MA329 Statistical linear models

Assignment 4 (Due date: Nov 26, 11pm. For late submission, each day costs 10 percent)

1. (10 marks) Let

$$\begin{aligned} Y_1 &= \theta + \epsilon_1 \\ Y_2 &= 2\theta - \phi + \epsilon_2 \\ Y_3 &= \theta + 2\phi + \epsilon_3 \end{aligned}$$

   where $E[\epsilon_i] = 0$ $(i = 1; 2; 3)$. Find the least squares estimates of θ and ϕ.

2. (15 marks) In order to estimate two parameters θ and ϕ, it is possible to make observations of three types:

   (a) the first type have expectation θ,

   (b) the second type have expectation $\theta + \phi$, and

   (c) the third type have expectation $\theta - 2\phi$.

   All observations are subject to independent normal errors with zero means and common variance σ^2. If m observations of type (a), m observations of type (b) and n observations of type (c) are made, find the least squares estimates $\hat{\theta}$ and $\hat{\phi}$. Prove that these estimates are uncorrelated if $m = 2n$.

3. (15 marks) Consider the linear regression model

$$y = \boldsymbol{X_{n \times p}} \boldsymbol{\beta_{p \times 1}} + \boldsymbol{\epsilon}$$

   where $\boldsymbol{\epsilon} \sim N(\mathbf{0}; \sigma^2 \boldsymbol{I})$. Let $\hat{\boldsymbol{\beta}}$ be the least squares estimate of $\boldsymbol{\beta}$. Define $\tilde{\boldsymbol{\beta}} = c\hat{\boldsymbol{\beta}}$ where $c \leq 1$. The mean squared error (MSE) of $\tilde{\boldsymbol{\beta}}$ is

$$MSE(\tilde{\boldsymbol{\beta}}) = \mathrm{E}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}).$$

   (a) Prove that $MSE(\tilde{\boldsymbol{\beta}}) = c^2\sigma^2 \mathrm{tr}(\boldsymbol{X}'\boldsymbol{X})^{-1} + (c-1)^2 \boldsymbol{\beta}'\boldsymbol{\beta}$.

   (b) Let c^* be the value of c such that $MSE(\tilde{\boldsymbol{\beta}})$ is a minimum. Find c^*.

   (c) Let $p = 5$, $\sigma^2 = 1$, $\boldsymbol{\beta}' = (1; 2; 3; 4; 5)$ and the eigenvalues of $\boldsymbol{X}'\boldsymbol{X}$ be 1, 2, 3, 4, 5. Evaluate c^*.

4. (20 marks) Let

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}, \qquad \boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix}$$

   and assume that $\boldsymbol{X}$ and $\boldsymbol{X}_1$ have full column rank. Consider the linear model

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$. Let $\hat{\boldsymbol{\beta}}$ be the least squares estimator of $\boldsymbol{\beta}$ and $\hat{\boldsymbol{Y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{Y}}_1, \hat{\boldsymbol{Y}}_2)'$. Further, for the linear model

$$\boldsymbol{Y}_1 = \boldsymbol{X}_1 \boldsymbol{\beta}^* + \boldsymbol{\epsilon}^*$$

where $\boldsymbol{\epsilon}^* \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$, the least squares estimator of $\boldsymbol{\beta}^*$ is $\hat{\boldsymbol{\beta}}^*$. Let

$$\hat{\boldsymbol{Y}}^* = \boldsymbol{X}\hat{\boldsymbol{\beta}}^* = \begin{bmatrix} \hat{\boldsymbol{Y}}_1^* \\ \hat{\boldsymbol{Y}}_2^* \end{bmatrix}$$

Define

$$\boldsymbol{Y} - \hat{\boldsymbol{Y}} = \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix} - \begin{bmatrix} \hat{\boldsymbol{Y}}_1 \\ \hat{\boldsymbol{Y}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{bmatrix}$$

$$\boldsymbol{Y} - \hat{\boldsymbol{Y}}^* = \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix} - \begin{bmatrix} \hat{\boldsymbol{Y}}_1^* \\ \hat{\boldsymbol{Y}}_2^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1^* \\ \boldsymbol{e}_2^* \end{bmatrix}$$

(a) Prove that

$$\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}^* = \boldsymbol{M}_1^{-1} \boldsymbol{X}_2' \boldsymbol{e}_2$$

where $\boldsymbol{M}_1 = \boldsymbol{X}_1' \boldsymbol{X}_1$.

(b) Express $\boldsymbol{e}_2$ in terms of $\boldsymbol{e}_2^*$ and rewrite the expression of $\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}^*$ in Part (a)

(c) The following is a data set with sample size = 7

x	-3	-2	-1	0	1	2	3
y	14	7	·	·	·	·	-2

For the above data and with a simple linear regression model, the parameter estimate $\hat{\boldsymbol{\beta}}^* = (6, -2)'$.

Suppose an additional observation $(x, y) = (4, 4)$ is obtained (You now have 8 pairs of (x, y) in your updated dataset), compute the new parameter estimate $\hat{\boldsymbol{\beta}}$. (Hint: use Parts (a) and (b))