4 Random Vector and Matrices

- Expectation: Let **Y** and **X** be $p \times 1$ random vectors. The expected value of

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{pmatrix} \quad \text{is given by} \quad E(\boldsymbol{Y}) = \begin{pmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_p) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix} = \boldsymbol{\mu}$$

Note: $E(\boldsymbol{X}+\boldsymbol{Y}) = E(\boldsymbol{X}) + E(\boldsymbol{Y})$ (where $E(Y_i)=\mu_i$ is obtained as $E(Y_i)=\int y_i f_i(y_i)dy_i$)

- Covariance Matrix:

$$\boldsymbol{\Sigma} = Cov(\boldsymbol{Y}) = E\{[\boldsymbol{Y}-E(\boldsymbol{Y})][\boldsymbol{Y}-E(\boldsymbol{Y})]'\} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{pmatrix}$$

Note: 1°. The $\underset{\sim}{\Sigma}$ is symmetric since $\sigma_{ij}=\sigma_{ji}$

2°. In many applications, $\underset{\sim}{\Sigma}$ is assumed to be positive definite

$\boxed{\begin{array}{l} \text{— Define the } E \text{ of a random matrix } \underset{\sim}{Z} \text{ as the matrix of expected values:} \\ E(\underset{\sim}{Z}) = E\begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1n} \\ Z_{21} & Z_{22} & \cdots & Z_{2n} \\ \vdots & \vdots & & \vdots \\ Z_{n1} & Z_{n2} & \cdots & Z_{nn} \end{pmatrix} = \begin{pmatrix} E(Z_{11}) & E(Z_{12}) & \cdots & E(Z_{1n}) \\ E(Z_{21}) & E(Z_{22}) & \cdots & E(Z_{2n}) \\ \vdots & \vdots & & \vdots \\ E(Z_{n1}) & E(Z_{n2}) & \cdots & E(Z_{nn}) \end{pmatrix} \end{array}}$

Theorem 3.6(a). If $\underset{\sim}{a}$ is a $p\times1$ vector of constant and $\underset{\sim}{y}$ is a $p\times1$ random vector with mean vector $\underset{\sim}{\mu}$,

then. $\mu_a = E(\underset{\sim}{a}'\underset{\sim}{y}) = \underset{\sim}{a}'E(\underset{\sim}{y}) = \underset{\sim}{a}'\underset{\sim}{\mu}$

Theorem 3.6(b) Suppose that $\underset{\sim}{y}$ is a random vector, $\underset{\sim}{X}$ is a random matrix. $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are vectors of constants,

and $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are matrices of constants.

(i). $E(\underset{\sim}{A}\underset{\sim}{y}) = \underset{\sim}{A}E(\underset{\sim}{y})$

(ii) $E(\underset{\sim}{a}'\underset{\sim}{X}\underset{\sim}{b}) = \underset{\sim}{a}'E(\underset{\sim}{X})\underset{\sim}{b}$

(iii) $E(\underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{B}) = \underset{\sim}{A}E(\underset{\sim}{X})\underset{\sim}{B}$

$\Rightarrow$ Corollary: $E(\underset{\sim}{A}\underset{\sim}{y}+\underset{\sim}{b}) = \underset{\sim}{A}E(\underset{\sim}{y}) + \underset{\sim}{b}$

* sample mean and sample covariance matrix.

get some observations. $\underset{\sim}{x}_i = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{ip} \end{pmatrix}$ $i=1,\cdots,n$

$\Rightarrow \hat{\underset{\sim}{\mu}} = \overline{\underset{\sim}{x}} = \frac{1}{n}\sum_{i=1}^{n}\underset{\sim}{x}_i$

$\hat{\underset{\sim}{\Sigma}} = \underset{\sim}{S} = \frac{1}{n}\sum_{i=1}^{n}(\underset{\sim}{x}_i - \overline{\underset{\sim}{x}})(\underset{\sim}{x}_i - \overline{\underset{\sim}{x}})'$

Pf: $= E[(\underset{\sim}{A}\underset{\sim}{Y} - E(\underset{\sim}{A}\underset{\sim}{Y}))\cdot(\underset{\sim}{A}\underset{\sim}{Y} - E(\underset{\sim}{A}\underset{\sim}{Y}))']$
$= E[\underset{\sim}{A}(\underset{\sim}{Y} - E(\underset{\sim}{Y}))\cdot(\underset{\sim}{Y} - E(\underset{\sim}{Y}))'\underset{\sim}{A}']$
$= \underset{\sim}{A}E[(\underset{\sim}{Y} - E(\underset{\sim}{Y}))\cdot(\underset{\sim}{Y} - E(\underset{\sim}{Y}))']\underset{\sim}{A}'$
$= \underset{\sim}{A}Cov(\underset{\sim}{Y})\underset{\sim}{A}'$

- Let $\boldsymbol{A}$ be a constant matrix, then

$$Cov(\boldsymbol{AY}) = \boldsymbol{A}[Cov\boldsymbol{Y}]\boldsymbol{A}'$$

Pf: $= E[(\underset{\sim}{A}\underset{\sim}{X} - E(\underset{\sim}{A}\underset{\sim}{X}))\cdot(\underset{\sim}{B}\underset{\sim}{Y} - E(\underset{\sim}{B}\underset{\sim}{Y}))']$
$= E[\underset{\sim}{A}(\underset{\sim}{X} - E(\underset{\sim}{X}))\cdot(\underset{\sim}{Y} - E(\underset{\sim}{Y}))'\underset{\sim}{B}']$
$= \underset{\sim}{A}Cov(\underset{\sim}{X},\underset{\sim}{Y})\underset{\sim}{B}'$

- Let $\boldsymbol{A}, \boldsymbol{B}$ be constant matrices, then

$$Cov(\boldsymbol{AX}, \boldsymbol{BY}) = \mathbf{A}Cov(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{B}'$$

- Generalized variance: overall measure of variability can be defined as the determinant of $\underset{\sim}{\Sigma}$.

$$\text{i.e. Generalized variance} = |\boldsymbol{\Sigma}|$$

- Correlation matrices

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{pmatrix}$$

where

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}}$$

for $i \neq j$.

- Partitioned random vectors

$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{pmatrix}$$

$$\boldsymbol{\mu} = E(\boldsymbol{V}) = E\begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} E(\boldsymbol{Y}) \\ E(\boldsymbol{X}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{\boldsymbol{Y}} \\ \boldsymbol{\mu}_{\boldsymbol{X}} \end{pmatrix}$$

$$\boldsymbol{\Sigma} = Cov(\boldsymbol{V}) = Cov\begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{YY}} & \boldsymbol{\Sigma}_{\boldsymbol{YX}} \\ \boldsymbol{\Sigma}_{\boldsymbol{XY}} & \cancel{\boldsymbol{\Sigma}_{\boldsymbol{YX}}} \end{pmatrix}$$

Σ_{XX}

$$= E\left[\begin{pmatrix} \vec{Y}-E(\vec{Y}) \\ \vec{X}-E(\vec{X}) \end{pmatrix}\begin{pmatrix} \vec{Y}-E(\vec{Y}) \\ \vec{X}-E(\vec{X}) \end{pmatrix}'\right]$$

$$= E\left[\begin{pmatrix} \vec{Y}-E(\vec{Y}) \\ \vec{X}-E(\vec{X}) \end{pmatrix}\left((\vec{Y}-E(\vec{Y}))' \ (\vec{X}-E(\vec{X}))'\right)\right]$$

Σ^{-1} (use formulation discussed in ch3)

- Let $\boldsymbol{Y}$ be a random vector with mean $\boldsymbol{\mu} = E(\boldsymbol{Y})$ and $\boldsymbol{\Sigma} = Cov(\boldsymbol{Y})$, then $E(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}) = tr(\boldsymbol{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$ where $\boldsymbol{A}$ is a symmetric matrix.

$$\left(\begin{aligned} &= E[(\underset{\sim}{Y}'\underset{\sim}{A} - \underset{\sim}{\mu}'\underset{\sim}{A})(\underset{\sim}{Y} - \underset{\sim}{\mu})] \\ &= E(\underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y} - \underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{Y} - \underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{\mu} + \underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu}) \\ &= E(\underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y}) - \mu'A\mu - \underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu} + \underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu} \\ &= E(\underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y}) - \underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu} \end{aligned}\right)$$

since $E(\underset{\sim}{Y} - \underset{\sim}{\mu})'A(\underset{\sim}{Y} - \underset{\sim}{\mu}) = E(\underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y}) - \underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu}$

then

$$E(\underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y}) = E(\underset{\sim}{Y} - \underset{\sim}{\mu})'\underset{\sim}{A}(\underset{\sim}{Y} - \underset{\sim}{\mu}) + \underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu}$$

$\underbrace{E(\underset{\sim}{Y} - \underset{\sim}{\mu})'}_{1\times p}\underbrace{\underset{\sim}{A}}_{p\times p}\underbrace{(\underset{\sim}{Y} - \underset{\sim}{\mu})}_{p\times 1}$ — quadratic form

$\boxed{tr(AB) = tr(BA)}$

$$\begin{aligned} &= tr[E(\underset{\sim}{Y} - \underset{\sim}{\mu})'\underset{\sim}{A}(\underset{\sim}{Y} - \underset{\sim}{\mu})] \\ &= E\ tr[(Y - \mu)'\underset{\sim}{A}(\underset{\sim}{Y} - \underset{\sim}{\mu})] \\ &= E\ tr[\underset{\sim}{A}(\underset{\sim}{Y} - \underset{\sim}{\mu})(\underset{\sim}{Y} - \underset{\sim}{\mu})'] \\ &= tr\{\underset{\sim}{A}\,E[(\underset{\sim}{Y} - \underset{\sim}{\mu})(\underset{\sim}{Y} - \underset{\sim}{\mu})']\} \\ &= tr\,\underset{\sim}{A}\underset{\sim}{\Sigma} \end{aligned}$$

- MGF: The moment generating function of a random vector $\boldsymbol{Y}$ is given by

$$M_{\mathbf{Y}}(\mathbf{t}) = E(e^{\mathbf{t}'\boldsymbol{Y}})$$

where $\mathbf{t} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}$ if the expectation exists

for $-h < t_i < h$ where $h > 0$ and $i = 1, \cdots, n$

- Theorem
Let $g_1(\boldsymbol{Y}_1), \cdots, g_m(\boldsymbol{Y}_m)$ be m functions of the random vectors $\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_m$, respectively. If $\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_m$ are mutually independent, then $g_1, \cdots, g_m$ are mutually independent.