10 ANOVA

10.1 Models not of full rank

Example 10.1: Weights of 6 plants

Type of Plant		
Normal	Off-Type	Aberrant
101	84	32
105	88	
94		

Let y_{ij} = weight of the j^{th} plant of the i^{th} type, $i = 1, 2, 3.$

$j = 1, \dots, n_i$

$\Rightarrow$ the linear model is

μ – overall effect

$$\begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{31} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \underset{\sim}{\varepsilon}$$

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \boldsymbol{\varepsilon} \qquad (y_{ij} = \mu + \alpha_i + \varepsilon_{ij})$$

$\left.\begin{matrix}\alpha_1 \\ \alpha_2 \\ \alpha_3\end{matrix}\right\}$ effects for the i-th group

$$\boldsymbol{\varepsilon} \sim N(\,\mathbf{0},\ \sigma^2\mathbf{I}\,)$$

where

$$\mathbf{y} = \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{31} \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$\underset{\sim}{y} = \underset{\sim}{X}\,\underset{\sim}{b} + \underset{\sim}{\varepsilon}$

lm

treatment effects

($i=1$) group 1 $\mu + \alpha_1$

2 $\mu + \alpha_2$

3 $\mu + \alpha_3$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

column rank of $\mathbf{X} = 3$

$\Rightarrow$ non full column rank.

$\Rightarrow$ rank $(\mathbf{X}'\mathbf{X}) = 3$ $\Rightarrow$ inverse does not exist.

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} 6 & 3 & 2 & 1 \\ 3 & 3 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$(X'X)^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{y} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{31} \end{pmatrix}$$

$$= \begin{pmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ y_{3.} \end{pmatrix} = \begin{pmatrix} 504 \\ 300 \\ 172 \\ 32 \end{pmatrix}$$

LSE: $\underset{b}{} \; (y - Xb)'(y - Xb)$

$$0 = \frac{\partial}{\partial b} = (X'X)b - X'y$$

The normal equation is

$$(\mathbf{X}'\mathbf{X})\mathbf{b}^0 = \mathbf{X}'\mathbf{y} \qquad (*)$$

$b^0 = (X'X)^- X'y$

$\hat{b} = (X'X)^{-1} X'y$

if rank $(X'X)$ = full rank

Let $\mathbf{G}$ be any generalized inverse of $\mathbf{X}'\mathbf{X}$, then

$$\mathbf{b}^0 \; = \; \mathbf{G}\mathbf{X}'\mathbf{y}$$

$G = (X'X)^-$

is a solution of $(*)$ because

L.H.S. of $(*)$

$$= \; (\mathbf{X}'\mathbf{X})\mathbf{G}\mathbf{X}'\mathbf{y} \; = \; \mathbf{X}'\mathbf{y} = \; \text{R.H.S. of } (*)$$

However, $\mathbf{b}^0$ is not unique.

Example 10.1 (continued) We can take

$$\mathbf{G} \; = \; \mathbf{G}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{G} \; = \; \mathbf{G}_2 = \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & \frac{4}{3} & 1 & 0 \\ -1 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that

$$\mathbf{b}_1^0 = \mathbf{G}_1\mathbf{X}'\mathbf{y} \; = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 504 \\ 300 \\ 172 \\ 32 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 100 \\ 86 \\ 32 \end{pmatrix}$$

$\hat{y}_i = 0 + 100 \cdot x_{i1} + 86\, x_{i2} + 32\, x_{i3}$

treatment effects — normal 100 — off-type 86 — aberrant 32

$\hat{y}_i = 32 + 68x_{i1} + 54x_{i2} + 0 \cdot x_{i3}$

normal off-type aberrant

treatment effects: $32+68 = 100$, $54+32 = 86$, 32

$$\mathbf{b}_2^0 = \mathbf{G}_2\mathbf{X}'\mathbf{y} = \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & \frac{4}{3} & 1 & 0 \\ -1 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 504 \\ 300 \\ 172 \\ 32 \end{pmatrix}$$

$$= \begin{pmatrix} 32 \\ 68 \\ 54 \\ 0 \end{pmatrix}$$

$$\Rightarrow \qquad \mathbf{b}_1^0 \neq \mathbf{b}_2^0$$

Remark 10.1

1°. Although $\hat{\underset{\sim}{b}}$ is not unique, the estimates of treatment effects are unique.

2°. $\underset{\sim}{b} = (b_0, b_1, b_2, b_3)'$ are not estimable, but b_0+b_1, b_0+b_2, b_0+b_3 are estimable. in addition, b_3-b_1, b_2-b_1, b_3-b_2 are estimable.

3°. In applications, we need some constraints for $\underset{\sim}{b}$.

Remedy over-parametrization

Remark 10.2

$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$

(1) $y_{ij} = \alpha_i + \varepsilon_{ij}$, no-intercept

treatment effect for the i-th group (level) — α_i

$(\alpha_1+\alpha_2+\alpha_3) = 0$

(2) $y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$ $\quad \sum \alpha_i = 0$

overall effect: μ

treatment effect for 1st group: $\mu+\alpha_1$

2nd ": $\mu+\alpha_2$

3rd ": $\mu+\alpha_3$

(3) $y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$

$\alpha_1 = 0$

treatment effect for the 1st group: μ

2nd group: $\mu+\alpha_2$

3rd ...: $\mu+\alpha_3$

$y = Xb + \varepsilon$

Notes:

$b^0 = (X'X)^- X'y$

$G = (X'X)^-$ not unique!

(1) $$E(\mathbf{b}^0) = E(\mathbf{G}\mathbf{X}'\mathbf{y}) \quad (\ \mathbf{G} \text{ is the generalized inverse of } (\mathbf{X}'\mathbf{X})\ )$$
$$= \mathbf{G}\mathbf{X}'E(\mathbf{y})$$
$$= \mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b}$$
$$= \mathbf{A}\mathbf{b} \qquad (\ \text{Let } \mathbf{A} = \mathbf{G}\mathbf{X}'\mathbf{X}\ )$$

$A = (X'X)^-(X'X)$

$(X'X)(X'X)^-(X'X) = (X'X)$

$\Rightarrow$ $\mathbf{b}^0$ is an unbiased estimator of $\mathbf{A}\mathbf{b}$

NOT $\mathbf{b}$

(2) $$\mathrm{Var}(\mathbf{b}^0) = \mathrm{Var}(\mathbf{G}\mathbf{X}'\mathbf{y})$$
$$= \mathbf{G}\mathbf{X}'\mathrm{Var}(\mathbf{y})\mathbf{X}\mathbf{G}'$$
$$= \mathbf{G}\mathbf{X}'(\sigma^2\mathbf{I})\mathbf{X}\mathbf{G}'$$
$$= \mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{G}'\sigma^2$$

(3) $$\hat{\mathbf{y}} = \mathbf{x}\mathbf{b}^0$$
$$= \mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{y}$$

unique

$XGX' = X(X'X)^- X'$ is invariant for any g-inverse of $(X'X)$ / Chap 3

Since $\mathbf{X}\mathbf{G}\mathbf{X}'$ is invariant to the choice of $\mathbf{G}$

$\Rightarrow$ consistent values of $\hat{\mathbf{y}}$ with different $\mathbf{G}$

(4) $$E(\hat{\mathbf{y}}) = E(\mathbf{X}\mathbf{b}^0)$$
$$= \mathbf{X}E(\mathbf{b}^0)$$
$$= \mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b}$$
$$= \mathbf{X}\mathbf{b} \qquad (\ \Rightarrow \text{invariant to } \mathbf{G}\ )$$

$X(X'X)^- X'X = X$

Chap 3

$$(5) \qquad \text{SSE} = (\mathbf{y}-\widehat{\mathbf{y}})'(\mathbf{y}-\widehat{\mathbf{y}})$$

$$= (\mathbf{y}-\mathbf{X}\mathbf{b}^0)'(\mathbf{y}-\mathbf{X}\mathbf{b})$$

$$= (\mathbf{y}-\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{y})'(\mathbf{y}-\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{y})$$

$$= \mathbf{y}'(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')'(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{y}$$

$X(X'X)^- X'$

$$= \mathbf{y}'(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')'\mathbf{y} \qquad (\mathbf{X}\mathbf{G}\mathbf{X}' \text{ is symmetric})$$

$$(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}' \text{ is idempotent})$$

$$(\Rightarrow \text{ invariant to } \mathbf{G})$$

$$(6) \qquad \text{SSR} = \text{SST} - \text{SSE}$$

$J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$

$$= \mathbf{y}'(\mathbf{I}-\tfrac{\mathbf{J}}{n})\mathbf{y} - \mathbf{y}'(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{y}$$

$$= \mathbf{y}'(\mathbf{X}\mathbf{G}\mathbf{X}'-\tfrac{\mathbf{J}}{n})\mathbf{y} \qquad (\text{ invariant to } \mathbf{G})$$

$y \sim N(\; , \;)$

$$(7) \qquad \text{E(SSE)} = \text{E}[\mathbf{y}'(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{y}]$$

$$= tr[(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{I}\sigma^2] + \mathbf{b}'\mathbf{X}'\underbrace{(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{X}}_{=0}\mathbf{b}$$

chap 6

$$= \sigma^2 tr(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')$$

$$= \sigma^2 r(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')$$

$$= \sigma^2(n - r(\mathbf{X}))$$

$$\Rightarrow \quad E\left(\frac{SSE}{n-r(\mathbf{X})}\right) = \sigma^2$$

$$\Rightarrow \quad \widehat{\sigma}^2 = \frac{SSE}{n-r(\mathbf{X})} \text{ is an unbiased estimator of } \sigma^2$$

$$(8) \qquad \text{SST} = \mathbf{y}'\mathbf{y} - \mathbf{y}'\frac{\mathbf{1}\mathbf{1}'}{n}\mathbf{y}$$

$$= \mathbf{y}'(\mathbf{I}-\tfrac{\mathbf{1}\mathbf{1}'}{n})\mathbf{y}$$

10.2 Distributional Properties

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \boldsymbol{\varepsilon} \; , \qquad \boldsymbol{\varepsilon} \sim N(\; \mathbf{0}, \; \sigma^2 \mathbf{I} \;)$$

(1) $\mathbf{y} \sim N(\; \mathbf{X}\mathbf{b}, \; \sigma^2 \mathbf{I} \;)$

(2) $\mathbf{b}^0 = \mathbf{G}\mathbf{X}'\mathbf{y} \sim N(\; \mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b}, \; \mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{G}'\sigma^2 \;)$

(3) $\mathbf{b}^0$ and $\widehat{\sigma}^2$ are independent

$$\begin{aligned} \mathbf{b}^0 &= \mathbf{G}\mathbf{X}'\mathbf{y} \\ SSE &= \mathbf{y}'(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{y} \end{aligned}$$

$$\begin{aligned} &\mathbf{G}\mathbf{X}'(\mathbf{I}\sigma^2)(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}') \\ &= \sigma^2 \mathbf{G}\mathbf{X}'(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}') \;=\; 0 \end{aligned}$$

Theorem 6.2

$\Rightarrow$ $\mathbf{b}^0$ and $\widehat{\sigma}^2$ are independent.

(4) $$\frac{SSE}{\sigma^2} \sim \chi^2$$

$$\frac{SSE}{\sigma^2} = \frac{\mathbf{y}'(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{y}}{\sigma^2}$$

Theorem 6.1

But

$$\frac{(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')}{\sigma^2}\,\mathbf{I}\sigma^2 = \mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}' \quad \text{is idempotent}$$

and

$$\begin{aligned} rank\Big(\frac{\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}'}{\sigma^2}\Big) &= rank(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}') \\ &= n - r(\mathbf{X}) \; , \end{aligned}$$

$$\Rightarrow \quad \frac{SSE}{\sigma^2} \sim \chi^2_{(n-r(\mathbf{X}), \;\; \frac{1}{2\sigma^2}\boldsymbol{\beta}'\mathbf{X}'(\mathbf{I}-\mathbf{X}\mathbf{G}\mathbf{X}')\mathbf{X}\mathbf{b})}$$

But $$\frac{1}{2\sigma^2}\mathbf{b}'\mathbf{X}(\mathbf{I}-\mathbf{XGX}')\mathbf{Xb} = 0$$

$\Rightarrow$ $$\frac{SSE}{\sigma^2} \sim \chi^2_{(n-r(\mathbf{X}))}$$

(5) $$\text{SSR} = \mathbf{y}'(\mathbf{XGX}' - \tfrac{\mathbf{J}}{n}))\mathbf{y}$$

$\Rightarrow$ $$\frac{SSR}{\sigma^2} = \mathbf{y}'\frac{(\mathbf{XGX}'-\frac{\mathbf{J}}{n})}{\sigma^2}\mathbf{y}$$

and $$\frac{(\mathbf{XGX}'-\frac{\mathbf{J}}{n})}{\sigma^2}\,\sigma^2\mathbf{I} \quad \text{is idempotent.}$$

$$\text{rank}\Big(\frac{(\mathbf{XGX}'-\frac{\mathbf{J}}{n})}{\sigma^2}\Big) = r(\mathbf{X}) - 1$$

$$\frac{1}{2\sigma^2}\,\mathbf{b}'\mathbf{X}'(\mathbf{XGX}' - \tfrac{\mathbf{J}}{n})\mathbf{Xb} = \frac{1}{2\sigma^2}\,\mathbf{b}'\mathbf{X}'(\mathbf{I} - \tfrac{\mathbf{J}}{n})\mathbf{Xb}$$

$\Rightarrow$ $$\frac{SSR}{\sigma^2} \sim \chi^2_{(\ r(\mathbf{X})-1,\ \frac{1}{2\sigma^2}\mathbf{b}'\mathbf{X}'(\mathbf{I}-\frac{\mathbf{J}}{n})\mathbf{Xb})}$$

(6) SSE and SSR are independent

Since $(\mathbf{XGX}' - \frac{\mathbf{J}}{n})\mathbf{I}\sigma^2(\mathbf{I} - \mathbf{XGX}') = 0$

$\Rightarrow$ SSE and SSR are independent

(7)

$$F(R) = \frac{SSR/(r(\mathbf{X})-1)}{SSE/(n-r(\mathbf{X}))} \sim F_{(\ r(\mathbf{X})-1,\ n-r(\mathbf{X}),\ \frac{1}{2\sigma^2}\boldsymbol{\beta}'\mathbf{X}'(\mathbf{I}-\frac{\mathbf{11}'}{n})\mathbf{Xb})}$$

$\tilde{q}'\tilde{b}$ is estimable $\Longleftrightarrow$ there is $\tilde{t}$, s.t. $E(\tilde{t}'\tilde{y}) = \tilde{q}'\tilde{b}$

ex) 10.1 $\mu+\alpha_1 = (1\ 1\ 0\ 0)\begin{pmatrix}\mu\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$

$\tilde{q}' = (1\ 1\ 0\ 0)$, $\tilde{t}' = (1\ 0\ 0\ 0\ 0\ 0\ 0)$

$E(\tilde{t}'\tilde{y}) = E(y_1) = \mu+\alpha_1 = \tilde{q}'\cdot\tilde{b}$

$\Rightarrow \mu+\alpha_1$ is estimable !

10.3 Estimable Functions

– The parametric function $\mathbf{q}'\mathbf{b}$ is said to be estimable if it has a linear unbiased estimate, $\mathbf{t}'\mathbf{y}$ say.

$\Rightarrow$ if $\mathbf{q}'\mathbf{b}$ is estimable, there exist $\mathbf{t}$ such that

$$\begin{aligned} & E(\mathbf{t}'\mathbf{y}) &=& \ \mathbf{q}'\mathbf{b} \\ \Rightarrow\quad & \mathbf{t}'E(\mathbf{y}) &=& \ \mathbf{q}'\mathbf{b} \\ \Rightarrow\quad & \mathbf{t}'\mathbf{X}\mathbf{b} &=& \ \mathbf{q}'\mathbf{b} \qquad (*) \end{aligned}$$

$\mu+\alpha_1$
$\mu+\alpha_2$
$\mu+\alpha_3$
$\alpha_3-\alpha_1$
$\alpha_3-\alpha_2$
$\alpha_2-\alpha_1$
estimable

Since $(*)$ is true for all $\mathbf{b}$,

$\tilde{q}'\tilde{b}$ is estimable $\Longleftrightarrow$

$$\Rightarrow \quad \mathbf{t}'\mathbf{X} = \mathbf{q}'$$

– The b.l.u.e. of the estimable function $\mathbf{q}'\mathbf{b}$ is $\mathbf{q}'\mathbf{b}^0$

(i) $\mathbf{q}'\mathbf{b}^0 = \mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{y}$

$\Rightarrow$ linear function of y_i

$\tilde{b}^0 = (\tilde{X}'\tilde{X})^- \tilde{X}'\tilde{Y}$

(ii)
$$\begin{aligned} E(\mathbf{q}'\mathbf{b}^0) &= \mathbf{q}'E(\mathbf{b}^0) \\ &= \mathbf{q}'\mathbf{G}\mathbf{X}'E(\mathbf{y}) \\ &= \mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b} \\ &= \mathbf{t}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{t}'\mathbf{X}\mathbf{b} = \mathbf{q}'\mathbf{b} \end{aligned}$$

$\Rightarrow$ unbiased estimator

(iii) Minimum variance. (Best) BLUE

$$
\begin{aligned}
var(\mathbf{q}'\mathbf{b}^0) &= \mathbf{q}'var(\mathbf{b}^0)\mathbf{q} \\
&= \mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{G}'\mathbf{q}\sigma^2 \\
&= \mathbf{t}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{G}'\mathbf{X}'\mathbf{t}\sigma^2 \\
&= \mathbf{t}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{t}\sigma^2 \\
&= \mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2
\end{aligned}
$$

Suppose $\mathbf{k}'\mathbf{y}$ is another linear unbiased estimator of $\mathbf{q}'\mathbf{b}$ different from $\mathbf{q}'\mathbf{b}^0$.

$$\Rightarrow \quad E(\mathbf{k}'\mathbf{y}) = \mathbf{q}'\mathbf{b} \qquad \Rightarrow \quad \mathbf{k}'\mathbf{X} = \mathbf{q}'$$

$$
\begin{aligned}
\Rightarrow \quad & cov(\mathbf{q}'\mathbf{b}^0,\ \mathbf{k}'\mathbf{y}) \\
&= cov(\mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{y},\ \mathbf{k}'\mathbf{y}) \\
&= \mathbf{q}'\mathbf{G}\mathbf{X}'(\mathbf{I}\sigma^2)\mathbf{k} \\
&= \mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2
\end{aligned}
$$

Now, $var(\mathbf{q}'\mathbf{b}^0 - \mathbf{k}'\mathbf{y})$

$$
\begin{aligned}
&= var(\mathbf{q}'\mathbf{b}^0) + var(\mathbf{k}'\mathbf{y}) - 2cov(\mathbf{q}'\mathbf{b}^0, \mathbf{k}'\mathbf{y}) \\
&= var(\mathbf{k}'\mathbf{y}) + \mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2 - 2\mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & var(\mathbf{q}'\mathbf{b}^0 - \mathbf{k}'\mathbf{y}) \\
&= var(\mathbf{k}'\mathbf{y}) - var(\mathbf{q}'\mathbf{b}^0) \ \geq \ 0
\end{aligned}
$$

$$\Rightarrow \quad var(\mathbf{k}'\mathbf{y}) \ \geq \ var(\mathbf{q}'\mathbf{b}^0)$$

$\Rightarrow$ $\mathbf{q}'\mathbf{b}^0$ is B.L.U.E. of $\mathbf{q}'\mathbf{b}$

<u>Note</u> $\mathbf{q}'\mathbf{b}^0 \sim N(\ \mathbf{q}'\mathbf{b},\ \mathbf{q}'\mathbf{G}\mathbf{q}\sigma^2)$

10.3.1 Test of Estimability

$\tilde{A} = (\tilde{X}'\tilde{X})^{-}(\tilde{X}'\tilde{X})$ for any g-inverse.

$\mathbf{q}'\mathbf{b}$ is estimable if and only if $\mathbf{q}'\mathbf{A} = \mathbf{q}'$ where $\mathbf{A} = \mathbf{G}\mathbf{X}'\mathbf{X}$

Proof: If $\mathbf{q}'\mathbf{b}$ is estimable, there exist a vector $\mathbf{t}$ such that $\mathbf{t}'\mathbf{X} = \mathbf{q}'$

$$
\begin{aligned}
\Rightarrow \quad \mathbf{q}'\mathbf{A} &= \mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{X} \\
&= \mathbf{t}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X} \\
&= \mathbf{t}'\mathbf{X} = \mathbf{q}'
\end{aligned}
$$

Ex 10.1

$$(\tilde{X}'\tilde{X}) = \begin{pmatrix} 6 & 3 & 2 & 1 \\ 3 & 3 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

If $\mathbf{q}'\mathbf{A} = \mathbf{q}'$

$\Rightarrow \quad \mathbf{q}' = \mathbf{q}'\mathbf{G}\mathbf{X}'\mathbf{X}$

$$G_1 = (\tilde{X}'\tilde{X})^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow \quad$ take $\mathbf{t}' = \mathbf{q}'\mathbf{G}\mathbf{X}'$

we have $\mathbf{q}' = \mathbf{t}'\mathbf{X}$ $\quad \mu + \alpha_1 = (1\ 1\ 0\ 0)\begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad \tilde{q}' = (1\ 1\ 0\ 0)$

$\Rightarrow \quad \mathbf{q}'\mathbf{b}$ is estimable.

$$\tilde{q}'G_1 = (1\ 1\ 0\ 0)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0\ \tfrac{1}{3}\ 0\ 0)$$

$$\tilde{q}'\tilde{A} = \tilde{q}'G_1(\tilde{X}'\tilde{X}) = (1\ 1\ 0\ 0) = \tilde{q}'$$

$\Rightarrow \tilde{q}'\tilde{b} = \mu + \alpha_1$ is estimable.

Example 10.2: Consider the normal equations

$$(\mathbf{X}'\mathbf{X})\mathbf{b} = \mathbf{X}'\mathbf{y}$$

$$\text{where} \qquad (\mathbf{X}'\mathbf{X}) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad , \quad \mathbf{X}'\mathbf{y} = \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix}$$

One possible generalized inverse is

$$\Rightarrow \qquad \mathbf{G} \; = \; (\mathbf{X}'\mathbf{X})^{-1} \; = \; \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{b}^0 \; &= \; \mathbf{G}\mathbf{X}'\mathbf{y} \; = \; \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ -2 \\ 0 \end{pmatrix} \end{aligned}$$

Another generalized inverse is

$$\mathbf{G}_1 \; = \; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{b}_1^0 \; = \; \mathbf{G}_1\mathbf{X}'\mathbf{y} \; = \; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix} \; = \; \begin{pmatrix} 0 \\ 6 \\ 8 \end{pmatrix}$$

– SSR (by $\mathbf{G}$)

$$= \mathbf{y}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{y}$$

$$= \begin{pmatrix} 14 & 6 & 8 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix}$$

$$= 100$$

– $\mathbf{A} = \mathbf{G}_1\mathbf{X}'\mathbf{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

– Is $\beta_1 - \beta_2$ estimable?

$$\mathbf{b} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\mathbf{q}' = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{q}'\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} = \mathbf{q}'$$

$\Rightarrow$ $\beta_1 - \beta_2$ is estimable.

– Is $\beta_1 + \beta_2$ estimable?

$$\mathbf{q}' = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{q}'\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \neq \mathbf{q}'$$

$\Rightarrow \quad \beta_1 + \beta_2$ is not estimable.

– Is $3\beta_0 - \beta_1 - 2\beta_2$ estimable?

$$\mathbf{q}' = \begin{pmatrix} 3 & -1 & -2 \end{pmatrix}$$

$$\mathbf{q}'\mathbf{A} = \begin{pmatrix} 3 & -1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -1 & -2 \end{pmatrix} \neq \mathbf{q}'$$

$\Rightarrow \quad 3\beta_0 - \beta_1 - 2\beta_2$ is not estimable.

10.4 Testable Hypothesis

$\underset{\sim}{a} = \underset{\sim}{b}$

- A hypothesis that can be expressed in terms of estimable functions.

- Let $H_0 : \mathbf{K}'\mathbf{a} = \boldsymbol{\mu}$ $\quad$ $[\mathbf{K}'$ is $r \times (k+1)]$

  $\underset{\sim}{y} = \underset{\sim}{X}\,\underset{\sim}{a} + \underset{\sim}{\varepsilon}$

  and $\mathbf{K}'\mathbf{a}$ is estimable

  $\mathbf{K}' = \mathbf{S}'\mathbf{X}'\mathbf{X}$ for some full row rank $\mathbf{S}'$

  $\mathbf{K}'\mathbf{a}^0$ is used to estimate $\mathbf{K}'\mathbf{a}$

  $E(\mathbf{K}'\mathbf{a}^0) = \mathbf{K}'\mathbf{a}$

  and $Var(\mathbf{K}'\mathbf{a}^0) = \mathbf{K}'\mathbf{G}\mathbf{K}\sigma^2$

  Note that $rank(\mathbf{K}') = rank(\mathbf{S}') = rank(\mathbf{S}'\mathbf{X}') = r$

  Also

$$\underset{\sim}{K}'(\underset{\sim}{X}'\underset{\sim}{X})^{-}(\underset{\sim}{X}'\underset{\sim}{X}) = K'$$

$$\begin{aligned} \mathbf{K}'\mathbf{G}\mathbf{K} &= \mathbf{S}'\mathbf{X}'\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X}\mathbf{S} \\ &= \mathbf{S}'\mathbf{X}'\mathbf{X}\mathbf{S} \\ &= (\mathbf{S}'\mathbf{X}')(\mathbf{S}'\mathbf{X}')' \end{aligned}$$

$= \underset{\sim}{S}'$

  $rank(\mathbf{K}'\mathbf{G}\mathbf{K}) = rank(\mathbf{S}'\mathbf{X}') = r$

  $\Rightarrow$ $\mathbf{K}'\mathbf{G}\mathbf{K}$ is nonsingular

$r = 2$

Ex 10.1.

$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$

$H_0: \alpha_1 = \alpha_2 = \alpha_3$

$\Leftrightarrow H_0: \alpha_1 - \alpha_2 = 0,\ \alpha_1 - \alpha_3 = 0$

$\Leftrightarrow \underset{\sim}{K}' = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

$a = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

$H_0: \underset{\sim}{K}'\underset{\sim}{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hypothesis testing

$\underset{\sim}{a}^0 = (\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{Y}$

$K'a$ is estimable

$\Rightarrow \underset{\sim}{K}'\underset{\sim}{a}^0$ is unique.

$H_0 : \mathbf{K}'\mathbf{a} = \boldsymbol{\mu}$ (let # of rows in $\mathbf{K}' = s$)

$\mathbf{y} \sim N(\mathbf{Xa}, \boldsymbol{\sigma}^2\mathbf{I})$

$\mathbf{a}^0 \sim N(\mathbf{GX}'\mathbf{Xa}, \mathbf{GX}'\mathbf{XG}'\boldsymbol{\sigma}^2)$

and $\mathbf{K}'\mathbf{a}^0 - \boldsymbol{\mu} \sim N(\mathbf{K}'\mathbf{a} - \boldsymbol{\mu}, \mathbf{K}'\mathbf{GK}\boldsymbol{\sigma}^2)$

Take

$$Q = (\mathbf{K}'\mathbf{a}^0 - \boldsymbol{\mu})'(\mathbf{K}'\mathbf{GK})^{-1}(\mathbf{K}'\mathbf{a}^0 - \boldsymbol{\mu})$$

Unique!

then

$$\frac{\mathbf{Q}}{\boldsymbol{\sigma}^2} \sim \chi^2_{(s,\ (\mathbf{K}'\mathbf{a}-\boldsymbol{\mu})'(\mathbf{K}'\mathbf{GK})^{-1}(\mathbf{K}'\mathbf{a}-\boldsymbol{\mu})/2\boldsymbol{\sigma}^2)}$$

Theorem 6.1

It is straightforward to show that

$$F(H) = \frac{Q/s}{SSE/(n - r(\mathbf{X}))} \sim F_{(s,\ n-r(\mathbf{X}),\ (\mathbf{K}'\mathbf{a}-\boldsymbol{\mu})'(\mathbf{K}'\mathbf{GK})^{-1}(\mathbf{K}'\mathbf{a}^0-\boldsymbol{\mu})/2\boldsymbol{\sigma}^2)}$$

Under $H_0 : \mathbf{K}'\mathbf{a} = \boldsymbol{\mu}$,

$$F(H) = \frac{Q/s}{SSE/(n - r(\mathbf{X}))} \sim F_{(s,\ n-r(\mathbf{X}))}$$

E.g. 6.1 $H_0: \alpha_1 = \alpha_2 = \alpha_3$, $\underset{\sim}{K}' = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

$\Leftrightarrow H_0: \underset{\sim}{K}'\underset{\sim}{a} = 0$ $\quad \underset{\sim}{a} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

$$Q = (\underset{\sim}{K}'\underset{\sim}{a}^0)'(\underset{\sim}{K}'(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{K})^{-1}(\underset{\sim}{K}'\underset{\sim}{a}^0)$$

$(\underset{\sim}{X}'\underset{\sim}{X})^{-} = \underset{\sim}{G}_1$ $\quad \underset{\sim}{a}^0 = (\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{Y}$

$F(H) = \dfrac{Q/2}{SSE/(n-3)} \overset{H_0}{\sim} F_{2,3}$

Reject H_0, if $F(H) \geq F_{0.05,2,3}$ OR

if P-value $= P(F_{2,3} \geq F(H)) < 0.05$

Under $H_0 : \mathbf{K}'\mathbf{a} = \boldsymbol{\mu}$

$$\mathbf{a}_H^0 = \mathbf{a}^0 - \mathbf{GK}(\mathbf{K}'\mathbf{GK})^{-1}(\mathbf{K}'\mathbf{a}^0 - \boldsymbol{\mu})$$

$$SSE_H = SSE + Q$$

where SSE = $\mathbf{y}$'($\mathbf{I}$- $\mathbf{XGX}$')$\mathbf{y}$

10.4.1 One - Way Layout (One factor (fixed) design)

Model :

$$y_{ij} = \mu + \alpha_i + \boldsymbol{\varepsilon}_{ij}, \qquad i = 1, \cdots, a; \quad j = 1, \cdots, n_i.$$

In matrix notation $\mathbf{y} = \mathbf{X}\mathbf{a} + \boldsymbol{\varepsilon}$, $\qquad \boldsymbol{\varepsilon} \sim N(0, \boldsymbol{\sigma}^2\mathbf{I})$

$$\mathbf{y} = \begin{pmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1n_1} \\ \cdots \\ \vdots \\ \cdots \\ y_{a_1} \\ y_{a_2} \\ \vdots \\ y_{an_a} \end{pmatrix} \qquad \mathbf{a} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_a \end{pmatrix} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \boldsymbol{\varepsilon}_{11} \\ \boldsymbol{\varepsilon}_{12} \\ \vdots \\ \boldsymbol{\varepsilon}_{1n_1} \\ \cdots \\ \vdots \\ \cdots \\ \boldsymbol{\varepsilon}_{a1} \\ \vdots \\ \boldsymbol{\varepsilon}_{an_a} \end{pmatrix}$$

$$
\mathbf{X} = \begin{array}{c}
\begin{matrix} \mu & \alpha_1 & \alpha_2 & \cdots & \alpha_a \end{matrix} \\
\begin{pmatrix}
1 & 1 & 0 & & 0 \\
1 & 1 & 0 & & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 1 & 0 & & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 1 & & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 0 & 0 & & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & & 1 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 0 & 0 & & 1
\end{pmatrix}
\end{array}
$$

$$
n = \sum_{i=1}^{a} n_i
$$

$$
\mathbf{X}'\mathbf{X} = \begin{pmatrix}
n & n_1 & n_2 & \cdots & n_a \\
n_1 & n_1 & 0 & \cdots & 0 \\
n_2 & 0 & n_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n_a & 0 & 0 & \cdots & n_a
\end{pmatrix}
$$

A generalized inverse is

$$\mathbf{G} = (\mathbf{X}'\mathbf{X})^- = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{n_1} & & 0 \\ \vdots & 0 & \frac{1}{n_2} & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{n_a} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \mathbf{D}\{\frac{1}{n_i}\} \end{pmatrix}$$

$$\begin{aligned} \mathbf{H} &= \mathbf{G}\mathbf{X}'\mathbf{X} \\ &= \begin{pmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \mathbf{D}\{\frac{1}{n_i}\} \end{pmatrix} \begin{pmatrix} n & n_1 & n_2 & \cdots & n_a \\ n_1 & n_1 & 0 & \cdots & 0 \\ n_2 & 0 & n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_a & 0 & 0 & \cdots & n_a \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{0}' \\ \mathbf{1}_a & \mathbf{I}_a \end{pmatrix} \\ \mathbf{X}'\mathbf{y} &= \begin{pmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ \vdots \\ y_{a.} \end{pmatrix} \end{aligned}$$

$$
\mathbf{a}^0 \;=\; \mathbf{G}\mathbf{X}'\mathbf{y} \;=\; \begin{pmatrix} 0 \\ \bar{y}_{1.} \\ \bar{y}_{2.} \\ \vdots \\ \bar{y}_{a.} \end{pmatrix}
$$

<u>Analysis of Variance</u>

1.

$$
\mathbf{a}^{0\prime}\mathbf{X}'\mathbf{y} \;=\; (0 \quad \bar{y}_{1.} \quad \bar{y}_{2.} \quad \cdots \quad \bar{y}_{a.}) \begin{pmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ \vdots \\ y_{a.} \end{pmatrix} = \mathbf{S}_1 \frac{y_{i.}^2}{n_i}
$$

2.

$$
SSR = \mathbf{a}^{0\prime}\mathbf{X}'\mathbf{y} - \mathbf{y}'\frac{\mathbf{J}}{n}\mathbf{y} = \mathbf{S}_1 \frac{y_{i.}^2}{n_i} - n\bar{y}_{..}^2
$$

3.

$$
SST = \mathbf{S}_1 \sum_{j=1}^{n_i} y_{ij}^2 - n\bar{y}_{..}^2
$$

4.

$$
\begin{aligned}
SSE \;&=\; SST - SSR \\
&=\; \mathbf{S}_1 \sum_{j=1}^{n_i} y_{ij}^2 - \mathbf{S}_1 \frac{y_{i.}^2}{n_i}
\end{aligned}
$$

Example 10.3

Three different treatment methods for removing organic carbon from tar sand wastewater are to be compared. The methods are airflotation (AF), foam seperation (FS), and ferric - chloride coagulation (FCC). The data are given as follows

AF(I)	FS(II)	FCC(III)
34.6	38.8	26.7
35.1	39.0	26.7
35.3	40.1	27.0
35.8	40.9	27.1
36.1	41.0	27.5
36.5	43.2	28.1
36.8	44.9	28.1
37.2	46.9	28.7
37.4	51.6	30.7
37.7	53.6	31.2

Model:

$$y_{ij} = \mu + T_i + \boldsymbol{\varepsilon}_{ij}, \qquad \boldsymbol{\varepsilon}_{ij} \ \sim \ N(0, \boldsymbol{\sigma}^2)$$

T_i: Effect of treatment i $\qquad\qquad$ $i = 1, 2, 3;\ \ j = 1, 2, \cdots, 10$

$$\mathbf{a} = \begin{pmatrix} \mu \\ T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} 30 & 10 & 10 & 10 \\ 10 & 10 & 0 & 0 \\ 10 & 0 & 10 & 0 \\ 10 & 0 & 0 & 10 \end{pmatrix}$$

A generalized inverse of $\mathbf{X}'\mathbf{X}$ is

$$(\mathbf{X}'\mathbf{X})^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{pmatrix}$$

$$\mathbf{a}^0 = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y} = \begin{pmatrix} 0 \\ 36.25 \\ 44 \\ 28.18 \end{pmatrix}$$

SST = 1530.19367
SSR =1251.53267
SSE = 278.661
MSE = $\frac{278.661}{27} = 10.32 = \hat{\boldsymbol{\sigma}}^2$

Let us test

$$H_0 : T_1 = T_2 = T_3$$

with reference to p.91 ,

$$\mathbf{K}' = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \qquad \boldsymbol{\mu} = \mathbf{0}$$

$$\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{K}$$
$$= \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \frac{1}{10} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{K})^{-1} = \frac{10}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

The test statistics is

$$
\begin{aligned}
F(H) &= \frac{Q/s}{\hat{\boldsymbol{\sigma}}^2} \\
&= \frac{(\mathbf{K}'\mathbf{a} - \boldsymbol{\mu})'(\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{K})^{-1}(\mathbf{K}'\mathbf{a} - \boldsymbol{\mu})/2}{10.32} \\
&= 60.63 \qquad\qquad (Q = 1251.533)
\end{aligned}
$$

Since p-value < 0.01, have strong evidence to reject H_0

Anova Table

Source of Variation	SS	df	MS	F
Regression	1251.53267	2	625.7665	60.63
Residual	278.661	27	10.32	
Total	1530.19367	29		