Statistical Linear Model

牛孟杰 11910901

Assignment 4.

1. Note that by the problem. we have

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \underset{\sim}{y} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix} + \underset{\sim}{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

Denote $X \stackrel{d}{=} \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}$. since $E(\varepsilon_i) = 0$, $i = 1, 2, 3$.

then we have that the least squares estimate:

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{y}$$

$$\Rightarrow \underset{\sim}{X}'\underset{\sim}{X} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix} \Rightarrow (\underset{\sim}{X}'\underset{\sim}{X})^{-1} = \frac{1}{30} \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$$

then $\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix} \underset{\sim}{y} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \underset{\sim}{y} = \begin{pmatrix} \frac{1}{6}(Y_1 + 2Y_2 + Y_3) \\ \frac{1}{5}(2Y_3 - Y_2) \end{pmatrix}$

$\Rightarrow$ the least square estimate: $\begin{cases} \hat{\theta} = \frac{1}{6}(Y_1 + 2Y_2 + Y_3) \\ \hat{\phi} = \frac{1}{5}(2Y_3 - Y_2) \end{cases}$

2. By the problem. we have that the following observations.

type (a): $Y_{1i} = \theta + \varepsilon_{1i}$, $i = 1, \cdots, m$, $\varepsilon_{1i} \stackrel{iid}{\sim} N(0, \sigma^2)$

type (b): $Y_{2i} = \theta + \phi + \varepsilon_{2i}$, $i = 1, \cdots, m$. $\varepsilon_{2i} \stackrel{iid}{\sim} N(0, \sigma^2)$

type (c). $Y_{3i} = \theta - 2\phi + \varepsilon_{3i}$, $i = 1, \cdots, n$ $\varepsilon_{3i} \stackrel{iid}{\sim} N(0, \sigma^2)$

$\Rightarrow$ Consider the least Square estimate:

Then we need to find:

$$\min \left(\sum_{i=1}^{m} (Y_{1i} - \hat{Y}_{1i})^2 + \sum_{i=1}^{m} (Y_{2i} - \hat{Y}_{2i})^2 + \sum_{i=1}^{n} (Y_{3i} - \hat{Y}_{3i})^2 \right)$$

Let $\left(\sum_{i=1}^{m} (Y_{1i} - \theta)^2 + \sum_{i=1}^{m} (Y_{2i} - \theta - \phi)^2 + \sum_{i=1}^{n} (Y_{3i} - \theta + 2\phi)^2 \stackrel{d}{=} SSE \right.$

$$\Rightarrow \frac{\partial SSE}{\partial \theta} = \sum_{i=1}^{m} -(Y_{1i} - \theta) + \sum_{i=1}^{m} -(Y_{2i} - \theta - \phi) + \sum_{i=1}^{n} -(Y_{3i} - \theta + 2\phi)$$

$$= -\sum_{i=1}^{m} Y_{1i} + m\theta - \sum_{i=1}^{m} Y_{2i} + m\theta + m\phi - \sum_{i=1}^{n} Y_{3i} + n\theta - 2n\phi$$

let $\frac{\partial SSE}{\partial \theta} = 0 \Rightarrow \hat{\theta} = \dfrac{\sum_{i=1}^{m} Y_{1i} + \sum_{i=1}^{m} Y_{2i} + \sum_{i=1}^{n} Y_{3i} + (2n - m)\hat{\phi}}{2m + n}$

$$\frac{\partial SSE}{\partial \phi} = \sum_{i=1}^{m} (Y_{2i} - \phi - \theta) + \sum_{i=1}^{n} -2(Y_{3i} - \theta + 2\phi)$$

$$= \sum_{i=1}^{m} Y_{2i} - m\phi - m\theta - 2\sum_{i=1}^{n} Y_{3i} + 2n\theta - 4n\phi$$

let $\frac{\partial SSE}{\partial \phi} = 0 \Rightarrow \hat{\phi} = \frac{\sum_{i=1}^{m} Y_{2i} - 2\sum_{i=1}^{n} Y_{3i} + (2n-m)\hat{\theta}}{m+4n}$

Consider the problem. if $m=2n$. then we have

$$\begin{cases} \hat{\theta} = \frac{\sum_{i=1}^{m} Y_{1i} + \sum_{i=1}^{m} Y_{2i} + \sum_{i=1}^{n} Y_{3i}}{5n} \\ \hat{\phi} = \frac{\sum_{i=1}^{m} Y_{2i} - 2\sum_{i=1}^{n} Y_{3i}}{6n} \end{cases}$$

Since $\varepsilon \sim N(0, \sigma^2)$ for all errors, $\Rightarrow \begin{cases} Y_{1i} \overset{iid}{\sim} N(\theta, \sigma^2) & \text{for } i=1,\cdots,m \\ Y_{2i} \overset{iid}{\sim} N(\theta+\phi, \sigma^2) & \text{for } i=1,\cdots,m \\ Y_{3i} \overset{iid}{\sim} N(\theta-2\phi, \sigma^2) & \text{for } i=1,\cdots,n \end{cases}$

$\Rightarrow$ then we have $Cov(\hat{\theta}, \hat{\phi}) = \frac{Cov\left(\sum_{i=1}^{m} Y_{1i} + \sum_{i=1}^{m} Y_{2i} + \sum_{i=1}^{n} Y_{3i}, \sum_{i=1}^{m} Y_{2i} - 2\sum_{i=1}^{n} Y_{3i}\right)}{30n^2}$

By the property of independence. we have

$$= \frac{Cov\left(\sum_{i=1}^{m} Y_{2i}, \sum_{i=1}^{m} Y_{2i}\right) - 2Cov\left(\sum_{i=1}^{n} Y_{3i}, \sum_{i=1}^{n} Y_{3i}\right)}{30n^2}$$

$$= \frac{\sum_{i=1}^{m} Cov(Y_{2i}, Y_{2i}) - 2\sum_{i=1}^{n} Cov(Y_{3i}, Y_{3i})}{30n^2}$$

$$= \frac{m\,Var(Y_{2i}) - 2n\,Var(Y_{3i})}{30n^2}$$

$$= \frac{m\sigma^2 - 2n\sigma^2}{30n^2} = \frac{2n\sigma^2 - 2n\sigma^2}{30n^2} = 0$$

$\Rightarrow \hat{\theta}$ and $\hat{\phi}$ are uncorrelated. $\square$

3. (a). Note that $\hat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta+\varepsilon) = \beta + (X'X)^{-1}X'\varepsilon$. then we have

$MSE(\tilde{\beta}) = E(c\hat{\beta}-\beta)'(c\hat{\beta}-\beta) = E\left[((c-1)\beta + c(X'X)^{-1}X'\varepsilon)'((c-1)\beta + c(X'X)^{-1}X'\varepsilon)\right]$

$$= (c-1)^2\beta'\beta + E\left((c-1)\beta' \cdot c(X'X)^{-1}X'\varepsilon\right) + E\left((c-1)\beta\varepsilon'X(X'X)^{-1}c\right) + c^2E\left(\varepsilon'X(X'X)^{-1}(X'X)^{-1}X'\varepsilon\right) \cdots (*)$$

Note $E\left((c-1)\beta' \cdot c(X'X)^{-1}X'\varepsilon\right) = E\left((c-1)\beta\varepsilon'X(X'X)^{-1}c\right) = 0$ (Since $E(\varepsilon) = 0$)

$\Rightarrow (*) = (c-1)^2\beta'\beta + c^2E\left(\varepsilon'X(X'X)^{-1}(X'X)^{-1}X'\varepsilon\right) \cdots (**)$

Consider $c^2E\left(\varepsilon'X(X'X)^{-1}(X'X)^{-1}X'\varepsilon\right)$, $\varepsilon \sim N(0, \sigma^2 I)$ Since $E(X'AX) = tr(A\Sigma) + \mu'A\mu$.

$\Rightarrow E\left(\varepsilon'X(X'X)^{-1}(X'X)^{-1}X'\varepsilon\right) = \sigma^2 tr\left(X(X'X)^{-1}(X'X)^{-1}X'\right) = \sigma^2 tr\left((X'X)^{-1}X'X(X'X)^{-1}\right) = \sigma^2 tr(X'X)^{-1}$

$\Rightarrow (**) = (c-1)^2\beta'\beta + c^2\sigma^2 tr(X'X)^{-1}$

(b). From (a). we have that

$MSE(\beta) = (c-1)^2\beta'\beta + c^2\sigma^2 tr(X'X)^{-1}$

$= c^2(\beta'\beta + \sigma^2 tr(X'X)^{-1}) - 2c\,\beta'\beta + \beta'\beta$ which is quardratic form.

Note that $\beta'\beta + \sigma^2 tr(X'X)^{-1}$ must be positive.

$\Rightarrow$ the quardratic equation acquire its minuim value when $c = \dfrac{\beta'\beta}{\beta'\beta + \sigma^2 tr(X'X)^{-1}}$

(c). By the problem, $\beta'\beta = (1\ 2\ 3\ 4\ 5)(1\ 2\ 3\ 4\ 5)' = 55$

then consider $X'X \triangleq A$. if α is an eigenvalue of A $\Rightarrow$ there exist nonzero eigenvector x s.t. $Ax = \alpha x$

$\Rightarrow$ multiply A^{-1} on both sides $\Rightarrow$ $x = \lambda A^{-1}x \Rightarrow A^{-1}x = \frac{1}{\lambda}x \Rightarrow \frac{1}{\lambda}$ is an eigenvalue of $A^{-1} = (X'X)^{-1}$

Note that $p=5$, $A^{-1} = (X'X)^{-1}$ is 5×5 matrix $\Rightarrow$ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ are all eigenvalues of $(X'X)^{-1}$

$\Rightarrow tr(X'X)^{-1} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{137}{60}$

$\Rightarrow c^* = \dfrac{55}{55+\frac{137}{60}} = 0.96$

4. (a) $\hat\beta = (X'X)^{-1}X'Y$, $\hat\beta^* = (X_1'X_1)^{-1}X_1'Y_1$

Note that $e_2 = Y_2 - \hat Y_2 = Y_2 - X_2\hat\beta = Y_2 - X_2(X'X)^{-1}X'Y = Y_2 - X_2(X_1'X_1+X_2'X_2)^{-1}(X_1'Y_1+X_2'Y_2)$

$\Rightarrow (X_1'X_1)^{-1}X_2'e_2 = \underbrace{(X_1'X_1)^{-1}X_2'Y_2}_{1^\circ} - \underbrace{(X_1'X_1)^{-1}X_2'X_2(X_1'X_1+X_2'X_2)^{-1}(X_1'Y_1+X_2'Y_2)}_{2^\circ}$

And $\hat\beta - \hat\beta^* = (X'X)^{-1}X'Y - (X_1'X_1)^{-1}X_1'Y_1$

$= \underbrace{(X_1'X_1+X_2'X_2)^{-1}(X_1'Y_1+X_2'Y_2)}_{3^\circ} - \underbrace{(X_1'X_1)^{-1}X_1'Y_1}_{4^\circ}$

In order to prove $\hat\beta - \hat\beta^* = (X_1'X_1)^{-1}X_2'e_2$, we only need to prove $1^\circ+4^\circ = 2^\circ+3^\circ$.

Note that $3^\circ+2^\circ = (X_1'X_1+X_2'X_2)^{-1}(X_1'Y_1+X_2'Y_2) + (X_1'X_1)^{-1}X_2'X_2(X_1'X_1+X_2'X_2)^{-1}(X_1'Y_1+X_2'Y_2)$

$= [(X_1'X_1)^{-1}X_1'X_1 + (X_1'X_1)^{-1}(X_2'X_2)](X_1'X_1+X_2'X_2)^{-1}(X_1'Y_1+X_2'Y_2)$

$= (X_1'X_1)^{-1}(X_1'Y_1+X_2'Y_2) \cdots (*)$

At the same time, $1^\circ+4^\circ = (X_1'X_1)^{-1}X_2'Y_2 + (X_1'X_1)^{-1}(X_1'Y_1) = (X_1'X_1)^{-1}[X_1'Y_1 + X_2'Y_2] = (*)$

$\Rightarrow \hat\beta - \hat\beta^* = (X_1'X_1)^{-1}X_2'e_2$ $\square$

(b). Note that $e_2 = Y_2 - \hat Y_2 = Y_2 - X_2\hat\beta$, $Y_2 - X_2\hat\beta^* = e_2^*$.

$\Rightarrow e_2 = Y_2 - X_2\hat\beta = e_2^* + X_2\hat\beta^* - X_2\hat\beta = e_2^* + X_2(\hat\beta^* - \hat\beta)$

Then consider the $\hat\beta - \hat\beta^*$.

$\Rightarrow e_2 - e_2^* = X_2(\hat\beta^* - \hat\beta)$

$$\Rightarrow \hat{\beta} - \hat{\beta}^* = M_1^{-1} X_2' e_2 = M_1^{-1} X_2' \left[e_2^* + X_2 [\hat{\beta}^* - \hat{\beta}] \right]$$

$$\Rightarrow (I + M_1^{-1} X_2' X_2)(\hat{\beta} - \hat{\beta}^*) = M_1^{-1} X_2' e_2^*$$

$$\Rightarrow (\hat{\beta} - \hat{\beta}^*) = (I + M_1^{-1} X_2' X_2)^{-1} M_1^{-1} X_2' e_2^*$$

(c). Note that $X_1 = \begin{pmatrix} 1 & -3 \\ 1 & -2 \\ \vdots & \vdots \\ 1 & 3 \end{pmatrix}$, $X_2 = (1\ 4)$

$\Rightarrow$ then we have $M_1 = X_1' X_1 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -3 & -2 & \cdots & 3 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 1 & -2 \\ \vdots & \vdots \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 28 \end{pmatrix}$

And $M_1^{-1} = (X_1' X_1)^{-1} = \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{28} \end{pmatrix}$, $e_2^* = 6$.

$$\Rightarrow \hat{\beta} - \hat{\beta}^* = \left(I + \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{28} \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} (1\ 4) \right)^{-1} \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{28} \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} 6$$

$$= \begin{pmatrix} \frac{8}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{11}{7} \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{28} \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} 6$$

$$= \frac{1}{\frac{12}{7}} \begin{pmatrix} \frac{11}{7} & -\frac{4}{7} \\ -\frac{1}{7} & \frac{8}{7} \end{pmatrix} \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{28} \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} 6$$

$$= \frac{1}{2} \begin{pmatrix} 11 & -4 \\ -1 & 8 \end{pmatrix} \begin{pmatrix} \frac{1}{7} \\ \frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \hat{\beta} = \hat{\beta}^* + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 13 \\ -3 \end{pmatrix}$$