Matrix multiplication
(DP)

$A = (a_{ij})_{m \times n}$

$B = (b_{ij})_{n \times p}$

$C = AB = (c_{ij})_{m \times p}$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad - \quad O(n)$$

So computing C — $O(mnp)$

many matrix multiplications

$(AB)C = A(BC)$

$\underbrace{m \times n \quad n \times p}$ $\quad p \times q \qquad O(mnp)$

$\underbrace{m \times p \quad p \times q} \qquad O(mpq)$

$O(mnp + mpq)$

$m \times n \quad \underbrace{n \times p \quad p \times q} \qquad O(npq)$

$\downarrow$

$\underbrace{m \times n \qquad n \times q} \qquad O(mnq)$

$\downarrow$

$m \times q$

$O(npq + mnq)$

$m = 1 \qquad n = 100 \qquad p = 1 \qquad q = 100$

$mnp = 100$

$mpq = 100$

$\underline{200}$

$npq = 10000$

$mnq = 10000$

$\underline{20000}$

more efficient to find

$ABC = (AB)C$ than $A(BC)$

Problem Given $M_1 \quad M_2 \quad M_3 \quad \cdots \quad M_n$

* dim of M_i : $r_i \times c_i$
* $c_i = r_{i+1} \quad \forall \quad 1 \le i < n$

Find optimal order to compute product

$$(M_1 M_2 \cdots M_k)(M_{k+1} \cdots M_n)$$

last bracket

$$\Big[\, r_1 \times c_k \qquad r_{k+1} \times c_n \,\Big]$$

last multiplication cost

$r_1 c_k c_n$

$c_k = r_{k+1}$

Total cost $= r_1 c_k c_n + \text{cost}(M_1 \cdots M_k) + \text{cost}(M_{k+1} \cdots M_n)$

if last bracket is separated @ k

$$\text{Optimal cost} = \min_{1 \le k < n} \Big[r_1 c_k c_n + C(M_1 \cdots M_k) + C(M_{k+1} \cdots M_n) \Big]$$

As you solve subproblems, you'll have to find $C(M_i \cdots M_j) \ldots$

So DP – find $C(i,j) := \text{Cost}(M_i M_{i+1} \cdots M_j)$

$$C(i,i) = 0 \qquad \text{(no multiplication)}$$

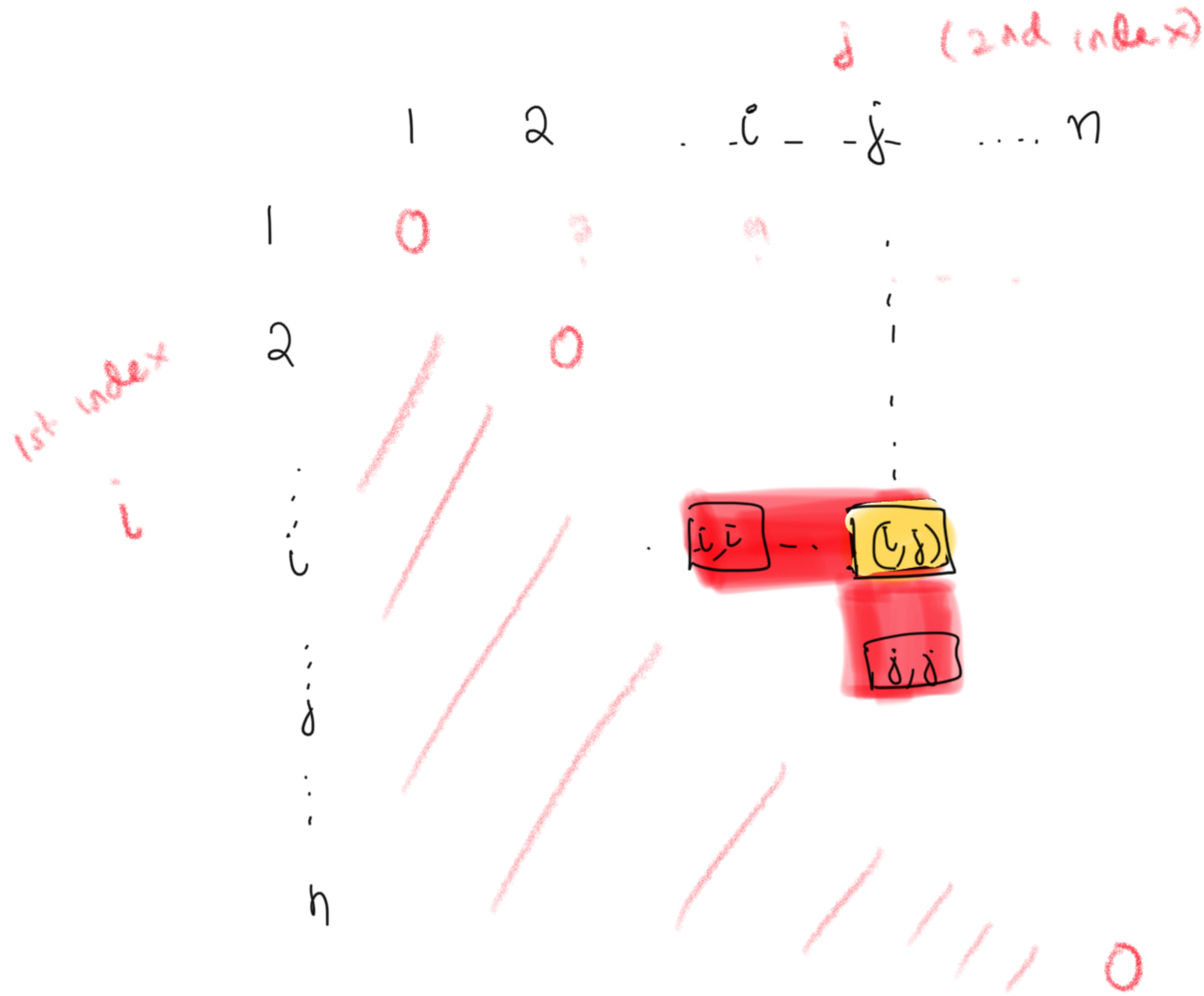
$$C(i,j) = \min_{i \le k < j} \left[C(i,k) + C(k+1,j) + r_i c_k c_j \right]$$

Only want $C(i,j)$ $i < j$

eg $C(i,i+1) = \min_{\substack{k=i \\ i \le k < i+1}} \left[C(i,k) + C(k+1,i+1) + r_i c_k c_{i+1} \right]$

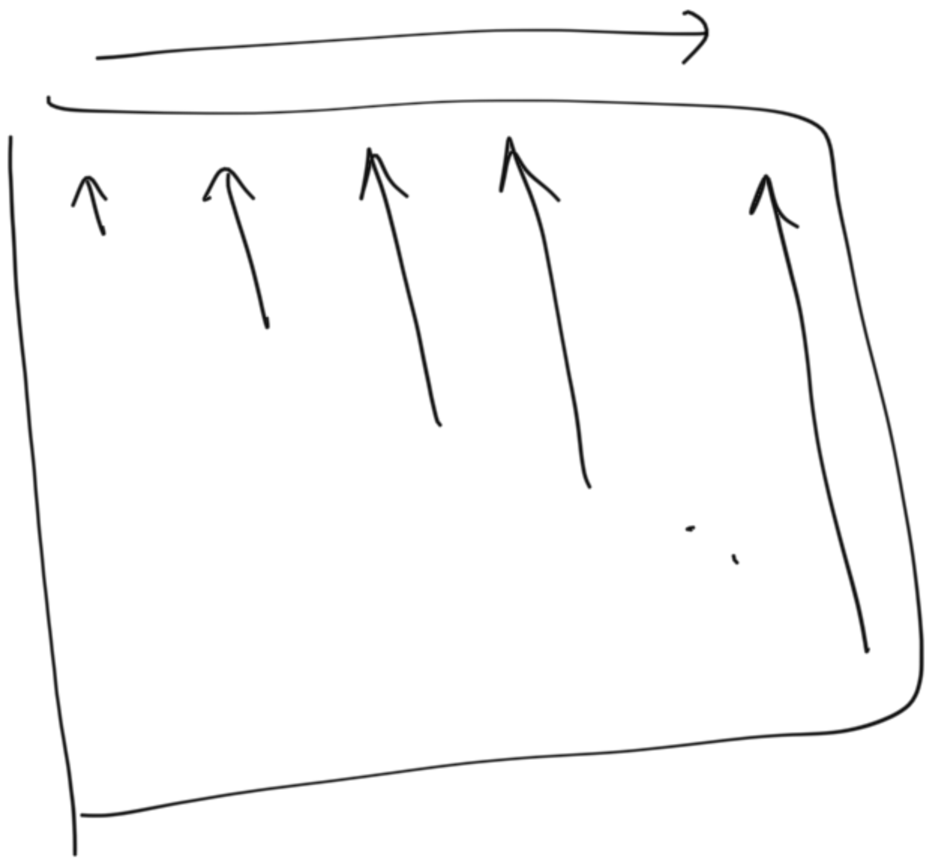
$$= C(i,i) + C(i+1,i+1) + r_i c_i c_{i+1}$$

$$= r_i c_i c_{i+1}$$

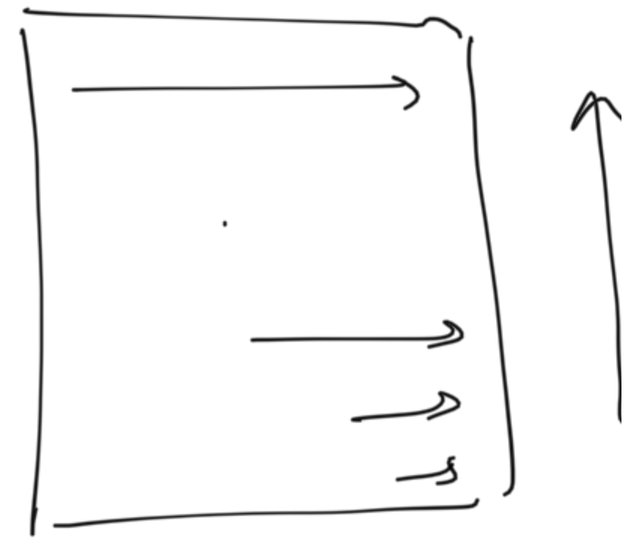


$C(i,j)$ uses $C(i,k)$, $C(k+1,j)$ for $i \le k < j$

Fill up

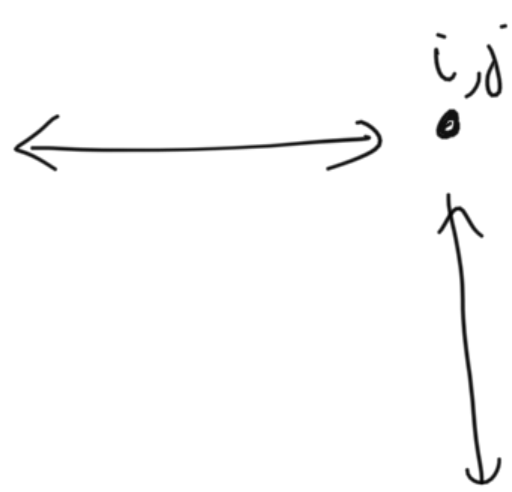


or



<u>Complexity analysis</u> : Filling up $O(n^2)$ table

Filling up 1 entry



$O(n)$

so $O(n^3)$ overall complexity