A neglected class of magnetic field calculations

Richard G. Fowler
Department of Physics and Astronomy, University of Oklahoma, Norman, Oklahoma 73069
(Received 27 October 1976; accepted 8 August 1977)

MAGNETIC ANALOG OF FARADAY'S LAW

Faraday's law of induction, in the well-known form

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \Phi}{\partial t},$$

is widely used as an elementary basis for calculating electric fields in slowly changing magnetic fields. Less well known is the possibility of calculating magnetic fields in time-changing situations, such as slow charge convection, by magnetic analog of Faraday's law.

Thus in a region where $\rho = 0$ and $\sigma = 0$, we have curl $\mathbf{H} = \partial \mathbf{D}/\partial t$. It should be noted that ρ is not zero somewhere *outside* the region. Integrating over a surface lying wholly in this region,

$$\oint \text{curl}\, \mathbf{H} \cdot d\mathbf{A} = \int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{A}.$$

If the surface over which integration is performed is not moving, then we can apply Stokes' theorem to shift the left-hand integration to the perimeter of the surface, and interchange operations on the right to obtain

$$\oint \mathbf{H} \cdot d\mathbf{l} = \frac{\partial}{\partial t} \int \mathbf{D} \cdot d\mathbf{A} = \frac{\partial \Phi_D}{\partial t},$$

where Φ_D is the flux of the electric induction.

APPLICATION TO A PROBLEM

The utility of this relation depends as always on the existence of suitable symmetry. Let us consider one special example from many possibilities: a charge ring, total charge Q, in the equatorial plane of a system of spherical coordinates, with a radius a which is growing at speed v as shown in Fig. 1. We require first the electric field of the ring. The potential along the z axis is

$$V = Q(a^2 + z^2)^{-1/2}/4\pi\epsilon_0.$$

We will follow only the outside problem, $z > a$, but the inside problem is wholly analogous. Expanding

$$V = \frac{Q}{4\pi\epsilon_0 z}\left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} \cdots\right), \qquad z > a.$$

Then at all points of spherical space, using the Legendre polynomials $P_n(\cos\theta)$

$$V = \frac{Q}{4\pi\epsilon_0 r}\left(P_0 - \frac{a^2}{2r^2} P_2 + \frac{3a^4}{8r^4} P_4 \cdots\right), \qquad r > a.$$

Now in vacuum, $\mathbf{D} = -\epsilon_0 \nabla V$, so

$$\mathbf{D} = \frac{Q}{4\pi r^2}\left[\left(P_0 - \frac{3a^2}{2r^2} P_2 + \frac{15a^4}{8r^4} P_4 \cdots\right) \mathbf{r}_1 + \left(-\frac{a^2}{2r^2} P_2' + \frac{3a^4}{8r^4} P_4' \cdots\right) \sin\theta\, \boldsymbol{\theta}_1\right], \qquad r > a.$$

One can quickly convince himself from symmetry that the magnetic field has only an azimuthal component. The most convenient path and surface combination for the calculation of Φ_D and $\int \mathbf{H} \cdot d\mathbf{l}$ is then the circular edge of a spherical cap of radius r. Here only D_r contributes. (One can also calculate Φ_D for the surface of the cone extending from r to infinity, using D_θ with identical results.)

$$\Phi_D = \frac{Q}{2}\left(\int_0^\theta P_0 \sin\theta\, d\theta - \frac{3a^2}{2r^2}\int_0^\theta P_2 \sin\theta\, d\theta + \frac{15a^4}{8r^4}\int_0^\theta P_4 \sin\theta\, d\theta\right), \qquad r > a.$$

Now $a = a_0 + vt$, so $\partial\Phi_D/\partial t = v\partial\Phi_D/\partial a$, and by symmetry

$$\int \mathbf{H} \cdot d\mathbf{l} = H_\phi \int dl = 2\pi r H_\phi.$$

Then

$$H_\phi = -\frac{3Qva}{16\pi r^3}\sin\theta \sin 2\theta \times \left(1 - \frac{5a^2}{8r^2}(7\cos 2\theta - 3) + \cdots\right), \qquad r > a.$$

Inside the circle of convergence, carrying out the same process, one finds

$$H_\phi = -\frac{3Qvr^2}{16\pi a^4}\sin\theta \sin 2\theta \times \left(1 - \frac{5r^2}{8a^2}(7\cos 2\theta - 3) + \cdots\right), \qquad r < a.$$

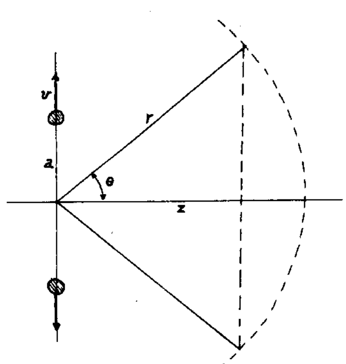

Fig. 1. A ring of net charge is expanding with a radial velocity v.

Many other problems can be attacked by this analog of Faraday's law. Disks, rings, cylinders, both expanding and in translation, present little difficulty. The method seems as though it should be a standard textbook tool, and the absence of any reference to the possibilities of $\partial\Phi_D/\partial t$ in all but one[1] out of forty textbooks is somewhat surprising. Perhaps it is because so few examples of single sign convection of distributed charges are seen in nature. This article was prompted by the need to solve[2] certain problems in the propagation of lightning where charge convection does play an important role.

[1] R. P. Winch, *Electricity and Magnetism* (Prentice-Hall, Englewood Cliffs, NJ, 1963).
[2] R. G. Fowler, Adv. Electron. Electron Phys. **41**, 1–72 (1976).