^{-1})'$

2°. $(A^{-1})^T = (A^T)^{-1} = A^{-1}$.

$$(y-\mu)'\Sigma^{-1}(y-\mu) - 2t'y$$

$$= y'\Sigma^{-1}y - 2\mu'\Sigma^{-1}y + \mu'\Sigma^{-1}\mu - 2t'y$$

$$= y'\Sigma^{-1}y - 2(\Sigma t + \mu)'\Sigma^{-1}y + \mu'\Sigma^{-1}\mu$$

$(y-\mu)'\Sigma^{-1}(y-\mu)$

$= (y'\Sigma^{-1} - \mu'\Sigma^{-1})(y-\mu)$

$= y'\Sigma^{-1}y - \mu'\Sigma^{-1}y - y'\Sigma^{-1}\mu + \mu'\Sigma^{-1}\mu$

$\mu'\Sigma^{-1}y \overset{?}{=} y'\Sigma^{-1}\mu$

$\Sigma t + \mu \overset{d}{=} \mu^*$

$$= (y-\mu^*)'\Sigma^{-1}(y-\mu^*) - \mu^{*\prime}\Sigma^{-1}\mu^* + \mu'\Sigma^{-1}\mu$$

$$= (y-\mu^*)'\Sigma^{-1}(y-\mu^*) - t'\Sigma t - 2t'\mu \quad \cdots (*)$$

$$= \underbrace{|\Sigma|^{-\frac{1}{2}} (2\pi)^{-\frac{p}{2}} \int e^{-\frac{1}{2}(y-\mu^*)'\Sigma^{-1}(y-\mu^*)} dy}_{\text{pdf of } N(\mu^*, \Sigma)} \; e^{t'\mu + \frac{1}{2}t'\Sigma t}$$

$$= \exp\{t'\mu + \frac{1}{2}t'\Sigma t\}$$

- Let $\boldsymbol{B}$ be a constant matrix and $\boldsymbol{C}$ be a constant vector

Remark 5.2 $$\boldsymbol{BY}+\boldsymbol{C} \sim N(\boldsymbol{B\mu}+\boldsymbol{C}, \boldsymbol{B\Sigma B'})$$

- Marginal Distribution, Condition Distribution and independence

  Let
$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix} \sim N\left[\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}\right]$$
  then

  (i) $\boldsymbol{Y}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$

  (ii) $\boldsymbol{Y}_1|\boldsymbol{Y}_2 = \boldsymbol{y}_2 \sim N(\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{y}_2-\boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})$

  (iii) $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are independent iff $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$

Remark 5.3

Remark 5.2

1°. $BY + c \sim N(\quad)$, if $Y \sim N(\quad)$.

2°. $E(BY + c) = B\mu + c$

3°. $Var(BY + c) = Var(BY) = B\,Var(Y)B' = B\Sigma B'$

4°. Proof: use MGF

$$M_{BY+c}(t) = E\left(e^{t'(BY+c)}\right)$$

$$= E\left(e^{t'BY + t'c}\right)$$

$$= E\left(e^{t^{*\prime}Y}\right) \cdot e^{t'c} \qquad t'B \triangleq t^{*\prime}$$

$$= \exp\{t^{*\prime}\mu + \tfrac{1}{2}t^{*\prime}\Sigma t^*\} \cdot \exp\{t'c\}$$

$$= \exp\{t'B\mu + \tfrac{1}{2}t'B\Sigma B't + t'c\}$$

$$= e^{t'(B\mu + c) + \frac{1}{2}t'B\Sigma B't}$$

$$= e^{t'\mu^* + \frac{1}{2}t'\Sigma^* t} \qquad \mu^* \triangleq B\mu + c$$

$$\sim N(\mu^*, \Sigma^*) \qquad \Sigma^* \triangleq B\Sigma B'$$

Remark 5.3

偷懒xl

$$P(Y_1 | Y_2) = \frac{P(y_1, y_2)}{P(y_2)} \propto \exp\left\{-\frac{1}{2}\begin{pmatrix} y_1 - \mu_1 \\ y_2 - \mu_2 \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} y_1 - \mu_1 \\ y_2 - \mu_2 \end{pmatrix}\right\}$$

关心y_1，其余可以当作 constant.

$$\Rightarrow \Sigma^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} B^{-1} & -B^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{22}^{-1}\Sigma_{21}B^{-1} & * \end{pmatrix}$$

where $B^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$$\begin{pmatrix} y_1-\mu_1 \\ y_2-\mu_2 \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} y_1-\mu_1 \\ y_2-\mu_2 \end{pmatrix} = (y_1-\mu_1)'B^{-1}(y_1-\mu_1) - 2(y_1-\mu_1)'B^{-1}\Sigma_{12}\Sigma_{22}^{-1}(y_2-\mu_2) + C$$

偷懒x2.

$$= (y_1-\mu_1)'B^{-1}(y_1-\mu_1) - 2(y_1-\mu_1)'B^{-1}\mu^* + C$$

跳跃

$$= (y_1-\mu_1-\mu^*)'B^{-1}(y_1-\mu_1-\mu^*) + C^*$$

$$= y_1^{*\prime}B^{-1}y_1^* - 2y_1^{*\prime}B^{-1}\mu^* + c^*$$

$$= (y_1^*-\mu^*)'B^{-1}(y_1^*-\mu^*) + c^*$$

$\mu^* = \Sigma_{12}\Sigma_{22}^{-1}(y_2-\mu_2)$

$y_1^* = y_1-\mu_1$

$$P(y_1|y_2) \propto \exp\{-\tfrac{1}{2}(y_1-\mu_1-\mu^*)'B^{-1}(y_1-\mu_1-\mu^*)\}$$

$$\sim N(\mu_1+\mu^*, B) = N(\mu_1+\Sigma_{12}\Sigma_{22}^{-1}(y_2-\mu_2),\ \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

$$= (y_1-\mu_1)'B^{-1}(y_1-\mu_1) - 2(y_1-\mu_1)'B^{-1}\mu^* + C$$

understand: 当给定你爸身高时，你爷爷的身高的信息量几乎包含在你爸提供的信息中了.

- Partial Correlation Conditional correlation

Let $\boldsymbol{v} \sim N_q(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{x} \end{pmatrix}; \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_y \\ \boldsymbol{\mu}_x \end{pmatrix}; \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{yy} & \boldsymbol{\Sigma}_{yx} \\ \boldsymbol{\Sigma}_{xy} & \boldsymbol{\Sigma}_{xx} \end{pmatrix}$$

where $\boldsymbol{y} = (y_1, y_2, ..., y_{r-1})'$ and $\boldsymbol{x} = (x_r, ..., x_q)'$. Let $\rho_{ij.r...q}$ be the partial correlation between y_i and y_j, $1 \leq i < j \leq r-1$, in the conditional distribution of $\boldsymbol{y}$ given $\boldsymbol{x}$. By the definition of correlation, we have

Remark 5.4

$$\rho_{ij.r...q} = \frac{\sigma_{ij.r...q}}{\sqrt{\sigma_{ii.r...q}\sigma_{jj.r...q}}}.$$

Matrix of partial correlations

$$\boldsymbol{\Omega}_{y.x} = \boldsymbol{D}_{y.x}^{-1} \boldsymbol{\Sigma}_{y.x} \boldsymbol{D}_{y.x}^{-1}$$

where $\boldsymbol{\Sigma}_{y.x} = \boldsymbol{\Sigma}_{yy} - \boldsymbol{\Sigma}_{yx}\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xy}$ and $\boldsymbol{D}_{y.x} = [diag(\boldsymbol{\Sigma}_{y.x})]^{1/2}$.

$\Omega_{y.x} = D_{y.x}^{-1} \Sigma_{y.x} D_{y.x}^{-1}$

Remark 5.4

Correlation and partial correlation

Y — blood pressure.

X_1 — amount of food eaton

X_2 — weight

$\rho_{y.X_1} = Corr(Y. X_1)$

In general. $Corr(X_1. X_2) \neq 0$. $Corr(y. X_2)$ — need to consider confounder X_1.

— If we need to add X_2 in the model.

We need to consider partial correlation between Y and X_2. given X_1

— how to do ?

residual

$y = a + bX_1 + \varepsilon. \Rightarrow r_{y.X_1} = y - (\hat{a} + \hat{b}X_1)$

$X_2 = C + dX_1 + \varepsilon_1 \Rightarrow r_{X_2 X_1} = X_2 - (\hat{c} + \hat{d}X_1)$

$Corr(r_{y.X_1}, r_{X_2.X_1}) = \rho_{(y.X_2) X_1}$ partial correlation between y and X_2 given X_1

— if $\rho_{(y.X_2).X_1} = 0$. all the information in X_2 to explain the variation of y is included in X_1.

we don't need to add X_2 to the model if X_1 is already used in the model.

Ch5. Remark 5.4

$$y = a + bx_1 + \varepsilon \Rightarrow r_{y.x_1} = y - (\hat{a} + \hat{b}x_1)$$

$$x_2 = c + dx_1 + \varepsilon \Rightarrow r_{x_2.x_1} = x_2 - (\hat{c} + \hat{d}x_1)$$

$Corr(r_{y.x_1}, r_{x_2.x_1}) = \rho_{(y.x_2).x_1}$ partial correlation between y and x_2 given x_1

$$\begin{pmatrix} y \\ x_1 \\ x_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \right)$$

从上一节课

$$\begin{pmatrix} y \\ x_2 \end{pmatrix} \Big| x_1 \sim N\left(\underset{\sim}{\mu}_{(y,x_2|x_1)}, \underset{\sim}{\Sigma}_{(y,x_2|x_1)} \right)$$

用线性模型解释一下,方便理解.

where $\underset{\sim}{\mu}_{(y,x_2|x_1)} = \begin{pmatrix} \mu_{y|x_1} \\ \mu_{x_2|x_1} \end{pmatrix}$, $\underset{\sim}{\Sigma}_{(y,x_2|x_1)} = \begin{pmatrix} \sigma_{yy|x_1} & \sigma_{yx_2|x_1} \\ \sigma_{x_2y|x_1} & \sigma_{x_2x_2|x_1} \end{pmatrix}$

$$\Rightarrow \boxed{\frac{\sigma_{yx_2|x_1}}{\sqrt{\sigma_{yy|x_1} \cdot \sigma_{x_2x_2|x_1}}} = \rho_{(y,x_2).x_1}}$$ 可以证明

In general.

$$\underset{\sim}{V} = \begin{pmatrix} \underset{\sim}{y} \\ \underset{\sim}{x} \end{pmatrix}^{r\text{-dim}} \quad \underset{\sim}{V} \sim N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma}).\ \mu = \begin{pmatrix} \underset{\sim}{\mu_y} \\ \underset{\sim}{\mu_x} \end{pmatrix} \quad \underset{\sim}{\Sigma} = \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix}$$

$$Cov(\underset{\sim}{y}|\underset{\sim}{x})_{r\times r} = \underset{\sim}{\Sigma}_{yy} - \underset{\sim}{\Sigma}_{yx} \underset{\sim}{\Sigma}_{xx}^{-1} \underset{\sim}{\Sigma}_{xy}$$

$$= \begin{pmatrix} \sigma_{11.\underset{\sim}{x}} & \sigma_{12.\underset{\sim}{x}} & & \\ & \ddots & \sigma_{ii.\underset{\sim}{x}} & \\ & & \ddots & \sigma_{jj.x} \\ & & & \ddots \end{pmatrix}_{r\times r}$$

$jj.x$ ↑ means: given

Define $\rho_{ij.\underset{\sim}{x}} = \frac{\sigma_{ij.\underset{\sim}{x}}}{\sqrt{\sigma_{ii.x}} \cdot \sqrt{\sigma_{jj.x}}}$ partial correlation between y_i and y_j given $\underset{\sim}{x}$

partial correlation matrix:

$$\rho_{\underset{\sim}{y}\underset{\sim}{x}} = \begin{pmatrix} 1 & \rho_{12.x} & \cdots & \rho_{1r.x} \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ \rho_{r1.x} & \rho_{r2.x} & \cdots & 1 \end{pmatrix}$$

$\frac{33}{14}$ $\frac{1}{14}$ $-\frac{9}{14}$
$\frac{1}{14}$ $\frac{3}{14}$ $\frac{1}{14}$
$-\frac{9}{14}$ $\frac{1}{14}$ $\frac{5}{14}$

$\Rightarrow \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$= 9 - (0,3,3)\left(\quad\right)\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$

Example 5.1

$$\underset{\sim}{V} = \begin{pmatrix} y \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{pmatrix}\right)$$

sample covariance
$n \to \infty$. $S \to \Sigma$

$$\begin{pmatrix} \mu_y \\ \mu_x \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix} \quad \underset{\sim}{\Sigma} = \begin{pmatrix} \sigma_y^2 & \sigma_{yx}' \\ \sigma_{yx} & \Sigma_{xx} \end{pmatrix}$$

$y|\underset{\sim}{x} \sim N(E(y|\underset{\sim}{x}), Var(y|\underset{\sim}{x}))$

$E(y|\underset{\sim}{x}) = \mu_y + \sigma_{yx}'\Sigma_{xx}^{-1}(x-\mu_x)$

$= 2 + (0,3,3)\begin{pmatrix} 1 & -1 & 2 \\ -1 & 6 & -3 \\ 2 & -3 & 7 \end{pmatrix}^{-1}\begin{pmatrix} x_1-5 \\ x_2+2 \\ x_3-1 \end{pmatrix}$

$= \cdots$

$= \frac{95}{7} - \frac{12}{7}x_1 + \frac{6}{7}x_2 + \frac{9}{7}x_3$

$Var(y|\underset{\sim}{x}) = \sigma_y^2 - \sigma_{yx}'\Sigma_{xx}^{-1}\sigma_{xy}$

$= 9 - (0\ 3\ 3)\left(\quad\right)^{-1}\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \frac{18}{7} < 9$ ✓

$(-\frac{12}{7}, \frac{6}{7}, \frac{9}{7})\begin{pmatrix} 9 \\ 3 \\ 3 \end{pmatrix}$ $\frac{63}{7}$

$\frac{4}{14}$ $\frac{2}{7}$ $\frac{6}{14}$ $\frac{3}{7}$ $\frac{(6+9)3}{7}$

$\frac{45}{7}$ $\frac{15 \times 3}{7}$

$2 + (0,3,3)\begin{pmatrix} 1 & -1 & 2 \\ -1 & 6 & -3 \\ 2 & -3 & 7 \end{pmatrix}^{-1}\begin{pmatrix} x_1-5 \\ x_2+2 \\ x_3-1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ -1 & 6 & -3 & 0 & 1 & 0 \\ 2 & -3 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -1 & 1 & 1 & 0 \\ 0 & -1 & 3 & -2 & 0 & 1 \end{array}\right)$$

$-\frac{1}{5}+3$
$\frac{1}{5}$ $\frac{1}{5}$ 0

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -1 & 1 & 1 & 0 \\ 0 & 0 & \frac{14}{5} & -\frac{9}{5} & \frac{1}{5} & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{9}{14} & \frac{1}{14} & \frac{5}{14} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{14} & \frac{3}{14} & \frac{1}{14} \\ 0 & 0 & 1 & -\frac{9}{14} & \frac{1}{14} & \frac{5}{14} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{15}{14} & \frac{3}{14} & \frac{1}{14} \\ 0 & 1 & 0 & \frac{1}{14} & \frac{3}{14} & \frac{1}{14} \\ 0 & 0 & 1 & -\frac{9}{14} & \frac{1}{14} & \frac{5}{14} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{33}{14} & \frac{1}{14} & -\frac{9}{14} \\ 0 & 1 & 0 & \frac{1}{14} & \frac{3}{14} & \frac{1}{14} \\ 0 & 0 & 1 & -\frac{9}{14} & \frac{1}{14} & \frac{5}{14} \end{array}\right)$$

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Remarks for ex.5.1

$E(y|\underset{\sim}{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$

$(\beta_0, \beta_1, \beta_2, \beta_3) = (\frac{95}{7}, -\frac{12}{7}, \frac{6}{7}, \frac{9}{7})$

Suppose $(y, \underset{\sim}{x})'$ obey multivariate normal distribution

$\Rightarrow$ Conditional mean of $y|\underset{\sim}{x}$ is linear regression model

conditional distribution
↓
con. mean and con. variance

$2 + (-\frac{12}{7}\ \frac{6}{7}\ \frac{9}{7})$

$\frac{60}{7}$ $\frac{72}{7}$ $\frac{12}{7}$ $\frac{3}{7}$

$2 + -\frac{12}{7}(x_1-5)$
$+ \frac{6}{7}(x_2+2)$
$+ \frac{9}{7}(x_3-1)$

$\frac{3}{7}$

$2 - \frac{12}{7}x_1 + \frac{60}{7} + \cancel{\frac{12}{7} - \frac{9}{7}}$ $\frac{63}{7} + \cancel{8}7$

$\frac{63}{7} + 2$ 14 $\frac{8\,75}{7}$

Example 5.2

$$\underset{\sim}{V} = \begin{pmatrix} y \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{pmatrix} \right)$$

sample covariance

$n \to \infty$. $S \to \Sigma$

$42 - (9)$ 33

$$\begin{pmatrix} \underset{\sim}{\mu_y} \\ \underset{\sim}{\mu_x} \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix} \qquad \underset{\sim}{\Sigma} = \begin{pmatrix} \sigma_y^2 & \sigma'_{yx} \\ \sigma_{yx} & \Sigma_{xx} \end{pmatrix}$$

$$\begin{aligned} \mathrm{Cov}(\underset{\sim}{y} \mid \underset{\sim}{x}) &= \Sigma_{yy} - \Sigma_{yx} \Sigma_{xx}^{-1} \Sigma_{xy} \\ &= \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ -3 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -1 \\ 3 & 2 \end{pmatrix} \\ &= \frac{1}{33} \begin{pmatrix} 126 & -24 \\ -24 & 14 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &\begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ -3 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -1 \\ 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{33} \begin{pmatrix} 3 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{33} \begin{pmatrix} 30 & 27 \\ -1 & 9 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 3 & 2 \end{pmatrix} \end{aligned}$$

如何解释

$$\rho_{yx_1.(x_2,x_3)} = \frac{-24/33}{\sqrt{126/33} \cdot \sqrt{14/33}} = -.571$$

$$\rho_{yx_1} = 0$$

Sub-group analysis.

mediation analysis

$$\begin{tikzcd} & x_2 \arrow[dr] & \\ x_1 \arrow[ur] \arrow[rr] & & y \end{tikzcd}$$