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Magnetic field calculation for arbitrarily shaped planar wires

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It is common to use the Biot–Savart law as a tool to explicitly calculate the magnetic field due to currents flowing in simply shaped wires such as circular loops and straight lines. In this work, by using the Biot–Savart law and its inherent geometric properties, we derive a very simple integral expression that allows a straightforward computation of the magnetic field due to arbitrarily shaped planar current-carrying wires, at a point that lies in the same plane as the current filament. Such an expression is conveniently written in terms of the wire's shape $r=r(\theta)$. We illustrate the usefulness of our result by calculating the magnetic field at specific points in the wire's plane due to currents flowing in conic curves, spirals, and harmonically deformed circular circuits. Relevant asymptotic behavior is calculated in various limits of interest.

I. INTRODUCTION

During their first exposure to electricity and magnetism, students learn how to calculate magnetic fields by using the Biot–Savart law.[1,2] This law allows one to determine the total magnetic field **B** at a given point in space as the superposition of infinitesimal contributions $d\mathbf{B}$, caused by the flow of current I through an infinitesimal current path segment $d\mathbf{s}$. Nevertheless, students soon realize that only a few simple current distributions provide an analytically manageable evaluation of the magnetic field. The vectorial nature of the Biot–Savart law can lead to very awkward calculations, depending on the problem's geometry. Because of this fact, introductory,[1,2] intermediate,[3] and advanced[4] level textbook examples and exercises generally restrict themselves to the evaluation of magnetic fields due to simply shaped current-carrying wires. Students are often asked to calculate the magnetic field at the center of circular current loops, or to determine **B** at a given distance from infinite/finite straight wires. Many other similar textbook exercises are created based on the superposition of these two basic geometric wire configurations. To the best of our knowledge a systematic extension of such kinds of magnetic field calculations, generalized to currents flowing in wires of more general shape, has never been examined.

In this paper we go beyond the existing textbook material[1–4] and derive a general expression which permits us to calculate, with great facility, the magnetic field due to an arbitrarily shaped (not self-intersecting), planar, current-carrying wire at a point lying in the plane of the wire. We obtain a very simple and compact expression for the field, written in terms of the wire shape $r=r(\theta)$. We emphasize that our derivation has the advantage of being fairly simple, only requiring the knowledge of basic geometric concepts from elementary calculus and analytic geometry.[5] This basic result expands the degree of applicability of the Biot–Savart law as a useful tool to exactly solve a whole new set of elementary problems.

We illustrate the usefulness of our result by explicitly evaluating the magnetic field due to currents flowing in ellipses, parabolas, hyperbolas, spirals, and harmonically deformed circular wires, at specific points located in the plane defined by each of these assumed shapes. Relevant asymptotic limits are consistently discussed for all these new examples. We trivially recover, using our field expression, the field due to circular and straight wires.

II. MAGNETIC FIELD CALCULATION

Let us begin by stating the Biot–Savart law in a more quantitative fashion. If $d\mathbf{s}$ is an element of length (pointing in the direction of current flow) of a wire which carries a current I and $\hat{r}$ is the unit vector pointing from the element of length to the observation point O, then the infinitesimal field contribution due to this element at O is given by[1,2]

$$d\mathbf{B}=\frac{\mu_0 I}{4\pi}\,\frac{(d\mathbf{s}\times\hat{r})}{r^2}, \tag{1}$$

where μ_0 is the permeability of free space.

At first glance, the vectorial nature of Eq. (1) seems to indicate serious difficulties if we wish to compute simple, closed form expressions for the total magnetic field **B** at point O due to an *arbitrarily shaped* current-carrying wire. Fortunately, this is not always the case: If the wire lies in a plane, and if we only want to compute the total magnetic field at a point in this plane, then even if the wire shape is complicated, we show that the field calculation may be easily carried out. Our purpose in this work is to derive a general and simple expression for the total magnetic field **B** at the point O for the case in which the field is due to an arbitrarily shaped, planar current-carrying wire.

When the basic differential form of the Biot–Savart law (1) is applied to a finite length of wire, and then to a closed circuit (ultimately, the case of genuine physical interest), it becomes a closed *line* integral

$$\mathbf{B}=\frac{\mu_0 I}{4\pi}\oint\frac{(d\mathbf{s}\times\hat{r})}{r^2}, \tag{2}$$

where the path of integration is along the wire. In general, the observation point O need not be in the plane of the loop, and for that matter the loop itself need not lie in a plane. However, for the purposes of this work, we consider loops that lie in a single plane and concentrate our attention on the calculation of the magnetic field at observation points that lie in the plane of the loop. Considering these conditions, we want to obtain a general expression for the magnitude of the total magnetic field at O, conveniently written in terms of r

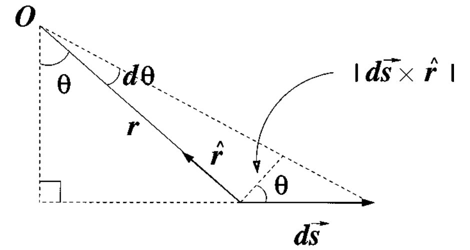


Fig. 1. Schematic representation of the plane diagram showing the main geometrical components related to the magnetic field evaluation at the observation point O. The line element of the wire is denoted by $d\mathbf{s}$, vector $\hat{r}$ is the unit vector pointing from $d\mathbf{s}$ to O, and θ represents the polar angle. Note that $r\,d\theta = ds \cos\theta = |d\mathbf{s}\times\hat{r}|$.

$=r(\theta)$, where $r(\theta)$ denotes the polar equation of the curve that describes the wire shape, and θ is the polar angle.

A surprisingly simple result for the magnitude of $\mathbf{B}$ can be obtained with the help of the plane diagram depicted in Fig. 1. Figure 1 sketches the basic geometric elements required for calculating the magnetic field at O due to a current element $I d\mathbf{s}$. Denoting the angle between the vectors $d\mathbf{s}$ and $\hat{r}$ by φ and inspecting Fig. 1, we readily see that $\theta = \varphi - \pi/2$. Consequently, an infinitesimal change in θ can be written as

$$d\theta = \frac{|d\mathbf{s}\times\hat{r}|}{r}. \tag{3}$$

By combining Eq. (2) and (3) we immediately obtain the following useful expression for the magnitude of the magnetic field at point O:

$$B = \frac{\mu_0 I}{4\pi} \oint \frac{d\theta}{r}. \tag{4}$$

Equation (4) is our central result; it is a simple and compact expression yielding explicit evaluation of the magnetic field produced by a generally shaped planar wire $r = r(\theta)$ when the observation point O is in the same plane as the current filament. We would like to stress that our line integral expression (4) is valid for observation points O located either inside or outside the current loop. In addition, Eq. (4) works for plane loops of any shape, including those described by curves $r = r(\theta)$ which are not single-valued functions of θ. As far as Eq. (4) is concerned, we are allowed to compute the field due to a finite segment of wire. Naturally, a finite segment by itself would not support a steady current.[3] However, we can always consider the finite segment as being a portion of an infinite or closed circuit. In this case, Eq. (4) would give the contribution of the finite segment (arbitrarily shaped) to the total field.

We point out that, even though our result (4) is quite simple, it does not seem to be well known. After extensively searching the existing physics literature, we have not been able to find Eq. (4) explicitly stated in any standard electromagnetism book or journal publication. To the best of our knowledge, despite its simplicity, our basic result (4) has not been published elsewhere. On the other hand, in Sec. III we do acknowledge the fact that, for some particular geometric shapes such as elliptical and logarithmic spiral wires, other authors have calculated the magnetic field at the center of these planar circuits.[6–8] Below we apply our result (4) to rederive some of these known results and to obtain new ones.

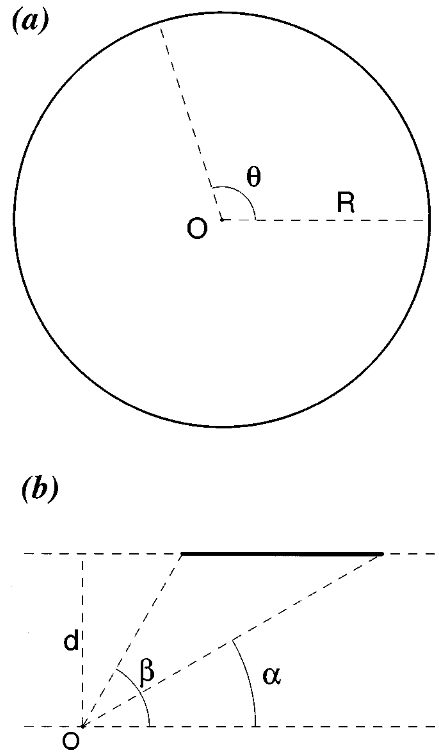


Fig. 2. (a) Circular wire of radius R; (b) finite straight wire (represented by the solid line segment) at distance d from the horizontal axis passing through the coordinate system origin O. The angular location of the wire's end points is given by α and β ($\alpha \leqslant \theta \leqslant \beta$).

III. ILLUSTRATIVE EXAMPLES

A. Trivial examples: Circular and straight wires

We initiate this section by applying our general expression (4) to rederive results that are common textbook exercises.[1–4] First, let us compute the magnitude of the magnetic field at the center of a circular wire of radius R, carrying a current I [Fig. 2(a)]. Since the polar equation for the circular wire is trivial, $r(\theta) = R$, the magnetic field obtained from (4) is

$$B = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\theta}{R} = \frac{\mu_0 I}{2R}. \tag{5}$$

As a second example we evaluate the magnetic field due to a finite straight wire, which is parallel to the x axis and carries a current I. The wire is at distance d from the horizontal axis passing through O [see Fig. 2(b)]. The end points of the wire are identified by the angles α and β. Even though textbook solutions of this problem require some algebra, we obtain the value of the field B very easily from Eq. (4). Since the wire is parallel to the x axis, we write its polar equation as $r(\theta) = d/\sin\theta$. Then, from (4) the magnetic field at the origin is given by

$$B = \frac{\mu_0 I}{4\pi d} \int_\alpha^\beta \sin\theta\, d\theta, \tag{6}$$

or

$$B = \frac{\mu_0 I}{4\pi d} [\cos\alpha - \cos\beta]. \tag{7}$$

The field due to an infinite straight wire is readily obtained from (7) by setting $\alpha = 0$ and $\beta = \pi$.

B. Elliptic, parabolic, and hyperbolic wires

Let us show, using Eq. (4), how to calculate explicitly the magnetic field in the case in which the wire has the shape of a conic curve.[9] We point out that, since the general equation in polar coordinates for conics involves only trigonometric functions,[9] the evaluation of the magnetic field from our Eq. (4) is quite simple.

First consider an elliptic wire, centered at the origin O, for which the lengths of major and minor axes are $2a$ and $2b$, respectively. Assume that the major axis is parallel to the x axis [see Fig. 3(a)]. The equation in polar coordinates for such an elliptic wire is[9]

$$r(\theta) = \frac{ab}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}. \tag{8}$$

Using our general expression (4) we obtain the total magnetic field at the point O,

$$B = \frac{\mu_0 I}{\pi a} E(k), \tag{9}$$

where $E(k)$ denotes the complete elliptic integral of the second kind,[10]

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta, \tag{10}$$

and

$$k = \sqrt{1 - a^2/b^2}. \tag{11}$$

Our result (9) agrees with the formulas for the field at the center of an elliptical circuit previously obtained in Refs. 6 and 7, which are written in terms of the ratio of the perimeter to the area of the ellipse.

It is interesting to compute two particular asymptotic limits of Eq. (9). (a) For $a \approx b$, the parameter $k \to 0$ and, since $E(0) = \pi/2$, Eq. (9) reduces to $B \approx \mu_0 I/2a$, the field at the center of a circular (radius a) wire [see Eq. (5)]. (b) Another situation of interest happens when $a \to \infty$, for fixed b. In this case we obtain $B \approx \mu_0 I/\pi b$, the field produced by two parallel infinite wires (distant $2b$ from each other) carrying equal currents I, which flow in opposite directions [see Eq. (7)].

We proceed by calculating the magnetic field due to a parabolic shape wire, carrying a current I. This parabolic wire has its axis parallel to the x axis, with focus located at the origin O [see Fig. 3(b)]. The distance from the parabola's vertex to its focus is f. The equation for this parabolic wire, in polar coordinates, is[9]

$$r(\theta) = \frac{2f}{1 - \cos \theta}. \tag{12}$$

Since in Eq. (4) we just need to compute the angular integral of $1/r(\theta)$, we easily calculate a very simple expression for the magnetic field at the point O,

$$B = \frac{\mu_0 I}{4f}. \tag{13}$$

The field varies as the inverse of the vertex–focus distance f. If $f \to 0$ the parabola becomes increasingly narrow and the wire would eventually cross the point O, making B diverge at this point. In the limit of large f the parabola spreads out, approaching the form of a straight wire. Simultaneously, its distance from O increases, so B tends to zero when $f \to \infty$.

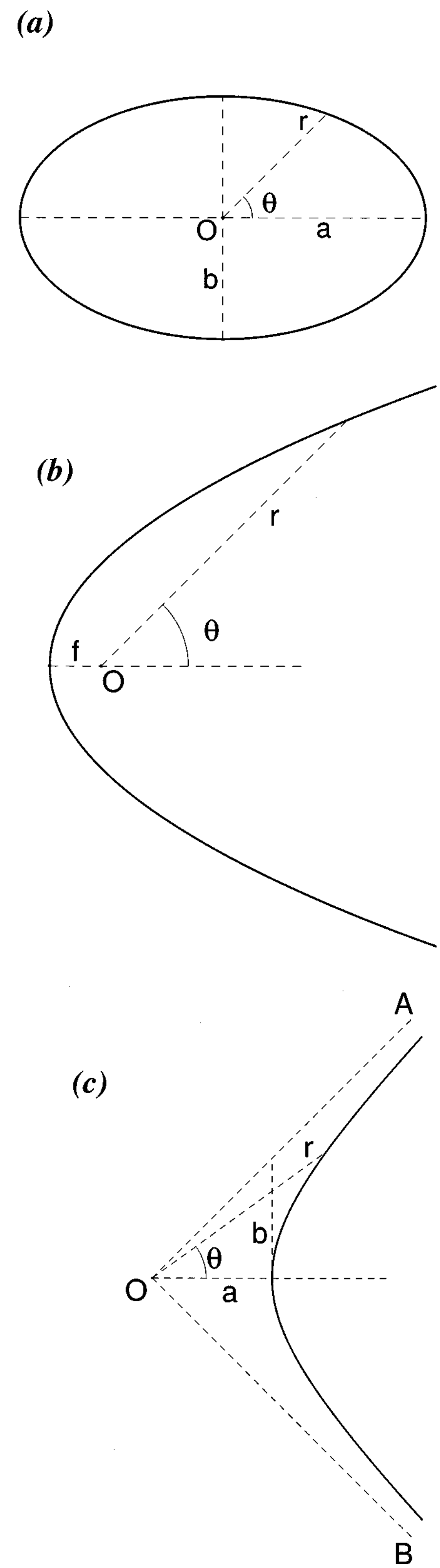


Fig. 3. (a) Elliptic wire, presenting major axis $2a$ and minor axis $2b$, centered at the origin O; (b) parabolic wire, with focus at the origin O. The vertex–focus distance is denoted by f and $r = r(\theta)$; (c) hyperbolic wire with major axis $2a$ and minor axis $2b$, centered at the origin O. The slopes of asymptotes OA and OB are $\pm b/a$.

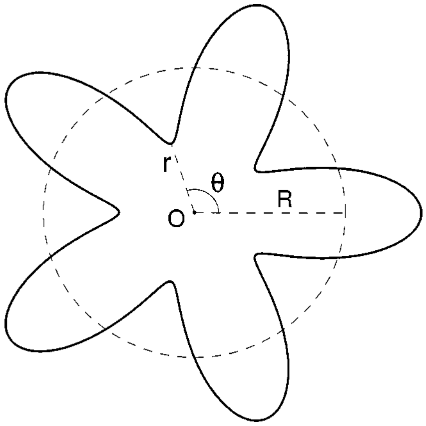


Fig. 4. Harmonically deformed wire $r(\theta)=R[1+\epsilon\cos(n\theta)]$, centered at O. We depict a fivefold ($n=5$) ''flower-like'' shape, taking $\epsilon=0.5$. When $\epsilon=0$ (no deformation) a circle of radius R (dashed curve) is obtained.

To close this section, we consider the field produced by a wire that has the shape of one branch of a hyperbola. The hyperbola-shaped wire is centered at the origin O. Assume that the major axis is parallel to the x axis [see Fig. 3(c)]. The equation in polar coordinates for such a hyperbolic wire is[9]

$$r(\theta)=\frac{ab}{\sqrt{b^2\cos^2\theta-a^2\sin^2\theta}}. \tag{14}$$

By our general expression (4) we know that the total magnetic field at point O is given by

$$B=\frac{\mu_0 I}{2\pi a}\int_0^{\phi}\sqrt{1-\kappa^{-2}\sin^2\theta}\,d\theta, \tag{15}$$

where $\phi=\arctan(b/a)$ and

$$\kappa=\left(1+\frac{a^2}{b^2}\right)^{-1/2}. \tag{16}$$

With the assistance of a table of integrals,[11] we solve the integral in (15), and get

$$B=\frac{\mu_0 I}{2\pi a}\left[\frac{E(\kappa)}{\kappa}+\left(\frac{\kappa^2-1}{\kappa}\right)K(\kappa)\right], \tag{17}$$

where $K(\kappa)$ denotes the complete elliptic integral of the first kind,[10]

$$K(\kappa)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\kappa^2\sin^2\theta}}, \tag{18}$$

while $E(\kappa)$ is defined as in Eq. (10). It is worth observing the limit of Eq. (17) for $b\rightarrow\infty$, and fixed a: In this case $\kappa\rightarrow 1$, and we recover the field due to an infinite straight wire, parallel to the y axis, at a distance a from it. Note that the field produced at point O by the hyperbola $y=c^2/x$ [obtained from Eq. (14), by setting $a=b$, through a counterclockwise rotation around O by $\pi/4$ rad], can be obtained directly from Eq. (17), considering $a=\sqrt{2}c$.

C. Harmonically deformed circular wires

In this section, we compute the value of the magnetic field at the center of a harmonically deformed, regularly undulating wire. The shape of such wire is described by (see Fig. 4)

$$r(\theta)=R[1+\epsilon\cos(n\theta)], \tag{19}$$

where the mode $n\geqslant 2$ (an integer) denotes the discrete azimuthal wave number (or the number of wire undulations) and $\epsilon(0\leqslant\epsilon<1)$ represents a dimensionless parameter that determines the degree of distortion from a perfect circularly shaped wire of radius R. At $\epsilon\approx 0$ the wire is nearly circular, presenting n small undulations; at $\epsilon\approx 1$ the wire tends to pinch off at the origin, assuming an n-leaf ''flower-like'' shape (Fig. 4). In reality, the single-mode patterns described by (19) are simple examples of fairly general Fourier expansions of more complex shapes, written in terms of many modes n. Harmonically distorted forms are particularly useful to model shape transitions which arise in very interesting physical systems such as magnetic liquids[12] and lipid monolayer domains.[13]

Again, using (4) we get the magnetic field at the center of the wire

$$B=\frac{\mu_0 I}{2R\sqrt{1-\epsilon^2}}. \tag{20}$$

In order to obtain expression (20) from Eq. (4), we perform the substitution of variables $\gamma=n\theta$ and make use of the fact that $r(\theta)$ is a periodic function of γ, with period 2π: The resulting integral is solved with the help of Gradshteyn and Ryzhik's book.[14] We point out that, by this procedure, the n dependence drops off. This way, the magnetic field calculated in (20) is independent of the number of undulations n present in the wire. For $\epsilon=0$ (no distortion) we obtain the usual field due to a circular wire of radius R [Eq. (5)]. Stronger deformation ϵ from circular shape leads to larger field values at the center.

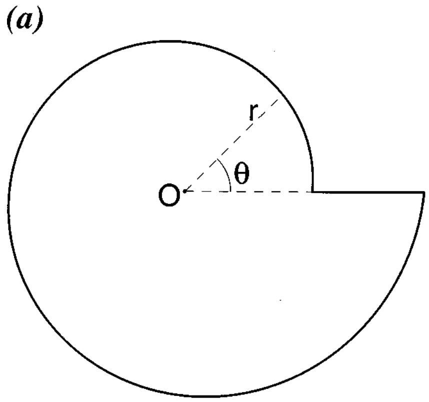
D. Spiral wires

We conclude by evaluating the magnetic field due to spiral[15] wires. Even though spirals are curves that wind indefinitely, from our discussion at the end of Sec. II, we can always consider the calculation of the magnetic field due to a finite spiral segment. Ultimately, the finite segment must be hooked up to some other wires so that a steady current could flow. This fact is no reason for concern, since the finite wire field contribution is still evaluated by Eq. (4). Alternatively, we can also perform the field calculation considering a modified spiral circuit, built by combining a one-turn spiral segment with a finite or infinite straight wire, provided the magnetic field due to the straight portion is zero at the point of interest. We adopt the latter in order to illustrate how to complete the circuit and let the joint wire (spiral+straight) support a steady current I.

First, consider the case of a logarithmic spiral[15] wire [Fig. 5(a)] described by

$$r(\theta)=qe^{p\theta}, \tag{21}$$

where $q>0$ and $p>0$ are spiral parameters, and $0\leqslant\theta\leqslant 2\pi$. Note that a small value of the parameter p makes the wire spiral more tightly. To complete the circuit we connect the finite spiral end points by a straight wire segment [Fig. 5(a)]. The magnetic field contribution at point O due to the straight wire is obviously zero. Incidentally, the problem of determining the magnetic field due to this particular spiral wire (for the case $p=q=1$) is a textbook exercise.[1,8] Nevertheless, in Ref. 1, a very specific solution (distinct from ours)



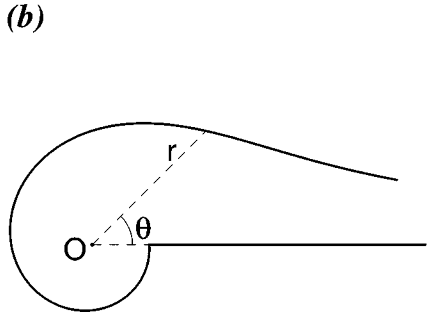


Fig. 5. (a) Logarithmic spiral wire [Eq. (21)]; (b) *lituus* spiral wire [Eq. (23)]. In both cases a complete current-carrying circuit is constructed by connecting the spirals to finite (a) or semi-infinite (b) straight wires. The straight wires' field contributions at O are zero, since $d\mathbf{s}\times\hat{r}=0$ [see Biot–Savart equation (1)].

is suggested, and the general problem of determining the magnetic field due to arbitrarily shaped, planar wires is not explored.

The field at the origin O, due to the spiral wire (21), is trivially obtained with the help of our general expression (4), yielding

$$B=\frac{\mu_0 I}{4\pi q}\left[\frac{1-e^{-2\pi p}}{p}\right]. \tag{22}$$

Note that the magnetic field gets larger for decreasing values of the parameter p. In the limit $p\rightarrow 0$, $B\approx\mu_0 I/2q$ (the field due to a circular wire of radius q), the upper bound field intensity associated with this spiral current configuration.

Another interesting example is the one in which the field is produced by a special kind of spiral, called *lituus* [15] [see Fig. 5(b)],

$$r(\theta)=\frac{\eta}{\sqrt{\theta}}, \tag{23}$$

where $\eta>0$ is the spiral parameter. For this specific case we consider an infinite circuit carrying a current I, built by connecting the end point $\theta=2\pi$ of a one-turn *lituus* ($0\leqslant\theta\leqslant 2\pi$) to a semi-infinite straight wire which lies in the direction passing through the origin O [Fig. 5(b)]. Similar to our previous example, the magnetic field contribution at point O due to the straight wire is zero. By Eq. (4), the magnetic field due to the *lituus* spiral shaped wire can be written as

$$B=\frac{\mu_0 I}{3}\,\frac{\sqrt{2\pi}}{\eta}. \tag{24}$$

This last example and all others discussed throughout this work make it clear that, in a number of cases, despite the wire's specific geometry, the corresponding B field calculation may be promptly performed by using our Eq. (4). The difficulty of having a complicated wire shape is not a serious obstacle in order to compute B in closed form, as long as $1/r(\theta)$ is not terribly hard to integrate.

IV. CONCLUDING REMARKS

In this work, we derive a simple and compact expression to compute the magnetic field produced by arbitrarily shaped, not self-intersecting, planar, current-carrying wires, at a point lying in the plane of the wire. The calculation is performed using the Biot–Savart law as the starting point, and takes full advantage of the fact that it is straightforward to rewrite the law's geometric parameters in terms of the wire shape $r(\theta)$. The calculated expression for the field turns out to be very simple, involving the line integral of $1/r(\theta)$. The simplicity of the field expression we derived allows us to easily evaluate the magnetic field at points lying in the plane of current filaments assuming the shape of conics, spirals, and regularly undulating wires. Asymptotic behavior in several interesting limits is investigated. The derivation of our main result [Eq. (4)] involves only basic geometric concepts, making it accessible to the majority of introductory electricity and magnetism students. We hope our results will be helpful for future studies on the topic of magnetic field evaluation, using the Biot–Savart law.

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