Report on Second Order Solvers for the 1D Wave Equation

Girish Nivarti (50951094)

March 9, 2012

Problem: Solve the one dimensional wave equation given by:

$$\frac{\partial T(x,t)}{\partial t} + u\frac{\partial T(x,t)}{\partial x} = 0 \tag{1}$$

using the finite volume method with given initial condition, T(x, 0) = sin(2πx). Compare the computed solution to the analytically solved exact solution:

$$T(x,y) = sin(2\pi(x-2t)) \tag{2}$$

Solution:

I The error in numerical solutions using 20, and 40 cells has been plotted in Fig. 2. The solution profiles have been plotted in Fig. 1. The L_2 norms of the solutions are, 0.298335722822351 for 20 cells, and 0.0783745912112995 for solution using 40 cells. The L_2 norm reduces with increasing mesh size. It drops below 10^{-3} (value of norm 0.000990862339100897)for 357 control volumes, and below 10^{-4} (value of norm 0.0000999594805847065) for 1124 cells. The accuracy of the code is reasonable, however, there is a phase lead in the numerical solution which is a sizeable fraction (5%) for low cell numbers. A log plot of error norms versus time step for 40 control volume mesh, as well as a log plot of error norm versus cell size for CFL 0.3 were plotted. The trend plotted in Fig. 3 and exhibited a power law fit of second order. The solution norm shows a low order dependence on time step variation and has not been plotted.

II A graph showing various stability profiles has been plotted. For purpose of clarity and to demonstrate what happens around that point, a value that is slightly beyond borderline is shown in Fig. 4 to highlight the behaviour of numerical solution near stability boundary. A borderline value (of CFL) for stability was established by hit and trial. About 153 time steps ($\Delta t = 0.0065625$) were taken for the highest possible CFL number that produced a solution that had no recognizable changes from the exact solution. However, the L_2 norm changes abruptly at CFL = 0.5 - even a small increase beyond this number (which happens to be the analytically calculated maximum CFL) changes the norm by a sizable fraction. However, the plots show a 0.525 CFL number as being borderline based on observable properties of the graph. L_2 norms of these different solutions have been tabulated in Table 1. This is roughly consistent with an eigenvalue analysis.

III The Superbee limiter was implemented to evaluate fluxes in the domain for both smooth solution (sine), and a discontinuous solution (square wave). A CFL number of $0.8 \times CFL_{max} = 0.42$ (where, $CFL_{max} = 0.525$ was used, as it was the observable maximum) was used for these calculations for a mesh with 100 control volumes. Plots of smooth solution (Fig. 6), and discontinuous solution (Fig. 7) have been made showing calculations using limited (Superbee) and non-limited flux evaluation.

CFL Number	Time Step (Δt)	L_2 Norm	Stability Character
0.3	0.00375	0.138003604480734	Surely Stable
0.525	0.0065625	0.177735260766837	Borderline Stable
0.528	0.0066	0.221828943126969	Slightly Unstable
0.54	0.00675	7521.55871620826	Quite Unstable

Table 1: Stability characteristics of numerical solutions with different CFL numbers.

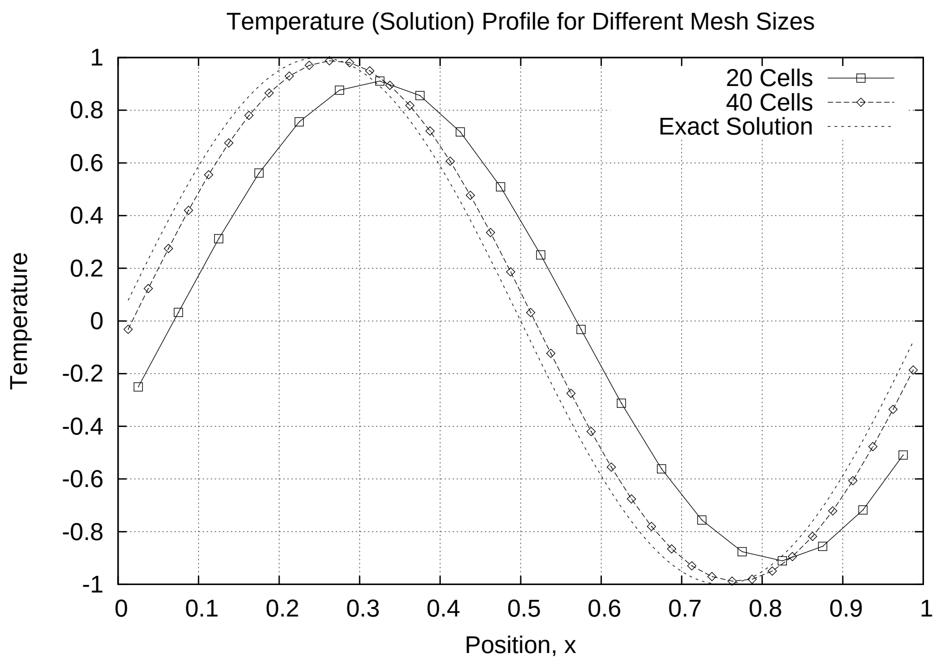

Figure 1: Numerical solutions of the 1D wave equation using second order upwind spatial discretization, and two-stage Runge Kutta time marching scheme with different mesh sizes (20, and 40 control volumes) compared with the exact solution after 1 time unit. The L_2 norms of the solutions are, 0.298335722822351 for 20 cells, and 0.0783745912112995 for solution using 40 cells.

CFL Number	Exact Solution Character (Δt)	Flux Evaluation	L_2 Norm
0.42	Smooth	Unlimited	0.0369757641117924
0.42	Smooth	Superbee	0.0276493801287461
0.42	Discontinuous	Unlimited	0.336394727433405
0.42	Discontinuous	Superbee	0.13283223625804

Table 2: Solution error characteristics for smooth, and discontinuous solutions with, and without limiting. Interestingly the Superbee solution has lower norms in both smooth and discontinuous solutions. This is due to the phase lead that makes the unlimited solution have a bigger norm.

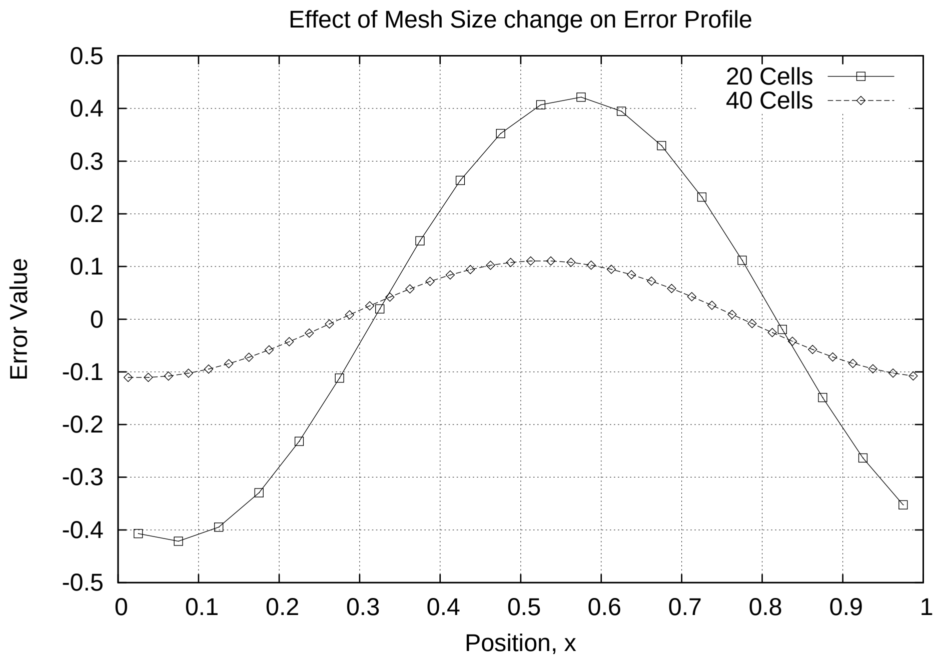

Figure 2: Errors in the numerical solution for 1D wave equation for meshes with 20, and 40 control volumes. Errors subside quickly with increasing control volumes yet oscillate about values making it difficult to find threshold mesh sizes for given L_2 norm order.

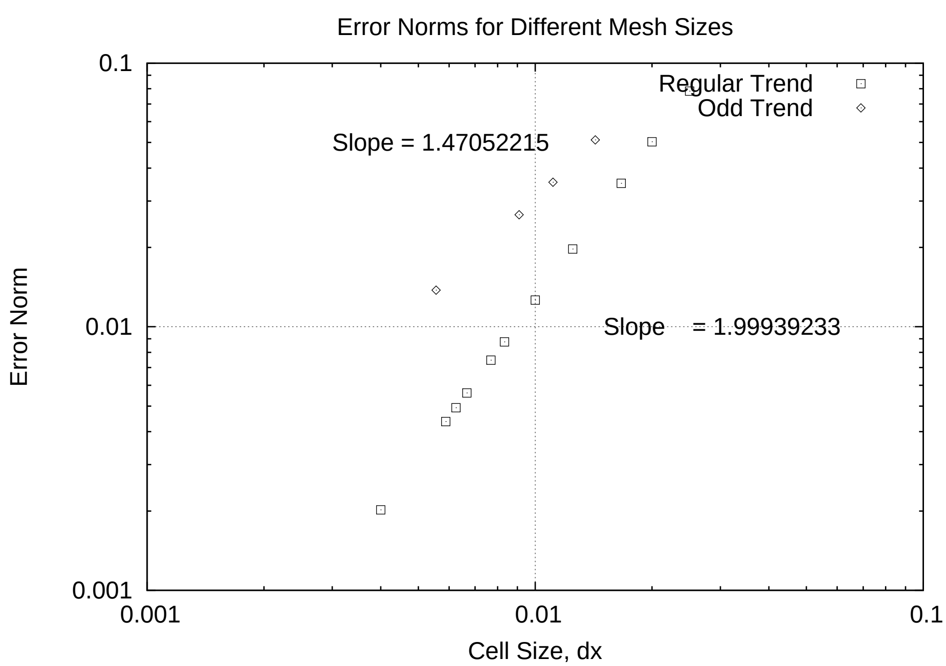

Figure 3: Errorn norms (L_2) of the numerical solution using second order upwind spatial discretization, and two-stage Runge Kutta time marching scheme has been plotted on a log scale for a range of mesh sizes. The L_2 norms fit the general trend with slope 1.99939233, while for some odd points, another trend with a slightly lower order (1.47052215) seems to exist.

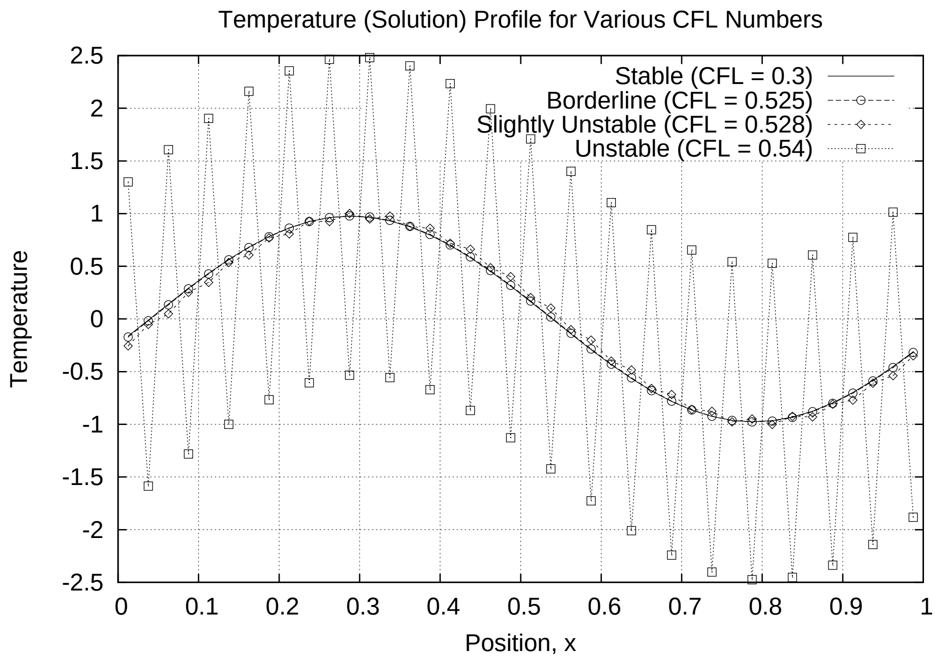

Figure 4: Numerical solutions with different CFL numbers have been compared to highlight the behaviour of profiles with varying time step sizes. The solution is sensitive near the borderline making it difficult (not impossible) to pin-point borderline value. The profile for a near borderline yet unstable solution has been plotted to show the changes in solution near the borderline.

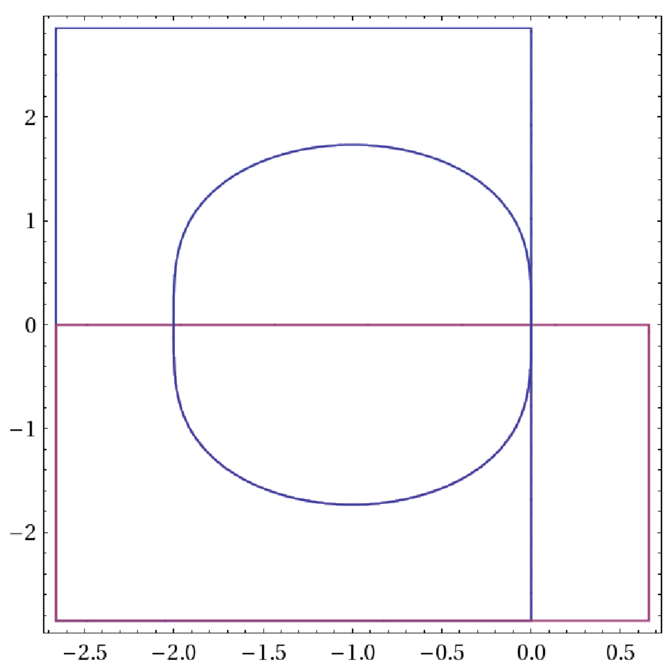

Figure 5: Two stage Runge-Kutta time marching scheme has the given contour for amplification factor, $\sigma = 1$ with the minimum real value being $\lambda\Delta t = -2$. The domain for second order upwind has a contour that is primarily dictated by the real values of CFL number. The minimum value is $-4\frac{u\Delta t}{\Delta x}$. Hence, that gives $CFL_{max} \leq 0.5$.

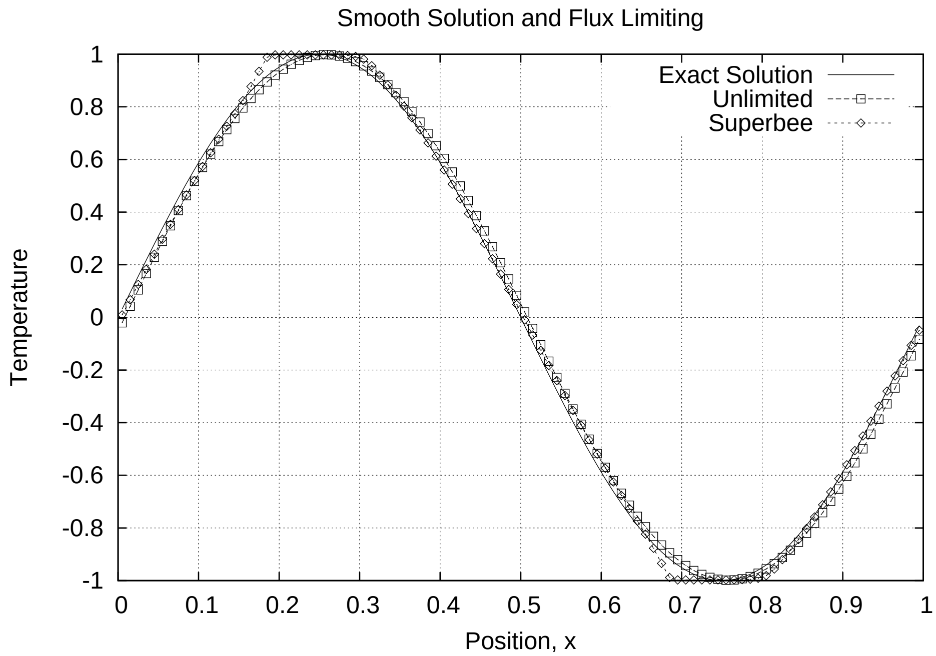

Figure 6: Profiles of numerical solutions with (Superbee), and without flux limiting for a 100 control volume mesh size have been shown to demonstrate failure of flux limiting which is based on purely discontinuous solutions. ENO schemes prove useful for solution exhibiting mixed character.

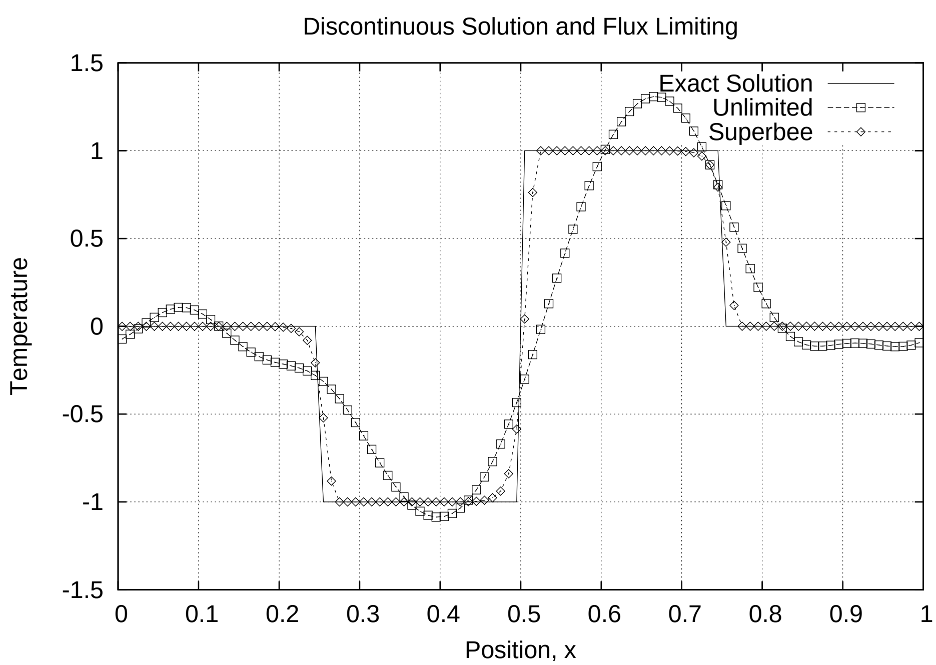

Figure 7: The Superbee flux limiting proves to be highly accurate in comparison to a non-limited scheme (for a 100 control volume mesh size) in solving a square wave.

Comments:

- A phase lead is seen in the RK2 solution of the smooth function (unnoticeable in the discontinuous case due to other predominant errors). In the solution with 20 control volumes, a single time step (5%) lead was noticed, and with 40 control volumes the solution leads by 3 time steps (3%).

- While implementing the Superbee flux limiting, as an experiment, the corner points in the Ψ - r curve, were assigned to different parts of the curve. For example the point (r = 0.5, Ψ = 1) can be assigned to curves Ψ = 2r, or Ψ = 1. To my disappointment, there is no significant change by changing where this corner point lies.

- The odd trend of norms can be attributed to the irregular values of dt obtained for certain mesh sizes that make the errors in floating point propagate and finally cause the time marching to carry to a time greater than required (inspite of running the loop based on number of time steps rather than on total time, and implementing a dt/2.0 correction factor mechanism). These errors are not 1^{st} order due to this correction factor, but are $\approx$ 1.5 order. It is difficult to eliminate such error.

- For the last few control volumes, the value of r for limiting schemes depends on cells outside the domain. To handle this, ghost cells were created that contained values of the first few control volumes. In effect, the values along the wave are tabulated beyond the mesh. This maintains periodicity and also allows us to use limiting for the last 2 cells. To add a caption of glory to our success, the L_2 norm decreases as well!

- The L_2 norms provide an interesting picture of numerical solution as tabulated in Table 2. Although the non-limited flux evaluation is considerably accurate with smooth solutions, the Superbee clearly fails by losing slope near extrema (Fig. 6). But the L_2 norm in the discontinuous Superbee solution is greater than that for the smooth Superbee solution (looks - Fig. 7 - can be deceiving)! Indeed, the limited solution fails near the step change (in the square wave solution) by a greater amount than its failure near extrema in smooth solutions.

- A crucial change was made on your suggestion, to force the time loop to stop nearest to the final time, T ($($tFinal) in code). This was done by counting number of integer time steps rather than calculating floating-point time. This makes a huge difference, because with comparing floating point numbers although there is a considerable drop in error on doubling the number of control volumes, but, it is noticed that the error norm fluctuates as the cells are increased further. It is difficult to pin-point a number for control volumes, for which error decreases steadily with increasing cells. In fact, there seem to be two modes of errors that steadily decrease with increasing control volumes but have different orders (1^{st} and 2^{nd}) of magnitude.