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Research Article

Extension of The Best Approximation Operator in Orlicz Spaces

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Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $\mathcal{L} \subset \mathcal{A}$ a sub-σ-lattice of the σ-algebra $\mathcal{A}$. We study an extension of the best ϕ-approximation operator from an Orlicz space L^{ϕ} to the space $L^{\phi'}$, where ϕ' denotes the derivative of the convex, but not necessarily a strictly convex function ϕ. We obtain convergence results when a sequence of σ-algebras $\mathcal{B}_n$ converges to $\mathcal{B}_\infty$ in a suitable way.

1. Introduction and notations

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathcal{M} = \mathcal{M}(\Omega, \mathcal{A}, \mu)$ be the set of all $\mathcal{A}$-measurable real-valued functions defined on Ω. Given a C^1 convex function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(0) = 0$, $\phi(t) > 0$, when $t > 0$, let $L^{\phi} = L^{\phi}(\Omega, \mathcal{A}, \mu)$ be the space of all functions $f \in \mathcal{M}$ such that

$$\int_{\Omega} \phi(\lambda |f|) d\mu < \infty, \tag{1.1}$$

for some $\lambda > 0$. Since we only deal with a Δ_2 function ϕ, that is, there exists a constant $K > 0$ such that $\phi(2x) \leq K\phi(x)$ for all $x \geq 0$, the space L^{ϕ} can be defined as the space of all function $f \in \mathcal{M}$, where (1.1) holds for every positive number λ. The space $L^{\phi'}$ is analogously defined, where ϕ' is the derivative of the function ϕ. Besides, observe that for a Δ_2 function ϕ it holds the next inequality

$$\phi(x) \leq x\phi'(x) \leq \phi(2x) \leq K\phi(x) \tag{1.2}$$

for all $x \geq 0$, and therefore $L^{\phi} \subset L^{\phi'}$. Moreover φ is a Δ_2 function if and only if φ' is a Δ_2 function.

We say, according to [1], that a collection $\mathcal{L}$ of sets in $\mathcal{A}$ is a σ-lattice if it is closed under countable unions and intersections and contains $\varnothing$ and Ω. Given a σ-lattice $\mathcal{L}$, we denote by $\overline{\mathcal{L}}$ the σ-lattice of all the complementary sets of $\mathcal{L}$, that is, $\overline{\mathcal{L}} = \{A^c : A \in \mathcal{L}\}$. Denote by $L^{\phi}(\mathcal{L})$ all $\mathcal{L}$-measurable functions in L^{ϕ}.

A set $C \subset L^{\phi}$ is called ϕ-closed if and only if $f_n \in C$ and $f_n \nearrow f \in L^{\phi}$ or $(f_n \searrow f \in L^{\phi})$ then $f \in C$. Then $L^{\phi}(\mathcal{L})$ is a ϕ-closed convex set and a lattice, that is closed for the maximum and minimum of functions.

We will use the notation $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$. A set $C \subset L^{\phi}$ is called a σ-complete lattice if and only if $\bigwedge_{n\in\mathbb{N}} f_n = \inf_{n\in\mathbb{N}} f_n \in C$ and $\bigvee_{n\in\mathbb{N}} f_n = \sup_{n\in\mathbb{N}} f_n \in C$ for all sequence $\{f_n\}_{n\in\mathbb{N}} \subset C$.

It is well known, see [1], that for every $f \in L^{\phi}$ there exists an element $g \in L^{\phi}(\mathcal{L})$ such that

$$\int_{\Omega} \phi(|f-g|)d\mu = \inf_{h\in L^{\phi}(\mathcal{L})} \int_{\Omega} \phi(|f-h|)d\mu. \tag{1.3}$$

Denote by $\mu_{\phi}^{\mathcal{L}}(f)$ the set of all $g \in L^{\phi}(\mathcal{L})$ satisfying (1.3). Each element of $\mu_{\phi}^{\mathcal{L}}(f)$ will be called a best ϕ-approximation of f given $L^{\phi}(\mathcal{L})$, and we will refer to the mapping $f \to \mu_{\phi}^{\mathcal{L}}(f)$ defined on L^{φ} as the best approximation operator.

It is showed in [1] that for $f \in L^{\phi}$ the set $\mu_{\phi}^{\mathcal{L}}(f)$ is a nonvoid σ-complete lattice. Also it was proved that if $f \le g$, both in L^{ϕ}, $f_1 \in \mu_{\phi}^{\mathcal{L}}(f)$ and $g_1 \in \mu_{\phi}^{\mathcal{L}}(g)$ then we have $f_1 \wedge f_2 \in \mu_{\phi}^{\mathcal{L}}(f)$ and $f_1 \vee f_2 \in \mu_{\phi}^{\mathcal{L}}(g)$. In this case, we say that the multivalued operator $\mu_{\phi}^{\mathcal{L}}(f)$ is a monotone operator.

The main purpose of this paper is to extend the best approximation operator to the set $L^{\varphi'}$. The case $\varphi(t) = t^p$, $p > 1$, was extensively treated in [2] and the best L^1 approximation operator is extended to all measurable functions in [3]. The extension from L^{φ} to $L^{\varphi'}$ is considered in [4] for a C^1 function φ which is strictly convex and $\varphi'(0) = 0$.

Now in this paper we consider a C^1 convex function ϕ, $\phi'(0) = 0$, but not necessarily a strictly convex function. Extension of best approximation operator when the approximation classes are the constants is treated in [5–7].

The extension of the best approximation operator is $\mu_{\phi}^{\mathcal{L}}(f)$; for $f \in L^{\phi'}$, we will be denoted by $\tilde{\mu}_{\phi}^{\mathcal{L}}(f)$. In Theorem 2.12, we prove that $\tilde{\mu}_{\phi}^{\mathcal{L}}(f) \neq \varnothing$ for every $f \in L^{\phi'}$, while in Theorem 2.16 it is proved that it is indeed an extension, that is, $\tilde{\mu}_{\phi}^{\mathcal{L}}(f) = \mu_{\phi}^{\mathcal{L}}(f)$ for $f \in L^{\phi}$. Additional properties are obtained for the set $\tilde{\mu}_{\phi}^{\mathcal{L}}(f)$ when the σ-lattice $\mathcal{L}$ is a σ-algebra (see Theorem 2.17) and similar results hold when $\mathcal{L}$ is the class of monotone functions in $L^{\phi'}$ (see Theorem 3.2). A martingale-type result is given in Theorem 4.1 which generalizes [8, Theorem 2.8] for the particular family of measures considered in this paper.

2. Extension of the best ϕ-approximation operator

We begin with some definitions and auxiliary results. The proof of the next two lemmas can be found in [4].

Lemma 2.1. *A necessary and sufficient condition for a continuously differentiable and convex function ϕ to satisfy the Δ_2 condition is that there exists a constant $\alpha > 1$ such that*

$$u\phi'(u) \le \alpha\, \phi(u) \quad \forall u \ge 0. \tag{2.1}$$

Lemma 2.2. *Let f, g be in L^{ϕ}. Then $\phi'(|f|)g$ is an integrable function.*

According to Brunk and Johansen [8], we set the following definitions.

Definition 2.3. Let ν be a signed measure on $\mathcal{A}$ and let $\mathcal{L}$ be a σ-lattice contained in $\mathcal{A}$. Say that $P \in \mathcal{L}$ is a ν-positive set, if for all $D \in \overline{\mathcal{L}}$, then $\nu(P \cap D) \geq 0$. A set $N \in \overline{\mathcal{L}}$ is called ν-negative if for all $C \in \mathcal{L}$ one has $\nu(N \cap C) \leq 0$.

Definition 2.4. Let $\{\nu_a\}_{a\in\mathbb{R}}$ be a family of measures on $\mathcal{A}$, and let $\mathcal{L}$ be a σ-lattice contained in $\mathcal{A}$. An $\mathcal{L}$-measurable function g is called a Lebesgue-Radon-Nikodym function (LRN function) for $\{\nu_a\}$ given $\mathcal{L}$ if and only if the set $\{g > a\}$ is ν_a-positive for all $a \in \mathbb{R}$ and the set $\{g < a\}$ is ν_a-negative for all $a \in \mathbb{R}$.

Remark 2.5. We note that in Definition 2.4 it is sufficient to impose the conditions for all a in a dense set in $\mathbb{R}$, see [8, page 588].

For $f \in L^{\phi}$, $g \in L^{\phi}(\mathcal{L})$, and $a \in \mathbb{R}$, we define the following measures on $\mathcal{A}$:

$$\mu_g(A) = \int_A \underline{\phi}'(f-g)d\mu, \qquad \mu_a(A) = \int_A \underline{\phi}'(f-a)d\mu, \tag{2.2}$$

where $\underline{\phi}'(x) = \phi'(|x|)\text{sign}(x)$. Note that when $f \in L^{\phi'}$ and $g \in L^{\phi'}(\mathcal{L})$, the measure μ_g and μ_a are well defined.

The next theorem is a characterization of $\mu_{\phi}^{\mathcal{L}}(f)$, see [9, Theorem 3.2].

Theorem 2.6. *Let $f \in L^{\phi}$, $\mathcal{L} \subset \mathcal{A}$ be a σ-lattice and $g \in L^{\phi}(\mathcal{L})$. Then the following statement are equivalent.*

(1) $g \in \mu_{\phi}^{\mathcal{L}}(f)$.

(2) (a) *a set $\{g > a\}$ is μ_g-positive for all $a \in \mathbb{R}$; and*

(b) *a set $\{g < a\}$ is μ_g-negative for all $a \in \mathbb{R}$.*

(3) *g is an LRN function for the family $\{\mu_a\}_{a\in\mathbb{R}}$ given $\mathcal{L}$.*

Now we extend the operator $\mu_{\phi}^{\mathcal{L}}(\cdot)$ to the space $L^{\phi'}$.

Definition 2.7. Let $\mathcal{L}$ be a σ-lattice and let $f \in L^{\phi'}$. Then g is an extended best ϕ-approximation if and only if $g \in L^{\phi'}(\mathcal{L})$ and

(i) the set $\{g > a\}$ is μ_g-positive for all $a \in \mathbb{R}$;

(ii) the set $\{g < a\}$ is μ_g-negative for all $a \in \mathbb{R}$.

For $f \in L^{\phi'}$ we denote by $\tilde{\mu}_{\phi}^{\mathcal{L}}(f)$ the set of all extended best ϕ-approximation functions.

Remark 2.8. Let $f \in L^{\phi'}$ and let g be a function in $L^{\phi'}(\mathcal{L})$ such that the set $\{g > a\}$ is μ_g-positive and the set $\{g < a\}$ is μ_g-negative for all $a \in \mathbb{R}$. Then we have the following.

(i) For all $h \in L^{\infty}(\overline{\mathcal{L}})$ and $h \geq 0$,

$$\int_{\{g>a\}} \underline{\phi}'(f-g)h\, d\mu \geq 0. \tag{2.3}$$

For all $h \in L^\infty(\mathcal{L})$ and $h \geq 0$,

$$\int_{\{g<a\}} \underline{\phi}'(f-g)h\,d\mu \leq 0. \tag{2.4}$$

(ii)

$$\int \underline{\phi}'(f-g)d\mu = 0. \tag{2.5}$$

Proof. We prove inequality (2.3). Since the set $\{g > a\}$ is μ_g-positive, that is, for each $D \in \overline{\mathcal{L}}$ and $a \in \mathbb{R}$, we have

$$\int_{\{g>a\}\cap D} \underline{\phi}'(f-g)d\mu \geq 0. \tag{2.6}$$

For $h = \sum_{k=1}^{N} c_k \mathcal{X}_{D_k}$, where $D_k \in \overline{\mathcal{L}}$ and $c_k \geq 0$, $k = 1, \dots, N$, then by (2.6), we have

$$\int_{\{g>a\}} \underline{\phi}'(f-g)h\ d\mu = \sum_{k=1}^{N} c_k \int_{\{g>a\}\cap D_k} \underline{\phi}'(f-g)d\mu \geq 0. \tag{2.7}$$

All nonnegative $h \in L^\infty(\overline{\mathcal{L}})$ can be obtained as a limit of functions of the above type. The proof of inequality (2.4) is similar.

The equality (2.5) is obtained using Lebesgue's theorem when $a \to \infty$ with $h = 1$ in (2.4) and if $a \to -\infty$ consider in (2.3) also $h = 1$. □

As a reference, we note that (2) is equivalent to (3) in Theorem 2.6 for $f \in L^{\phi'}$, $g \in L^{\phi'}$. We have the next remark.

Remark 2.9. For $f \in L^{\phi'}$ and $g \in L^{\phi'}(\mathcal{L})$, the following statements are equivalent:

(1) $g \in \tilde{\mu}_\phi^{\mathcal{L}}(f)$;

(2) g is an LRN function for the family $\{\mu_a\}_{a\in\mathbb{R}}$ given $\mathcal{L}$.

The next lemma is a particular case of [8, Theorem 1.8].

Lemma 2.10. *Let $f \in L^{\phi'}$ and $g \in L^{\phi'}(\mathcal{L})$. Then the following statements are equivalent.*

(1) *g is an LRN function for the family $\{\mu_a\}_{a\in\mathbb{R}}$ given $\mathcal{L}$.*

(2) *There exists a countable set D such that $\{g \leq a\}$ is μ_a-negative for all $a \in D^c$ and the set $\{g \geq a\}$ is μ_a-positive for all $a \in D^c$.*

We need the following auxiliary result.

Lemma 2.11. *Let $f_n \in L^{\phi'}$ be a sequence of functions such that $f_n \nearrow f (f_n \searrow f)$, where $f \in L^{\phi'}$. Let $g_n \in \tilde{\mu}_\phi^{\mathcal{L}}(f_n)$ be such that $g_n \nearrow g (g_n \searrow g)$. Then $g \in \tilde{\mu}_\phi^{\mathcal{L}}(f)$.*

Proof. We will prove the result just for the increasing case, the proof for the decreasing case follows the same pattern. The function $g = \lim_{n\to\infty} g_n$ is obviously $\mathcal{L}$-measurable function. Now, we prove that $g \in L^{\phi'}$, and it satisfies (i) and (ii) of Definition 2.7. We have that

$$f_n - g_n \le f - g_n \le f - g_1. \tag{2.8}$$

Using (2.5),

$$\int_\Omega \underline{\phi}'(f_n - g_n)\, d\mu = 0 \quad \text{for each } n \in \mathbb{N}. \tag{2.9}$$

According to (2.8) and (2.9), we have, since $\underline{\phi}'(x)$ is an increasing function,

$$\int_\Omega \underline{\phi}'(f - g_n)\, d\mu \ge 0. \tag{2.10}$$

Since ϕ' is a continuous function and $\phi'(0) = 0$, we have by (2.8) and Lebesgue's theorem

$$\int_\Omega (\underline{\phi}'(f - g) \vee 0)\, d\mu \le \int_\Omega (\underline{\phi}'(f - g_1) \vee 0)\, d\mu. \tag{2.11}$$

Now by (2.10) it holds

$$\int_\Omega (-\underline{\phi}'(f - g_n) \vee 0)\, d\mu \le \int_\Omega (\underline{\phi}'(f - g_n) \vee 0)\, d\mu. \tag{2.12}$$

Using Fatou in (2.12), we have

$$\int_\Omega (-\underline{\phi}'(f - g) \vee 0)\, d\mu \le \int_\Omega (\underline{\phi}'(f - g) \vee 0)\, d\mu. \tag{2.13}$$

Therefore, using (2.11) and (2.12), we get $g \in L^{\phi'}(\Omega)$ and $\int_\Omega \underline{\phi}'(f - g)\, d\mu \ge 0$.

Let $a \in \mathbb{R}$, $D \in \overline{\mathcal{L}}$, and $C \in \mathcal{L}$ we know for each n

$$\int_{\{g_n > a\} \cap D} \underline{\phi}'(f_n - g_n)\, d\mu \ge 0, \qquad \int_{\{g_n < a\} \cap C} \underline{\phi}'(f_n - g_n)\, d\mu \le 0. \tag{2.14}$$

Since $\{g_n\}_{n\in\mathbb{N}}$ is an increasing sequence, we get $\bigcup_{n\in\mathbb{N}} \{g_n > a\} = \{g > a\}$, and by (2.14), we have

$$\int_{\{g > a\} \cap D} \underline{\phi}'(f - g)\, d\mu \ge 0. \tag{2.15}$$

Hence, the set $\{g > a\}$ is μ_g-positive for all $a \in \mathbb{R}$.

Now, for $a \in \mathbb{R}$ and $k, n \in \mathbb{N}$, we define $B_n = \{g_n < a\}$ and $B_{n,k} = \{g_n < a - 1/k\}$. We have that, for $n \to \infty$, $B_{n,k} \searrow A_k$ for some $\mathcal{A}$-measurable set such that

$$\left\{ g < a - \frac{1}{k} \right\} \subset A_k \subset \left\{ g \le a - \frac{1}{k} \right\}. \tag{2.16}$$

We observe that $\mathcal{X}_{A_k} \to \mathcal{X}_{\{g<a\}}$, a.e. Then taking limit as $k \to \infty$, $n \to \infty$ and using Lebesgue's theorem, we obtain

$$\int_{\{g<a\}\cap C} \underline{\phi}'(f-g)\,d\mu = \lim_{k\to\infty}\int \underline{\phi}'(f-g)\mathcal{X}_{A_k\cap C}\,d\mu = \lim_{k\to\infty}\lim_{n\to\infty}\int_{B_{n,k}\cap C}\underline{\phi}'(f-g)d\mu \le 0. \tag{2.17}$$

□

Theorem 2.12. *Let $\mathcal{L}$ be a σ-lattice and $f \in L^{\phi'}$, then $\widetilde{\mu}_\phi^{\mathcal{L}}(f) \neq \emptyset$.*

Proof. For $f \in L^{\phi'}$ we can define the following sequences. For each $m \in \mathbb{N}$, let $f_m = f \vee (-m)$ and when $m \to \infty$ we have $f_m \searrow f$. Set $f_m^n = (f \vee (-m)) \wedge n$ for all n, m in $\mathbb{N}$, then we have $f_{mn} \nearrow f_m$, when $n \to \infty$. Since for each $m, n \in \mathbb{N}$ we have $f_m^n \in L^\phi$, there exist $g_m^n \in \mu_\phi^{\mathcal{L}}(f_m^n)$. As $\mu_\phi^{\mathcal{L}}(\cdot)$ is a mono-tone operator over L^ϕ we can take a new sequence that we call again $g_m^n \in \mu_\phi^{\mathcal{L}}(f_m^n)$, such that $g_m^n \le g_m^{n+1}$ for all $n \in \mathbb{N}$.

Since $f_m^n \ge f_{m+1}^n$ and using again that $\mu_\phi^{\mathcal{L}}(\cdot)$ is a monotone operator, we have $\widetilde{g}_{m+1}^n \le \widetilde{g}_m^n$, where $\widetilde{g}_m^n \in \mu_\phi^{\mathcal{L}}(f_m^n)$ is the sequence defined by $\widetilde{g}_1^n = g_1^n$ and $\widetilde{g}_{m+1}^n = \widetilde{g}_m^n \wedge g_{m+1}^n$. Furthermore, it is easy to check that $\widetilde{g}_m^n \le \widetilde{g}_m^{n+1}$.

Then, we have that for each $m \in \mathbb{N}$ that $f_m^n \nearrow f_m$, when $n \to \infty$, and since $\mu_\phi^{\mathcal{L}}(f_m^n) \subset \widetilde{\mu}_\phi^{\mathcal{L}}(f_m^n)$ we have $\widetilde{g}_m^n \in \widetilde{\mu}_\phi^{\mathcal{L}}(f_m^n)$ and if we define $g_m = \lim_{n\to\infty}\widetilde{g}_m^n$ by Lemma 2.11 we obtain $g_m \in \widetilde{\mu}_\phi^{\mathcal{L}}(f_m)$ and $g_m \ge g_{m+1}$ for all $m \in \mathbb{N}$. If we take $m \to \infty$, we have $f_m \searrow f$ and by Lemma 2.11 we get $g \in \widetilde{\mu}_\phi^{\mathcal{L}}(f)$, where $g = \lim_{m\to\infty} g_m$. □

To see that the extended best ϕ-approximation is an extension of the best ϕ-approximation operator, we must prove $\widetilde{\mu}_\phi^{\mathcal{L}}(f) = \mu_\phi^{\mathcal{L}}(f)$ for every $f \in L^\phi$. First, we need to prove the following lemmas.

Lemma 2.13. *Let ϕ be a C^1 convex function and assume that it satisfies the Δ_2-condition. Then*

$$\phi(a) + \frac{K}{2}\,a\,\underline{\phi}'(x-a) \le \frac{K+2}{2}\,\phi(x), \tag{2.18}$$

for $a, x \ge 0$, where K is the constant for the Δ_2 condition.

Proof. We consider two cases. First, we assume $0 < x \le a$. Since ϕ is Δ_2-convex function, we have that $\phi(a) = \phi(a - x + x) \le (K/2)(\phi(a-x) + \phi(x))$. Using $x\phi'(x) \ge \phi(x)$ for all x, we get

$$\begin{aligned} \phi(x) + a\phi'(a-x) &= \phi(x) + (a - x + x)\phi'(a-x) \\ &\ge \phi(x) + \phi(a-x) + x\phi'(a-x) \\ &\ge \frac{2}{K}\phi(a) + x\phi'(a-x) \ge \frac{2}{K}\phi(a). \end{aligned} \tag{2.19}$$

Then we obtain

$$\phi(a) - \frac{K}{2}a\phi'(a-x) \le \frac{K}{2}\phi(x). \tag{2.20}$$

For $0 \le a < x$, we have

$$\phi(a) + a\frac{K}{2}\phi'(x-a) \le \phi(a) + \frac{K}{2}\int_{x-a}^{x}\phi'(t)dt \le \phi(x) + \frac{K}{2}\phi(x) = \frac{K+2}{2}\phi(x). \tag{2.21}$$

□

Lemma 2.14. *Let $f \in L^{\phi}$ and $g \in \tilde{\mu}^{\mathcal{L}}_{\phi}(f)$, then*

$$\int_{\{g>0\}} \underline{\phi}'(f-g)g\,d\mu \geq 0. \tag{2.22}$$

Proof. Since $\{g > a\}$ is μ_g-positive for all $a \in \mathbb{R}$, then for all $D \in \overline{\mathcal{L}}$, we have that

$$\int_{\{g>a\}\cap D} \underline{\phi}'(f-g)d\mu \geq 0. \tag{2.23}$$

In particular, it holds that for all $a \in \mathbb{R}$,

$$\int_{\{g>a\}} \underline{\phi}'(f-g)d\mu \geq 0, \tag{2.24}$$

that is,

$$\int_{\{g>a\}\cap\{f>g\}} \phi'(|f-g|)d\mu \geq \int_{\{g>a\}\cap\{f\leq g\}} \phi'(|f-g|)d\mu. \tag{2.25}$$

Now, we have

$$\int_0^\infty \int_{\{g>a\}\cap\{f>g\}} \phi'(|f-g|)d\mu\,da \geq \int_0^\infty \int_{\{g>a\}\cap\{f\leq g\}} \phi'(|f-g|)d\mu\,da. \tag{2.26}$$

By the Fubini's theorem, we get

$$\int_{\{g>0\}} \int_0^g \phi'(|f-g|)\mathcal{X}_{\{f>g\}}\,da\,d\mu \geq \int_{\{g>0\}} \int_0^g \phi'(|f-g|)\mathcal{X}_{\{f\leq g\}}\,da\,d\mu. \tag{2.27}$$

Thus

$$\int_{\{g>0\}} \phi'(|f-g|)\mathcal{X}_{\{f>g\}}g\,d\mu \geq \int_{\{g>0\}} \phi'(|f-g|)\mathcal{X}_{\{f\leq g\}}g\,d\mu. \tag{2.28}$$

To see that inequality (2.22) is equivalent to (2.28) we will prove that $\phi'(f-g)\mathcal{X}_{\{f>g\}}g \in L^1(\{g > 0\})$. In fact

$$\int_{\{g>0\}} \phi'(f-g)\mathcal{X}_{\{f>g\}}g\,d\mu \leq \int_{\{g>0\}} \phi'(f-g)\mathcal{X}_{\{f>g\}}f\,d\mu \leq \int_{\{g>0\}} \phi'(f)f. \tag{2.29}$$

Since $f \in L^{\phi}$ by Lemma 2.2, the last integral is finite. □

The following properties of the set $\tilde{\mu}^{\mathcal{L}}_{\phi}(f)$ can be easily proved.

Proposition 2.15. *Let $f \in L^{\phi'}$, then*

(1) $-\tilde{\mu}^{\overline{\mathcal{L}}}_{\phi}(-f) = \tilde{\mu}^{\mathcal{L}}_{\phi}(f)$,

(2) $\tilde{\mu}^{\mathcal{L}}_{\phi}(f+t) = \tilde{\mu}^{\mathcal{L}}_{\phi}(f) + t$ *for all $h \in \mathbb{R}$.*

Now we prove that the operator $\tilde{\mu}^{\mathcal{L}}_{\phi}(f)$ is in fact an extension of the operator $\mu^{\mathcal{L}}_{\phi}(f)$.

Theorem 2.16. *Let $f \in L^{\phi}$, then $\tilde{\mu}^{\ell}_{\phi}(f) = \mu^{\ell}_{\phi}(f)$.*

Proof. For $f \in L^{\phi}$, we will prove only that $\tilde{\mu}^{\ell}_{\phi}(f) \subset \mu^{\ell}_{\phi}(f)$. The other inclusion follows from Theorem 2.6. Let $g \in \tilde{\mu}^{\ell}_{\phi}(f)$ and again using Theorem 2.6 it remains to prove that $g \in L^{\phi}$. Recall that $\phi(0) = 0$, then

$$\int_{\Omega} \phi(|g|)d\mu = \int_{\{g>0\}} \phi(g)d\mu + \int_{\{g<0\}} \phi(-g)d\mu. \tag{2.30}$$

By Lemma 2.13 we obtain the following inequality:

$$\int_{\{g>0\}} \phi(g)d\mu + \frac{K}{2}\int_{\{g>0\}} \underline{\phi}'(|f|-g)g\,d\mu \leq \frac{K+2}{2}\int_{\{g>0\}} \phi(|f|)d\mu. \tag{2.31}$$

Applying Lemma 2.14 and taking into account that $\underline{\phi}'(x)$ is an increasing function, we get

$$0 \leq \int_{\{g>0\}} \underline{\phi}'(f-g)g\,d\mu \leq \int_{\{g>0\}} \underline{\phi}'(|f|-g)g\,d\mu. \tag{2.32}$$

Thus using (2.32) in (2.31), we have

$$\int_{\{g>0\}} \phi(g)d\mu \leq \frac{K+2}{2}\int_{\Omega} \phi(|f|)d\mu. \tag{2.33}$$

For the set $\{g<0\}$, again by Lemma 2.13, we obtain

$$\int_{\{g<0\}} \phi(-g)d\mu + \frac{K}{2}\int_{\{g<0\}} \underline{\phi}'(|f|+g)(-g)\,d\mu \leq \frac{K+2}{2}\int_{\{g<0\}} \phi(|f|)d\mu. \tag{2.34}$$

Since $-f+g \leq |f|+g$, we have $\underline{\phi}'(-f+g) \leq \underline{\phi}'(|f|+g)$. Thus

$$\int_{\{g<0\}} \underline{\phi}'(-f+g)(-g)d\mu \leq \int_{\{g<0\}} \underline{\phi}'(|f|+g)(-g)d\mu. \tag{2.35}$$

By (1) in Proposition 2.15, $-g \in \tilde{\mu}^{\overline{\ell}}_{\phi}(-f)$, and by Lemma 2.14, we have that

$$\int_{\{g<0\}} \underline{\phi}'(-f+g)(-g)\,d\mu \geq 0. \tag{2.36}$$

Therefore, $\int_{\{g<0\}} \phi(-g)d\mu \leq ((K+2)/2)\int \phi(|f|)d\mu$. By (2.33), we have

$$\int_{\Omega} \phi(|g|)d\mu \leq \frac{K+2}{2}\int_{\Omega} \phi(|f|)d\mu, \tag{2.37}$$

and therefore $g \in L^{\phi}$. □

Now, if consider a σ-subalgebra $\mathcal{B}$ instead of a σ-lattice $\mathcal{L}$, the extended best ϕ-approximation operator has the following properties.

Theorem 2.17. *Let f, f_1, and f_2 be in $L^{\phi'}$, if $\mathcal{B}$ is a sub-σ-algebra of the σ-algebra $\mathcal{A}$, then the following hold.*

(1) *The set-valued function $\tilde{\mu}_\phi^{\mathcal{B}}(f)$ is a monotone operator.*

(2) *The set $\tilde{\mu}_\phi^{\mathcal{B}}(f)$ is a σ-complete lattice, and there exist $U_{\mathcal{B}}, V_{\mathcal{B}} \in \tilde{\mu}_\phi^{\mathcal{B}}(f)$ such that $U_{\mathcal{B}} \le g \le V_{\mathcal{B}}$ a.e. for every $g \in \tilde{\mu}_\phi^{\mathcal{B}}(f)$.*

Proof. To prove (1), recall that this set-valued operator is monotone if $f_1 \le f_2$; then if $g_1 \in \tilde{\mu}_\phi^{\mathcal{B}}(f_1)$ and $g_2 \in \tilde{\mu}_\phi^{\mathcal{B}}(f_2)$, we have that $g_1 \wedge g_2 \in \tilde{\mu}_\phi^{\mathcal{B}}(f_1)$ and $g_1 \vee g_2 \in \tilde{\mu}_\phi^{\mathcal{B}}(f_2)$. Since $L^\phi(\mathcal{B})$ is a lattice, we know $g_1 \wedge g_2 \in L^\phi(\mathcal{B})$ and $g_1 \vee g_2 \in L^\phi(\mathcal{B})$. We will prove first that $g_1 \wedge g_2 \in \tilde{\mu}_\phi^{\mathcal{B}}(f_1)$. Set $\mu_a^{f_i}(A) = \int_A \underline{\phi}'(f_i - a)d\mu$, where $a \in \mathbb{R}$ and $i = 1, 2$. We will see that $g_1 \wedge g_2$ is an LRN function for the family of measures $\{\mu_a^{f_1}\}_{a\in\mathbb{R}}$ given $\mathcal{B}$. First, we will see that for each $a \in \mathbb{R}$ and for all $B \in \mathcal{B}$, we have

$$\mu_a^{f_1}(\{g_1 \wedge g_2 > a\} \cap B) \ge 0. \tag{2.38}$$

Since $\{g_1 \wedge g_2 > a\} \cap B = \{g_1 > a\} \cap \{g_2 > a\} \cap B$ and $\{g_2 > a\} \cap B \in \mathcal{B}$ and using that g_1 is an LRN function of the family $\{\mu_a^{f_1}\}_{a\in\mathbb{R}}$, we obtain that for all $B \in \mathcal{B}$

$$\int_{\{g_1\wedge g_2>a\}\cap B} \underline{\phi}'(f_1 - a)d\mu = \int_{\{g_1>a\}\cap\{g_2>a\}\cap B} \underline{\phi}'(f_1 - a)d\mu \ge 0. \tag{2.39}$$

Now, we see that $\{g_1 \wedge g_2 < a\}$ is $\mu_a^{f_1}$-negative for all $a \in \mathbb{R}$. For $B \in \mathcal{B}$, we have

$$\{g_1 \wedge g_2 < a\} \cap B = (\{g_1 < a\} \cap B) \cup (\{g_1 \ge a\} \cap \{g_2 < a\} \cap B). \tag{2.40}$$

Using $f_1 \le f_2$ and that $\underline{\phi}'(\cdot)$ is a nondecreasing function, we obtain

$$\begin{aligned}\int_{\{g_1\wedge g_2<a\}\cap B} \underline{\phi}'(f_1 - a)d\mu &= \int_{\{g_1<a\}\cap B} \underline{\phi}'(f_1 - a)d\mu + \int_{\{g_1\ge a\}\cap\{g_2<a\}\cap B} \underline{\phi}'(f_1 - a)d\mu \\ &\le \int_{\{g_1<a\}\cap B} \underline{\phi}'(f_1 - a)d\mu + \int_{\{g_1\ge a\}\cap\{g_2<a\}\cap B} \underline{\phi}'(f_2 - a)d\mu \le 0.\end{aligned} \tag{2.41}$$

Thus

$$\int_{\{g_1\wedge g_2<a\}\cap B} \underline{\phi}'(f_1 - a)d\mu \le 0. \tag{2.42}$$

By (2.39) and (2.42), we have $g_1 \wedge g_2 \in \tilde{\mu}_\phi^{\mathcal{B}}(f_1)$.

Now we show $g_1 \vee g_2 \in \tilde{\mu}_\phi^{\mathcal{B}}(f_2)$. Since $f_1 \le f_2$ and $\{g_1 > a\}$ is a $\mu_a^{f_1}$-positive for all $a \in \mathbb{R}$ and for all $B \in \mathcal{B}$, we have

$$\begin{aligned}\int_{\{g_1\vee g_2>a\}\cap B} \underline{\phi}'(f_2 - a)d\mu &= \int_{\{g_2>a\}\cap B} \underline{\phi}'(f_2 - a)d\mu + \int_{\{g_1>a\}\cap\{g_2\le a\}\cap B} \underline{\phi}'(f_2 - a)d\mu \\ &\ge \int_{\{g_2>a\}\cap B} \underline{\phi}'(f_2 - a)d\mu + \int_{\{g_1>a\}\cap\{g_2\le a\}\cap B} \underline{\phi}'(f_1 - a)d\mu \ge 0.\end{aligned} \tag{2.43}$$

Since

$$\int_{\{g_1\vee g_2<a\}\cap B}\underline{\phi}'(f_2-a)\,d\mu=\int_{\{g_1<a\}\cap\{g_2<a\}\cap B}\underline{\phi}'(f_2-a)\,d\mu\le 0, \tag{2.44}$$

the inequalities (2.43) and (2.44) prove that $g_1\vee g_2\in\tilde{\mu}_\phi^{\mathcal{B}}(f_2)$.

As the statement (1) proves in particular that $\tilde{\mu}_\phi^{\mathcal{B}}(f)$ is a lattice, we will see that the set is a σ-complete lattice. Given a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $\tilde{\mu}_\phi^{\mathcal{B}}(f)$, we have that $\bigvee_1^n g_k\in\tilde{\mu}_\phi^{\mathcal{B}}(f)$; then, from Lemma 2.11 we obtain that $\bigvee_{n\in\mathbb{N}}g_n=\lim_{n\to\infty}\bigvee_1^n g_n\in\tilde{\mu}_\phi^{\mathcal{B}}(f)$. The proof $\bigwedge_{n\in\mathbb{N}}g_n\in\tilde{\mu}_\phi^{\mathcal{B}}(f)$ is similar.

By [10, Proposition II.4.1] there exists a sequence $g_n\in\tilde{\mu}_\phi^{\mathcal{B}}(f)$ such that $\inf g_n\le g\le\sup g_n$, for every $g\in\tilde{\mu}_\phi^{\mathcal{B}}(f)$. Set $U_{\mathcal{B}}=\inf g_n$ and $V_{\mathcal{B}}=\sup g_n$, then $U_{\mathcal{B}}$ and $V_{\mathcal{B}}$ are in $\tilde{\mu}_\phi^{\mathcal{B}}(f)$ since this set is a σ-complete lattice. □

3. Extended best ϕ-approximation with nondecreasing functions

When the approximation class is the monotone functions defined on $[0,1]$ we can obtain similar results as those of Theorem 2.17. Now $\Omega=[0,1]$, μ is Lebesgue measure on the measurable sets, and $\mathcal{L}=\{(a,1),[a,1),\varnothing,\mathbb{R}\}_{a\in\mathbb{R}}$. Therefore, $L^{\phi'}(\mathcal{L})$ is the set of nondecreasing functions in $L^{\phi'}[0,1]$.

Remark 3.1. Let g be a nondecreasing function on $[0,1]$. Given $a\in\mathbb{R}$, the set $\{g<a\}$ is one of the intervals $[0,\alpha_a)$ or $[0,\alpha_a]$ and similarly the set $\{g>a\}$ is $(\beta_a,1]$ or $[\beta_a,1]$ with $0\le\alpha_a\le\beta_a\le 1$. Then $H_g=\{a\in\mathbb{R}:\alpha_a=\beta_a\}$ is a dense set in $\mathbb{R}$. In fact, the complement set of H_g is a countable set.

Note that each $C\in\mathcal{L}$ is of the form $(c,1]$ or $[c,1]$ and $D\in\overline{\mathcal{L}}$ is $D=[0,d)$ or $[0,d]$. Thus

$$\int_{\{g>a\}\cap D}\underline{\phi}'(f-a)\,d\mu=\int_{\beta_a}^{d}\underline{\phi}'(f-a)\,d\mu,\qquad \int_{\{g<a\}\cap C}\underline{\phi}'(f-a)\,d\mu=\int_{c}^{\alpha_a}\underline{\phi}'(f-a)\,d\mu. \tag{$\bigstar$}$$

Theorem 3.2. *Let $L^{\phi}(\mathcal{L})$ be the class of the ϕ'-integrable nondecreasing functions in $[0,1]$. Then the following hold.*

(1) *The set mapping $\tilde{\mu}_\phi^{\mathcal{L}}(f)$ is a monotone operator.*

(2) *For every $f\in L^{\phi'}$, the set $\tilde{\mu}_\phi^{\mathcal{L}}(f)$ is a σ-complete lattice.*

Proof. First we prove (1), that is given f_1, f_2 in $L^{\phi'}$ with $f_1\le f_2$, for each $g_i\in\tilde{\mu}_\phi^{\mathcal{L}}(f_i)$, $i=1,2$ we will see that $g_1\wedge g_2\in\tilde{\mu}_\phi^{\mathcal{L}}(f_1)$ and $g_1\vee g_2\in\tilde{\mu}_\phi^{\mathcal{L}}(f_2)$. Let H be $H_{g_1}\cap H_{g_2}$, where H_{g_i} is the set given in Remark 3.1. Recall that $g_1\wedge g_2\in\tilde{\mu}_\phi^{\mathcal{L}}(f_1)$ if and only if for each $a\in H$ and $c,d\in\mathbb{R}$

$$\int_{\{g_1\wedge g_2>a\}\cap(0,d)}\underline{\phi}'(f_1-a)\,dx\ge 0,\qquad \int_{\{g_1\wedge g_2<a\}\cap(c,1)}\underline{\phi}'(f_1-a)\,dx\le 0. \tag{3.1}$$

Also $g_1\vee g_2\in\tilde{\mu}_\phi^{\mathcal{L}}(f_2)$ if and only if for each $a\in H$, and $c,d\in\mathbb{R}$,

$$\int_{\{g_1\vee g_2>a\}\cap(0,d)}\underline{\phi}'(f_2-a)\,dx\ge 0,\qquad \int_{\{g_1\vee g_2<a\}\cap(c,1)}\underline{\phi}'(f_2-a)\,dx\le 0. \tag{3.2}$$

First we prove (3.1). Now we see that

$$\int_{\{g_1\wedge g_2>a\}\cap(0,d)} \underline{\phi}'(f_1-a)dx \geq 0, \tag{3.3}$$

with $\{g_1 \wedge g_2 > a\} = (\beta_1^a, 1] \cap (\beta_2^a, 1]$. Since $\int_{(\beta_1^a,d)} \underline{\phi}'(f_1-a)dx \geq 0$ to prove (3.3), we have to see that

$$\int_{(\beta_2^a,d)} \underline{\phi}'(f_1-a)dx \geq 0, \tag{3.4}$$

where $\beta_1^a \leq \beta_2^a$. Indeed by (★), we get

$$0 \leq \int_{\beta_1^a} a^d \underline{\phi}'(f_1-a)dx = \int_{\beta_1^a}^{\beta_2^a} \underline{\phi}'(f_1-a)dx + \int_{\beta_2^a}^{d} \underline{\phi}'(f_1-a)dx. \tag{3.5}$$

Since $\underline{\phi}'(\cdot)$ is a nondecreasing function, we have

$$0 \leq \int_{\beta_1^a}^{\beta_2^a} \underline{\phi}'(f_2-a)dx + \int_{\beta_2^a}^{d} \underline{\phi}'(f_1-a)dx. \tag{3.6}$$

As $\int_{\beta_1^a}^{\beta_2^a} \underline{\phi}'(f_2-a)dx \leq 0$ $(\beta_2^a = \alpha_2^a)$, we have $\int_{\beta_2^a}^{d} \underline{\phi}'(f_1-a)dx \geq 0$, that is (3.4).

Now we will prove that

$$\int_{\{g_1\wedge g_2<a\}\cap(c,1)} \underline{\phi}'(f_1-a)dx \leq 0. \tag{3.7}$$

In fact

$$\begin{aligned}\int_{\{g_1\wedge g_2<a\}\cap(c,1)} \underline{\phi}'(f_1-a)dx &= \int_{\{g_1<a\}\cap(c,1)} \underline{\phi}'(f_1-a)dx + \int_{\{g_1\geq a\}\cap\{g_2<a\}\cap(c,1)} \underline{\phi}'(f_1-a)dx \\ &\leq \int_{\{g_1<a\}\cap(c,1)} \underline{\phi}'(f_1-a)dx + \int_{\{g_2<a\}\cap[\{g_1\geq a\}\cap(c,1)]} \underline{\phi}'(f_2-a)dx.\end{aligned} \tag{3.8}$$

The last two integrals in (3.8) are less or equal than zero, so (3.7) holds. A similar argument shows that $g_1 \vee g_2 \in \widetilde{\mu}_\phi^{\mathcal{L}}(f_2)$. Therefore, the extended best ϕ-approximation operator is a monotone operator. By (1), we have that $\widetilde{\mu}_\phi^{\mathcal{L}}(f)$ is a lattice, just setting $f = f_1 = f_2$. Now by Lemma 2.11 we obtain that $\widetilde{\mu}_\phi^{\mathcal{L}}(f)$ is a σ-complete lattice. □

4. A limit theorem for extended best ϕ-approximations

Given a sequence $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ of σ-algebras contained in the σ-algebra $\mathcal{A}$, we consider two cases, $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ for all $n \in \mathbb{N}$ and we set $\mathcal{B}_\infty$ for the σ-algebra generated by $\bigcup_n \mathcal{B}_n$, and if $\mathcal{B}_n \supseteq \mathcal{B}_{n+1}$ for all $n \in \mathbb{N}$, we set $\mathcal{B}_\infty = \bigcap_{n\in\mathbb{N}} \mathcal{B}_n$.

The next result is a particular case of [8, Theorem 2.8] when ϕ is a strictly convex function. This assumption on the function ϕ assures that the family of measures $\{\mu_a\}_{a\in\mathbb{R}}$ decreases at zero as required by Brunk and Johansen in that theorem.

Theorem 4.1. *Let $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ be an increasing or decreasing sequence of σ-algebras in $\mathcal{A}$, and let $\mathcal{B}_\infty$ be the limit of the sequence. If $f \in L^{\phi'}$, then we have for all $g_n \in \tilde{\mu}_\phi^{\mathcal{B}_n}(f)$, $n \in \mathbb{N}$, that $\liminf_{n\to\infty} g_n$, and $\limsup_{n\to\infty} g_n$ are in $\tilde{\mu}_\phi^{\mathcal{B}_\infty}(f)$.*

Proof. Define $\overline{g} = \limsup_{n\to\infty} g_n$ and $\underline{g} = \liminf_{n\to\infty} g_n$, then we only prove that $\underline{g}, \overline{g} \in \tilde{\mu}_\phi^{\mathcal{B}_\infty}(f)$, when $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ is an increasing sequence of σ-algebras, the proof for the decreasing case is similar.

First, we prove for each $f \in L^{\phi'}$ that the set $\{\overline{g} \geq a\}$ is μ_a-positive for all $a \in \mathbb{R}$. Let $B \in \mathcal{B}_m$, and for $H_n = \bigcup_{k\geq n}\{g_k > a - \epsilon_n\}$, where ϵ_n decreases to zero, we have that $\{\overline{g} \geq a\} = \bigcap_{n\geq 1} H_n$, and for all $n \in \mathbb{N}, H_{n+1} \subset H_n$. Now for each $n \in \mathbb{N}$, we define the following disjoint sets. For $p \geq n$, set $H_{n,n} = \{g_n > a - \epsilon_n\}, \dots ,\ H_{n,p} = \{g_p > a - \epsilon_n\} \cap \{g_{p-1} \leq a - \epsilon_n\} \cap \cdots \cap \{g_n \leq a - \epsilon_n\}$. Thus, $H_n = \bigcup_{p\geq n} H_{n,p}$, and then

$$\int_{H_n\cap B} \underline{\phi}'(f - (a - \epsilon_n))\,d\mu = \sum_{p=n}^{\infty} \int_{H_{n,p}\cap B} \underline{\phi}'(f - (a - \epsilon_n))\,d\mu. \tag{4.1}$$

As $\mathcal{B}_n \subset \mathcal{B}_{n+1}$, and for $p \geq n$, we have that $\{g_{p-1} \leq a - \epsilon_n\} \cap \cdots \cap \{g_n \leq a - \epsilon_n\} \in \mathcal{B}_p$. As $B \in \mathcal{B}_m$ then $B \in \mathcal{B}_p$ and $\int_{H_{n,p}\cap B} \underline{\phi}'(f - (a - \epsilon_n))d\mu \geq 0$ for $m \leq n \leq p$. Thus $\int_{H_n\cap B} \underline{\phi}'(f - (a - \epsilon_n))d\mu \geq 0$, and by Lebesgue's theorem, we get

$$\int_{\{\overline{g}\geq a\}\cap B} \underline{\phi}'(f - a)\,d\mu = \lim_{n\to\infty} \int_{H_n\cap B} \underline{\phi}'(f - (a - \epsilon_n))\,d\mu \geq 0 \tag{4.2}$$

for all $B \in \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$. Now we have (4.2) for all $B \in \mathcal{B}_\infty$. In fact, the set $D = \{B \in \mathcal{A} : \int_{\{\overline{g}\geq a\}\cap B} \underline{\phi}'(f - a)d\mu \geq 0\}$ is a monotone class, that is, the set D is closed for increasing and decreasing sequences of sets. As $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n \subset D$ and this union is an algebra of sets, the monotone class generated by it is $\mathcal{B}_\infty$, that is, $\mathcal{B}_\infty \subset D$.

Now let us prove that the set $\{\overline{g} \leq a\}$ is μ_a-negative for all $a \in \mathbb{R}$. As $\{\overline{g} \leq a\} = \bigcap_{n\in\mathbb{N}}\{\overline{g} < a + 1/n\}$ and

$$\int_{\{\overline{g}\leq a\}\cap B} \underline{\phi}'(f - a)\,d\mu = \lim_{n\to\infty} \int_{\{\overline{g}< a+1/n\}\cap B} \underline{\phi}'(f - a)\,d\mu, \tag{4.3}$$

then we have to prove that for all $a \in \mathbb{R}$, the set $\{\overline{g} < a\}$ is μ_a-negative. Since $\{\overline{g} < a\} = \bigcup_{n\geq 1}\bigcap_{k\geq n}\{g_k < a - \epsilon_n\} = \bigcup_{n\geq 1} H_n$, where $H_n = \bigcap_{k\geq n}\{g_k < a - \epsilon_n\}$ and $\epsilon_n \searrow 0$, we have for all $n \in \mathbb{N}$ that $H_n \subset H_{n+1}$. Then

$$\int_{\{\overline{g}< a\}\cap B} \underline{\phi}'(f - a)\,d\mu = \lim_{n\to\infty} \int_{H_n\cap B} \underline{\phi}'(f - (a - \epsilon_n))\,d\mu, \tag{4.4}$$

for a fixed set $B \in \mathcal{B}_m$.

Set $G_n = \bigcap_{k\geq n+1}\{g_k < a - \epsilon_n\}$ and note that $H_n = G_n \cap \{g_n < a - \epsilon_n\}$. Then for $m \leq n$, we have

$$\begin{aligned}\int_{\{g_n< a-\epsilon_n\}\cap B\cap G_n} \underline{\phi}'(f - (a - \epsilon_n))\,d\mu &+ \int_{\{g_n< a-\epsilon_n\}\cap B\cap G_n^c} \underline{\phi}'(f - (a - \epsilon_n))\,d\mu \\ &= \int_{\{g_n< a-\epsilon_n\}\cap B} \underline{\phi}'(f - (a - \epsilon_n))\,d\mu \leq 0.\end{aligned} \tag{4.5}$$

Now, we prove the following inequality:

$$\int_{\{g_n<a-\epsilon_n\}\cap B\cap G_n^c} \underline{\phi}'(f-(a-\epsilon_n))d\mu \geq 0. \tag{4.6}$$

We can see that

$$\{g_n < a-\epsilon_n\}\cap B\cap G_n^c = \bigcup_{k\geq n+1} A_k, \tag{4.7}$$

where A_k are the following disjoint sets

$$\begin{aligned} A_{n+1} &= \{g_{n+1} \geq a-\epsilon_n\}\cap B\cap \{g_n < a-\epsilon_n\}, \\ &\vdots \\ A_k &= \{g_k \geq a-\epsilon_n\}\cap \bigcap_{i=n+1}^{k-1}\{g_i < a-\epsilon_n\}\cap\{g_n<a-\epsilon_n\}\cap B. \end{aligned} \tag{4.8}$$

Then

$$\int_{\{g_n<a-\epsilon_n\}\cap B\cap G_n^c} \underline{\phi}'(f-(a-\epsilon_n))d\mu = \sum_{k\geq n+1}\int_{A_k} \underline{\phi}'(f-(a-\epsilon_n))d\mu. \tag{4.9}$$

Since $A_k = \{g_k \geq a-\epsilon_n\}\cap B_k$, where $B_k \in \mathcal{B}_k$, we have (4.6). Therefore by (4.5), we have

$$\int_{H_n\cap B} \underline{\phi}'(f-(a+\epsilon_n))d\mu \leq 0, \tag{4.10}$$

for all $B \in \bigcup_{n\in\mathbb{N}}\mathcal{B}_n$. Thus by (4.10), (4.4), and (4.3), we get

$$\int_{\{\overline{g}\leq a\}\cap B} \underline{\phi}'(f-a)d\mu \leq 0 \tag{4.11}$$

for all $B \in \bigcup_{n\in\mathbb{N}}\mathcal{B}_n$. Therefore, the result is satisfied for all $B\in\mathcal{B}_\infty$. Thus $\overline{g}\in\tilde{\mu}_\phi^{\mathcal{B}_\infty}(f)$.

We have $\{\underline{g}\leq a\} = \bigcap_{n\geq 1} H_n$, where $H_n = \bigcup_{k\geq n}\{g_k < a+\epsilon_n\}$ and $\epsilon_n \searrow 0$, then $H_{n+1}\subset H_n$ for all $n\in\mathbb{N}$. Since $f\in L^{\phi'}$ we have for all $B\in\mathcal{B}_\infty$ that

$$\int_{\{\underline{g}\leq a\}\cap B} \underline{\phi}'(f-a)d\mu = \lim_{n\to\infty}\int_{H_n\cap B} \underline{\phi}'(f-(a+\epsilon_n))d\mu. \tag{4.12}$$

For $p>n$ define the following disjoint sets $H_{n,n} = \{g_n < a+\epsilon_n\}$ and $H_{n,p} = \{g_p < a+\epsilon_n\}\cap\{g_{p-1}\geq a+\epsilon_n\}\cap\cdots\cap\{g_n \geq a+\epsilon_n\}$. Then for $B\in\mathcal{B}_m$ we have

$$\int_{H_n\cap B} \underline{\phi}'(f-(a+\epsilon_n))d\mu = \sum_{p\geq n}\int_{H_{n,p}\cap B} \underline{\phi}'(f-(a+\epsilon_n))d\mu. \tag{4.13}$$

Now if $m \le n \le p$, $H_{n,p} \cap B = \{g_k < a + \epsilon_n\} \cap B^\star$, where $B^\star \in \mathcal{B}_p$. Thus

$$\int_{H_{n,p}\cap B} \underline{\phi}'(f - (a + \epsilon_n))d\mu \le 0. \tag{4.14}$$

Then by (4.12) and (4.14), we have for all $B \in \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ that

$$\int_{\{\underline{g}\le a\}\cap B} \underline{\phi}'(f - a)d\mu \le 0. \tag{4.15}$$

Therefore, we have (4.15) for all $B \in \mathcal{B}_\infty$.

Let us see now that

$$\int_{\{\underline{g}\ge a\}\cap B} \underline{\phi}'(f - a)d\mu \ge 0 \tag{4.16}$$

for all $B \in \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$.

As $\{\underline{g} \ge a\} = \bigcap_{n\in\mathbb{N}}\{\underline{g} > a - 1/n\}$, we have to prove that for all $a \in \mathbb{R}$, the set $\{\underline{g} > a\}$ is μ_a-positive. We have that $\{\underline{g} > a\} = \bigcup_{n\ge1}\bigcap_{k\ge n}\{g_k \ge a + \epsilon_n\} = \bigcup_{n\ge1} H_n$, where H_n is the increasing sequence $H_n = \bigcap_{k\ge n}\{g_k \ge a + \epsilon_n\}$ and $\epsilon_n \searrow 0$. Then we have

$$\int_{\{\underline{g}>a\}\cap B} \underline{\phi}'(f - a)d\mu = \lim_{n\to\infty} \int_{H_n\cap B} \underline{\phi}'(f - (a + \epsilon_n))d\mu. \tag{4.17}$$

Set $G_n = \bigcap_{k\ge n+1}\{g_k \ge a + \epsilon_n\}$ and note that $H_n = G_n \cap \{g_n \ge a + \epsilon_n\}$. Then for $B \in \mathcal{B}_m$, $m \le n$, we have

$$\begin{aligned} 0 &\le \int_{\{g_n\ge a+\epsilon_n\}\cap B} \underline{\phi}'(f - (a + \epsilon_n))d\mu \\ &= \int_{\{g_n\ge a+\epsilon_n\}\cap B\cap G_n} \underline{\phi}'(f - (a + \epsilon_n))d\mu + \int_{\{g_n\ge a+\epsilon_n\}\cap B\cap G_n^c} \underline{\phi}'(f - (a + \epsilon_n))d\mu. \end{aligned} \tag{4.18}$$

Now, we prove

$$\int_{\{g_n\ge a+\epsilon_n\}\cap B\cap G_n^c} \underline{\phi}'(f - (a + \epsilon_n))d\mu \le 0. \tag{4.19}$$

We can see that

$$\{g_n \ge a + \epsilon_n\} \cap B \cap G_n^c = \bigcup_{k\ge n+1} A_k, \tag{4.20}$$

where A_k are the following disjoint sets:

$$\begin{aligned} A_{n+1} &= \{g_{n+1} < a + \epsilon_n\} \cap B \cap \{g_n \ge a + \epsilon_n\}, \\ &\ \vdots \\ A_k &= \{g_k < a + \epsilon_n\} \cap \bigcap_{i=n+1}^{k-1} \{g_i \ge a + \epsilon_n\} \cap \{g_n \ge a + \epsilon_n\} \cap B. \end{aligned} \tag{4.21}$$

Then

$$\int_{\{g_n \geq a+\epsilon_n\}\cap B\cap G_n^c} \underline{\phi}'(f-(a+\epsilon_n))d\mu = \sum_{k\geq n+1}\int_{A_k}\underline{\phi}'(f-(a+\epsilon_n))d\mu. \tag{4.22}$$

Since $A_k = \{g_k < a+\epsilon_n\} \cap B_k$, where $B_k \in \mathcal{B}_k$, we have (4.19). Therefore by (4.18), we have

$$\int_{H_n\cap B} \underline{\phi}'(f-(a+\epsilon_n))d\mu \geq 0 \tag{4.23}$$

for all $B \in \bigcup_{n\in\mathbb{N}}\mathcal{B}_n$. Thus by (4.23) and (4.17), we get

$$\int_{\{\underline{g}\geq a\}\cap B} \underline{\phi}'(f-a)d\mu \geq 0 \tag{4.24}$$

for all $B \in \bigcup_{n\in\mathbb{N}}\mathcal{B}_n$. Therefore, the result is satisfied for all $B \in \mathcal{B}_\infty$. Thus $\underline{g} \in \tilde{\mu}_\phi^{\mathcal{B}_\infty}(f)$. □

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