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L. Lerner

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Magnetic field of a finite solenoid with a linear permeable core

L. Lerner
School of Chemical and Physical Sciences, Flinders University, Adelaide 5001, Australia

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The integral form of Ampere's law applied to an infinitely long solenoid gives a good first approximation to the magnetic field of finite air-cored solenoids with length L to width w ratios (aspect ratios) significantly greater than 1. It is less well known that this approximation requires much larger aspect ratios to be applicable to solenoids whose relative core permeability μ_r is much greater than that of the surrounding medium. An exact expression is derived for the magnetic field and inductance of a linear permeable core spheroidal solenoid, and an inequality is obtained for the validity of the infinite solenoid approximation. It is demonstrated that the magnetic field and inductance of a spheroidal solenoid with a linear core approximate well with those of a cylindrical solenoid of the same length and width.
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I. INTRODUCTION

The nature of the magnetic field generated in free space by current distributions,[1] and especially by air-cored solenoids, has received considerable attention.[2] The simplest case of this type is an infinitely long circularly symmetric uniform solenoid. It is well known that the integral form of Ampere's law,

$$\oint \frac{1}{\mu}\mathbf{B}.dl = \int \mathbf{J}.dA, \tag{1}$$

applied to such a solenoid with uniform surface current density K, predicts a constant magnetic field inside the solenoid, parallel to the solenoid axis,

$$B_0 = \mu_0 K, \tag{2}$$

and zero magnetic field outside. The assumption of a uniform azimuthal surface current, contrasts with a typical physical situation where the solenoid is excited by a current-carrying wound wire, introducing both an axial current component and the current inhomogeneities. However, the correction due to these effects is usually small.[3]

The major shortcoming of Eq. (2) is the assumption that the solenoid is infinitely long, without which there is no axial symmetry, and the integral in Eq. (1) cannot be easily performed. Equation (2) is still useful because it provides a good first approximation for solenoids with substantial length to width ratios. Figure 1 shows the ratio of the magnetic field at the center of a finite air-cored solenoid, given by the expression[4] $B = \mu_0 KL/\sqrt{L^2 + w^2}$ to that of the infinite solenoid B_0. Figure 1 shows that for $L/w > 2$, Eq. (2) introduces less than a 10% error.

The coil inductance depends on the total magnetostatic field energy W and is more difficult to calculate accurately. Equation (2) predicts $W_0 = \mu_0 K^2 V/2$, where V is the solenoid volume. This prediction compares well with the magnetostatic energy evaluated numerically with a magnetostatic solver, shown in Fig. 1 scaled by W_0, as a function of the aspect ratio L/w, where L is the length and w is the width. The inductance I, can easily be obtained from the ratio of W to W_0, $I = \mu_0 n^2 VW/W_0$, where n is the number of turns per unit length.

It is well known that Eq. (2) is also applicable to solenoids with a constant core permeability different from that of free space. Usually, the core consists of a ferromagnetic material such as iron, in which case $\mu(\mathbf{H})$ is a function of the exciting field $\mathbf{H}$. This dependence is often further complicated by a dependence on the history of $\mathbf{H}$-field variation. Far from saturation, which for iron cores can correspond to μ_r up to 10^4, the core material can be approximated as linear and μ as constant.[5] To avoid unnecessary complications we make this assumption here.

Equation (2) then predicts an internal magnetic field B which is a factor of $\mu_r = \mu/\mu_0$ greater than in the air-cored case. Because the problem is axially symmetric, this solution must be correct for an infinite permeable core solenoid. However, it is rarely emphasized that the condition for the infinite solenoid approximation to apply to permeable core solenoids of finite extent is much more restrictive than for air-cored solenoids, to the extent that, even without considering saturation, Eq. (2) predicts a magnetic field which is an order of magnitude too large for the typical iron-cored solenoids.

We can obtain a qualitative understanding of this discrepancy by reconsidering the derivation of Eq. (2) from Eq. (1). If we apply Ampere's law as usual, and integrate along the sides of a small rectangular contour parallel to the solenoid, we obtain $B_0/\mu_r - B_1 = \mu_0 K$, where B_1 is the axial magnetic field just outside the solenoid, which is usually neglected. We may equivalently set B_1 equal to a small fraction β of the internal field, so that $\beta B_0 = -B_1$, the negative sign being due to the opposite polarity of the external field. The solution for B_0 is $B_0 = \mu K/(1 + \beta\mu_r)$, which shows that the contribution of β, and thus B_1, becomes significant as μ_r increases and limits the magnetic field to $1/\beta$ of its free-space value. This result is incomplete because β may also depend on μ_r. However, Eq. (25), which is derived using more exact methods in Sec. IV, has the identical form and exhibits how β depends on L/w. That is, the magnetic field in the permeable core solenoid depends on two dimensionless quantities, μ_r and the demagnetization,[6] due to an "open circuited" magnetic flux. As the demagnetization increases and its effect becomes larger than the relative permeability of the core, the magnetic field and magnetostatic energy approach $1/\beta$ times that of the air-cored solenoid. The transition occurs at correspondingly smaller L/w ratios as the permeability increases. This behavior is not acknowledged in most textbooks.[7]

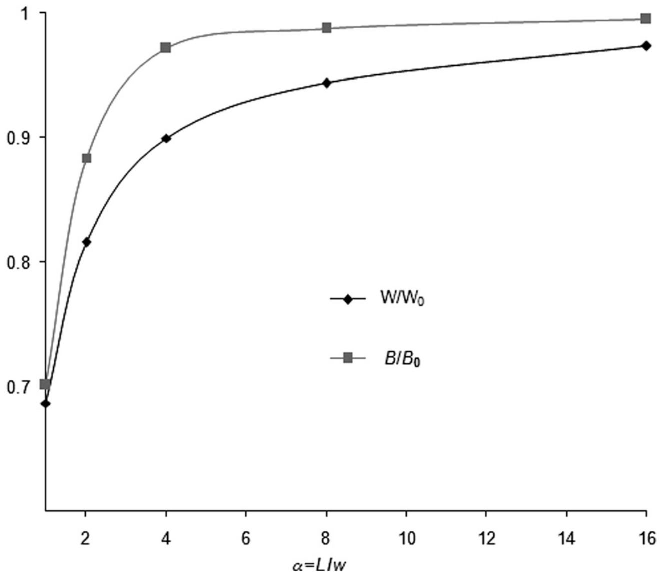

Fig. 1. The central magnetic field B and magnetostatic energy W for a finite cylindrical air-cored solenoid, normalized by their infinite solenoid values, as a function of the aspect ratio. The magnetic field is calculated by an analytical expression (Ref. 4), and the magnetostatic field energy is computed numerically.

The aim of this paper is to derive a criterion for when Eq. (2) is a useful approximation for a cylindrical solenoid whose core permeability differs from that of free space. We also provide a way to estimate the magnetic field of a solenoid, as well as its magnetostatic energy and inductance, when this approximation does not hold.

II. APPROXIMATE EXPRESSION

In the well-known problem of a uniform toroidal solenoid with small air gap[8] (see Fig. 2), we neglect the flux loss through the core sides by assuming a large permeability, and large radius of curvature R compared to the core diameter, $w \ll R$. We also assume the air gap is sufficiently small compared to its width, so that B_z does not substantially vary within it. Maxwell's equation for the magnetic field, $\nabla \cdot \mathbf{B} = 0$, implies that B_z is constant across the air-core interface, which combined with the previous assumptions, implies that the field is constant throughout the solenoid, including the air gap. The uniformity of the magnetic field

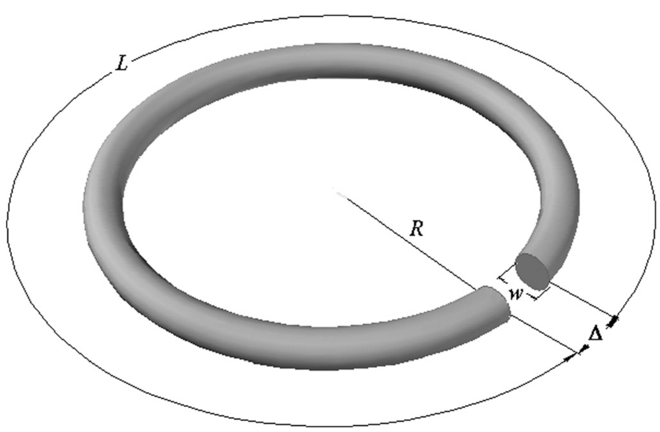

Fig. 2. A toroidal solenoid with a small core diameter w to radius R, ratio w, and a small air gap Δ, compared to the length of the magnetic circuit L.

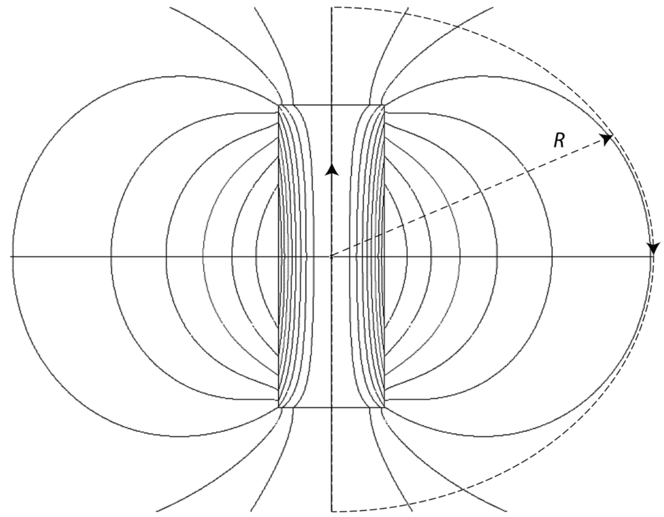

Fig. 3. Integration contour for approximating the reluctance of a cylindrical solenoid. In Eq. (4) the limit is taken as R approaches infinity.

introduces a symmetry, which enables us to calculate this field by applying Eq. (1) to a circular contour around the toroid,

$$B = \mu K \frac{L}{L + \mu_r \Delta}, \tag{3}$$

where the nominator is related to the magnetomotance, which is the current linking the magnetic circuit, and the denominator relates to the magnetic reluctance[9] around the path. The field reduces to that of the infinite solenoid, $B = \mu K$, when the air gap is small, $\Delta \ll L/\mu_r$. We see that that the air gap needs to be smaller for a toroid with large μ for its magnetic field to approach the limiting value.

The effect of increasing the permeability for a cylindrical solenoid can be demonstrated approximately using the same approach. The contour shown in Fig. 3 enables the line integral in Eq. (1) to be approximated analytically. Because the magnetic field at large distances R from a dipole distribution varies as $B \sim 1/R^3$, the integral over the circular contour taken to infinity is zero, and we are left with integrating along the straight line. If we assume that the spatial variation of the magnetic field on the axis and outside a permeable core solenoid is similar to that of an air-cored solenoid, we can use the well-known expressions for the latter[10] to evaluate the line integral in Eq. (1),

$$KL = B\left(\frac{L}{\mu} + \frac{2}{\mu_0}\int_a^\infty dz\left(\frac{z+a}{\left(b^2+(z+a)^2\right)^{1/2}} - \frac{z-a}{\left(b^2+(z-a)^2\right)^{1/2}}\right)\right), \tag{4}$$

where b = w/2, and a = L/2, giving

$$B = \mu K \frac{L}{L + \mu_r\left(L + b - (L^2+b^2)^{1/2}\right)}. \tag{5}$$

Figure 4 shows a good agreement of Eq. (5) with the numerical solution of the magnetostatic equation, Eq. (9), for $L/w = 1$. From Eq. (5) the condition for the infinite solenoid

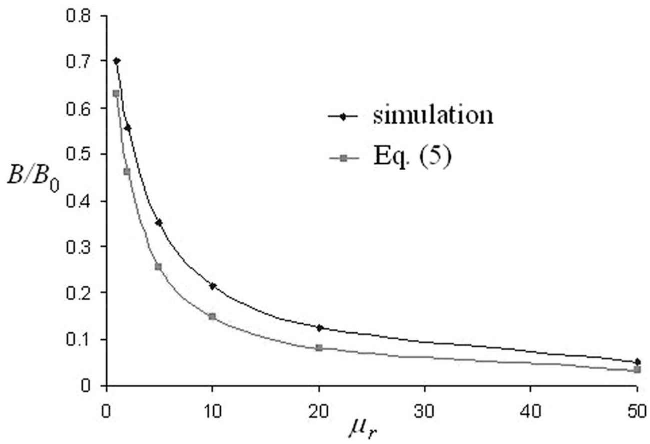

Fig. 4. The central magnetic field B scaled by the infinite solenoid field B_0 as a function of the relative permeability μ_r, for $L/w = 1$, obtained from Eq. (5) and a numerical calculation.

approximation to hold is $\mu_r w \ll L$, the same as in the toroidal case. However, due to the assumptions made in Eq. (4), the field given in Eq. (5) is a good approximation only when μ_r or L/w is small. It is not a good approximation when μ_r and L/w are large. The more accurate treatment of Sec. IV places a less stringent limit on the aspect ratio, $\mu_r w^2 \ln(L/w) \ll L^2$, for the infinite solenoid approximation to be applicable.

III. MAGNETOSTATIC EQUATION WITH VARYING μ

The results in Sec. II demonstrate that consideration of the external field is much more important for solenoids with a permeable core than for air-cored solenoids. A magnetostatic equation is now derived when μ is a function of position, and an exact result is obtained for a spheroidal solenoid, which is the only geometry for which a closed analytic solution is possible.

A magnetostatic problem with inhomogeneous $\mu(r)$ is most readily solved using potential theory. Conservation of current, $\nabla \cdot \mathbf{J} = 0$, implies we can always find a function $\mathbf{M}$ such that,

$$\mathbf{J} = \nabla \times \mathbf{M}. \tag{6}$$

Equation (6) can be combined with the differential form of Ampere's law (1) to give

$$\nabla \times (\mathbf{B}/\mu - \mathbf{M}) = 0, \tag{7}$$

so that,

$$\mathbf{B} = \mu \nabla \Psi + \mu \mathbf{M}, \tag{8}$$

for some potential distribution Ψ, whose existence is guaranteed by the vanishing of the curl in Eq. (7). $\mathbf{M}$ is not determined uniquely by Eq. (6), and $\mathbf{M}$ distributions differing from each other by a conservative field give the same current $\mathbf{J}$. By applying $\nabla \cdot \mathbf{B} = 0$ to Eq. (8), we obtain the field equation[11] for Ψ,

$$\nabla \cdot \mu \nabla \Psi = -\nabla \cdot \mu \mathbf{M}. \tag{9}$$

The effect of Eq. (8) is to cancel the conservative component of $\mu\mathbf{M}$, setting the magnetic field equal to its solenoidal component. Thus, $\mathbf{M}$ distributions related to each other by an additive conservative field $\mu\mathbf{M}' = \mu\mathbf{M} + \nabla\Psi'$, for any Ψ', lead to the same magnetic field. Hence, we are free to make any convenient choice of $\mathbf{M}$ satisfying $\mathbf{J} = \nabla \times \mathbf{M}$. This freedom can also be used to simplify the calculation of the total magnetostatic energy W,

$$W = \frac{1}{2} \int_{\text{all space}} \frac{1}{\mu} \mathbf{B} \cdot \mathbf{B} dV = \frac{1}{2} \int_{\text{solenoid}} \mathbf{M} \cdot \mathbf{B} dV, \tag{10}$$

because $\int \nabla\Psi \cdot \mathbf{B} dV = 0$ for finite source distributions.[5]

Given a solution Ψ to Eq. (9), we can rewrite Eq. (8) as

$$\mathbf{B} = \mu_0 \nabla \Psi + (\mu - \mu_0) \nabla \Psi + \mu \mathbf{M}, \tag{11}$$

which shows that the magnetic field generated by a source $\mathbf{M}$ in a medium of varying μ is equivalent to that generated by a magnetic moment distribution $\mathbf{M}' = (\mu - \mu_0)\nabla\Psi + \mu\mathbf{M}$ in a medium of constant μ_0, which is zero outside the solenoid, as is the original distribution $\mathbf{M}$. However, if the original $\mathbf{M}$ is uniform except at the boundaries, $\mathbf{M}'$ is not generally so. A spheroid of uniform magnetization $\mathbf{M}$ is an exception, as we shall see in Sec. IV.

The most convenient choice of $\mathbf{M}$ for a cylindrical solenoid is $\mathbf{M} = (0, 0, M)$ inside the solenoid, and zero elsewhere, and clearly generates the correct solenoid current by Eq. (6). If we substitute this choice into Eq. (9), the magnetostatic problem becomes

$$\nabla^2 \Psi + \frac{1}{\mu} \nabla \cdot \mu \nabla \Psi = M \Theta(b - \rho) \times [\delta(z - a) - \delta(z + a)], \tag{12}$$

where $\Theta(r)$ is the Heavyside step function and $\delta(z)$ is the Dirac delta function. Thus, for the cylindrical solenoid with a uniform core, the effect of varying μ in the second term on the left-hand side of Eq. (12) is to introduce a discontinuity into the normal component of the potential gradient on the core sides. The discontinuity is therefore responsible for the difference between permeable core and air-cored solenoids.

The problem is formally identical to the well-known case[12] of the field of a cylindrical capacitor with dielectric permittivity ε and plates of diameter b separated by length $2a$,

$$\nabla \cdot \varepsilon \mathbf{E} = \sigma_0 \Theta(b - \rho)[\delta(z - a) - \delta(z + a)], \tag{13}$$

with equal and opposite charges on the two capacitor plates, which is the equivalent of the two poles of a magnetic dipole. However, while in the electrostatic problem the fringing field forms only a minor correction to the simple solution given by the integral form of Gauss's law, in magnetostatics the fringing field is the essence of the problem. Thus, the electric field outside the capacitor plates approaches zero as the extent of the plates increases, whereas in magnetostatics $\nabla \cdot \mathbf{B}$ ensures that the normal component of the magnetic field is equally large inside and outside the solenoid. That is, the constant field between the charges, which forms the main part of the electrostatic problem, is subtracted out in determining $\mathbf{B}$ by Eq. (8). Furthermore, in electrostatics we are interested in small aspect ratios, which exhibit the highest capacitance and lowest fringe fields, whereas in magnetostatics the field energy approaches zero in this limit.

IV. EXACT SOLUTION FOR A SPHEROIDAL SOLENOID

Equation (9) has been solved by separation of variables in ellipsoidal coordinates[13] for an elliptical core of uniform magnetization $\mathbf{M}$ and constant $\mu = \mu_0$. Here, we present a solution for uniform $\mathbf{M}$ and variable μ for a spheroidal solenoid so that $\mu = \mu_1$ inside the spheroid and μ_0 outside. The uniform magnetization is equivalent to a solenoid with core permeability μ_1 excited by a surface current $K = |\mathbf{M} - \mathbf{M}\cdot\hat{\mathbf{n}}|\hat{\boldsymbol{\phi}}$, where $\hat{\mathbf{n}}$ is the normal to the spheroid surface, and $\hat{\boldsymbol{\phi}}$ is the azimuthal unit vector. The solution gives the exact dependence of the internal magnetic field on the ratios $\alpha = L/w$ and $\mu_r = \mu/\mu_0$. This problem can be transformed into that of an ellipsoid in a uniform external magnetic field with no source current, which has been solved in Ref. 6. However, it is more instructive to solve the finite source problem directly in spheroidal coordinates.

Laplace's equation $\nabla^2\Psi = 0$ with elliptical boundaries of revolution about an axis (say $\hat{\mathbf{z}}$) is separable in spheroidal coordinates. I choose the prolate spheroidal coordinates $(p, q, \cos\phi)$, as shown in Fig. 5, because $L/w > 1$. Spheroidal coordinates are related to Cartesian coordinates by,[9]

$$\begin{aligned} z &= \cosh\varepsilon\cos\theta &&\equiv pq \\ x &= \sinh\varepsilon\sin\theta\cos\phi &&\equiv (p^2-1)^{1/2}(1-q^2)^{1/2}\cos\phi\,, \\ y &= \sinh\varepsilon\sin\theta\sin\phi &&\equiv (p^2-1)^{1/2}(1-q^2)^{1/2}\sin\phi \end{aligned} \tag{14}$$

where $p \equiv \cosh\varepsilon$ and $q = \cos\theta$ for convenience. It follows that

$$\frac{z^2}{p^2} + \frac{x^2+y^2}{\sqrt{p^2-1}} = 1, \tag{15}$$

so that surfaces $p = \text{constant}$ describe ellipses with $L/w = \coth\varepsilon > 1$. The coordinates, θ and ϕ, are the axial and azimuthal angles, respectively, about the z-axis.

The limit $p \to \infty$ corresponds to spatial infinity where the surfaces $p = \text{constant}$ approach spheres. Equipotential surfaces cannot consist entirely of points near the origin, which is therefore described by the dual limits $p \to 1$ and $q \to 0$. The azimuthally symmetric gradient and Laplacian, which exhibit a singularity in this limit, are given by

$$\nabla = \left(\frac{1-q^2}{p^2-q^2}\right)^{1/2}\hat{\mathbf{p}}\frac{\partial}{\partial p} + \left(\frac{p^2-1}{p^2-q^2}\right)^{1/2}\hat{\mathbf{q}}\frac{\partial}{\partial q}, \tag{16}$$

$$\nabla^2 = \frac{1}{(p^2-q^2)}\left(\frac{\partial}{\partial p}\left((p^2-1)\frac{\partial}{\partial p}\right) + \frac{\partial}{\partial q}\left((1-q^2)\frac{\partial}{\partial q}\right)\right). \tag{17}$$

In the standard procedure, we expand the solution to Laplace's equation as a sum of products $\Psi(p,q) = \sum A_n P_n(p) Q_n(q)$, where $P_n(p)$ and $Q_n(q)$ each satisfies a second-order ordinary differential equation derivable from Eq. (17). In this case $P_n(p)$ are the spheroidal harmonics.[9] The boundary conditions on the spheroid have to be satisfied independently at each order n. In the present case I choose a surface current density $K(p,q) = |\mathbf{M} - \mathbf{M}\cdot\hat{\mathbf{n}}|\hat{\boldsymbol{\phi}}$, which leads only to a single term in the sum. Such a current is generated by a constant magnetization $\mathbf{M} = (0, 0, M)$ inside the spheroid through $\nabla \times \mathbf{M} = \mathbf{J}$.

Because a constant magnetic field with potential $\Psi = z$ is a solution to Laplace's equation in Cartesian coordinates, $\Psi = pq$ is a solution of the Laplacian (17). As usual, a given angular dependence $Q(q)$ admits two radial solutions $P(p)$. The second solution can be found by substituting the trial solution $\Psi = P(p)q$ into Eq. (16) to obtain a second-order differential equation for $P(p)$,

$$\frac{d}{dp}\left((p^2-1)\frac{dP}{dp}\right) - 2P = 0. \tag{18}$$

We know one solution, $P(p) = p$. By substituting $P(p) = pg(p)$, we find the second solution,

$$g(p) = p\coth^{-1}p - 1. \tag{19}$$

It is easily established that $g(p)$ approaches zero at spatial infinity ($g(p) \sim 1/3p^2$ as $p \to \infty$), and diverges as $p \to 1$. Hence, the boundary conditions at the origin and at infinity impose that $\Psi^- = Cpq$ inside the spheroid, and $\Psi^+ = Dg(p)q$ outside, where C and D are constants to be determined. This determination is made by requiring continuity of the potential and discontinuity of its gradient along the normal, in accordance with Eq. (9), on the spheroid surface $p_0 = L/\sqrt{(L^2-w^2)}$,

$$Cp_0 = Dg(p_0), \tag{20}$$

$$q\left(\frac{p_0^2-1}{p_0^2-q^2}\right)(\mu_0\dot{g}(p_0)D - \mu_1 C) = \mu_1 M\hat{\mathbf{n}}.\hat{\mathbf{z}}, \tag{21}$$

where we have chosen $\mathbf{M} = M\hat{\mathbf{z}}$ inside the spheroid. Hence, integrating about a small pillbox with sides parallel to the solenoid boundary gives $\int \mu\mathbf{M}.dA = -\mu_1 M\hat{\mathbf{n}}\cdot\hat{\mathbf{z}}\Delta A$. This choice of $\mathbf{M}$ gives a single term solution, as can be checked by calculating the normal to the ellipse $\hat{\mathbf{n}}\cdot\hat{\mathbf{z}}$ in terms of (p,q), which leads to the q-dependence cancelling on both sides of Eq. (21). The surface current density for this choice of $\mathbf{M}$ is given by

$$K(q) = M|1 - \hat{\mathbf{n}}\cdot\hat{\mathbf{z}}|\hat{\boldsymbol{\phi}} = Mp\left(\frac{1-q^2}{p^2-q^2}\right)^{1/2}, \tag{22}$$

which for elongated spheroids varies little along the long sides. The smooth variation of this current enables our single term expansion of the solution, whereas for a cylindrical solenoid a single term solution is not possible due to the sharp variation of K at the edges.

If we solve Eqs. (20) and (21) for C, we find the potential Ψ^- which corresponds to a constant magnetic field inside the solenoid.

$$\Psi^- = -\frac{\mu_1 M}{\mu_1 + \mu_0 h(L/w)}z, \tag{23}$$

where

$$\begin{aligned} h(\alpha) \equiv -p\frac{\dot{g}(p)}{g(p)} &= \frac{1}{(p^2-1)(p\coth^{-1}(p)-1)} - 1 \\ &= \frac{(\alpha^2-1)^{3/2}}{\alpha\cosh^{-1}(\alpha) - (\alpha^2-1)^{1/2}} - 1, \end{aligned} \tag{24}$$

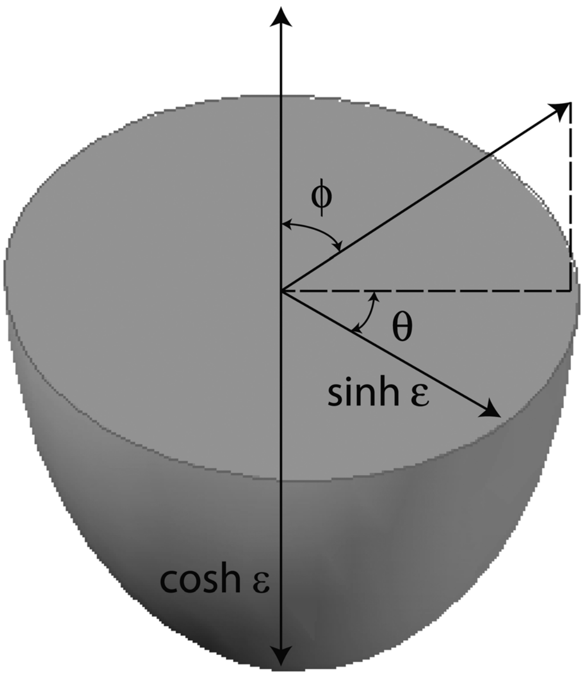

Fig. 5. An equipotential surface cosh $\varepsilon =$ constant for the spheroidal coordinates of Eq. (14), showing the two angular (surface) variables and ϕ.

and we have defined $L/w \equiv \alpha$. Here $h(L/w \to 1) = 2$ corresponds to the sphere, and $h(L/w \to \infty) \sim \alpha^2/(\log(2\alpha) - 1)$ is the long solenoid limit.

If we substitute Eq. (23) into Eq. (8), we find the final result: the dependence of the magnetic field inside the solenoid on the two dimensionless quantities μ_r and L/w,

$$B = \mu_0 M \frac{\mu_r h(L/w)}{\mu_r + h(L/w)}. \tag{25}$$

For long solenoids $h(L/w) \gg \mu_r$, and Eq. (25) reduces to Eq. (2) $B \sim \mu M$. In the other limit $B \sim \mu_0 M h(L/w)$, which is independent of the core permeability, as explained in Sec. I.

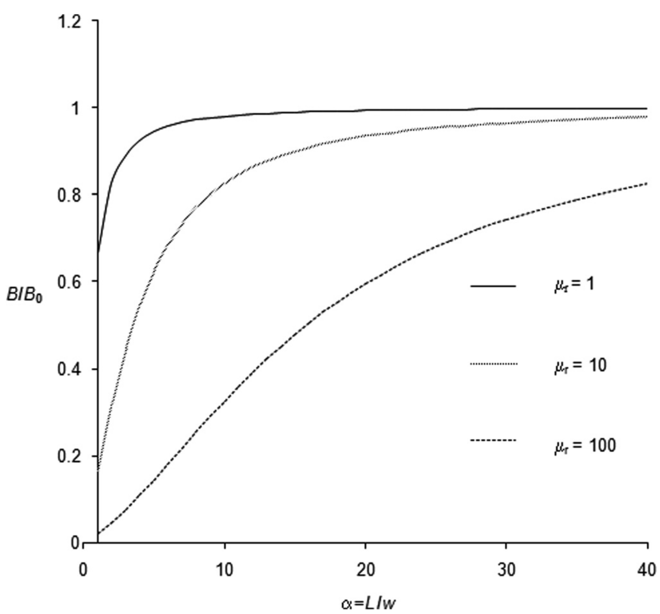

Fig. 6. The magnetic field of a spheroidal solenoid as a function of the aspect ratio. The deviation from the infinite solenoid approximation increases with increasing core permeability.

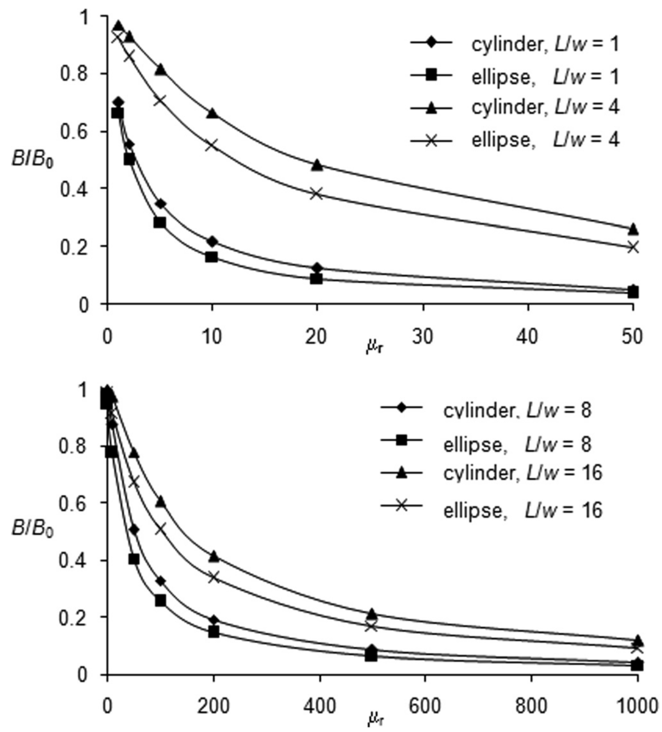

Fig. 7. The central magnetic field of the spheroidal and cylindrical solenoids as a function of core permeability for various aspect ratios.

From Eq. (25) we see that the condition for a solenoid to attain its maximum field (the long solenoid limit) is $h(L/w) \gg \mu_r$. When the equality holds, the field assumes half of the infinite solenoid value. This condition can be simplified because the inequality is satisfied at aspect ratios

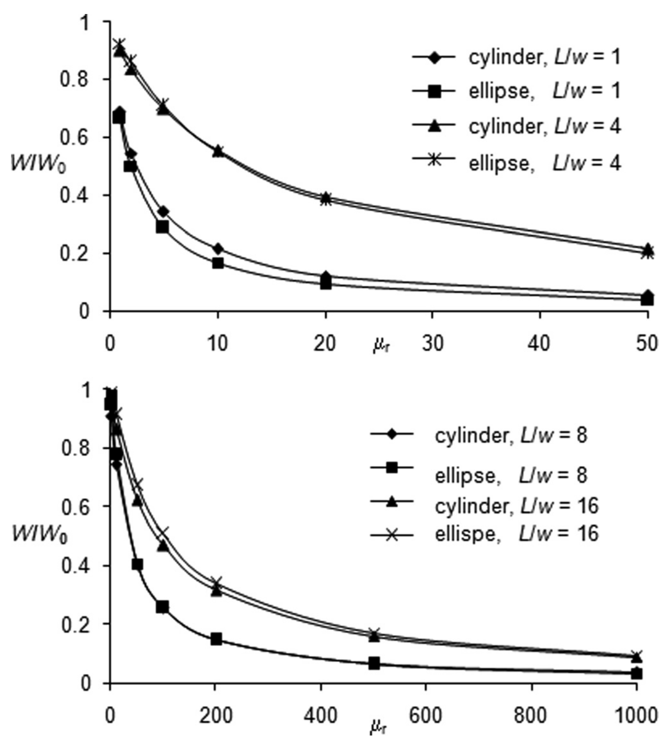

Fig. 8. The ratio of the total magnetostatic energy W of the spheroidal and cylindrical solenoids to that of the infinite solenoid approximation W_0, as a function of the core permeability for various aspect ratios.

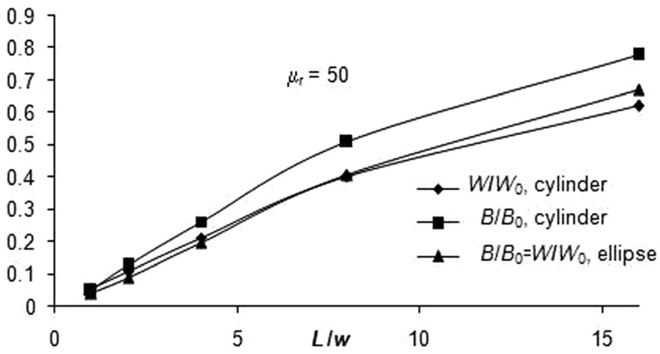

Fig. 9. The central magnetic field and magnetostatic energy for the spheroidal and cylindrical solenoids as a function of the aspect ratio with $\mu_r = 50$.

significantly greater than unity (μ_r is usually greater than 1), and we can approximate $h(L/w)$ for large aspect ratios in the fashion described following Eq. (24). The long solenoid limit becomes

$$L^2 > \mu_r w^2 \ln(L/w). \tag{26}$$

This approximation for $h(L/w)$ gives better than 90% accuracy for the magnetic field at $L/w > 3$. It is also the case that when the equality holds, the magnetic field is between 44% and 50% of its infinite solenoid limit for aspect ratios greater than 2.

Figure 6 shows the magnetic field inside the spheroidal solenoid, scaled by the limit μM, as a function of L/w, for various μ. As expected, the magnetic field of an air-cored solenoid varies least with L/w. The well-known case of a uniformly magnetized sphere,[7] $B = 2\mu_0 M/3$, corresponds to the limit $L/w = 1$, and the infinite solenoid field, $B = \mu_0 M$, forms the other limit. The magnetic field for the sphere, $B = 2\mu M/(2 + \mu_r)$, corresponds to the smallest field for prolate solenoids for any μ. At the crossover point $\mu_r = h(L/w)$, the field assumes half of its limiting value.

A. The field of a finite cylindrical solenoid

The results for the central magnetic field and magnetostatic energy of a cylindrical solenoid, which can only be obtained numerically, are very similar to the analytical results of Sec. IV for a spheroidal solenoid with the same aspect ratio, permeability, and maximal surface current. This similarity is due to the fact that much of the surface of a prolate spheroid is nearly parallel to the magnetic field where the surface current, which depends on the angle between the surface normal and the axial unit vector by Eq. (22), is nearly constant.

Figures 7 and 8 show the dependence of the central magnetic field and total magnetostatic energy of cylindrical and spheroidal solenoids as a function of L/w for various μ. For a spheroid the normalized energy and magnetic field are equal, due to its constant magnetization, $\int \mathbf{B}^2/\mu dV = \int \mathbf{M}.\mathbf{B}dV$. We see that although the magnetic field is always greater in the cylindrical case by up to 30% for large μ, the energy and thus the inductance differ by less than 10% if $L/w > 4$ for all μ_r.

Figure 9 shows a plot of W/W_0 and B/B_0 as a function of L/w for $\mu_r = 50$ for both cylindrical and spheroidal solenoids, showing that the magnetostatic field energy of the two geometries differs little over a wide range of L/w.

Curves similar to those of Figs. 7 and 8 for the dependence of the magnetic field of elliptical magnets on the L/w ratio for materials with varying permeability are available.[8] However, in this paper they are a consequence of the analytical result in Eq. (25); and hence, we can clearly identify the criterion for which the infinite solenoid approximation is valid.

V. CONCLUSION

We have demonstrated that the infinite solenoid approximation becomes significantly less accurate when applied to finite solenoids whose core permeability is much different from that of free space. An exact expression has been derived for spheroidal solenoids that support this conclusion. This expression implies that the infinite solenoid approximation holds for solenoids with a permeable core if $L^2 \gg \mu_r w^2 \ln(L/w)$, with the field attaining between 44% and 50% of its limiting value when the equality holds. Due to the similarity of the spheroidal results to those for the cylindrical solenoid, the former can be used to estimate the magnetic field and inductance of cylindrical solenoids with the same aspect ratio and permeability.

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