# The external magnetic field created by the superposition of identical parallel finite solenoids

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# The external magnetic field created by the superposition of identical parallel finite solenoids

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We use superposition and numerical methods to show that the external magnetic field generated by parallel identical solenoids can be nearly uniform and substantial, even when the solenoids have lengths that are large compared to their radii. We examine both a ring of solenoids and a large hexagonal array of solenoids. In both cases, we discuss how the magnitude and uniformity of the external field depend on the length of and the spacing between the solenoids. We also discuss some novel properties of a single solenoid, e.g., that even for short solenoids the energy stored in the internal magnetic field exceeds the energy stored in the spatially infinite external magnetic field. These results should be broadly interesting to undergraduates learning about electricity and magnetism. 
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## I. INTRODUCTION

In introductory physics courses on electricity and magnetism[1] and in upper-level undergraduate courses on electrodynamics,[2,3] students learn a striking fact which is that the magnetic field **B** generated by a sufficiently long cylindrical solenoid is highly uniform inside and of small magnitude outside. An implication of this fact is that the external magnetic field generated by a solenoid can usually be ignored as small.

But this well-known result raises some questions that are often not discussed in undergraduate courses. For example, how long must a solenoid actually be for the external magnetic field to be less than some specified value, at different points in space? Further, is there a circumstance in which the external magnetic fields of several solenoids could combine to be become substantial in magnitude, even for long solenoids? If so, might some configuration of parallel solenoids be able to generate an approximately uniform magnetic field in some region of space, like a Helmholtz coil?[2]

In this paper, we investigate these questions for two geometric arrangements of identical parallel solenoids (see Fig. 1). The first is a ring of solenoids, each of which is tangent to its nearest neighbors and to a common inner cylindrical surface. The second is an extended periodic hexagonal array of solenoids. In both cases, we find that there is a range of parameters (spacing of the solenoids and length of the solenoids for a given current density) for which the external magnetic field is comparable to the magnitude of the field at the center of a long solenoid. Further, for a large hexagonal array of solenoids, the magnetic field between the solenoids is highly uniform and decreases slowly in magnitude with increasing solenoid length.

As part of our analysis, we also discuss some calculations and insights regarding the magnetic field of a single finite solenoid. In particular, we calculate how the external magnetic field of a single solenoid decreases with increasing solenoid length for points close to and far from the solenoid's surface. We find that a simple analytical expression derived for the far-field of a solenoid on its symmetry plane gives a surprisingly accurate approximation of the magnetic field even close to the surface of the solenoid, at least for solenoids whose lengths exceed about four radii. We also show that the energy associated with the internal magnetic field of a solenoid exceeds the energy associated with the infinitely extended external magnetic field, and find that comparing the internal and external magnetic energies does not lead to a new length scale that could be used to distinguish long from short solenoids.

While most of our results are new, parts of Sec. II overlap with some previous papers. Derby and Olbert[4] express the magnetic field of a finite solenoid in terms of two one-dimensional integrals, and then provide a short computer code to evaluate these integrals. Brown and Flax[5] use the same integrals to calculate the magnetic field of a thick solenoid. These integrals form the starting point of our investigation. Farley and Price[6] show that the magnetic field strength just outside of, and on the midplane of a long finite solenoid of length $L$ whose cross-section is some constant shape, falls off as $L^{-2}$ with increasing $L$. However, these authors do not discuss (as we do for solenoids of circular cross section) how the magnetic field behaves for short solenoids or for points that are at intermediate distances from the solenoid's surface. Muniz *et al.*[7] used Taylor expansions of the magnetic scalar potential to calculate the off-axis internal magnetic field of a finite solenoid. This complements our direct approximation of two one-dimensional integrals using numerical integration, although our method works for any point in space and no assumption needs to be made about the domain of convergence. However, these earlier papers do not mention several of our single solenoid results such as how the external magnetic energy compares with the internal energy, and how a simple expression provides an accurate approximation of the magnetic field on the midplane of the solenoid for all points, not just close to a solenoid's surface.

Our results should be of broad interest to students and to instructors of undergraduate physics courses, as a moderately more complicated example of superposition of magnetic fields. The example we discuss below for a ring of parallel solenoids shows that just having a high symmetry of sources (many identical solenoids tangent to an inner cylinder) is not

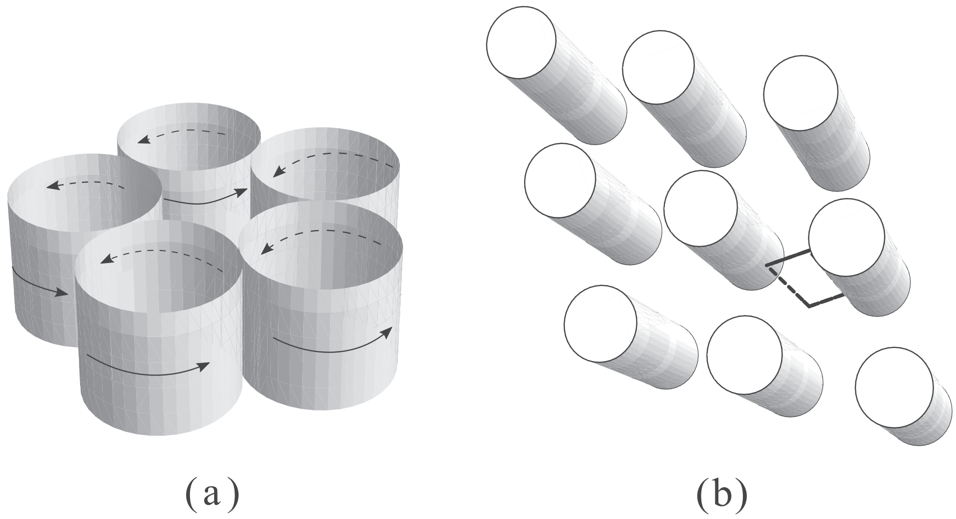



Fig. 1. Perspective drawings of two geometric arrangements of parallel identical solenoids. (a) A ring of five solenoids, each of which is tangent to nearest neighbors and also tangent to a common inner cylindrical surface. The direction of the constant current density $K$ on the surface of the solenoids is indicated by the arrows. (b) A section of an extended hexagonal array of parallel solenoids. The black dashed rectangle corresponds to the region in which we compute the magnetic field in Sec. IV.

enough to ensure uniformity of the external magnetic field. The second example of a large hexagonal array of parallel solenoids also shows that the external magnetic field of solenoids can be substantial, even for solenoids whose lengths are large compared to their radii. Finally, this paper provides a useful example to share with students of how software such as MATHEMATICA,[8] MAPLE,[9] MATLAB,[10] PYTHON,[11] OCTAVE,[12] and SAGEMATH[13] make it straightforward for undergraduates to study superpositions of magnetic sources that are complicated by their number or geometry.

The remainder of this paper is organized as follows. In Sec. II, we discuss a previously published analytical result[4] for the magnetic field generated by an idealized cylindrical continuous solenoid and discuss some insights about this field based on numerical studies of the analytical solution. In Sec. III, we use the results of Sec. II to discuss the properties of the magnetic field generated by a ring of identical parallel continuous solenoids, while in Sec. IV we discuss the properties of the magnetic field generated by a large hexagonal grid of identical parallel continuous solenoids. After summarizing key points in Sec. V, we discuss in Appendix A how various numerical calculations were validated, and in Appendix B how the radial components of the magnetic fields from different solenoids were combined.

## II. THE MAGNETIC FIELD OF A SINGLE FINITE CONTINUOUS SOLENOID

The starting point for our study of the external magnetic field created by superimposing the fields of several parallel finite solenoids is an analytical expression[4] for the magnetic field $\mathbf{B}$ generated by a continuous solenoid. By continuous solenoid, we mean a cylindrical surface of radius $a$ and length $L$ such that a spatially uniform time-independent surface current density $K$ (with units of amperes per meter) flows azimuthally around the surface.

A continuous solenoid is an idealization of a real solenoid that is helically wound with wires of some finite thickness. The continuous solenoid has the advantage over real solenoids of requiring just three parameters to specify, a radius $a$, length $L$, and uniform current density $K$. It has the further advantage that the analytical expression for its magnetic field is easy to evaluate numerically, accurately, and quickly, at any point in space. The continuous solenoid is somewhat unphysical in that the zero-thickness of the surface causes the radial component of the magnetic field to diverge in magnitude at the top and bottom edges of the solenoids surface, while no such divergence occurs for physical solenoids of finite thickness.

In this section, we discuss some physical properties of this analytical expression. We characterize how the external magnetic field depends on its length $L$ for points close to and far from the solenoid's surface on the plane bisecting the solenoid. We also compare the energy associated with the external magnetic field, with the energy associated with the internal magnetic field as a function of $L/a$ and find that the two energies become comparable when $L/a \approx 1$. Thus, there is no new interesting scale for solenoids related to magnetic energy.

To describe the axisymmetric magnetic field of a continuous solenoid of radius $a$, length $L$, and current density $K$, we introduce a cylindrical coordinate system $(r, \theta, z)$ such that $r = 0$ defines the solenoid's axis and the plane $z = 0$ bisects the solenoid so that the ends of the solenoid lie at the coordinates $z = \pm L/2$. For this coordinate system, the current density $K$ flows counterclockwise (as seen from above), producing a magnetic field $\mathbf{B}$ that, internal to the solenoid, points mainly in the positive $\hat{\mathbf{z}}$ direction. From the azimuthal symmetry of the problem, we deduce that $B_\theta = 0$. Furthermore, $B_z$ and $B_r$ do not depend on $\theta$ so that the magnetic field has the form

$$\mathbf{B} = B_r(r, z)\,\hat{\mathbf{r}} + B_z(r, z)\,\hat{\mathbf{z}}. \tag{1}$$

Previous calculations then show[4] that the magnetic field at a point $(r, z)$ in space has components given in terms of the one-dimensional integrals

$$B_r = -\frac{a\mu_0 K}{2\pi}\int_0^{\pi} d\alpha \cos\alpha \left[\frac{1}{\sqrt{\xi^2 + r^2 + a^2 - 2ar\cos\alpha}}\right]_{\xi_-}^{\xi_+} \tag{2}$$

and

$$B_z = \frac{a\mu_0 K}{2\pi}\int_0^{\pi} d\alpha \frac{(a - r\cos\alpha)}{(r^2 + a^2 - 2ar\cos\alpha)} \times \left[\frac{\xi}{\sqrt{\xi^2 + r^2 + a^2 - 2ar\cos\alpha}}\right]_{\xi_-}^{\xi_+}, \tag{3}$$

with the lengths $\xi_\pm$ defined by

$$\xi_\pm = z \pm \frac{L}{2}, \tag{4}$$

and where the notation $[f(\xi)]_{\xi_-}^{\xi_+}$ stands for $f(\xi_+) - f(\xi_-)$.

Appendix A provides details of how we approximated Eqs. (2) and (3) numerically for given values of $r$, $z$, $a$, $L$, and $K$ using MATHEMATICA.[8] This Appendix also summarizes how we validated the numerical approximations of these integrals, as well as the results obtained for superpositions of solenoids. We found that MATHEMATICA's numerical approximations to Eqs. (2) and (3) had a relative accuracy that exceeded eight significant digits (which is more than adequate for our calculations), except for the component $B_r$ near the coordinates $(r, z) = (a, \pm L/2)$, where this component diverges to infinity.

Figure 2 shows a vector-field plot of the magnetic field $\mathbf{B}$ generated by Eqs. (2) and (3) for a solenoid of radius $a = 1$, length $L = 2$, and current density $K = 1$. Even for this relatively short length, the interior magnetic field is approximately uniform and the magnetic field at the solenoid's center $r = z = 0$ has attained a magnitude $B \approx 0.8\mu_0 K$ that nearly equals the magnitude

$$B_0 = \mu_0 K \tag{5}$$

of the uniform internal magnetic field of an infinitely long solenoid with the same current density $K$.

It is difficult to see from Fig. 2 that the magnetic field of a finite continuous solenoid has two kinds of discontinuities. Figure 3(a) shows how the radial component $B_r(r, z)$ varies

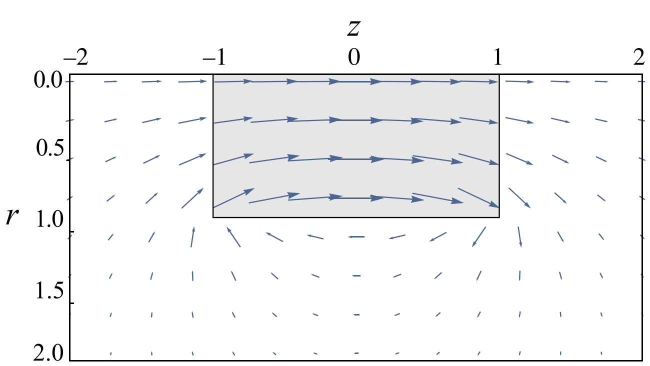



Fig. 2. Vector plot of the axisymmetric magnetic field $\mathbf{B}(r, z)$ generated by a finite continuous solenoid, given in Eqs. (1)–(3), for radius $a = 1$, length $L = 2$, and current density $K = 1$. Vectors are shown only for $r \geq 0$ so only the lower half of the solenoid (gray region) and half of the magnetic field are shown. The internal field is approximately uniform despite the solenoid's short length.

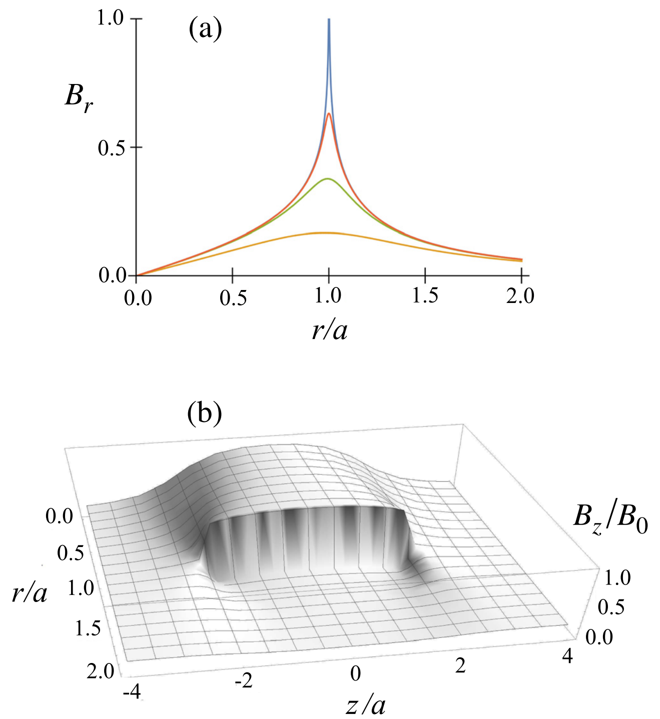



Fig. 3. The magnetic field $\mathbf{B}(r, z)$ of a finite continuous solenoid has two discontinuities, here shown for parameters $a = 1$, $L = 4$, and $K = 1$. (a) The radial component $B_r(r, z)$ diverges for $z = \pm L/2$ at $r = a$. Here, we show four curves of $B_r(r, z)$ for $z/(L/2) = 0.8$, 0.95, 0.99, and 1 (lowest to highest). The curve for $z = L/2$ (and a similar curve for $z = -L/2$, not shown) shows a logarithmic divergence of $B_r$ at $r = a$. (b) The axial component $B_z(r, z)$ has a jump discontinuity for $-L/2 \leq z \leq L/2$ at $r = a$ that represents the change in direction and magnitude of the magnetic field as one progresses radially from just inside to just outside the surface of the solenoid on a line of constant $z$.

with $r$ for fixed $z$, for several different values of $z$. We see that a singularity occurs at $r = a$ as $z \to \pm L/2$. (Note that this singularity does not occur for a solenoid of finite thickness with a corresponding volumetric current density $J$.) By setting $z = L/2$ in Eq. (2), one can evaluate the integral analytically over the range $0 \leq r \leq a - \epsilon$ and then find that the integral has a term proportional to $\ln(\epsilon)$ and so diverges logarithmically in the limit $\epsilon \to 0$. This implies that the total energy associated with the magnetic field is a finite quantity.

The surface plot of the axial component $B_z(r, z)$ in Fig. 3(b) shows a more familiar discontinuity, namely, that the $z$-component $B_z$ of the magnetic field abruptly changes direction and magnitude as one passes radially from just inside to just outside the surface of the solenoid.

We next discuss how the external magnetic field of the finite continuous solenoid decreases with increasing length $L$ for fixed radius $a$ and for fixed current density $K$. Understanding the dependence on $L$ will be important when we discuss how large the magnitude of the external magnetic field can be for multiple parallel continuous solenoids of some given length $L$. Here, we find some prior analytical insight via Exercise 5.61 on page 254 of Griffith's textbook on electrodynamics,[2] which states that the magnetic field on the solenoid's midplane ($z = 0$) far from the axis ($r \gg a$) asymptotically has the form $\mathbf{B} = -B\,\hat{\mathbf{z}}$, where the magnitude $B$ is

$$B \approx \left(\frac{\mu_0 K}{2}\right)\frac{L/2r}{4[(1 + (L/2r)^2]^{3/2}}\left(\frac{a}{r}\right)^2. \tag{6}$$

This is the leading-order term in an expansion in powers of the small quantity $a/r$, and so its validity when $a/r$ is not small (for example, near the surface of the solenoid, when $r \simeq a$) is not known.

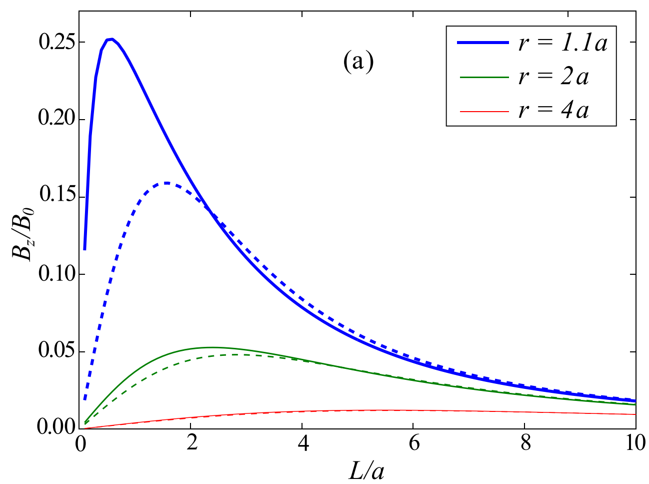
Equation (6) makes two predictions. First, for sufficiently large solenoid lengths $L \gg r$, $B_z \propto L^{-2}$ so $B_z$ decays rather slowly (algebraically rather than exponentially) to zero with increasing $L$. Second, if this expression has validity for general values of $r$, Eq. (6) predicts that at a fixed radial distance $r$ from the axis, $B$ does not decrease monotonically with increasing $L$, but first increases with a local maximum at $L_{max} = \sqrt{2}r$, and then asymptotically decreases to zero as $1/L^2$.

In Fig. 4, we compare these predictions with the analytical field of Eqs. (2) and (3) for radial distances that are close to and far from the solenoid's surface. In Fig. 4(a), we see that for the representative values $r = 1.1a$, $2a$, and $4a$ the $z$-component of the magnetic field in the plane $z = 0$ indeed first increases and then decreases with increasing $L$, as suggested by Eq. (6), even for relatively short solenoids for which $L/a \gtrsim 2$. Surprisingly, the $L$-dependence of the analytical field at some (fixed) $r$ values is accurately described in Eq. (6) even for distances close to the solenoid ($r \gtrsim 2a$). Figure 4(b) shows a complementary result, that even for modest fixed solenoid lengths ($L \gtrsim 4a$), Eq. (6) also describes the radial dependence $B_z(r, 0)$ quite well, even for $r$ values close to the outer surface of the solenoid. We conclude that Eq. (6) can be used as a straightforward way to determine the magnitude of the magnetic field on the symmetry plane $z = 0$ for configurations of parallel solenoids, without having to evaluate the integral in Eq. (3).

The analytical result for the magnetic field on the axis of a finite continuous solenoid [see Eq. (A1) of Appendix A] shows that solenoids whose lengths $L \gtrsim 4a$ are already "sufficiently long" in the sense that the magnetic field magnitude at the solenoid's center $|B_z(0,0)|$ exceeds 90% of its infinite-length value $\mu_0 K$. A possible alternative way to identify when a solenoid is "sufficiently long" is to find the length $L^*$ at which the energy $U_{ext}$ associated with the magnetic field external to the solenoid is one-half the total energy $U_{total}$ associated with the magnetic field over all of space. The external magnetic energy $U_{ext}$ is given by the volume integral

$$U_{ext} = \int \frac{B^2}{2\mu_0}\, d^3\mathbf{r} = \frac{1}{2\mu_0}\int\int 2\pi r \left(B_r^2 + B_z^2\right) dr\, dz, \qquad (7)$$

for coordinates $r$ and $z$ satisfying $r > a$ or $|z| > L/2$. By calculating the integral in Eq. (7) numerically, we show in Fig. 5 the ratio $U_{ext}/U_{tot}$ as a function of solenoid length $L$ for parameters $a = 1$ and $K = 1$. This ratio falls below the value 1/2 for $L \approx 0.69a$ so that, even for quite short solenoids, the energy associated with the magnetic field inside the solenoid exceeds the energy associated with the external field, even though the latter has infinite spatial extent.



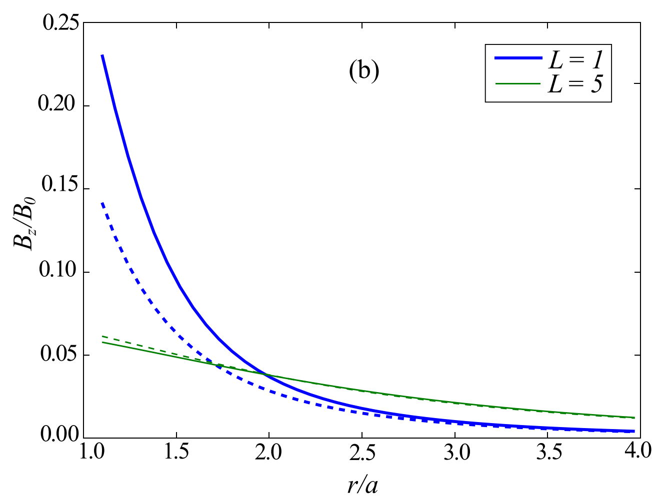




Fig. 4. (a) Comparison of Griffiths' result (dashed), given in Eq. (6), with the exact magnetic field in the plane $z = 0$ of a single finite continuous solenoid (solid), as a function of the solenoid length $L/a$. The results are normalized to $B_0$ and are shown for three representative radial values. Note how $B_z$ first increases and then decreases with increasing $L$. For $r = 4a$, Eq. (6) is essentially identical to the analytical result to the accuracy of the plot. (b) Equation (6) compared to the exact result as a function of $r$. Here, we see that Eq. (6) accurately describes the exact result even for $r$ close to the surface of the solenoid, provided that $L \gtrsim 4a$.

## III. THE MAGNETIC FIELD OF A RING OF IDENTICAL PARALLEL FINITE CONTINUOUS SOLENOIDS

We are interested in the question of whether the magnetic field external to multiple parallel finite solenoids can ever be

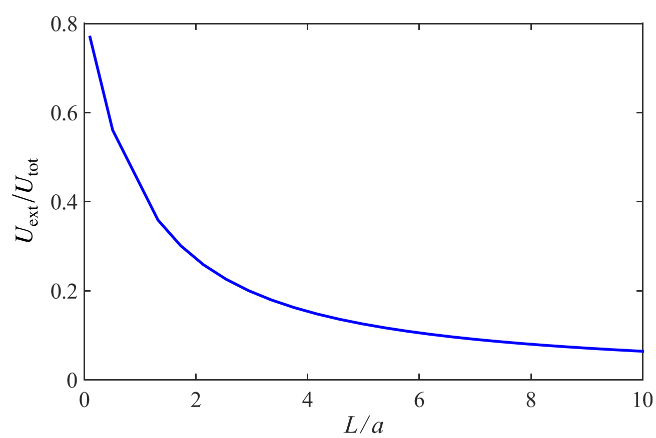



Fig. 5. Ratio of the energy $U_{ext}$, given in Eq. (7), associated with the magnetic field external to the solenoid ($r > a$ or $|z| > L/2$), to the total magnetic energy $U_{tot}$ for a single continuous solenoid of fixed radius $a = 1$ and fixed current density $K$, as a function of solenoid length $L$. For $L \gtrsim 0.7a$, the energy associated with the magnetic field inside the solenoid already exceeds the energy associated with the external magnetic field.

substantial in magnitude (comparable to the value $B_0 = \mu_0 K$ on the axis of an infinite solenoid) and approximately uniform. In this section, we consider an arrangement of $n \geq 3$ parallel identical finite continuous solenoids, each with current density $K = 1$ and of length $L$, that are arranged symmetrically in a ring as shown in Fig. 6 and in Fig. 1(a). Each solenoid is tangent to an inner common cylindrical surface of radius $R$ and also tangent to its neighbors. Because a substantial portion (nearly 50% in the case of many solenoids) of the external magnetic field of each solenoid passes through the common central cylindrical region, this ring geometry is a plausible candidate for producing an external magnetic field of substantial magnitude. We will show that this ring geometry can indeed produce a substantial magnetic field for solenoids that are long compared to their radii, but only for $n \leq 5$. But even for the best case of $n = 5$, the magnetic field in the common cylindrical region is not uniform.

To describe the external magnetic field in the central cylindrical region, we introduce a second cylindrical coordinate system $(\rho, \phi, z)$ such that $\rho = 0$ corresponds to the axis of the inner cylindrical surface. Here, $\rho = R$ corresponds to the cylindrical surface that all the solenoids are tangent to, and $z = 0$ is the common bisecting plane of the parallel solenoids. For $n$ parallel solenoids so described, the tangency conditions imply that the common solenoid radius $a_n$ is given by

$$a_n = \frac{\sin(\pi/n)}{1 - \sin(\pi/n)} R \qquad \text{for } n \geq 3, \tag{8}$$

so $a_3 \approx 6.5R$, $a_4 \approx 2.4R$, $a_5 \approx 1.4R$, $a_6 = R$, and $a_n \approx \pi R/n$ for large $n$.

At any point $(\rho, \phi, z)$ inside the central cylindrical region $\rho \leq R$, the $z$-component of the magnetic field $B_z(\rho, \phi, z)$ is obtained by directly adding the values $B_z(r_i, z)$ from each solenoid in the ring at that point, where $r_i$ is the distance of the point $(\rho, \phi, z)$ to the axis of the $i$th solenoid. Calculating the radial component $B_\rho(\rho, \phi, z)$ is more involved because the radial direction $\hat{\mathbf{r}}_i$ with respect to the $i$th solenoid at $(\rho, \phi, z)$ varies with $i$. The details are given in Appendix B.

We now discuss our calculations of the magnetic field in the central cylindrical region for rings with varying number $n$ of continuous solenoids, with varying lengths $L$, for a central cylindrical region of fixed radius $R = 1$. All solenoids have the same current density ($K = 1$).

For a ring with $n = 5$ solenoids, each of radius $a \approx 1.4$ [see Eq. (8)] and length $L = 4a$, Fig. 7(a) shows a vector plot of the magnetic field $\mathbf{B}(\rho, \phi, z)$ inside the central cylindrical region $0 \leq \rho \leq R$ along the plane $\phi = 0$, which corresponds to a rectangle that is perpendicular to and passes through the line segment $CB$ in Fig. 6(b). We see that the magnetic field is approximately uniform in direction and magnitude with a magnitude $B/B_0 \approx 0.3$ that is about one third the magnitude of the magnetic field $B_0 = \mu_0 K$ found on the axis of an infinitely long single solenoid with the same current density $K$. The magnetic field deviates substantially from that of a single solenoid (see Fig. 2) in the planes $z = \pm L/2 \approx \pm 2.8$, which are indicated by the horizontal black lines in Fig. 7(a). Here, the magnetic field is horizontal and of greatly decreased magnitude.

Further insight about the structure of the central magnetic field is provided in Fig. 7(b), which shows a surface plot of the field magnitude $B(\rho, z)$ for the same rectangle as for Fig. 7(a). Figure 7(b) confirms that the magnetic field is approximately uniform inside the central cylindrical region (the magnitude $B$ is approximately constant for $|z| \leq L/2$) but becomes non-uniform near the boundary of one of the solenoids because of the divergence of $B_r$ near the ends of the solenoid near $\rho/R \approx 1$ [see also Fig. 3(a)].

We found that the external magnetic field in the central cylindrical region becomes more nonuniform and decays radially in magnitude for larger numbers of solenoids ($n > 4$), so Fig. 7 is close to the best case scenario in terms of achieving a uniform magnetic field in a ring geometry. As the number of solenoids increases for fixed $R$ and fixed $L$, their radii $a_n$ [Eq. (8)] decrease in size and the solenoids effectively become longer compared to their radius. The magnetic field external to each solenoid therefore decreases in magnitude according to Eq. (6), being largest near the surfaces of the solenoids ($\rho \approx 1$) and decreasing towards the center of the common cylindrical region where $\rho = 0$. This behavior is illustrated in Fig. 8, which shows how the ratio $B_z/B_0(\rho, \phi = 0, z = 0)$ varies on the bisecting plane $z = 0$ for increasing values of solenoid number $n$, for a fixed length $L/a = 4$.

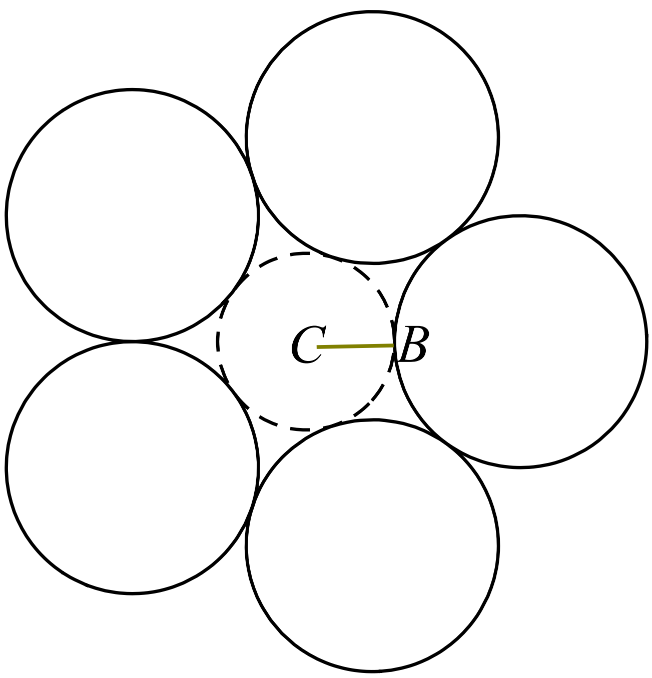
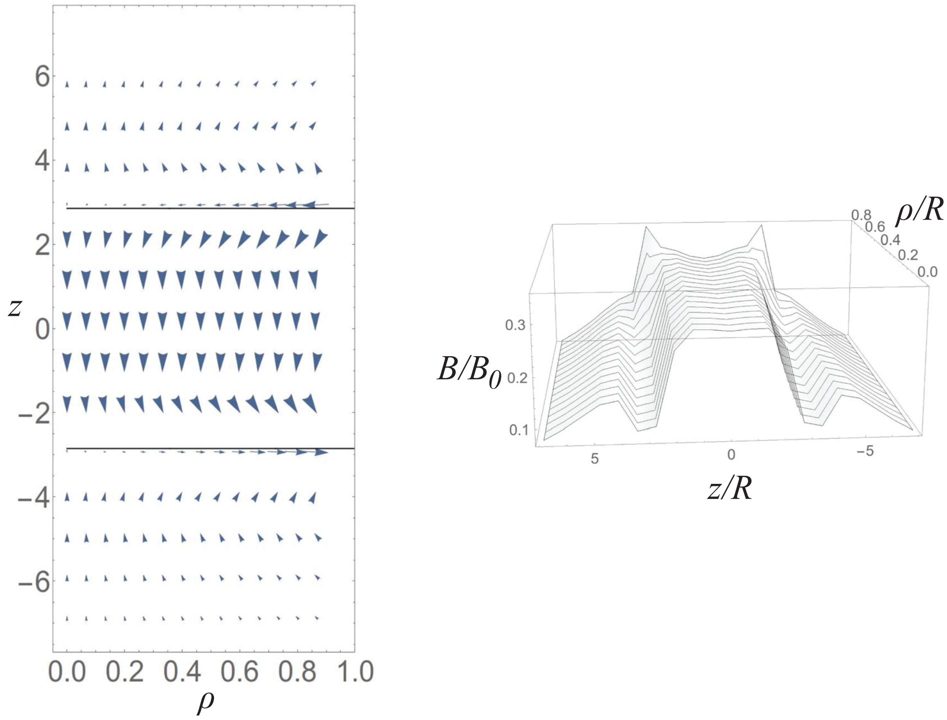



Fig. 6. Cross-section of a ring of $n = 5$ parallel continuous solenoids of length $L = 2$ and radius $r \approx 1.4$ that are tangent to neighboring solenoids and also to a common inner cylindrical surface of radius $R = 1$ (shown as a dotted dashed ring with center $C$). The line segment $CB$ denotes the location of the rectangular area, transverse to the plane of the figure, in which the magnetic field $\mathbf{B}$ is plotted in Fig. 7. See Fig. 1(a) for a perspective drawing.

## IV. THE MAGNETIC FIELD OF A HEXAGONAL ARRAY OF IDENTICAL PARALLEL CONTINUOUS SOLENOIDS

The second configuration of parallel finite solenoids that we explore is a large finite hexagonal lattice. Since we have already shown (see Fig. 4) that the external magnetic field of a single solenoid can be as large as $0.3B_0$ just outside the solenoid, this geometry is a potential candidate for producing an external magnetic field of substantial magnitude. In addition, the hexagonal lattice is the most densely packed of the five possible Bravais lattices in two dimensions,[14] and so is most likely to produce large external magnetic fields.




(a) (b)

Fig. 7. External magnetic field produced by $n = 5$ parallel continuous solenoids tangent to an inner cylinder of radius $R = 1$, for solenoid length $L = 4a \approx 5.6$ where the solenoid radius $a \approx 1.4$. (a) Vector plot of $\mathbf{B}(\rho, z)$ in the plane $\phi = 0$ [line CB of Fig. 6(b)] shows the direction and magnitude of the magnetic field inside the central cylindrical region. The lengths of the vector heads are proportional to the local magnitude $B$. (b) Surface plot of field magnitude $B/B_0$ normalized to the magnetic field magnitude $B_0$ inside an infinite single solenoid of the same current density $K$.

We define an hexagonal Bravais lattice[14] (see Fig. 9) such that the locations $\mathbf{r}_{mn}$ of each solenoid axis in the $z = 0$ plane are given by the two-dimensional vectors

$$\mathbf{r}_{mn} = s(m\mathbf{r}_1 + n\mathbf{r}_2) + \mathbf{r}_3, \tag{9}$$

where the coefficients $m$ and $n$ vary over all possible integers, and the positive parameter $s$ denotes the lattice spacing (distance between two nearest-neighbor solenoid axes). The basis vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are given by

$$\mathbf{r}_1 = \hat{\mathbf{x}}, \quad \text{and} \quad \mathbf{r}_2 = -\frac{1}{2}\,\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{y}}. \tag{10}$$

The vector $\mathbf{r}_3$, which here is $(1/2)\hat{\mathbf{x}} + (\sqrt{3}/4)\,\hat{\mathbf{y}}$, shifts the lattice relative to the origin of the coordinate system, and places the origin at the centroid of the triangle formed by three nearest-neighbour solenoids, which is convenient for plotting the external field. For a given lattice spacing $s$, the

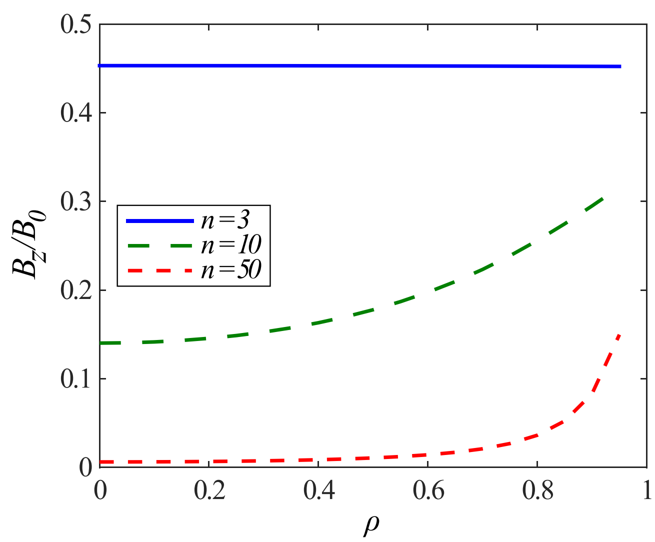



Fig. 8. Axial component $B_z(\rho, z = 0)$ on the bisection plane $z = 0$ for $n = 3$, 10, and 50 solenoids, for fixed solenoid length $L = 4a$ and fixed current density $K$. The increase in the number $n$ of sources cannot compensate for the $1/L^2$ decrease in magnetic field strength so the magnetic field in the central cylindrical region becomes small for large $n$.

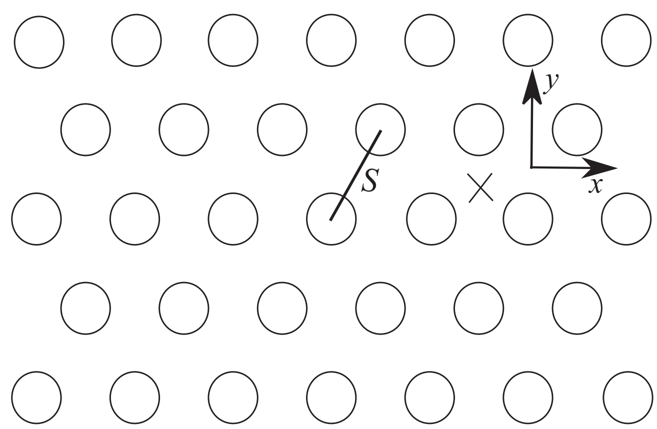



Fig. 9. Horizontal cross-section through a portion of an hexagonal lattice of parallel continuous solenoids for lattice spacing $s = 1$, as defined in Eqs. (9) and (10), see also Fig. 1(b). The common radius of the solenoids was chosen to be $a = 1/4$. Also shown are the directions of the coordinates $x$ and $y$. The origin of the coordinate system, marked with a cross, is chosen to be at the centroid of the triangle formed by three solenoids that are nearest neighbors.

solenoid radius $a$ must satisfy $s > 2a$, so that the solenoid surfaces do not intersect.

The array of solenoids is characterized by the lattice spacing $s$ and the length-to-radius ratio $L/a$ of each solenoids. We introduce an $(x, y, z)$ Cartesian coordinate system, such that the origin corresponds to the centroid of the triangle formed by the centers of three nearest-neighbour solenoids in a cell of the hexagonal lattice, and such that $z = 0$ corresponds to the bisecting plane of the solenoids.

We approximated the total magnetic field in a region about the origin for an infinite lattice by directly summing the fields from many solenoids in a large, finite, and approximately circular region centered on the point $(x, y, z) = (0, 0, 0)$. Figure 10 shows the component $B_z$ at $(0, 0, 0)$ as a function of the radius of the area from which fields of individual solenoids were included. Summing the fields from an area of the hexagonal array with radius $100\,s$ is sufficient to approximate the magnetic field due to the infinite array with a relative accuracy better than 0.05.

The magnetic field $B(x, y, z)$ in a triangular region surrounded by three nearest-neighbor solenoids in the center of the array was computed by individually summing the components of the magnetic field from the solenoids in the lattice from a region with radius $100\,s$ (see Appendix B). The field within each unit cell of the hexagonal lattice was found to be axisymmetric to high accuracy. Further details of the structure of the magnetic field within a single cell of the lattice are shown in Fig. 11, which is a vector field plot of the magnetic field in a plane defined by the rectangle shown in Fig. 1(b) and with axial coordinates between $z = \pm L$. The external magnetic field away from the edges of the solenoid is comparable in uniformity to the magnetic field inside the solenoid, with a magnitude $B_z \simeq 0.6 B_0$, despite the fact that the solenoid is long, with a length eight times its radius. This external magnetic field is quite uniform for most of $z \leq |L/2|$, unlike the central magnetic field described in Sec. III for a ring of solenoids.

We also found that the magnetic field $B_z(0, 0, 0)$ decreases approximately algebraically with increasing lattice size $s$ for fixed $a$ and $L$. Figure 12 shows how the magnetic field $B_z(0, 0, 0)$ decreases in magnitude with increasing lattice size $s$ for parameter values $K = 1$, $a = 1/4$, and $L = 0.1$, 5, 50, and 100. The decay of $B_z$ to 0 follows an approximate power law with a numerical exponent that approaches $-2$ with increasing solenoid length. Furthermore, we find that the magnitude of the magnetic field for any given lattice spacing depends nonmonotonically on the length of the solenoid. However, this dependence is small: for fixed $s$ we observe in Fig. 12 that changing the solenoid length $L$ by a factor of 20 decreases $B_z/B_0$ only by a factor of $\approx 0.5$.

This slow decay of $B_z/B_0$ towards zero as a function of $L$ for fixed $s$ can be seen more clearly in Fig. 13, which shows $B_z(0, 0, 0)/B_0$ as a function of $L\,/\,a$ for $a = 1/4$, $s = 1$,

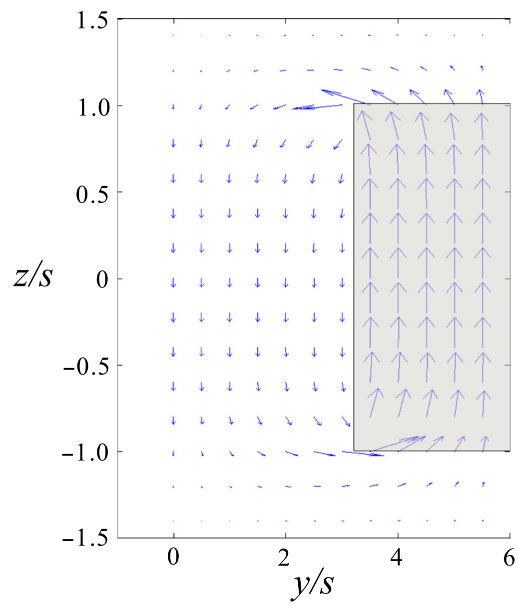



Fig. 11. Vector field plot of the magnetic field **B** generated between solenoids in a large hexagonal lattice in the black dashed rectangle shown in Fig. 1(b), with $s = 1$, $a = 1/4$, $K = 1$, and $L/a = 8$, summed over solenoids in a circular area with radius $R = 100s$. The location of the nearest solenoid in the $y$ direction is indicated by the shaded rectangle. Only the in-plane components of the magnetic field are non-zero. The magnetic field is highly uniform in the region external to the solenoids, for $|z| < L/2$.

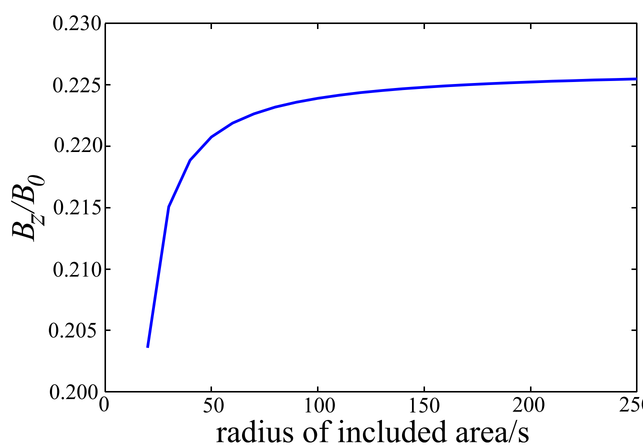



Fig. 10. Plot of the scaled magnetic field component $B_z$ at the centroid of three solenoids in a hexagonal lattice contained in a circle of radius $R$ as a function of $R$. For the largest radius $R = 500s$ (not shown), there are $\sim$750,000 solenoids enclosed within the circle. Note how over the range $30 < R < 500$, $B_z$ varies only by about 10%, and furthermore appears to asymptote to a finite value. A circle of radius $= 100s$ already approximates the infinite lattice case with a relative error better than 5%.

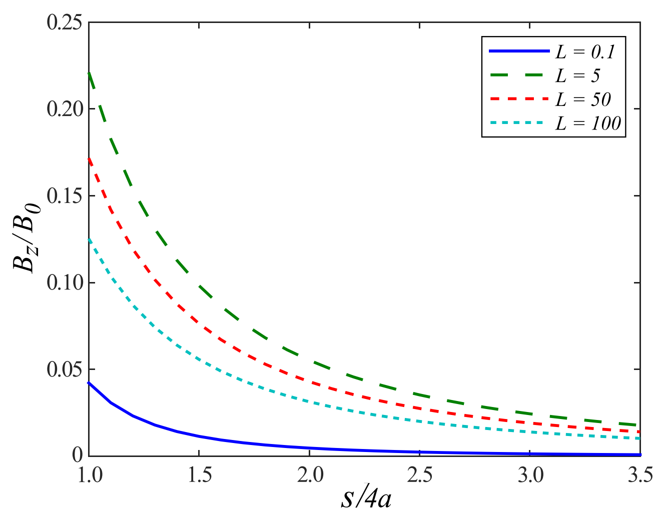



Fig. 12. Plot of the ratio $B_z(0, 0, 0)/B_0$ as a function of the lattice spacing $s$ for an hexagonal array of parallel identical solenoids of length $L = 0.1$, 5, 50, and 100, and with radius $a = 1/4$ and current density $K = 1$. The decay of $B_z$ towards 0 follows a power law with a numerically derived exponent that approaches $-2$ for increasing $L$, consistent with Eq. (6).

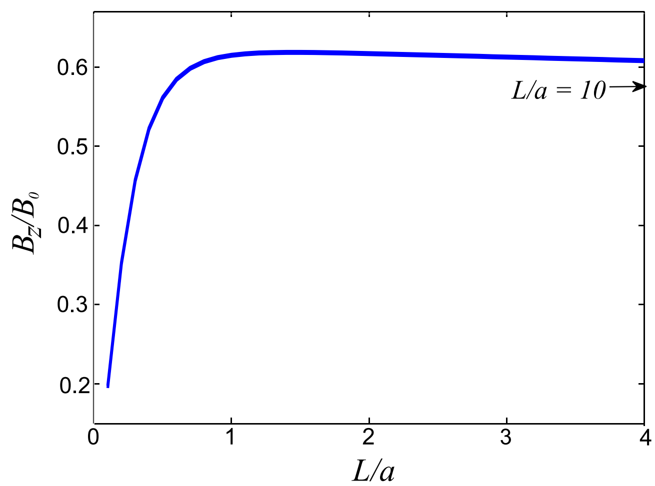



Fig. 13. Plot of the ratio $B_z(0,0,0)/B_0$ as a function of the solenoid length $L$ for a hexagonal array of solenoids with $s=1$. For $L/a = 10$, $B_z/B_0$ decreases to a value of 0.58, as marked by the arrow. The magnetic field attains a local maximum at $L/a = 1/2$ of about half the field of a single finite solenoid, after which it decays slowly towards the infinite length limit of $B_z = 0$.

and $K = 1$. As expected from our discussion of the external magnetic field of the single solenoid, the decay of $B_z$ is non-monotonic. However, after reaching a local maximum, $B_z$ decays slowly and approximately linearly with a numerically derived functional form $B_z = -0.0045L + 0.63$, from which we extrapolate that $B_z$ decays to 0 only for a solenoid length of $\sim 140$.

## V. CONCLUSIONS

In this paper, we have found that a magnetic field of substantial magnitude and uniformity can be produced by a finite number of parallel solenoids. Our motivation was to provide a deeper insight to the properties of the magnetic field external to a finite solenoid at the level an undergraduate student could understand.

We studied two configurations of identical parallel solenoids with high symmetry. One configuration consisted of a ring of $n$ parallel solenoids that each is tangent to nearest neighbors and to a common inner cylindrical surface [see Figs. 1(a) and 6]. The second configuration was a large, finite, roughly circular hexagonal array of parallel solenoids [see Figures 1(b) and 9] that we used to approximate the magnetic field generated by an infinite hexagonal array of parallel solenoids. For both cases, we investigated how the external magnetic field depended on the radius, length, and spacing of the solenoids, while the current density was kept fixed.

For the ring geometry and hexagonal array, we found that the external magnetic field could have a substantial magnitude, even for solenoids that are long compared to their radii. But only for the magnetic field generated by a large hexagonal array did we find that the external magnetic field could also be highly uniform, at least in regions of space that lies between the two planes that contain the ends of the solenoids.

In addition to providing new insights about the magnetic field generated by a single solenoid and by groups of solenoids, this paper provides a useful additional example to share with undergraduate physics students about how mathematics software environments like MATHEMATICA can be used to visually and efficiently explore physics problems.

## ACKNOWLEDGMENT


The authors thank Stephen Teithsworth for helpful discussions.


## APPENDIX A: VALIDATION OF THE NUMERICAL CALCULATIONS

The MATHEMATICA code needed to carry out the calculations discussed in Secs. II-IV is modest, about four pages in length. But it took some effort to determine the accuracy, and thus the correctness of the results. We summarize in this section some of the steps we took to validate the results.

We used the MATHEMATICA function `NIntegrate` to approximate the key one-dimensional integrals, Eqs. (2) and (3), that give the cylindrical components $B_r$ and $B_z$ of the magnetic field generated by a finite solenoid. This function takes as its arguments some integrand in symbolic form and the bounds of the definite integral (see Fig. 14) plus many possible optional parameters. (We used the default parameters and obtained acceptable accuracy as mentioned further below.) This function then uses an unspecified adaptive integration algorithm to numerically approximate the definite integral to some specified accuracy.

```
bzSolenoid[r_, z_, a_, L_] := Module[
  {ξp, ξn, K = 1., μ0 = 4.*π*10^-7},
  ξp = z + L/2;
  ξn = z - L/2;
  (a μ0 K)/(2 π) * NIntegrate[
    (a - r Cos[α])/(r^2 + a^2 - 2 a r Cos[α]) (ξp/(ξp^2 + r^2 + a^2 - 2 a r Cos[α])^0.5 - ξn/(ξn^2 + r^2 + a^2 - 2 a r Cos[α])^0.5)
    , {α, 0, π}
  ]
];
```

Fig. 14. MATHEMATICA code that uses the adaptive numerical integration function `NIntegrate` to approximate the magnetic field component $B_z(r,z,a,L)$ given in Eq. (3) at a point $(r, z)$ in space, for a solenoid of radius $a$, length $L$, and current density $K = 1$.

We validated our results with the following tests. First, we verified that Eq. (3) agrees accurately with the analytical expression for the magnetic field on the axis of a continuous solenoid with end surfaces at $z = \pm L/2$, given by

$$B_z(r=0,z) = \mu_0 K \frac{1}{2}\left(\frac{z+L/2}{\sqrt{(z+L/2)^2+a^2}} - \frac{z-L/2}{\sqrt{(z-L/2)^2+a^2}}\right). \tag{A1}$$

This expression includes the limiting case $L \to 0$ when the solenoid reduces to a current loop of radius $a$ and again the numerical approximations to the continuous solenoid were found to be accurate in that limit.[3]

Note that for $z=0$, Eq. (A1) predicts that $B_z \approx \mu_0 K[1-2(a/L)^2+\cdots]$ to lowest order in $(a/L)^2$, i.e., the relative error $[B_z(0,0)-\mu_0 K]/\mu_0 K$ decays as $1/L^2$ for large $L$, which is similar to what was found for the solenoid's external field by direct calculation.

Second, we verified that Eqs. (2) and (3) converge to the corresponding components

$$B_r(r,z) = \frac{\mu_0 m}{4\pi}\frac{3rz}{(r^2+z^2)^{5/2}},$$
$$B_z(r,z) = \frac{\mu_0 m}{4\pi}\frac{2z^2-r^2}{(r^2+z^2)^{5/2}} \tag{A2}$$

of the magnetic field of the continuous solenoid's equivalent point magnetic dipole at its center $r=z=0$, with magnetic moment

$$m = (\pi a^2)KL, \tag{A3}$$

when the distance $d=\sqrt{r^2+z^2}$ of the point $(r,z)$ of evaluation was large ($d \gg a, L$) compared to the solenoid's radius or length. That is, the solenoid's external magnetic field correctly converges to that of a point magnetic dipole far from the solenoid, with a $1/d^3$ decay in magnitude. Equation (A3) was obtained by comparing Eq. (6) with the magnetic field of a point dipole $\mu_0 m/4\pi r^3$ for $z=0$ in the far-field regime $r \gg L, a$. Lastly, we verified that the total magnetic field $\mathbf{B}$ at points of certain symmetry had zero components, even though the numbers added to obtain the component were nonzero.

## APPENDIX B: ADDING UP RADIAL COMPONENTS $B_r$ FROM DIFFERENT SOLENOIDS

For both the ring geometry and the hexagonal array, one has to add magnetic fields from different solenoids to get the total external magnetic field at a given point in space. We show the main details for the ring geometry by explaining how to calculate the total magnetic field $\mathbf{B}(\rho,\theta,z)$ at some point $P=(\rho,\theta,z)$ inside the common cylindrical surface $\rho=R$ of the ring of $n$ solenoids described in Sec. III. The key issue is that the radial unit vector $\hat{\mathbf{r}}_i$ associated with the $i$th solenoid varies with $i$ as shown in Fig. 15, so one has to accumulate the components of the vector $B_{r,i}\hat{\mathbf{r}}_i$ along the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\theta}}$ of the $(\rho,\theta,z)$ coordinate system. The

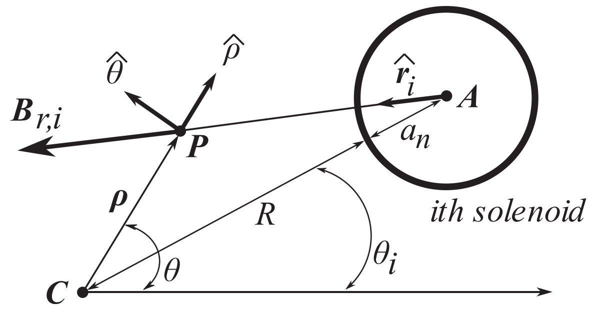



Fig. 15. Illustration showing how the total magnetic field $\mathbf{B}_{\rm tot}$ at a point $P=(\rho,\theta,z)$ inside the common cylindrical surface $\rho=R$ with axis at point $C$ is obtained by adding the vectors $\mathbf{B}_{r,i}=B_{r,i}(r_i,z)\hat{\mathbf{r}}_i$ of the magnetic field created by the $i$th solenoid whose axis at point $A$ has the coordinates $(\rho,\theta)=(R+a_n,\theta_i)$. The length $r_i$ is the radial coordinate of the point $P$ with respect to the solenoid's axis at $A$.

$z$-component $B_z$ is much easier to compute since the $z$ axes of the solenoids are all parallel to the $z$-axis of the cylindrical surface.

If the radius $a_n$ of each solenoid is given in Eq. (8), the axis of each solenoid lies on the circle $\rho=R+a_n$ and has angular coordinate

$$\theta_i = \frac{2\pi}{n}(i-1), \qquad i=1,\ldots,n. \tag{B1}$$

If we define the two-dimensional vectors

$$\hat{\boldsymbol{\rho}} = (\cos\theta,\ \sin\theta), \tag{B2}$$

$$\hat{\boldsymbol{\theta}} = (-\sin\theta,\ \cos\theta), \tag{B3}$$

$$\boldsymbol{\rho} = \rho\,\hat{\boldsymbol{\rho}}, \tag{B4}$$

and

$$\mathbf{a}_i = (R+a_n)(\cos\theta_i,\ \sin\theta_i), \tag{B5}$$

then the vector $\boldsymbol{\rho}-\mathbf{a}_i$ points from the axis $A$ of the $i$th solenoid to the point $P$. Its Euclidean length

$$r_i = \|\boldsymbol{\rho}-\mathbf{a}_i\| \tag{B6}$$

is the distance of the point $P$ to the axis of the $i$th solenoid, and so the radial unit vector $\hat{\mathbf{r}}_i$ along the radial coordinate centered on the $i$th solenoid is given by

$$\hat{\mathbf{r}}_i = \frac{\boldsymbol{\rho}-\mathbf{a}_i}{r_i}. \tag{B7}$$

With this notation, the radial and azimuthal components of the total magnetic field at the point $P$ are given by

$$B_\rho = \hat{\boldsymbol{\rho}}\cdot\sum_{i=1}^{n} B_{r,i}\hat{\mathbf{r}}_i, \tag{B8}$$

and

$$B_\theta = \hat{\boldsymbol{\theta}}\cdot\sum_{i=1}^{n} B_{r,i}\hat{\mathbf{r}}_i, \tag{B9}$$

where $B_{r,i} = B_r(r_i, z)$ is the radial component of the magnetic field [Eq. (2)] at a point a distance $r = r_i$ from the axis of the $i$th solenoid.

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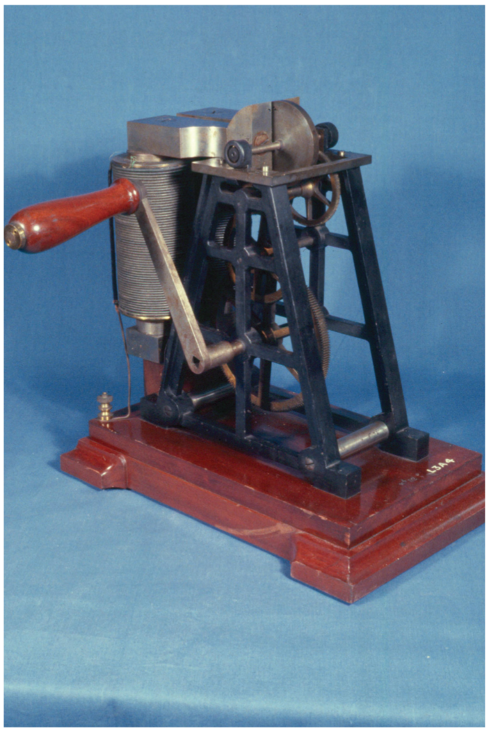

**Foucault's Disk**

This apparatus, designed by Leon Foucault (1819-1868) consists of a copper disk that can be spun rapidly in the field of a powerful electromagnet. When the magnet is not energized, the disk spins freely, but when it is turned on, turning the crank to spin the disk becomes difficult. From Adolphe Ganot, *Elementary Treatise on Physics*, eleventh edn, trans. and ed. By E. Atkinson (William Wood and Co., New York, 1883), pg 869: "In an experiment made by Foucault, the temperature of the disk rose from 10° [C] to 61° … The currents thus produced in solid conductors and which are converted into heat, are often spoken of as *Foucault currents*." This example is in the collection of the National Museum of American History at the Smithsonian Institution. (Notes and picture by Thomas B. Greenslade, Jr., Kenyon College)