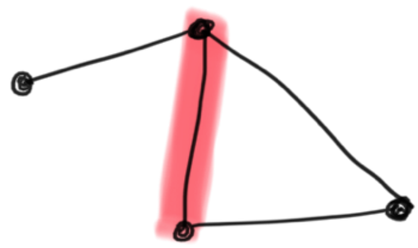
Max matching - Blossoms (theory)

Ref: CS 105 - winter 05 (CS Dartmouth).

matching in a graph G is a subset $M \subseteq$ Edges E so that

$$e \cap e' = \phi \text{ for } e \neq e' \text{ in } M.$$

maximal matching : Matching M maximal if $M \not\subset M'$ for any other matching M'



(can't add more edges)

maximal matching

maximum matching : matching M so that

$$|M| \geq |M'| \;\; \forall \text{ matchings } M'$$

maximum matching $\Rightarrow$ maximal matching

$$\not\Leftarrow$$

maximal match ⇐ greedy. keep adding edges that still keep it a match

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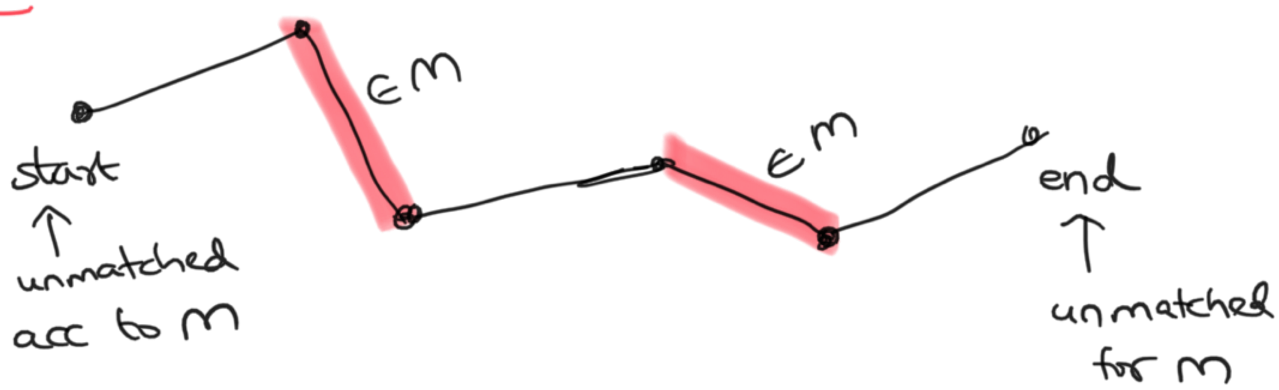
How to find maximum matching ?

* Given matching M, vertex $v \in V$ is called matched if $v \in e$ for some $e \in M$

* Augmenting path for M : path in G so that

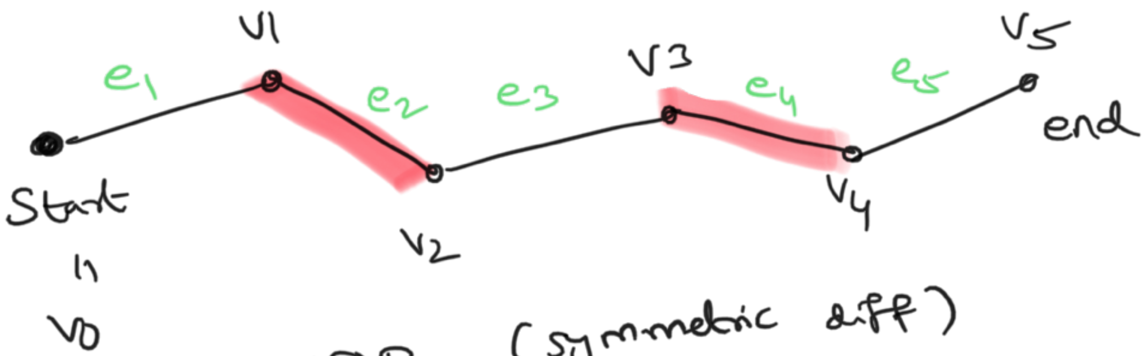


and alternate edges of path $\in M$.

Berge's lemma Let M be a matching.
M maximum $\Leftrightarrow$ no augmenting path for M

Pf

$\Rightarrow$ Let M be maximum. Suppose $\exists$ augmenting path for M



# of edges in path not in M > # of edges in path in M (actually exactly 1 more)

$M \oplus P$ (symmetric diff)

$\to M \setminus \{e_2, e_4\}$

$\cup \{e_1, e_3, e_5\} = M'$

is also a matching.

$|M'| > |M|$ $\to\leftarrow$

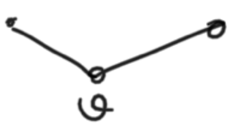
$\Leftarrow$ If no augmenting path, show M max.

Let M' be a maximum matching

show $|M| = |M'|$

$\to M \oplus M' = \{$collection of edges in $M \setminus M'$ $\cup$ $(M' \setminus M)\}$

$\to$ Look at subgraph H induced by edges $M \oplus M'$



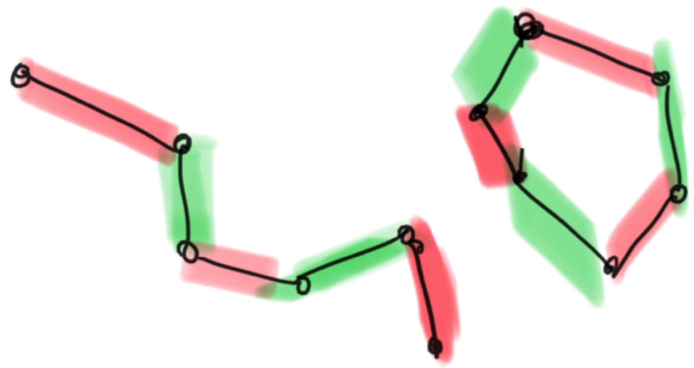
$\leftarrow \deg_H(v) \leq 2$ (1 from M, 1 from M' atmost)

FACT: any graph with all deg ≤ 2

$\to$ disjoint paths / cycles.

(Look at connected component and prove fact there ...)

Granting FACT, any path / cycle



$\in m \oplus m'$

must have alternate edges from m, m'

as both matchings

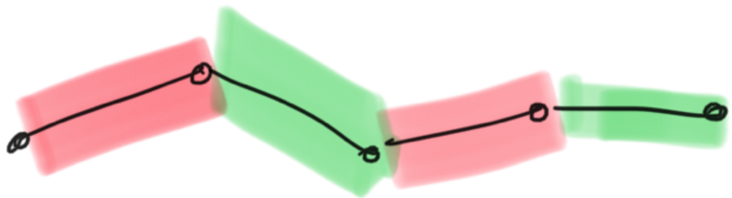
further any cycle has to be even length

??

so equal # of edges from m, m'

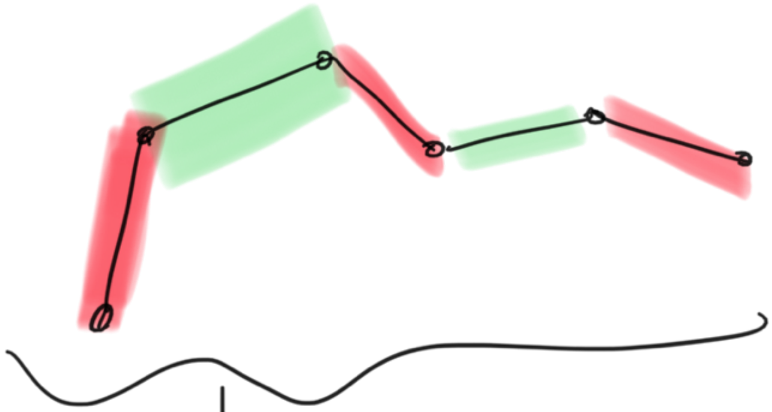
what about paths?

(a)



even path, again equal # of edges from m, m'

(b)

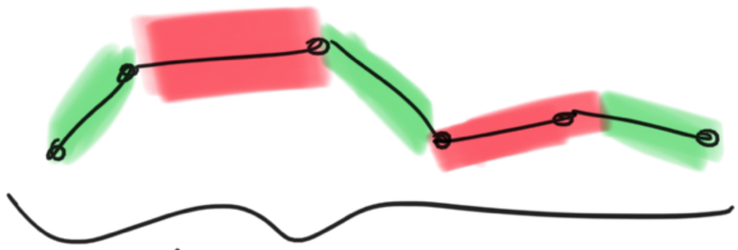


odd path more m than m'

augmenting path for m'

$\rightarrow\leftarrow$ as m' maximum

(c)



odd path more m' than m

augmenting path for m

$\rightarrow\leftarrow$ hypothesis

so $|m| = |m'|$

matching algorithm

$m = \emptyset$

Find an augmenting path for m

$m = m \oplus P$

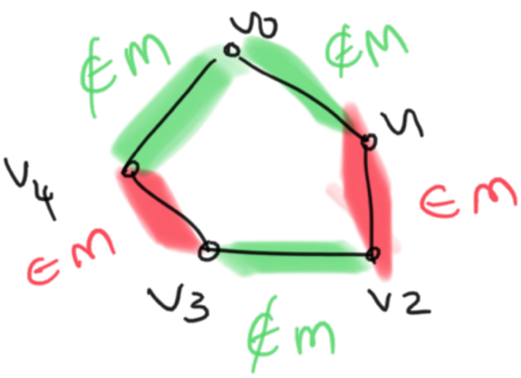
If no more augmenting paths, stop.

$M = M \oplus P \Rightarrow |M| \mathrel{+}= 1.$

If G has $|V|$ vertices, M can have atmost $|V|/2$ edges so can add augmenting paths at most $|V|/2$ times ...

How to find augmenting paths?

odd alternating cycle (Blossom)



$v_0 \notin M$ also.

$v_0 - v_1 \cdots - v_{2k}$ with $v_0 \notin M$

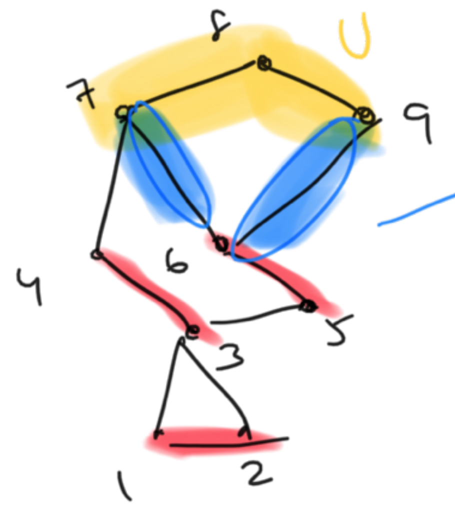
$v_1 v_2,\ v_3 v_4 \ldots \in M$

$v_0 v_1 \ldots,\ v_{2k} v_0 \notin M$

Graph G, M matching

$U \leftarrow$ set of all unmatched vertices

G/U : $U \rightarrow$ squeeze into 1 vertex

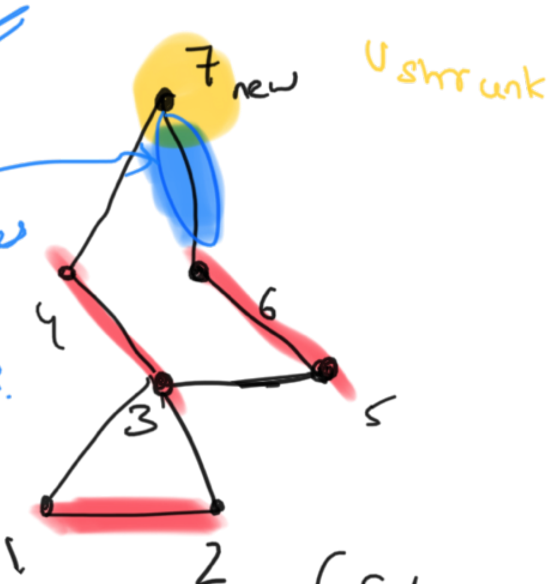


(G, m)

Ques

are we allowing multiple edges in H bet same 2 vertices ??

think.. yes



(G/U, m/U)

Observation

augmenting path in (G, m) starting and ending in U → odd alternating cycle in (G/U, m/U)

note that any such path looks like

→

Alternating breadth first tree

(H, m)   H = G/U

$s \notin m$

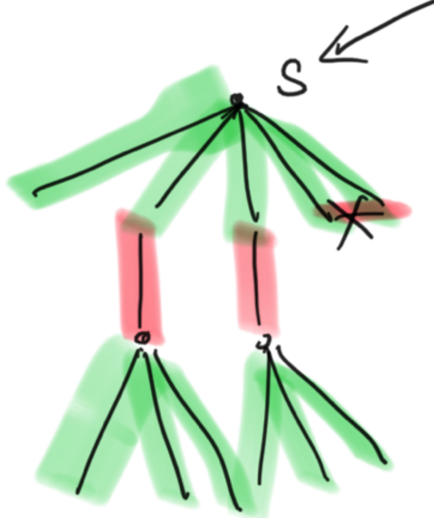
only unmatched vertex..

level 0

level 1 ← take all edges $\notin m$

level 2 ← take all matching edges

level 3 ← take all $\in m$



un matching edges ($\notin m$)

$\vdots$

level i vertices : U_i

Alternating BFS Tree

(1) $U_0 = \{s\}$ Level $[s] = 0$

$\forall \; v \in V \setminus U_0$, level$(v) = \infty$

$E_T = \phi$ (tree edges)

$i = 1$

(2) while $(U_{i-1} \neq \phi)$

$U_i = \phi$

foreach $v \in U_{i-1}$:

for each w adj to v

i odd

i even

If Level$[w] = \infty$ and $v-w \notin m$,

$U_i \cup \{w\} = U_i$

Level$[w] = i$

$E_T = E_T \cup \{v-w\}$

If Level$[w] = \infty$ and $v-w \in m$

$U_i \cup \{w\} = U_i$

Level$[w] = i$

$E_T = E_T \cup \{v-w$

U_∞ = not reachable .. $= V \setminus U_0 \setminus U_1 \ldots$

V

$U_0 \quad | \quad U_1 \quad | \quad \ldots \quad | \quad U_\infty$

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edges not in Alternating breadth first tree

Type E_∞: edges between 2 U_∞ vertices.

Type A: Level U_{2k+1}

non matching edges

$v \in$ any U_j $\quad (j = \infty)$ also

Type B: Level U_{2k+1} $\xrightarrow[\text{matching}]{}$ Level U_{2k+1}

Level U_{2k} $\xrightarrow[\text{unmatching}]{}$ Level U_{2k}

Type C: Level U_{2k}

unmatched edge

Level U_{2k+1}

<u>Note</u>

if $\exists\ v \in U_\infty$ with $v \overset{\in M}{\text{———}} u$

$\Rightarrow u \in U_\infty$

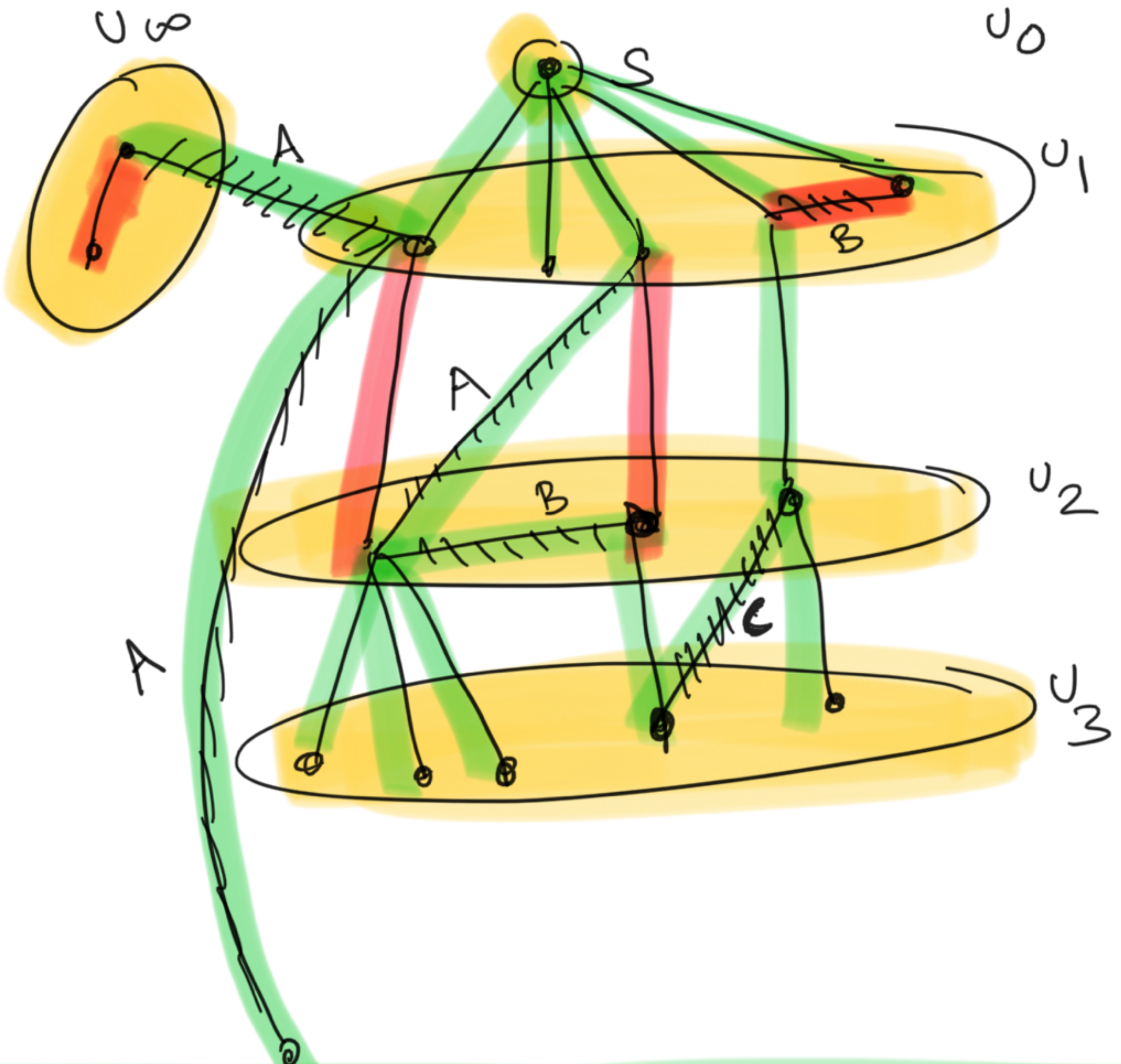
Pf: Suppose $u \notin U_\infty \Rightarrow u \leftarrow$ odd / even level

If u = odd level $\Rightarrow v \in U_{2k+2}$
$2k+1$

If u : even level $\Rightarrow v \in U_{2k-1}$.
$2k$

(note $s \in \{U_0\} \leftarrow$ unmatched)

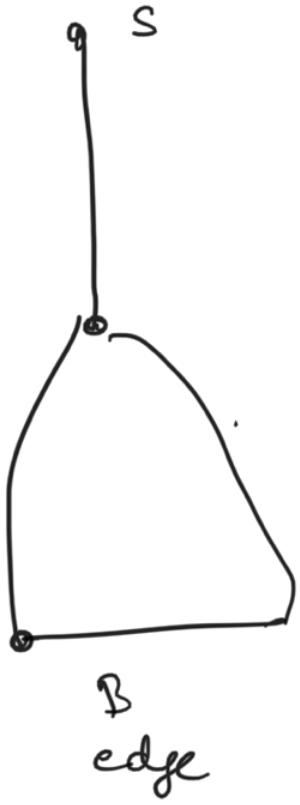
So Alternating Bread First Tree looks like



FACT

In $H = (G/U, M/U)$, every alternating cycle must pass through at least 1 type B edge

Granting this



alternating path + cycle..

use tree edges
and go up..

Type B can be identified in algo itself.

$v \in U_{i-1}$

$w \in \text{Adj}(v)$

if level $w = i-1 \Rightarrow v - w$ Type B

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augmenting path in (G, m, U) ← unmatched vertices

vs

alternating odd cycle in $(H = G/U, m/U, \{s\})$, U_0

$\phi^{-1}(C_1) \longleftarrow C_1$

augmenting

alternating odd

augmenting path in (G, m)

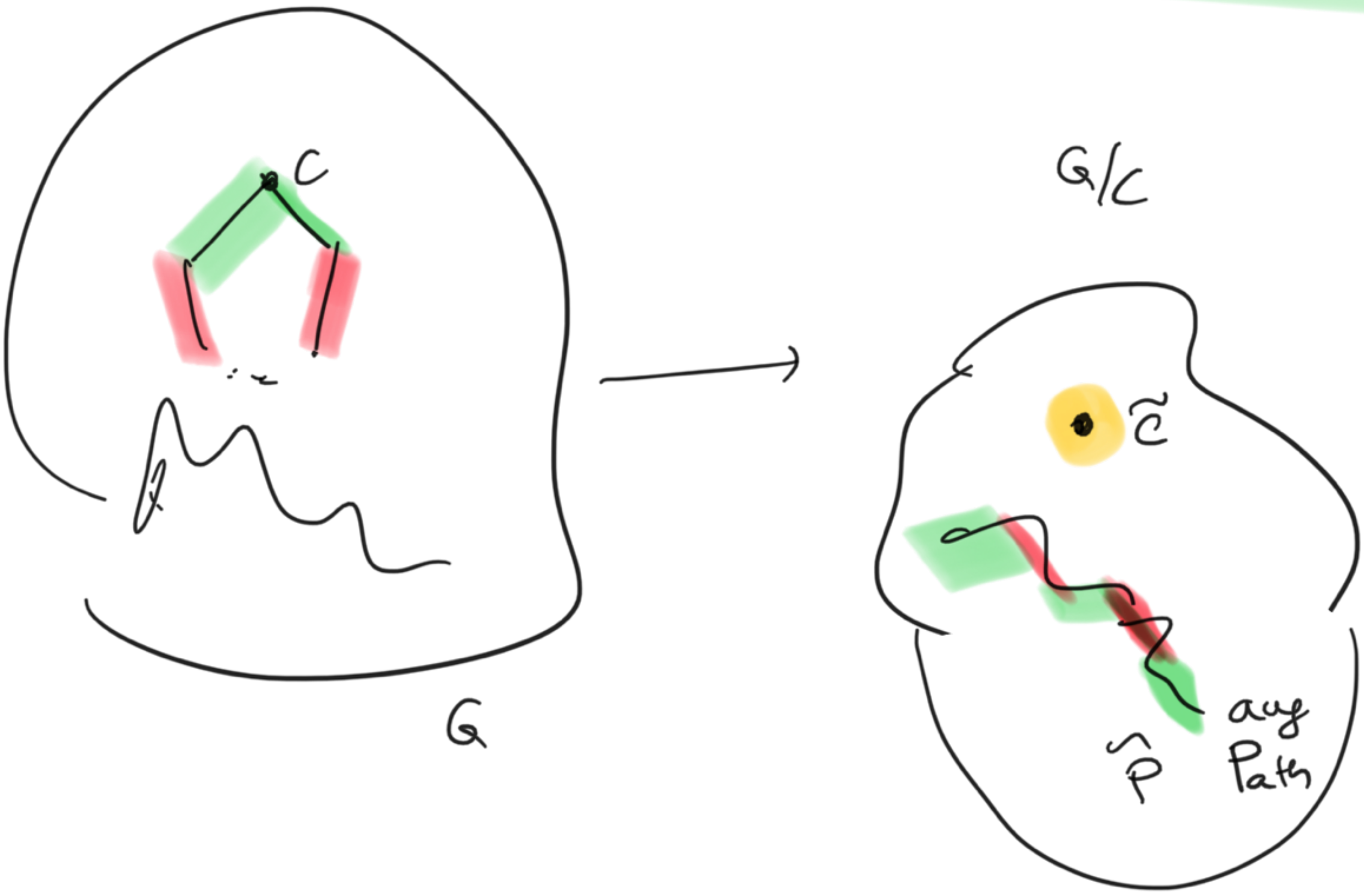
cycle in (G, m)

Thm if C = alternating odd cycle in G,

then G has an augmenting path

$\Leftrightarrow$ G/C has an augmenting path

Pf:

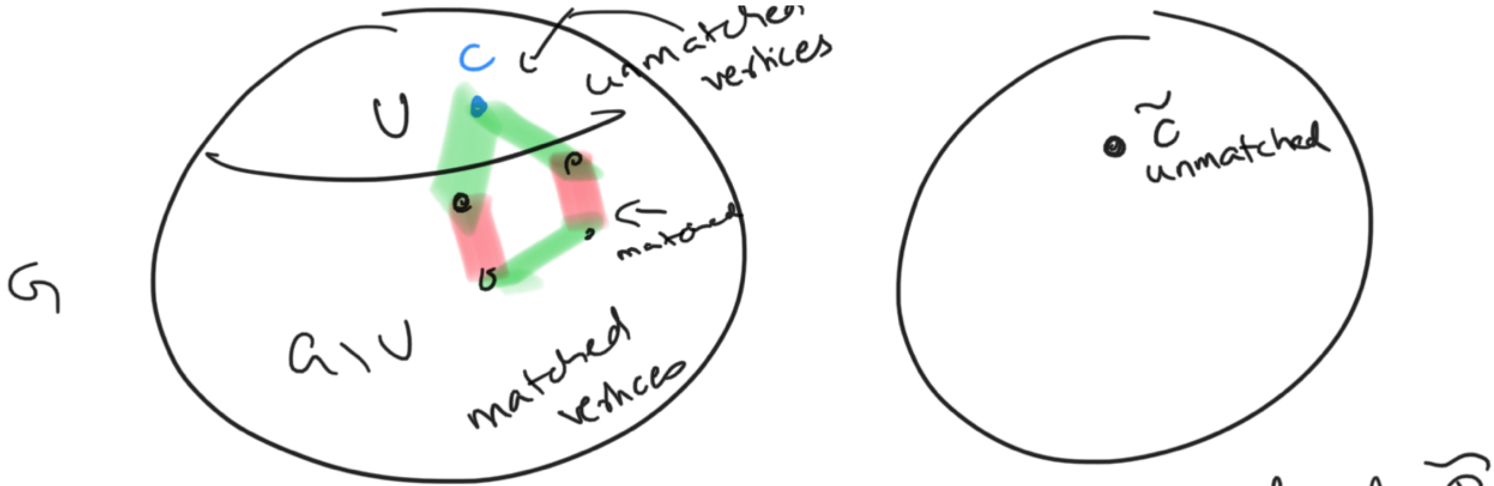


$\boxed{G/C \text{ augmenting path} \Rightarrow G \text{ augmenting path}}$

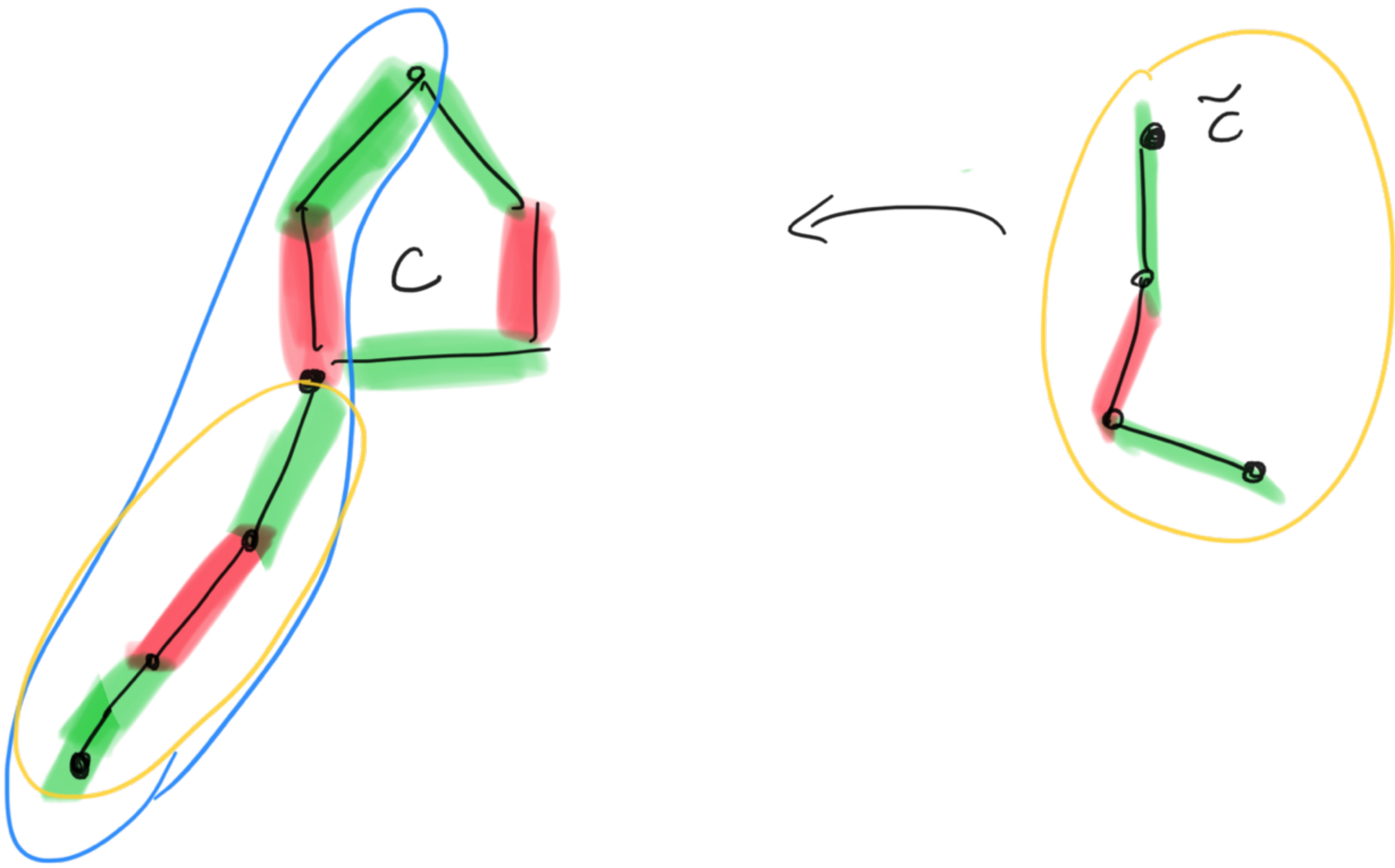
ⓐ If $\tilde{C} \notin \hat{P}$, $\hat{P}$'s inv image in G is augmenting path for G.

ⓑ if $\tilde{C} \in \hat{P}$,

G/C



$\tilde{c}$ unmatched $\Rightarrow$ $\tilde{c}$ occurs at end of $\tilde{P}$



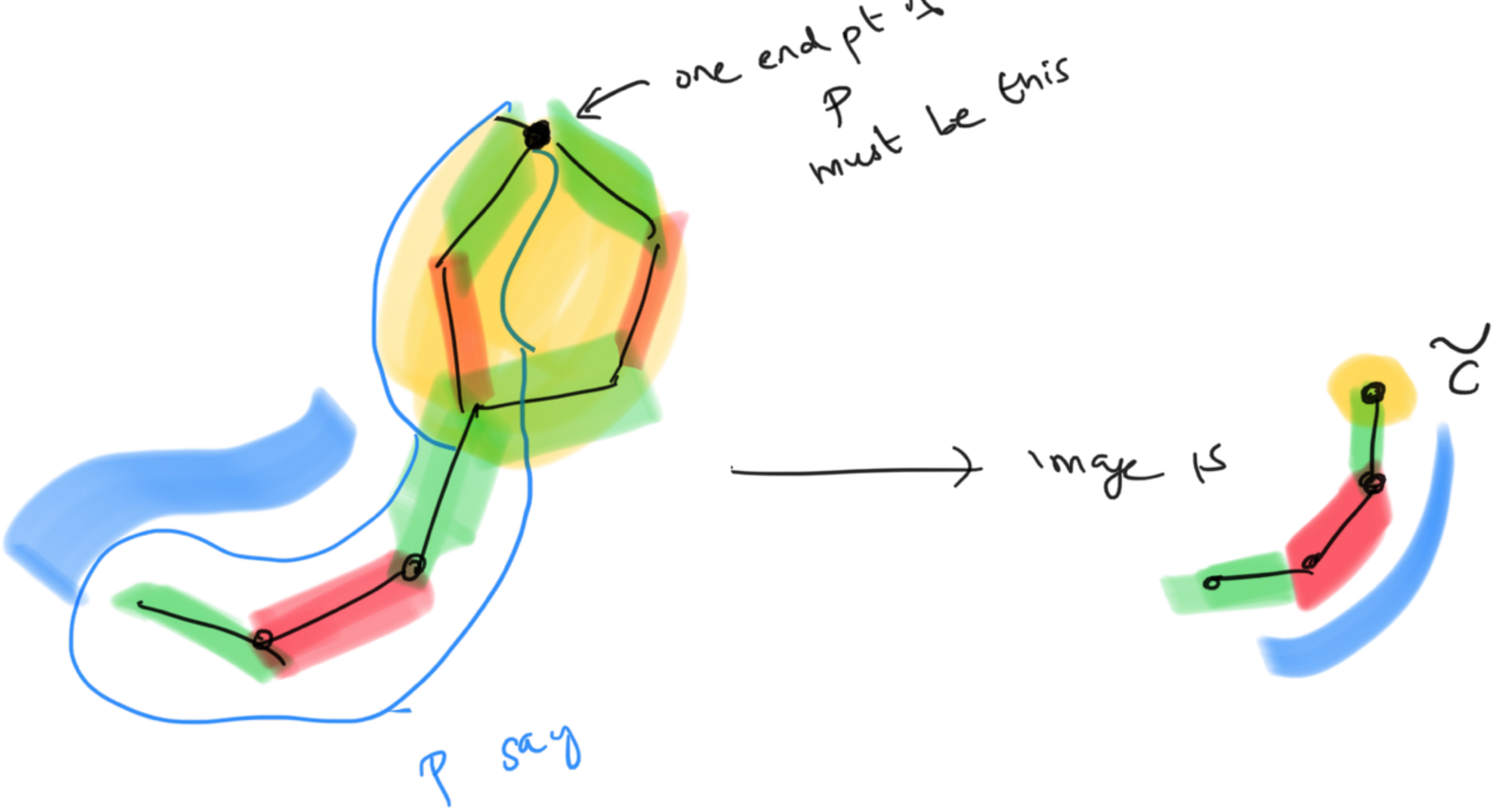
G augmenting path $\Rightarrow$ G/C augmenting path

pf

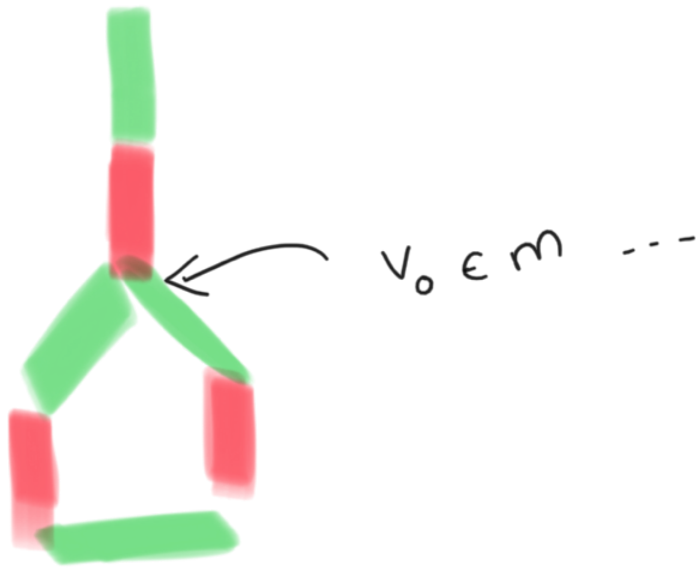
P aug. path in G

If P doesn't intersect C, $\phi(P) =$ aug path in G/C

If P intersects C,



(Discussion ignores blossoms with stems....)



can contract this also...

explained in algorithm written up in next pdf