In closing, it should be stressed that use of the complex functions $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ is limited to "two-dimensional" problems, or at least to three-dimensional problems that possess axial symmetry about one of the coordinate axes so that one dimension can be ignored. It is only then that one can construct an $\tilde{\mathbf{E}}$ or a $\tilde{\mathbf{B}}$ for which the Cauchy–Riemann conditions are nothing more than Maxwell's equations for **E** and **B** in two dimensions. This method, then, would not work for a system consisting of a single point charge placed at the origin. The problem possesses spherical symmetry, and one could not construct an $\tilde{\mathbf{E}}$ from E_x, E_y, and E_z that is analytic and consistent with Maxwell's equations.

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[1]A function of $f(z)$ of a complex variable $z = x + iy$ is analytic in a region R of the complex plane if and only if any of the following conditions are true; (a) $f(z)$ has a derivative at each point z_0 in R defined by $(df/dz)|_{z=z_0} = \lim_{\Delta z \to 0} f(z_0 + \Delta z) - f(z_0)/\Delta z$, where Δz approaches zero through any complex values; (b) the line integral $\int f(z)\, dz$ about any closed path in R vanishes, (c) $f(z)$ has a Taylor series expansion in powers of $(z - z_0)$ about each point z_0 in R; and (d) $f(z)$ can be written as $f(z) = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are real-valued functions of x and y and satisfy the Cauchy–Riemann differential equations, $(\partial u/\partial x) = (\partial v/\partial y)$ and $(\partial u/\partial y) = -(\partial v/\partial x)$.

[2]Ruel V. Churchill, James W. Brown, and Roger F. Verhey, *Complex Variables and Applications* (McGraw-Hill, New York, 1976), pp. 172–174.

[3]The convention adopted in most texts on complex analysis is the following: A curve C that forms a boundary for a closed region R is traversed in the positive sense if it is traversed such that the interior points of R lie to the left of the curve C.

[4]David Halliday and Robert Resnick, *Fundamentals of Physics* (Wiley, New York, 1981), pp. 561–563.

[5]John R. Reitz, Frederick J. Milford, and Robert W. Christy, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1979), pp. 172–173.

[6]Reference 4, p. 559.

[7]Reference 4, pp. 558–559.

[8]Reference 2, p. 127.

[9]Reference 2, p. 121.

[10]Reference 4, pp. 455–456.

[11]The "flux" of a vector field **A** through a two-dimensional surface S is defined as $\Phi_A = \int_s \mathbf{A} \cdot d\mathbf{S}$. The dot product picks out the component of **A** that is normal to the surface through the surface element dS.

[12]Gauss' law for electrostatics states that the surface integral of **E** through a closed surface S is given by $\int \mathbf{E} \cdot d\mathbf{S} = q_0/\epsilon_0$, where q_0 is the net charge enclosed by the surface, and the element $d\mathbf{S}$ points normal to the surface.

[13]It is customary to describe a current distribution by a current density **J** measured in units of positive charge crossing a unit area per unit time. The direction of **J** is defined by the direction of motion of the charges.

The magnetic field of a circular turn

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This article presents the numerical solution for the magnetic field of a current-carrying circular turn at any point. Textbook solutions usually confine themselves to the solution on the axis of the turn because the on-axis Biot–Savart law integration is easy to do in closed form. As the knowledge and use of hand-held calculators and computers has increased, the numerical solution of this problem for off-axis points has become possible by students in the introductory calculus-based course. A good way to do this problem is in the physics laboratory where actual measurements can be compared with the values obtained from a numerical solution.

I. INTRODUCTION

The use of the computer as a calculational tool has only barely begun in introductory physics courses. This is understandable because (1) the pressure of a very large and diverse subject matter does not leave much time for pursuing problems that require even the simplest of numerical techniques; and (2) students at the introductory physics level may lack the experience of working even simple numerical problems. However, as the years go by, the present picture will surely change and we will see an increased level of computer knowledge in our introductory physics students and a corresponding increase in the number of numerical problems in the introductory course.

The very first group of problems which are choice candidates for numerical solutions are those problems that students could always have been expected to set up, but whose solutions could not be achieved in closed form. This article describes such a problem and presents the numerical solution. It is the problem of the magnetic field of a circular current-carrying turn (or a flat coil). This problem is of particular interest because it also lends itself to experimental measurements in the laboratory. A recent article in this Journal by Gnanatilaka and Fernando[1] describes a laboratory experiment in which some of these measurements were made. It may be that the wave of the future is to introduce numerical solutions via laboratory experiments. This would solve the problem of having insufficient time to consider numerical problems in the lecture and recitation parts of the introductory course.

II. THE MAGNETIC FIELD IN THE PLANE OF A CIRCULAR, CURRENT-CARRYING TURN

In the student experiment described in Ref. 1, the magnetic field of a flat coil is measured for points in the plane of the coil that are inside the coil radius. The setup of the apparatus for this experiment (using the Earth's magnetic field as a reference) is easier than that for points outside the plane of the coil. Hence, it is convenient to divide the calculational problem into the specialized case of the magnetic field in the plane of the turn, and the general case of the magnetic field everywhere. Neither of these problems is currently treated in the introductory course. Indeed, even at the intermediate and advanced levels there seems to be little discussion of the problem. The text by Smythe[2] cited in Ref. 1 seems to be one of the few books that considers this problem. Smythe gives a general solution utilizing the vector potential. This solution is in terms of the complete elliptic integrals K and E.[3] Since neither the vector potential nor the elliptic integrals are familiar to students of the introductory course, we offer a solution based only on the Biot–Savart law and numerical integration.

We begin by considering the magnetic field in the plane of a circular, current-carrying turn (or flat coil). Figure 1 shows such a turn of radius R carrying a current I. A typical **ds** at point D give a **dB** at point P, which is into the plane of the diagram. We shall count this direction as negative. Another typical **ds** at point F gives a **dB** at point P, which is out of the plane of the diagram. From the Biot–Savart law,

$$\mathbf{dB} = (\mu_0 I/4\pi)(\mathbf{ds}\times\hat{\mathbf{r}}/r^2), \qquad (1)$$

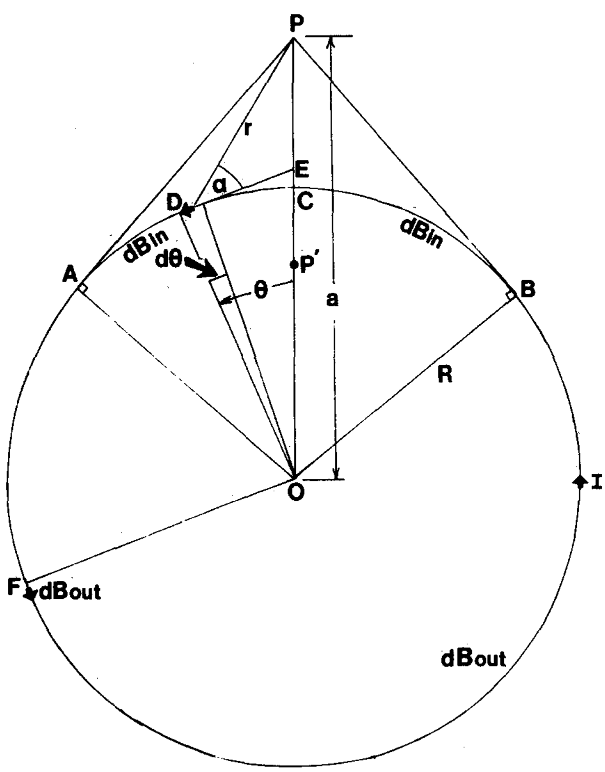

Fig. 1. Diagram for the determination of **B** in the plane of the turn.

it is clear that all the current elements along the arc ACB give **dB**s into the plane of the diagram, where the angles OAP and OBP are right angles. Current elements along the arc AFB give **dB**s out of the plane of the diagram.

The law of cosines applied to triangle OPD gives

$$r^2 = a^2 + R^2 - 2aR\cos\theta. \qquad (2)$$

The law of sines applied to triangle DEP gives

$$\sin\alpha = (a\cos\theta - R)/r. \qquad (3)$$

Substituting from Eqs. (2) and (3) into Eq. (1), we have

$$dB = \frac{\mu_0 I\,[1-(a/R)\cos\theta\,]d\theta}{4\pi R\,[1+(a/R)^2-2(a/R)\cos\theta\,]^{3/2}}. \qquad (4)$$

If we let

$$x = a/R,$$

and we use the familiar result that the magnetic field at the center of the loop is

$$B_c = \mu_0 I/2R,$$

then Eq. (4) can be rewritten as

$$dB = B_c(1 - x\cos\theta)d\theta/2\pi(1+x^2-2x\cos\theta)^{3/2}. \qquad (5)$$

We can integrate both sides of Eq. (5) to obtain

$$B_{\mathrm{rel}} = \frac{B}{B_c} = \frac{1}{\pi}\int_0^{\pi}\frac{(1-x\cos\theta)d\theta}{(1+x^2-2x\cos\theta)^{3/2}}, \qquad (6)$$

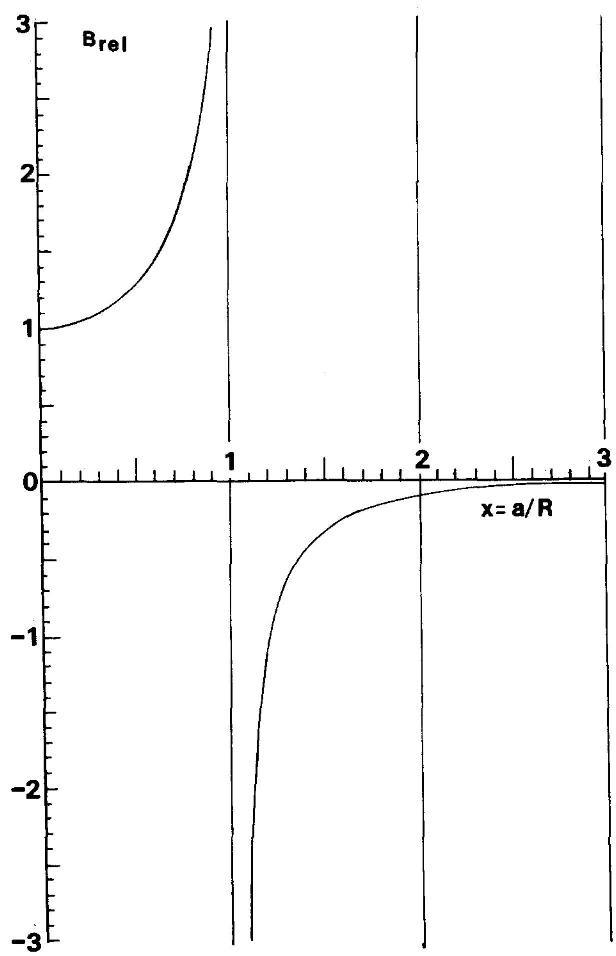

Fig. 2. The relative magnetic field in the plane of a circular, current-carrying turn.

where we have taken the integral from zero to π and doubled it since we have equal contributions from zero to π and from π to 2π. Although Eq. (6) was derived for exterior points P, it is also valid for interior points, such as P′ in Fig. 1. Equation (6) is, of course, not valid for $x = 1$, the locus of the circular, current-carrying turn.

The numerical solution of Eq. (6) should be an easy matter for students having some computer knowledge. Any decent integration routine can be used. The results obtained when a 24-point Gaussian integration[4] was used are plotted in Fig. 2. This numerical solution was checked against direct computations using tabulated values of K and E and Smythe's formula as given in Ref. 1, Eq. (1). The agreement was excellent.

III. THE NUMERICAL SOLUTION OF THE GENERAL PROBLEM

We can now proceed to the general solution for the magnetic field of a circular turn. Figure 3 illustrates the general problem. Given any point P′ a distance a from the axis of the current-carrying loop and a distance b from the plane of the current-carrying loop, the x axis of a Cartesian set is chosen so that it is parallel to AP′ as shown in Fig. 3. From symmetry, the y component of **B** at point P′ has to be zero, so the problem reduces to the determination of B_z and B_x.

The vector **r** from **ds** to P′ is given by

$$\mathbf{r} = (a - x)\hat{\mathbf{i}} - y\hat{\mathbf{j}} + b\,\hat{\mathbf{k}}, \qquad (7)$$

and the current element at P is

$$I\,\mathbf{ds} = I(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}).$$

Applying the Biot–Savart law, we find

$$\mathbf{dB} = (\mu_0 I/4\pi)\{b\,dy\,\hat{\mathbf{i}} - b\,dx\,\hat{\mathbf{j}} + [-y\,dx - (a - x)dy]\hat{\mathbf{k}}\}/r^3. \qquad (8)$$

This expression can be transformed into an expression in θ utilizing the equations $x = R\cos\theta$, and $y = R\sin\theta$, and we obtain

$$B_{x\,\mathrm{rel}} = \frac{B_x}{B_c} = \frac{1}{\pi}\int_0^{\pi} \frac{q\cos\theta\,d\theta}{(1 - 2p\cos\theta + p^2 + q^2)^{3/2}}, \qquad (9)$$

$$B_{z\,\mathrm{rel}} = \frac{B_z}{B_c} = \frac{1}{\pi}\int_0^{\pi} \frac{(1 - p\cos\theta)d\theta}{(1 - 2p\cos\theta + p^2 + q^2)^{3/2}}, \qquad (10)$$

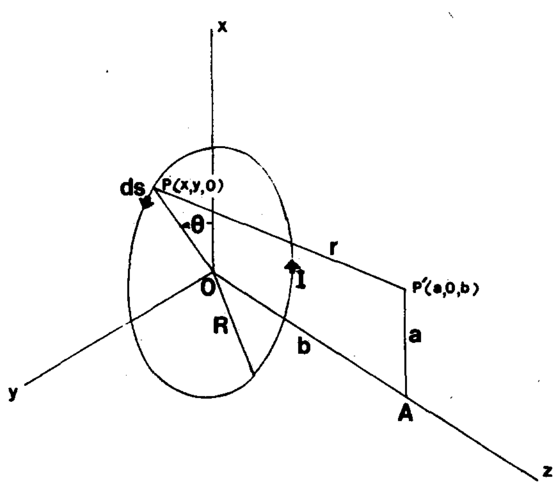

Fig. 3. Diagram for the general problem.

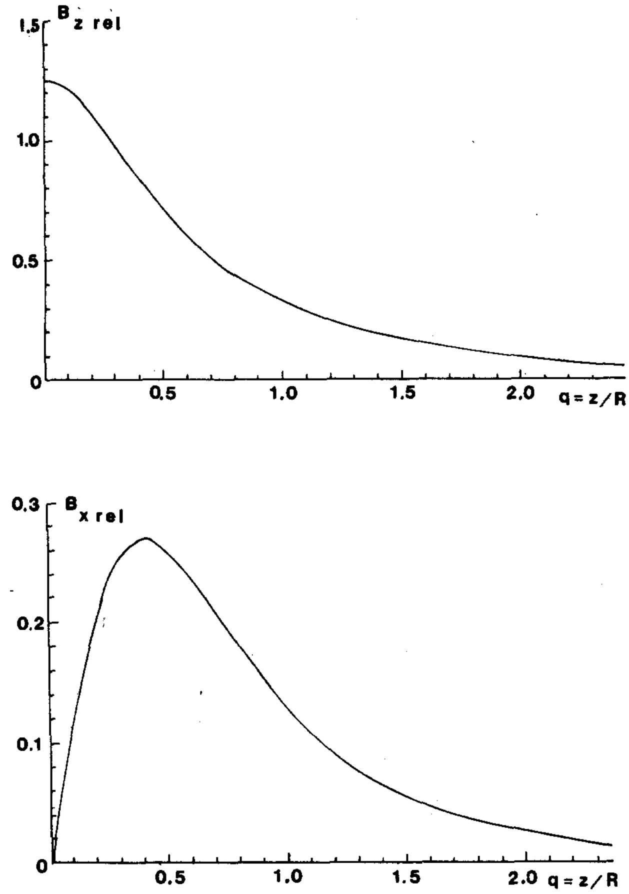

Fig. 4. The components of **B** for $a = R/2$ ($p = 0.5$).

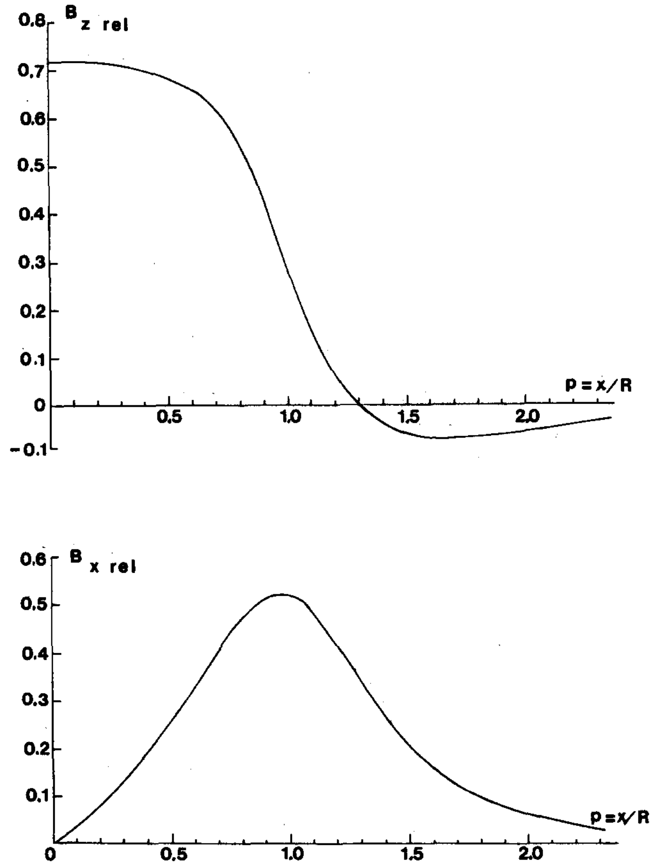

Fig. 5. The components of **B** for $b = R/2$ ($q = 0.5$).

where $p = a/r$ and $q = b/R$.

Equation (10) is the generalization of Eq. (6) and reduces to Eq. (6) when $q = 0$. On the z axis, where $p = 0$, Eq. (10) integrates to the familiar result for the on-axis B field of a circular turn,

$$B = \mu_0 I R^2 / 2(R^2 + z^2)^{3/2}. \tag{11}$$

Once again, the numerical solution to Eqs. (9) and (10) can be achieved by a numerical integration routine. Some of the results that were obtained using a 24-point Gaussian integration (DQG24) are shown in Figs. 4 and 5. Figure 4 displays the two components of $\mathbf{B}$ as one moves along a line parallel to the axis of the loop and at a distance of half the loop radius from the axis. Figure 5 shows the two components of $\mathbf{B}$ as one moves along a line parallel to the plane of the loop at a distance of half the loop radius.

IV. CONCLUSION

The numerical solution of the problem of the magnetic field of a circular, current-carrying turn has been presented in this article. The solution in integral form should be readily doable by students in the introductory course for science and engineering students. In addition, the numerical calculation of the integrals should be accessible to these students either through the use of a hand calculator or by a computer integration routine. This kind of problem is a good candidate for a laboratory exercise in which the students measure the field of the loop and compare it with their numerical solution.

ACKNOWLEDGMENT

The author thanks his colleague, Alan Benimoff, for the drawing of the figures.

[1]H. G. Gnanatilaka and P. C. B. Fernando, "An Investigation of the Magnetic Field in the Plane of a Circular Current Loop," Am. J. Phys. **55**, 341–344 (1987).

[2]W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York, 1950), pp. 270–271.

[3]The text by J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), pp. 141–145 gives an explicit solution for the vector potential in terms of K and E, but does not give an explicit general solution for the field components.

[4]The subroutine DQG24 from the IBM Scientific Subroutine package was used. For a reference on Gaussian numerical integration, see any standard work on numerical methods, for example, the reference on Gaussian integration suggested in the IBM manual is V. J. Krylov, *Approximate Calculation of Integrals* (Macmillan, New York/London, 1962), pp. 100–111 and 337–340.

Energy balance and the Abraham–Lorentz equation

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It is shown that the energy balance usually assumed for the derivation of the Abraham–Lorentz equation is incomplete because it does not take the change in the energy of the bound self-field into account. Starting from the energy balance implied by Poynting's theorem, a new derivation of the Abraham–Lorentz equation is presented. The energy balance is applied first to an extended charge distribution and then the point-charge limit is taken. This approach neither requires neglecting the so-called Schott energy term nor any further hypothesis. The calculations point to the dragged self-field as the origin of the "radiation" damping term.

I. INTRODUCTION

It is well known[1] that diverse conceptual difficulties impeded a clear understanding of the interaction between a charged particle and its self-field in the process of radiation. The classical nonrelativistic theory that supposedly takes the radiation reaction for a point charge into account is summed up in the Abraham–Lorentz (A–L) equation. In Gaussian units, this equation is

$$m\dot{\mathbf{a}} = \mathbf{F}_{\text{ext}} + m\tau\dot{\mathbf{a}}, \tag{1}$$

where $m\tau = 2e^2/3c^3$ and $m\tau\dot{\mathbf{a}}$ is the radiation reaction force.

While classically no force implies no acceleration, if we put $\mathbf{F}_{\text{ext}} = 0$ in this equation, we get

$$\mathbf{a}(t) = \mathbf{a}(0)e^{t/\tau} \tag{2}$$

and these diverging solutions imply an unexplained infinite growth of the kinetic energy of the particle.

One of the baffling aspects of this theory is precisely that under certain assumptions Eq. (1) can be derived from an energy balance,[2] while the general solution violates this balance.

In an attempt to reestablish energy conservation, it is sometimes argued that this infinite kinetic energy comes from the energy of the self-field, also infinite for a point (finite) charge such as the electron.[3] It must be noted, however, that since this infinite energy associated with the self-field of a point charge is the energy necessary to assemble a finite charge into a point, the liberation of this energy would mean the explosion of the charge and not its self-propulsion as a whole.

We intend to show in this article that the energy balance