9.2 More complex models

- Qualitative independent variables
- Interaction Model
- Polynomial Regression Models
- Summary of First-order and Second-order Models
- Coefficients of Partial Determination

Qualitative Predictors

- To quantify qualitative predictors, we use indicator variables (dummy variables).
- An indicator variable is a categorical explanatory variable with two levels:
  - yes or no, on or off, male or female
  - coded as 0 or 1
- If more than two levels, the number of indicator variables needed is (number of levels - 1)

Indicator-Variable Example (with 2 Levels)

$$\hat{Y} = b_0 + b_1 X_1 + b_2 X_2$$

Let:

Y = pie sales

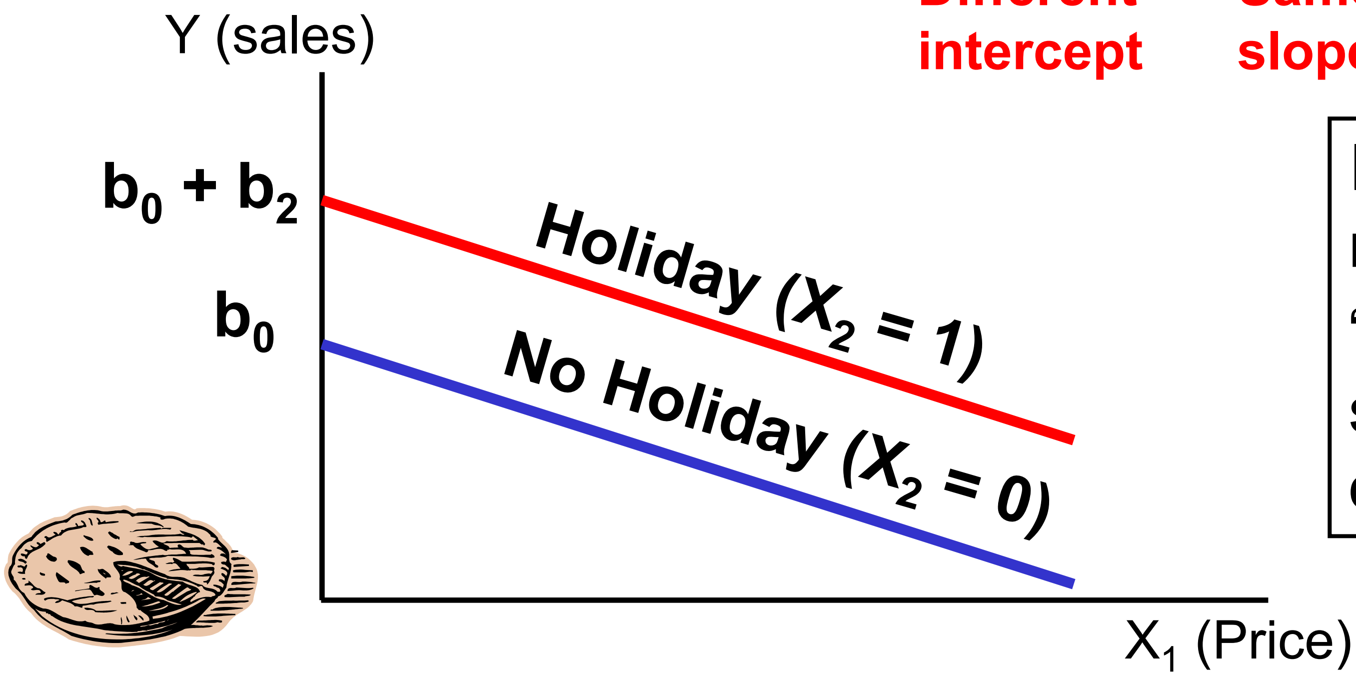
X_1 = price

X_2 = holiday (X_2 = 1 if a holiday occurred during the week)
(X_2 = 0 if there was no holiday that week)

Indicator-Variable Example (with 2 Levels)

$\hat{Y} = b_0 + b_1 X_1 + b_2(1) = (b_0 + b_2) + b_1 X_1$	**Holiday**
$\hat{Y} = b_0 + b_1 X_1 + b_2(0) = b_0 + b_1 X_1$	**No Holiday**

Different intercept **Same slope**



If H_0: $\beta_2 = 0$ is rejected, then "Holiday" has a significant effect on pie sales

Interpreting the Indicator Variable Coefficient (with 2 Levels)

Example: $\text{Sales} = 300 - 30(\text{Price}) + 15(\text{Holiday})$

Sales: number of pies sold per week

Price: pie price in $

$$\text{Holiday:} \begin{cases} 1 & \text{If a holiday occurred during the week} \\ 0 & \text{If no holiday occurred} \end{cases}$$

$b_2 = 15$: on average, sales were 15 pies greater in weeks with a holiday than in weeks without a holiday, given the same price

Indicator-Variable Models (more than 2 Levels)

- The number of dummy variables is **one less than the number of levels**
- Example: Y = house price ; X_1 = square feet

- If style of the house is also thought to matter:

  Style = ranch, split level, condo

Three levels, so two dummy variables are needed

Indicator-Variable Models (more than 2 Levels)

- Example: Let "condo" be the default category, and let X_2 and X_3 be used for the other two categories:

Y = house price

X_1 = square feet

X_2 = 1 if ranch, 0 otherwise

X_3 = 1 if split level, 0 otherwise

The multiple regression equation is:

$$\hat{Y} = b_0 + b_1 X_1 + b_2 X_2 + b_3 X_3$$

Interpreting the indicator Variable Coefficients (with 3 Levels)

Consider the regression equation: Remark 9.2

$$\hat{Y} = 20.43 + 0.045X_1 + 23.53X_2 + 18.84X_3$$

For a condo: $X_2 = X_3 = 0$

$$\hat{Y} = 20.43 + 0.045X_1$$

For a ranch: $X_2 = 1$; $X_3 = 0$

$$\hat{Y} = 20.43 + 0.045X_1 + 23.53$$

With the same square feet, a ranch will have an estimated average price of 23.53 thousand dollars more than a condo

For a split level: $X_2 = 0$; $X_3 = 1$

$$\hat{Y} = 20.43 + 0.045X_1 + 18.84$$

With the same square feet, a split-level will have an estimated average price of 18.84 thousand dollars more than a condo.

Interaction Regression Models

- Hypothesizes interaction between pairs of X variables
  - Response to one X variable may vary at different levels of another X variable

- Contains two-way cross product terms

For example

$X_2 = \begin{cases} 1 \\ 0 \end{cases}$

  - $$\hat{Y} = b_0 + b_1 X_1 + b_2 X_2 + b_3 X_3$$
    $$= b_0 + b_1 X_1 + b_2 X_2 + b_3 (X_1 X_2)$$

$x_2 = 1 \quad = b_0 + b_1 x_1 + b_2 + b_3 x_1 = b_0 + b_2 + (b_1 + b_3) x_1$

$x_2 = 0 \quad = b_0 + b_1 x_1$

Interaction Regression Models

Example: 3 predictor variables

$$Y_i = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 x_{i1} + \boldsymbol{\beta}_2 x_{i2} + \boldsymbol{\beta}_3 x_{i3} + \boldsymbol{\beta}_4 x_{i1} x_{i2} + \boldsymbol{\beta}_5 x_{i1} x_{i3} + \boldsymbol{\beta}_6 x_{i2} x_{i3} + \boldsymbol{\varepsilon}_i$$

$= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_6 x_{i6} + \varepsilon_i$

$H_0: \beta_5 = 0$?

$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$

$H_0: \beta_4 = \beta_5 = \beta_6 = 0$?

$H_0: \beta_2 = 0$?

Effect of Interaction

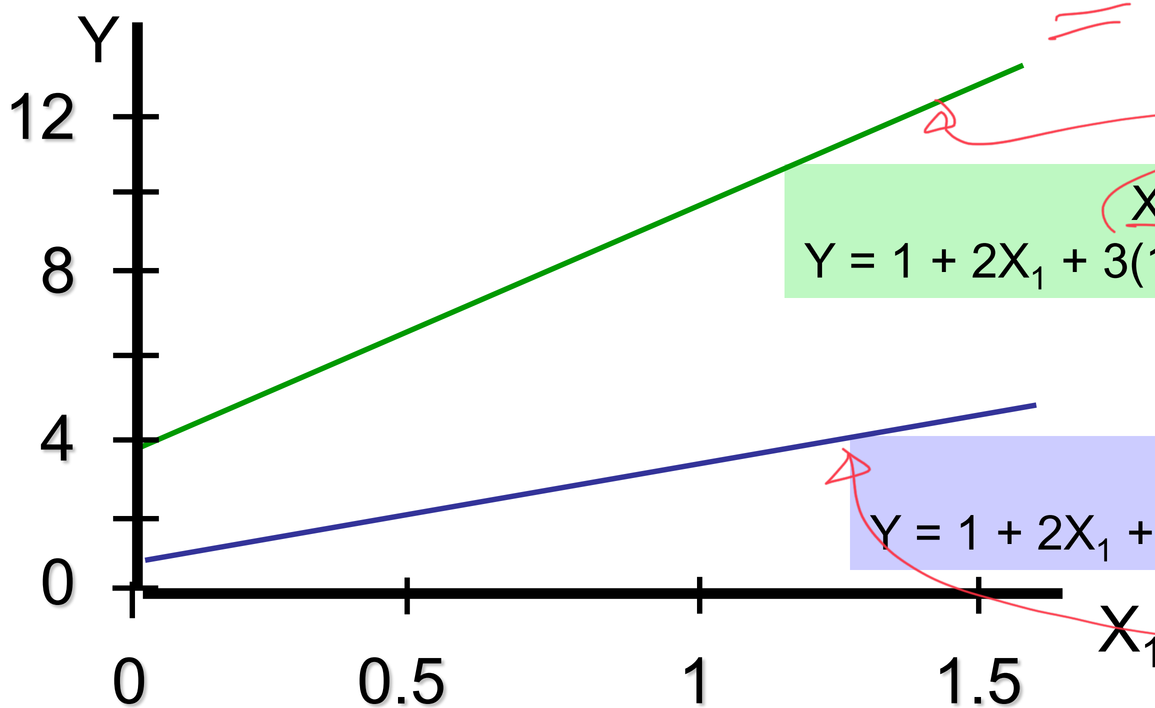
- Given: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + \varepsilon$

- Without interaction term, effect of X_1 on Y is measured by β_1
- With interaction term, effect of X_1 on Y is measured by $\beta_1 + \beta_3 X_2$
- Effect changes as X_2 changes

Effect of Interaction

Suppose X_2 is a dummy variable and the estimated regression equation is

$$\hat{Y} = 1 + 2X_1 + 3X_2 + 4X_1X_2$$



$X_2 = 1$:
$Y = 1 + 2X_1 + 3(1) + 4X_1(1) = 4 + 6X_1$

$X_2 = 0$:
$Y = 1 + 2X_1 + 3(0) + 4X_1(0) = 1 + 2X_1$

Slopes are different if the effect of X_1 on Y depends on X_2 value

Significance of Interaction Term

- Can perform a partial F-test for the contribution of a variable to see if the addition of an interaction term improves the model

- Multiple interaction terms can be included
  - Use a partial F-test for the simultaneous contribution of multiple variables to the model

Polynomial Regression Models

When are polynomial regression models being used?

- When the true curvilinear response function is indeed a polynomial function
- When the true curvilinear response function is unknown (or complex) but a polynomial function is a good approximation to the true function.

Polynomial Regression Models

Example: 1 predictor variable, second order

$$Y_i = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 x_i + \boldsymbol{\beta}_2 x_i^2 + \boldsymbol{\varepsilon}_i$$

where $= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}$

$$x_i = X_i - \overline{X}$$

$x_{i2} = x_i^2$

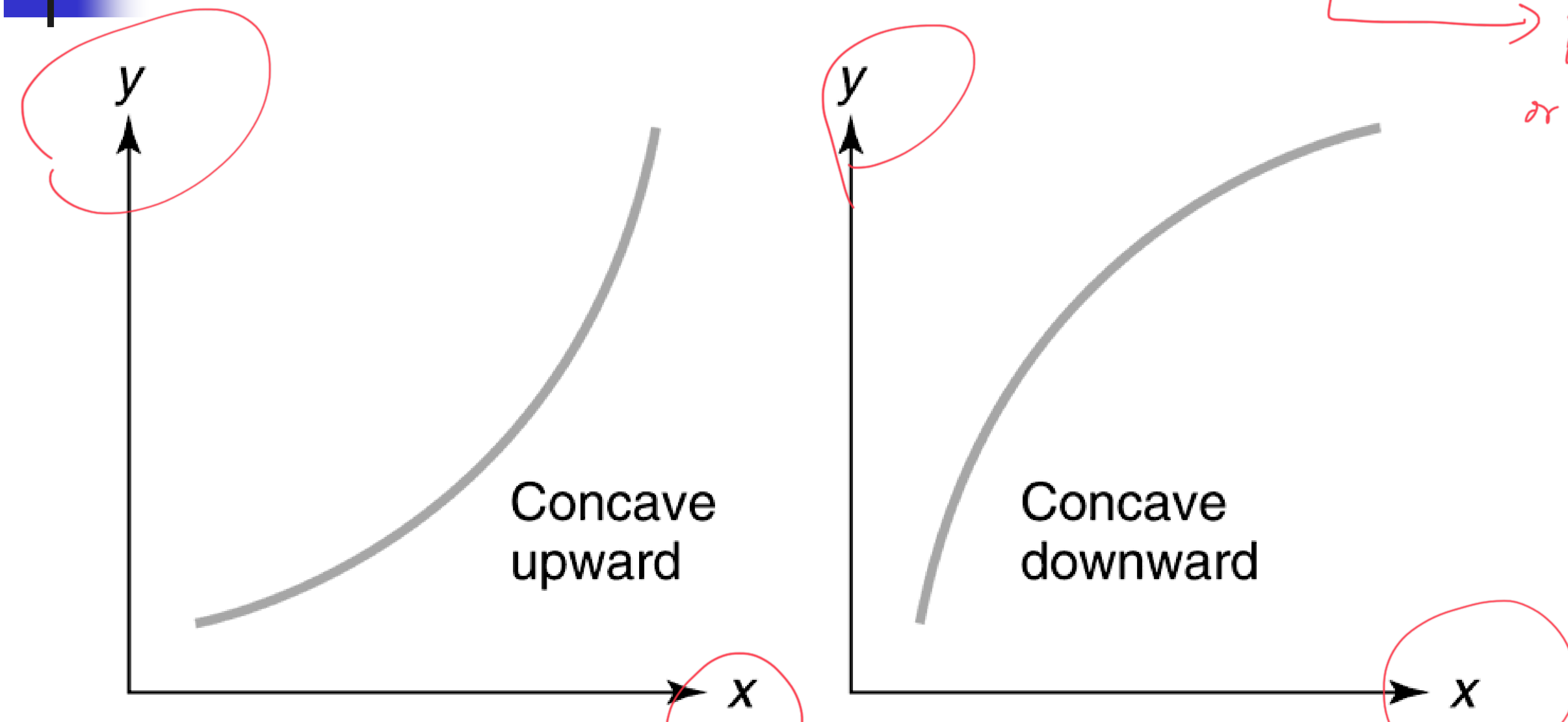
The reason for using a centered predictor variable in the polynomial regression model is that X and X2 often will be highly correlated. Centering the predictor variable often reduces the multicollinearity substantially, and tends to avoid computational difficulties.

$y_i = \beta_0 + \beta_1 x_{i1} + \varepsilon_i$

x_i

$\hat{y}_i$ or x_i

Graphs for two quadratic models



(a) $\beta_2 > 0$ (b) $\beta_2 < 0$

Polynomial Regression Models

Example: 2 predictor variables, second order

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_{11} x_{i1}^2 + \beta_{22} x_{i2}^2 + \beta_{12} x_{i1} x_{i2} + \varepsilon_i$$

where

$$x_{i1} = X_{i1} - \overline{X}_1$$

$$x_{i2} = X_{i2} - \overline{X}_2$$

Coefficients of partial determination

$$R^2_{Yj.(\text{all variables except } j)} = \frac{SSR(X_j \mid \text{all variables except } j)}{SSE(\text{all variables except } j)}$$

Remark 9.3

- Measures the proportion of variation in the dependent variable that is explained by X_j while controlling for (holding constant) the other explanatory variables
- Coefficients of partial correlation