10 ANOVA

10.1 Models not of full rank

X is not full-column rank

$\Rightarrow (X'X)^{-1}$ doesn't exist

Example 10.1: Weights of 6 plants

3 groups

Type of Plant		
Normal	Off-Type	Aberrant
101	84	32
105	88	
94		

$\rightarrow \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \end{pmatrix}$

Let y_{ij} = weight of the j^{th} plant of the i^{th} type, $i = 1, 2, 3.$ $j = 1, \cdots, n_2$

$\Rightarrow$ the linear model is

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \boldsymbol{\varepsilon} \qquad (\, y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \,)$$

$$\boldsymbol{\varepsilon} \sim N(\, \mathbf{0}, \ \sigma^2 \mathbf{I} \,)$$

where

$$\mathbf{y} = \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{31} \end{pmatrix} \begin{matrix} \rightarrow 101 \\ \rightarrow 105 \\ \rightarrow 94 \\ \rightarrow 84 \\ \rightarrow 88 \\ \rightarrow 32 \end{matrix} \qquad \mathbf{b} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \begin{matrix} \text{— overall effect} \\ \\ \text{— effect for the ith group} \\ \\ \end{matrix}$$

$$\Rightarrow \quad \begin{aligned} y_{11} &= \mu + \alpha_1 + \varepsilon_1 \\ y_{12} &= \mu + \alpha_1 + \varepsilon_2 \\ y_{13} &= \mu + \alpha_1 + \varepsilon_3 \\ y_{21} &= \mu + \alpha_2 + \varepsilon_4 \\ y_{22} &= \mu + \alpha_2 + \varepsilon_5 \\ y_{31} &= \mu + \alpha_3 + \varepsilon_6 \end{aligned}$$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

incidence matrix
(only 0 and 1)

column rank of $\mathbf{X} = 3$

$\Rightarrow$ non full column rank.

$\Rightarrow$ rank $(\mathbf{X}'\mathbf{X}) = 3$ $\qquad \Rightarrow$ inverse does not exist.

$$\mathbf{X}'\mathbf{X} = \left(\begin{array}{c|ccc} 6 & 3 & 2 & 1 \\ \hline 3 & 3 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow (\mathbf{X}'\mathbf{X})^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{y} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{31} \end{pmatrix}$$

$$= \begin{pmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ y_{3.} \end{pmatrix} = \begin{pmatrix} 504 \\ 300 \\ 172 \\ 32 \end{pmatrix}$$

The normal equation is

$$(\mathbf{X}'\mathbf{X})\mathbf{b}^0 = \mathbf{X}'\mathbf{y} \qquad (*)$$

Let $\mathbf{G}$ be any generalized inverse of $\mathbf{X}'\mathbf{X}$, then

$$\mathbf{b}^0 = \mathbf{G}\mathbf{X}'\mathbf{y}$$

is a solution of $(*)$ because

?

$$\begin{aligned} &\text{L.H.S. of } (*) \\ = &(\mathbf{X}'\mathbf{X})\mathbf{G}\mathbf{X}'\mathbf{y} = \mathbf{X}'\mathbf{y} = \text{R.H.S. of } (*) \end{aligned}$$

However, $\mathbf{b}^0$ is not unique.

Example 10.1 (continued) We can take

$$\mathbf{G} = \mathbf{G}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{G} = \mathbf{G}_2 = \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & \frac{4}{3} & 1 & 0 \\ -1 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that

$$\mathbf{b}_1^0 = \mathbf{G}_1\mathbf{X}'\mathbf{y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 504 \\ 300 \\ 172 \\ 32 \end{pmatrix}$$

or β_1^0

$$= \begin{pmatrix} 0 \\ 100 \\ 86 \\ 32 \end{pmatrix}$$

$$\mathbf{b}_2^0 = \mathbf{G}_2\mathbf{X}'\mathbf{y} = \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & \frac{4}{3} & 1 & 0 \\ -1 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 504 \\ 300 \\ 172 \\ 32 \end{pmatrix}$$

or β_2^0

$$= \begin{pmatrix} 32 \\ 68 \\ 54 \\ 0 \end{pmatrix}$$

$$\Rightarrow \qquad \mathbf{b}_1^0 \neq \mathbf{b}_2^0$$

Remedy over-parametrization

Notes:

(1) $$E(\mathbf{b}^0) = E(\mathbf{G}\mathbf{X}'\mathbf{y}) \quad (\ \mathbf{G} \text{ is the generalized inverse of } (\mathbf{X}'\mathbf{X})\ )$$

$$= \mathbf{G}\mathbf{X}'E(\mathbf{y})$$

$$= \mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b}$$

$$= \mathbf{A}\mathbf{b} \qquad (\text{ Let } \mathbf{A} = \mathbf{G}\mathbf{X}'\mathbf{X}\ )$$

$\Rightarrow$ $\mathbf{b}^0$ is an unbiased estimator of $\mathbf{A}\mathbf{b}$

<u>NOT</u> $\mathbf{b}$

$\underset{\sim}{A}$ depend on $\underset{\sim}{G}$

(2) $$\text{Var}(\mathbf{b}^0) = \text{Var}(\mathbf{G}\mathbf{X}'\mathbf{y})$$

$$= \mathbf{G}\mathbf{X}'\text{Var}(\mathbf{y})\mathbf{X}\mathbf{G}'$$

$$= \mathbf{G}\mathbf{X}'(\sigma^2\mathbf{I})\mathbf{X}\mathbf{G}'$$

$$= \mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{G}'\sigma^2$$

(3) $$\hat{\mathbf{y}} = \underset{\sim}{\mathbf{X}}\mathbf{b}^0$$

$$= \mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{y}$$

Since $\mathbf{X}\mathbf{G}\mathbf{X}'$ is invariant to the choice of $\mathbf{G}$

$\Rightarrow$ consistent values of $\hat{\mathbf{y}}$ with different $\mathbf{G}$

(4) $$E(\hat{\mathbf{y}}) = E(\mathbf{X}\mathbf{b}^0)$$

$$= \mathbf{X}E(\mathbf{b}^0)$$

$$= \mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b}$$

$$= \mathbf{X}\mathbf{b} \qquad (\ \Rightarrow \text{invariant to } \mathbf{G}\ )$$

ANOVA

SSR MSR F

SSE MSE

(5) $$\begin{aligned}\text{SSE} &= (\mathbf{y}-\hat{\mathbf{y}})'(\mathbf{y}-\hat{\mathbf{y}})\\ &= (\mathbf{y}-\mathbf{X}\mathbf{b}^0)'(\mathbf{y}-\mathbf{X}\mathbf{b})\\ &= (\mathbf{y}-\mathbf{XGX}'\mathbf{y})'(\mathbf{y}-\mathbf{XGX}'\mathbf{y})\\ &= \mathbf{y}'(\mathbf{I}-\mathbf{XGX}')'(\mathbf{I}-\mathbf{XGX}')\mathbf{y}\\ &= \mathbf{y}'(\mathbf{I}-\mathbf{XGX}')'\mathbf{y} \qquad (\mathbf{XGX}' \text{ is symmetric})\\ &\qquad\qquad (\mathbf{I}-\mathbf{XGX}' \text{ is idempotent})\end{aligned}$$

$(\Rightarrow$ invariant to $\mathbf{G}$ $)$

(6) $$\begin{aligned}\text{SSR} &= \text{SST} - \text{SSE}\\ &= \mathbf{y}'(\mathbf{I}-\tfrac{\mathbf{J}}{n})\mathbf{y} - \mathbf{y}'(\mathbf{I}-\mathbf{XGX}')\mathbf{y}\\ &= \mathbf{y}'(\mathbf{XGX}'-\tfrac{\mathbf{J}}{n})\mathbf{y} \qquad (\text{ invariant to } \mathbf{G}\ )\end{aligned}$$

(7) $$\begin{aligned}\text{E(SSE)} &= \text{E}[\mathbf{y}'(\mathbf{I}-\mathbf{XGX}')\mathbf{y}]\\ &= tr[(\mathbf{I}-\mathbf{XGX}')\mathbf{I}\sigma^2] + \mathbf{b}'\mathbf{X}'(\mathbf{I}-\mathbf{XGX}')\mathbf{X}\mathbf{b}\\ &= \sigma^2 tr(\mathbf{I}-\mathbf{XGX}')\\ &= \sigma^2 r(\mathbf{I}-\mathbf{XGX}')\\ &= \sigma^2(n-r(\mathbf{X}))\end{aligned}$$

$$\Rightarrow \quad E\left(\frac{SSE}{n-r(\mathbf{X})}\right) = \sigma^2$$

$$\Rightarrow \quad \hat{\sigma}^2 = \frac{SSE}{n-r(\mathbf{X})}$$ is an unbiased estimator of σ^2

(8) $$\begin{aligned}\text{SST} &= \mathbf{y}'\mathbf{y} - \mathbf{y}'\tfrac{\mathbf{11}'}{n}\mathbf{y}\\ &= \mathbf{y}'(\mathbf{I}-\tfrac{\mathbf{11}'}{n})\mathbf{y}\end{aligned}$$

10.2 Distributional Properties

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \boldsymbol{\varepsilon}\ , \qquad \boldsymbol{\varepsilon} \sim N(\ \mathbf{0},\ \sigma^2\mathbf{I}\ )$$

(1) $\mathbf{y} \sim N(\ \mathbf{X}\mathbf{b},\ \sigma^2\mathbf{I}\ )$

(2) $\mathbf{b}^0 = \mathbf{G}\mathbf{X}'\mathbf{y} \sim N(\ \mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b},\ \mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{G}'\sigma^2\ )$

(3) $\mathbf{b}^0$ and $\widehat{\sigma}^2$ are independent

$$\begin{aligned} \mathbf{b}^0 &= \mathbf{G}\mathbf{X}'\mathbf{y} \\ SSE &= \mathbf{y}'(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{y} \end{aligned}$$

$$\begin{aligned} &\mathbf{G}\mathbf{X}'(\mathbf{I}\sigma^2)(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}') \\ &= \sigma^2\mathbf{G}\mathbf{X}'(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}') = 0 \end{aligned}$$

$\Rightarrow$ $\mathbf{b}^0$ and $\widehat{\sigma}^2$ are independent.

(4) $$\frac{SSE}{\sigma^2} \sim \chi^2$$

$$\frac{SSE}{\sigma^2} = \frac{\mathbf{y}'(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{y}}{\sigma^2}$$

But

$$\frac{(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')}{\sigma^2}\ \mathbf{I}\sigma^2 = \mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}' \quad \text{is idempotent}$$

and

$$\begin{aligned} rank\Big(\frac{\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}'}{\sigma^2}\Big) &= rank(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}') \\ &= n - r(\mathbf{X})\ , \end{aligned}$$

$$\Rightarrow \qquad \frac{SSE}{\sigma^2} \sim \chi^2_{(n-r(\mathbf{X}),\ \frac{1}{2\sigma^2}\boldsymbol{\beta}'\mathbf{X}'(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{X}\mathbf{b})}$$

But $\qquad \dfrac{1}{2\sigma^2}\mathbf{b}'\mathbf{X}(\mathbf{I}-\mathbf{XGX}')\mathbf{Xb} = 0$

$\Rightarrow \qquad \dfrac{SSE}{\sigma^2} \sim \chi^2_{(n-r(\mathbf{X}))}$

(5) $\qquad$ SSR $= \mathbf{y}'(\mathbf{XGX}' - \frac{\mathbf{J}}{n}))\mathbf{y}$

$\Rightarrow \qquad \dfrac{SSR}{\sigma^2} = \mathbf{y}'\dfrac{(\mathbf{XGX}'-\frac{\mathbf{J}}{n})}{\sigma^2}\mathbf{y}$

and $\qquad \dfrac{(\mathbf{XGX}'-\frac{\mathbf{J}}{n})}{\sigma^2}\ \sigma^2\mathbf{I}$ is idempotent.

$$\text{rank}\Big(\frac{(\mathbf{XGX}'-\frac{\mathbf{J}}{n})}{\sigma^2}\Big) = r(\mathbf{X}) - 1$$

$$\frac{1}{2\sigma^2}\ \mathbf{b}'\mathbf{X}'(\mathbf{XGX}' - \tfrac{\mathbf{J}}{n})\mathbf{Xb} = \frac{1}{2\sigma^2}\ \mathbf{b}'\mathbf{X}'(\mathbf{I} - \tfrac{\mathbf{J}}{n})\mathbf{Xb}$$

$\Rightarrow \qquad \dfrac{SSR}{\sigma^2} \sim \chi^2_{(\ r(\mathbf{X})-1,\ \frac{1}{2\sigma^2}\mathbf{b}'\mathbf{X}'(\mathbf{I}-\frac{\mathbf{J}}{n})\mathbf{Xb})}$

(6) $\qquad$ SSE and SSR are independent

Since $(\mathbf{XGX}' - \frac{\mathbf{J}}{n})\mathbf{I}\sigma^2(\mathbf{I} - \mathbf{XGX}') = 0$

$\Rightarrow \qquad$ SSE and SSR are independent

(7)

$$F(R) = \frac{SSR/(r(\mathbf{X})-1)}{SSE/(n-r(\mathbf{X}))} \sim F_{(\ r(\mathbf{X})-1,\ n-r(\mathbf{X}),\ \frac{1}{2\sigma^2}\boldsymbol{\beta}'\mathbf{X}'(\mathbf{I}-\frac{\mathbf{11}'}{n})\mathbf{Xb})}$$

10.3 Estimable Functions

– The parametric function $\mathbf{q}'\mathbf{b}$ is said to be estimable if it has a linear unbiased estimate, $\mathbf{t}'\mathbf{y}$ say.

$\Rightarrow$ if $\mathbf{q}'\mathbf{b}$ is estimable, there exist $\mathbf{t}$ such that

$$
\begin{aligned}
E(\mathbf{t}'\mathbf{y}) &= \mathbf{q}'\mathbf{b} \\
\Rightarrow \quad \mathbf{t}'E(\mathbf{y}) &= \mathbf{q}'\mathbf{b} \\
\Rightarrow \quad \mathbf{t}'\mathbf{X}\mathbf{b} &= \mathbf{q}'\mathbf{b} \qquad (*)
\end{aligned}
$$

Since $(*)$ is true for all $\mathbf{b}$,

$$\Rightarrow \qquad \mathbf{t}'\mathbf{X} = \mathbf{q}'$$

– The b.l.u.e. of the estimable function $\mathbf{q}'\mathbf{b}$ is $\boxed{\mathbf{q}'\mathbf{b}^0}$

doesn't need to been unique

invariant to the choice of b^0

(i) $\quad \mathbf{q}'\mathbf{b}^0 = \mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{y}$

$\Rightarrow$ linear function of y_i

(ii)

$$
\begin{aligned}
E(\mathbf{q}'\mathbf{b}^0) &= \mathbf{q}'E(\mathbf{b}^0) \\
&= \mathbf{q}'\mathbf{G}\mathbf{X}'E(\mathbf{y}) \\
&= \mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b} \\
&= \mathbf{t}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{t}'\mathbf{X}\mathbf{b} = \mathbf{q}'\mathbf{b}
\end{aligned}
$$

$\Rightarrow$ unbiased estimator

(iii) Minimum variance. (Best)

$$
\begin{aligned}
var(\mathbf{q}'\mathbf{b}^0) &= \mathbf{q}' var(\mathbf{b}^0)\mathbf{q} \\
&= \mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{G}'\mathbf{q}\sigma^2 \\
&= \mathbf{t}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{G}'\mathbf{X}'\mathbf{t}\sigma^2 \\
&= \mathbf{t}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{t}\sigma^2 \\
&= \mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2
\end{aligned}
$$

Suppose $\mathbf{k}'\mathbf{y}$ is another linear unbiased estimator of $\mathbf{q}'\mathbf{b}$ different from $\mathbf{q}'\mathbf{b}^0$.

$$\Rightarrow \quad E(\mathbf{k}'\mathbf{y}) = \mathbf{q}'\mathbf{b} \qquad \Rightarrow \quad \mathbf{k}'\mathbf{X} = \mathbf{q}'$$

$$
\begin{aligned}
\Rightarrow \quad & cov(\mathbf{q}'\mathbf{b}^0,\ \mathbf{k}'\mathbf{y}) \\
&= cov(\mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{y},\ \mathbf{k}'\mathbf{y}) \\
&= \mathbf{q}'\mathbf{G}\mathbf{X}'(\mathbf{I}\sigma^2)\mathbf{k} \\
&= \mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2
\end{aligned}
$$

Now,

$$
\begin{aligned}
& var(\mathbf{q}'\mathbf{b}^0 - \mathbf{k}'\mathbf{y}) \\
&= var(\mathbf{q}'\mathbf{b}^0) + var(\mathbf{k}'\mathbf{y}) - 2cov(\mathbf{q}'\mathbf{b}^0, \mathbf{k}'\mathbf{y}) \\
&= var(\mathbf{k}'\mathbf{y}) + \mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2 - 2\mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & var(\mathbf{q}'\mathbf{b}^0 - \mathbf{k}'\mathbf{y}) \\
&= var(\mathbf{k}'\mathbf{y}) - var(\mathbf{q}'\mathbf{b}^0) \ \geq \ 0
\end{aligned}
$$

$$\Rightarrow \quad var(\mathbf{k}'\mathbf{y}) \ \geq \ var(\mathbf{q}'\mathbf{b}^0)$$

$\Rightarrow \quad \mathbf{q}'\mathbf{b}^0$ is B.L.U.E. of $\mathbf{q}'\mathbf{b}$

<u>Note</u> $\quad \mathbf{q}'\mathbf{b}^0 \ \sim \ N(\ \mathbf{q}'\mathbf{b},\ \mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2)$

10.3.1 Test of Estimability

$\mathbf{q}'\mathbf{b}$ is estimable if and only if $\mathbf{q}'\mathbf{A} = \mathbf{q}'$ where $\mathbf{A} = \mathbf{G}\mathbf{X}'\mathbf{X}$

Proof: If $\mathbf{q}'\mathbf{b}$ is estimable, there exist a vector $\mathbf{t}$ such that $\mathbf{t}'\mathbf{X} = \mathbf{q}'$

$$
\begin{aligned}
\Rightarrow \quad \mathbf{q}'\mathbf{A} &= \mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{X} \\
&= \mathbf{t}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X} \\
&= \mathbf{t}'\mathbf{X} \;=\; \mathbf{q}'
\end{aligned}
$$

If $\mathbf{q}'\mathbf{A} = \mathbf{q}'$

$$\Rightarrow \quad \mathbf{q}' = \overbrace{\mathbf{q}'\mathbf{G}\mathbf{X}'}^{\nearrow \underset{\sim}{t}'}\mathbf{X}$$

$\Rightarrow$ take $\mathbf{t}' = \mathbf{q}'\mathbf{G}\mathbf{X}'$

we have $\mathbf{q}' = \mathbf{t}'\mathbf{X}$

$\Rightarrow$ $\mathbf{q}'\mathbf{b}$ is estimable.

Example 10.2: Consider the normal equations

$$(\mathbf{X}'\mathbf{X})\mathbf{b} = \mathbf{X}'\mathbf{y}$$

where $$(\mathbf{X}'\mathbf{X}) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad , \quad \mathbf{X}'\mathbf{y} = \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix}$$

One possible generalized inverse is

$$\Rightarrow \quad \mathbf{G} = (\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{b}^0 = \mathbf{G}\mathbf{X}'\mathbf{y} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ -2 \\ 0 \end{pmatrix}$$

Another generalized inverse is

$$\mathbf{G}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{b}_1^0 = \mathbf{G}_1\mathbf{X}'\mathbf{y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 8 \end{pmatrix}$$

– SSR (by $\mathbf{G}$)

$$= \mathbf{y}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{y}$$

$$= \begin{pmatrix} 14 & 6 & 8 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix}$$

$$= 100$$

– $\mathbf{A} = \mathbf{G}_1\mathbf{X}'\mathbf{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

– Is $\beta_1 - \beta_2$ estimable?

$$\mathbf{b} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\mathbf{q}' = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{q}'\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} = \mathbf{q}'$$

$\Rightarrow$ $\beta_1 - \beta_2$ is estimable.

– Is $\beta_1 + \beta_2$ estimable?

$$\mathbf{q}' = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{q}'\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \neq \mathbf{q}'$$

$\Rightarrow \quad \beta_1 + \beta_2$ is not estimable.

– Is $3\beta_0 - \beta_1 - 2\beta_2$ estimable?

$$\mathbf{q}' = \begin{pmatrix} 3 & -1 & -2 \end{pmatrix}$$

$$\mathbf{q}'\mathbf{A} = \begin{pmatrix} 3 & -1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -1 & -2 \end{pmatrix} \neq \mathbf{q}'$$

$\Rightarrow \quad 3\beta_0 - \beta_1 - 2\beta_2$ is not estimable.

$H_0: \alpha_1-\alpha_2=0, \ \alpha_1+\alpha_2=0$

$\begin{pmatrix}\mu\\ \alpha_1\\ \alpha_2\end{pmatrix} \Rightarrow \underset{\sim}{K}' = \begin{pmatrix}0 & 1 & -1\\ 0 & 1 & 1\end{pmatrix} \quad \underset{\sim}{m} = \begin{pmatrix}0\\0\end{pmatrix}$

10.4 Testable Hypothesis

- A hypothesis that can be expressed in terms of estimable functions.

- Let $H_0 : \mathbf{K}'\mathbf{a} = \boldsymbol{\mu}$ $\qquad$ $[\mathbf{K}'$ is $r \times (k+1)]$

  and $\mathbf{K}'\mathbf{a}$ is estimable

  $\mathbf{K}' = \mathbf{S}'\mathbf{X}'\mathbf{X}$ for some full row rank $\mathbf{S}'$ $\qquad$ $\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{a} + \underset{\sim}{\varepsilon}$

  $\mathbf{K}'\mathbf{a}^0$ is used to estimate $\mathbf{K}'\mathbf{a}$

  $E(\mathbf{K}'\mathbf{a}^0) = \mathbf{K}'\mathbf{a}$

  and $Var(\mathbf{K}'\mathbf{a}^0) = \mathbf{K}'\mathbf{G}\mathbf{K}\sigma^2$

  Note that $rank(\mathbf{K}') = rank(\mathbf{S}') = rank(\mathbf{S}'\mathbf{X}') = r$

  Also

$$\begin{aligned}\mathbf{K}'\mathbf{G}\mathbf{K} &= \mathbf{S}'\mathbf{X}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{S}\\ &= \mathbf{S}'\mathbf{X}'\mathbf{X}\mathbf{S}\\ &= (\mathbf{S}'\mathbf{X}')(\mathbf{S}'\mathbf{X}')'\end{aligned}$$

  $rank(\mathbf{K}'\mathbf{G}\mathbf{K}) = rank(\mathbf{S}'\mathbf{X}') = r$

  $\Rightarrow$ $\mathbf{K}'\mathbf{G}\mathbf{K}$ is nonsingular

Example 10.1

$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$

$H_0: \alpha_1 = \alpha_2 = \alpha_3$

$\Leftrightarrow H_0: \alpha_1 - \alpha_2 = 0$

$\qquad \alpha_1 - \alpha_3 = 0$

$\Leftrightarrow \underset{\sim}{K}' = \begin{pmatrix}0 & 1 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix} \quad a = \begin{pmatrix}\mu\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$

$\Rightarrow H_0: \underset{\sim}{K}\underset{\sim}{a} = \begin{pmatrix}0\\0\end{pmatrix}$

Hypothesis testing

$H_0 : \mathbf{K'a} = \boldsymbol{\mu}$ (let # of rows in $\mathbf{K'} = s$)
$\mathbf{y} \sim N(\mathbf{Xa}, \boldsymbol{\sigma}^2\mathbf{I})$
$\mathbf{a}^0 \sim N(\mathbf{GX'Xa}, \mathbf{GX'XG'}\boldsymbol{\sigma}^2)$
and $\mathbf{K'a}^0 - \boldsymbol{\mu} \sim N(\mathbf{K'a} - \boldsymbol{\mu}, \mathbf{K'GK}\boldsymbol{\sigma}^2)$
Take

$$Q = (\mathbf{K'a}^0 - \boldsymbol{\mu})'(\mathbf{K'GK})^{-1}(\mathbf{K'a}^0 - \boldsymbol{\mu})$$

then

$$\frac{\mathbf{Q}}{\boldsymbol{\sigma}^2} \sim \chi^2_{(s,\ (\mathbf{K'a}-\boldsymbol{\mu})'(\mathbf{K'GK})^{-1}(\mathbf{K'a}-\boldsymbol{\mu})/2\boldsymbol{\sigma}^2)}$$

It is straightforward to show that

$$F(H) = \frac{Q/s}{SSE/(n - r(\mathbf{X}))} \sim F_{(s,\ n-r(\mathbf{X}),\ (\mathbf{K'a}-\boldsymbol{\mu})'(\mathbf{K'GK})^{-1}(\mathbf{K'a}^0-\boldsymbol{\mu})/2\boldsymbol{\sigma}^2)}$$

Under $H_0 : \mathbf{K'a} = \boldsymbol{\mu}$,

$$F(H) = \frac{Q/s}{SSE/(n - r(\mathbf{X}))} \sim F_{(s,\ n-r(\mathbf{X}))}$$

Under $H_0 : \mathbf{K}'\mathbf{a} = \boldsymbol{\mu}$

$$\mathbf{a}_H^0 = \mathbf{a}^0 - \mathbf{GK}(\mathbf{K}'\mathbf{GK})^{-1}(\mathbf{K}'\mathbf{a}^0 - \boldsymbol{\mu})$$

$$SSE_H = SSE + Q$$

where SSE = $\mathbf{y}$'($\mathbf{I}$- $\mathbf{XGX}$')$\mathbf{y}$

Treat 1 Treat 2 .... Treat a

n_1 n_2 n_a

10.4.1 One - Way Layout (One factor (fixed) design)

Model :

$$y_{ij} = \mu + \alpha_i + \boldsymbol{\varepsilon}_{ij}, \qquad i = 1, \cdots, a; \quad j = 1, \cdots, n_i.$$

In matrix notation $\mathbf{y} = \mathbf{X}\mathbf{a} + \boldsymbol{\varepsilon}$, $\qquad \boldsymbol{\varepsilon} \;\sim\; N(0, \boldsymbol{\sigma}^2\mathbf{I})$

$$\mathbf{y} = \begin{pmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1n_1} \\ \cdots \\ \vdots \\ \cdots \\ y_{a_1} \\ y_{a_2} \\ \vdots \\ y_{an_a} \end{pmatrix} \qquad \mathbf{a} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_a \end{pmatrix} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \boldsymbol{\varepsilon}_{11} \\ \boldsymbol{\varepsilon}_{12} \\ \vdots \\ \boldsymbol{\varepsilon}_{1n_1} \\ \cdots \\ \vdots \\ \cdots \\ \boldsymbol{\varepsilon}_{a1} \\ \vdots \\ \boldsymbol{\varepsilon}_{an_a} \end{pmatrix}$$

$$
\mathbf{X} = \begin{array}{c}
\begin{array}{ccccc} \mu & \alpha_1 & \alpha_2 & \cdots & \alpha_a \end{array} \\
\begin{pmatrix}
1 & 1 & 0 & & 0 \\
1 & 1 & 0 & & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 1 & 0 & & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 1 & & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 0 & 0 & & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & & 1 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 0 & 0 & & 1
\end{pmatrix}
\end{array}
$$

$$
n = \sum_{i=1}^{a} n_i
$$

$$
\mathbf{X}'\mathbf{X} = \left(\begin{array}{c|cccc}
n & n_1 & n_2 & \cdots & n_a \\
\hline
n_1 & n_1 & 0 & \cdots & 0 \\
n_2 & 0 & n_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n_a & 0 & 0 & \cdots & n_a
\end{array}\right)
$$

A generalized inverse is

$$\mathbf{G} = (\mathbf{X}'\mathbf{X})^- = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{n_1} & & 0 \\ \vdots & 0 & \frac{1}{n_2} & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{n_a} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \mathbf{D}\{\frac{1}{n_i}\} \end{pmatrix}$$

(A)
$$\begin{aligned}
\mathbf{H} &= \mathbf{G}\mathbf{X}'\mathbf{X} \\
&= \begin{pmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \mathbf{D}\{\frac{1}{n_i}\} \end{pmatrix} \begin{pmatrix} n & n_1 & n_2 & \cdots & n_a \\ n_1 & n_1 & 0 & \cdots & 0 \\ n_2 & 0 & n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_a & 0 & 0 & \cdots & n_a \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathbf{0}' \\ \mathbf{1}_a & \mathbf{I}_a \end{pmatrix}
\end{aligned}$$

$$\mathbf{X}'\mathbf{y} = \begin{pmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ \vdots \\ y_{a.} \end{pmatrix}$$

$$(\beta^0) \quad \mathbf{a}^0 = \mathbf{G}\mathbf{X}'\mathbf{y} = \begin{pmatrix} 0 \\ \bar{y}_{1.} \\ \bar{y}_{2.} \\ \vdots \\ \bar{y}_{a.} \end{pmatrix}$$

Analysis of Variance

1.

$$\mathbf{a}^{0\prime}\mathbf{X}'\mathbf{y} = (0 \quad \bar{y}_{1.} \quad \bar{y}_{2.} \quad \cdots \quad \bar{y}_{a.}) \begin{pmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ \vdots \\ y_{a.} \end{pmatrix} = \mathbf{S}_1 \frac{y_{i.}^2}{n_i}$$

sum of all obs

sum of obs in treat 1

2.

$$SSR = \mathbf{a}^{0\prime}\mathbf{X}'\mathbf{y} - \mathbf{y}'\frac{\mathbf{J}}{n}\mathbf{y} = \mathbf{S}_1 \frac{y_{i.}^2}{n_i} - n\bar{y}_{..}^2$$

3.

$$SST = \mathbf{S}_1 \sum_{j=1}^{n_i} y_{ij}^2 - n\bar{y}_{..}^2$$

4.

$$\begin{aligned} SSE &= SST - SSR \\ &= \mathbf{S}_1 \sum_{j=1}^{n_i} y_{ij}^2 - \mathbf{S}_1 \frac{y_{i.}^2}{n_i} \end{aligned}$$

Example 10.3

Three different treatment methods for removing organic carbon from tar sand wastewater are to be compared. The methods are airflotation (AF), foam seperation (FS), and ferric - chloride coagulation (FCC). The data are given as follows

AF(I)	FS(II)	FCC(III)
34.6	38.8	26.7
35.1	39.0	26.7
35.3	40.1	27.0
35.8	40.9	27.1
36.1	41.0	27.5
36.5	43.2	28.1
36.8	44.9	28.1
37.2	46.9	28.7
37.4	51.6	30.7
37.7	53.6	31.2

Model:

$$y_{ij} = \mu + T_i + \boldsymbol{\varepsilon}_{ij}, \qquad \boldsymbol{\varepsilon}_{ij} \sim N(0, \boldsymbol{\sigma}^2)$$

T_i: Effect of treatment i $\qquad i = 1, 2, 3; \quad j = 1, 2, \cdots, 10$

$$\mathbf{a} = \begin{pmatrix} \mu \\ T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} 30 & 10 & 10 & 10 \\ 10 & 10 & 0 & 0 \\ 10 & 0 & 10 & 0 \\ 10 & 0 & 0 & 10 \end{pmatrix}$$

A generalized inverse of $\mathbf{X}'\mathbf{X}$ is

$$(\mathbf{X}'\mathbf{X})^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{pmatrix}$$

$$\mathbf{a}^0 = (\mathbf{X}'\mathbf{X})^-\mathbf{X}'\mathbf{y} = \begin{pmatrix} 0 \\ 36.25 \\ 44 \\ 28.18 \end{pmatrix}$$

SST = 1530.19367
SSR =1251.53267
SSE = 278.661
MSE = $\frac{278.661}{27} = 10.32 = \hat{\boldsymbol{\sigma}}^2$

Let us test

$$H_0 : T_1 = T_2 = T_3$$

~~with reference to p.91 ,~~

$$\mathbf{K}' = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \qquad \boldsymbol{\mu} = \mathbf{0}$$

$$\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{K}$$
$$= \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \frac{1}{10} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{K})^{-1} = \frac{10}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

many possible solution

different $\underset{\sim}{\beta}^{o}$

$(\underset{\sim}{X}'\underset{\sim}{X})\underset{\sim}{\beta} = \underset{\sim}{X}'\underset{\sim}{Y}$

$(\underset{\sim}{X}'\underset{\sim}{X})_{4\times4}$ rank$[\underset{\sim}{X}'\underset{\sim}{X}] = 3$

The test statistics is

$$
\begin{aligned}
F(H) &= \frac{Q/s}{\hat{\boldsymbol{\sigma}}^2} \\
&= \frac{(\mathbf{K}'\mathbf{a} - \boldsymbol{\mu})'(\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{K})^{-1}(\mathbf{K}'\mathbf{a} - \boldsymbol{\mu})/2}{10.32} \\
&= 60.63 \qquad (Q = 1251.533)
\end{aligned}
$$

Since p-value < 0.01, have strong evidence to reject H_0

all of this table remain the same

Anova Table

Source of Variation	SS	df	MS	F
Regression	1251.53267	2	625.7665	60.63
Residual	278.661	27	10.32	
Total	1530.19367	29		