All pairs shortest paths

Let $V = \{1, 2, \ldots n\}$ in graph G

Goal : Find shortest paths bet (i, j) $\forall$ $i, j \in V \times V$

Observation

→ shortest path is without loops

→ so never visits any vertex twice

→ so length $\leq n-1$

(# of edges used)

Shortest path bet i, j

$$\left.\underset{i}{\bullet} \to v_1 \to v_2 \; \ldots \; \to v_m \to \underset{j}{\circ} \right\} \text{ looks like}$$

$v_k \neq v_l$

$v_k \neq i, j$

strategy : Find $W^k(i, j)$ = length of shortest path

bet i, j if in between vertices ($v_1, \ldots, v_m$) can only be from

$[1, 2, \ldots, k]$

(i, j need not be in $[1, 2, \ldots k]$)

Inductively find $W^k(i, j)$

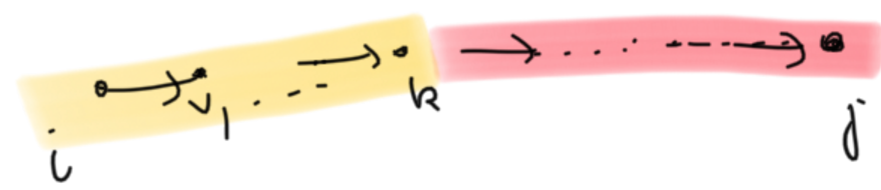
* $$W^0(i,j) = \begin{cases} w & \underset{i}{\bullet} \xrightarrow{w} \underset{j}{\bullet} \\ \infty & \text{if no edge bet } \underset{i}{\circ} \longrightarrow \underset{j}{\circ} \end{cases}$$

* If you know $W^{k-1}(i, j)$ $\forall i, j$, how to find $W^k(i, j)$

shortest path bet i and j using $[1,2,\ldots k]$ DOES NOT USE k

$\Rightarrow \; w^k(i,j) = w^{k-1}(i,j)$

shortest path bet i and j using $[1,2,\ldots k]$ USES k



$$w^k(i,j) = \underbrace{w^{k-1}(i,k)} + \underbrace{w^{k-1}(k,j)}$$

$$w^k(i,j) = \min \begin{bmatrix} w^{k-1}(i,j) \\ w^{k-1}(i,k) + w^{k-1}(k,j) \end{bmatrix}$$

<u>Floyd - Warshall algorithm</u>

* Adjacency matrix :

$$W_0 = \begin{bmatrix} w^0(i,j) \end{bmatrix}$$

* $k \in [1,2,\ldots n]$

Find $w^k(i,j) = \min \begin{bmatrix} w^{k-1}(i,j) \\ w^{k-1}(i,k) + w^{k-1}(k,j) \end{bmatrix}$

* $w^n(-,-) \leftarrow$ gives weights of shortest paths for all pairs

Algorithm

$O(n^3)$

for k in [1, 2, -- n]:
  for i in [1, 2 ... n]
    for j in [1, 2, ... n]

$$w^k(i,j) = \min \begin{cases} w^{k-1}(i,j) \\ w^{k-1}(i,k) + w^{k-1}(k,j) \end{cases}$$

update $w^k(i,j)$

$w^n(i,j) \leftarrow$ shortest path bet i and j

Space complexity

Naive: $W^k(i,j)$ $\quad 1 \le i, j, k \le n$

$O(n^3)$

less naive: For level k, only need level k-1

$w^k(-,-)$ $\quad w^{k-1}(-,-)$

So can keep only 2 levels data at any point and over write them

$O(n^2)$ space complexity

Warshall

'Algo for transitive closure'

edge relation $\longrightarrow$ path relation

$A(i,j)$ adjacency matrix $\longrightarrow$ $P(i,j)$ is there a path bet i, j

Floyd

Adapted warshall's algorithm to find length of shortest path as well

Warshall's algorithm

Given adjacency matrix: $\quad A(i,j) = \begin{cases} 1 & \text{if } i \to j \\ 0 & \text{else} \end{cases}$

Find path matrix $P(i,j)$

$$= \begin{cases} 1 & \text{if } i \xrightarrow{\text{path}} j \\ 0 & \text{otherwise} \end{cases}$$

Def : $P^k(i,j) = 1$ if $\exists$ path

$$i \underbrace{\to \to - - - }_{\text{vertices betw } \in\ [1,2,\dots k]} \to j$$

$$P^0(i,j) = A(i,j)$$

Compute $P^k(i,j)$ from $P^{k-1}(-,-)$

$$P^k(i,j) = \left(\begin{array}{l} P^{k-1}(i,j) \\ \quad \text{or} \\ P^{k-1}(i,k) \text{ and } P^{k-1}(k,j) \end{array} \right)$$

$i \overbrace{\frown}^{\text{vertices in } [1,..k-1]} j$

$$\underbrace{i \frown k}_{\text{vertices in } [1,..k-1]} \underbrace{\frown j}_{\text{vertices in } [1,..k-1]}$$