NOTES AND DISCUSSIONS

The Dirac equation with a confining potential

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Recently, Bhalerao and Ram examined the Dirac equation for a particle in one space dimension with a Lorentz scalar potential of the form of $g|x|$ where $g>0$.[1] Their intention was to devise a one-dimensional model that simulates a quark–antiquark bound system. They were able to find only one bound state of the particle. Their attempt triggered three publications in which it was found that the model does have an infinite number of bound states.[2–4] There are no continuum states, and the particle is confined.[5]

For the Dirac equation in three dimensions, a Lorentz scalar potential of the form of gr, where r is the radial variable, is a confining potential and allows an infinite number of bound states but no scattering states.[6,7] The purpose of this note is to draw attention to the fact that, as far as the confining mechanism is concerned, there is no essential difference between the one-, two-, and three-dimensional cases.

Let us start with the one-dimensional Dirac equation,

$$(H-E)\psi(x)=0, \tag{1a}$$

$$H=\alpha p+\beta[m+S(x)]+V(x), \tag{1b}$$

where $p=-i d/dx$, m is the rest mass of the free particle, and E is the energy eigenvalue. We are using units such that $\hbar=c=1$. The quantity $S(x)$ is a Lorentz scalar and $V(x)$ is the zeroth component of a Lorentz vector.[8] For the motivation of assuming $S(x)$, see Refs. 6 and 7 and references quoted therein. The $m+S(x)$ can be regarded as an effective mass that depends on x. The separation of the effective mass into the two terms of m and $S(x)$ is actually arbitrary. The ψ is a two-component wave function. The Dirac matrices α and β can be represented by any two of the 2×2 Pauli matrices. We use $\alpha=\sigma_y$ and $\beta=\sigma_z$. Then $\alpha p=-i\sigma_y d/dx$ is real. As a consequence ψ can be taken to be real. The representation of the Dirac matrices that we are using is different from the one used in Refs. 1–4 where $\alpha=\sigma_y$ and $\beta=\sigma_x$. The latter choice is convenient for seeing the supersymmetry aspect of the one-dimensional Dirac equation (with $V(x)=0$),[9,10] but not for seeing the relation between the one-, two-, and three-dimensional cases.

Let the upper and lower components of ψ be u and v, respectively. The one-dimensional Dirac equation can be written as

$$-v'+(m+S-E+V)u=0, \tag{2a}$$

$$u'-(m+S+E-V)v=0, \tag{2b}$$

where $u'=du/dx$. We define χ by

$$\chi=D^{-1/2}\,u, \tag{3}$$

with $D=m+S+E-V$. Then Eq. (2) can be reduced to

$$-\frac{\chi''}{2m}+W\chi=\frac{E^2-m^2}{2m}\chi. \tag{4}$$

Equation (4) is of the form of the nonrelativistic Schrödinger equation. Note, however, that E is the relativistic energy which includes the rest mass of the particle and can be positive or negative.[11] The effective potential W is given by

$$W=S+\frac{S^2-V^2+2EV}{2m}+\frac{3D'^2-2DD''}{8mD^2}. \tag{5}$$

The function $W(E,x)$ depends on energy E. If $m=0$, we multiply Eq. (4) by $2m$. Then $2mW$ is well defined for $m\to 0$.

We are interested in potentials such that $S(x)\to\pm\infty$ and $V(x)\to\pm\infty$ as $|x|\to\infty$. As an example let us consider

$$S(x)=g|x|,\quad V(x)=h|x|. \tag{6}$$

When $|x|$ becomes very large, the term $(S^2-V^2)/(2m)$ dominates in W, resulting in

$$W\to\frac{(g^2-h^2)x^2}{2m}. \tag{7}$$

This asymptotic behavior of W holds for all values of E. If $g^2-h^2>0$, W grows like a harmonic oscillator potential as $|x|$ increases and hence Eq. (4) allows nothing but bound states. The particle of the model is confined. However, if $g^2-h^2<0$, W behaves like an inverted harmonic oscillator potential. We are not interested in such cases.

Although we have assumed specific forms for $S(x)$ and $V(x)$, the analysis can easily be applied to other forms of $S(x)$ and $V(x)$. For example, they can be

$$S(x)=gx^{\nu},\quad V(x)=hx^{\nu}, \tag{8}$$

where ν is an arbitrary positive constant.[12] This potential is not necessarily an even function of x. If $S(x)=gx$ and $V(x)=0$, the Dirac equation has simple solutions. See Appendix B of Ref. 10. The functions $S(x)$ and $V(x)$ can also have an exponential form.

What we have done is nothing but a one-dimensional version of what Ram did a long time ago for three dimensions.[7] If we assume a central potential with $S(r)$ and $V(r)$ in $n=1$, 2, or 3 dimensions, the Dirac equation can again be reduced to Eq. (4), but with $\chi = r^{(1/2)(n-1)}D^{-1/2}u$ and a different W. The n-dimensional version of W, which we denote by $W^{(n)}$, is given by

$$W^{(n)} = W^{(1)} + \frac{k^{(n)}}{2mr}\left(\frac{k^{(n)}-1}{r} - \frac{D'}{D}\right), \tag{9}$$

where $W^{(1)}$ is the W of Eq. (5) with $S(x)$ and $V(x)$ replaced by $S(r)$ and $V(r)$, respectively. The $k^{(n)}$ is a dimensionless quantum number related to the angular momentum. It takes the values: $k^{(1)}=0$, $k^{(2)}=\pm\frac{1}{2},\pm\frac{3}{2},\ldots,$ and $k^{(3)}=\pm 1, \pm 2,\ldots$. For $k^{(2)}$ and $k^{(3)}$ see Refs. 13 and 14, respectively. The $W^{(3)}$ was derived in Ref. 7. As far as we know, however, $W^{(2)}$ is new. Its derivation would be a good (although somewhat tedious) exercise for students.

If $r\rightarrow\infty$, the term with $k^{(n)}$ of Eq. (9) becomes negligible in comparison to $W^{(1)}$ which contains $(S^2-V^2)/(2m)$. If $W^{(1)}$ is a confining potential, so are $W^{(2)}$ and $W^{(3)}$. The three-dimensional case with $S(r)=gr$ and $V(r)=0$ or a Coulombic $V(r)$ was examined in detail by Critchfield.[6] We have numerically reexamined Eq. (4) for the one-dimensional case with $S(x)=g|x|$ and $V(x)=0$ and confirmed that the energy spectrum is the same as the one obtained in Refs. 2–4. We also numerically examined the three-dimensional case with $S(r)=gr$ and $V(r)=0$ and compared the results with those of Ref. 6. The energy spectrum of the one-dimensional case of odd parity is very similar to that of the three-dimensional case with $k^{(3)}=1$.

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[1]R. S. Bhalerao and Budh Ram, ''Fun and frustration with quarkonium in a 1+1 dimension,'' Am. J. Phys. **69**, 817–818 (2001).

[2]Antonio S. de Castro, ''Comment on 'Fun and frustration with quarkonium in a 1+1 dimension,' by R. S. Bhalerao and B. Ram [Am. J. Phys. **69** (7), 817–818 (2001)],'' Am. J. Phys. **70**, 450–451 (2002).

[3]R. M. Cavalcanti, ''Comment on 'Fun and frustration with quarkonium in a 1+1 dimension,' by R. S. Bhalerao and B. Ram [Am. J. Phys. **69** (7), 817–818 (2001)],'' Am. J. Phys. **70**, 451–452 (2002).

[4]John R. Hiller, ''Solution of the one-dimensional Dirac equation with a linear scalar potential,'' Am. J. Phys. **70**, 522–524 (2002).

[5]The solutions of the Dirac equation for the model of Ref. 1 are given in terms of the Hermite function H_ν or equivalently, the parabolic cylinder function D_ν. The index ν was apparently assumed to be a positive integer in Ref. 1, which is why only one bound state was found. But ν is in general a noninteger as shown in Refs. 2–4.

[6]C. L. Critchfield, ''Scalar binding of quarks,'' Phys. Rev. D **12**, 923–925 (1975); ''Scalar potentials in the Dirac equation,'' J. Math. Phys. **17**, 261–266 (1976).

[7]Budh Ram, ''Quark confining potential in relativistic equations,'' Am. J. Phys. **50**, 549–551 (1982).

[8]The terminology such as ''Lorentz scalar'' in the present context is not strictly legitimate. For example, the ''scalar potential'' $S(x)=g|x|$ is not a Lorentz scalar because $|x|$ is not invariant under the Lorentz transformations. However, we adopt this commonly used terminology.

[9]F. Cooper, A. Khare, R. Musto, and A. Wipf, ''Supersymmetry and the Dirac equation,'' Ann. Phys. (N.Y.) **187**, 1–28 (1987).

[10]Y. Nogami and F. M. Toyama, ''Supersymmetry aspects of the Dirac equation in one dimension with a Lorentz scalar potential,'' Phys. Rev. A **47**, 1708–1714 (1993).

[11]In Eq. (4) v has been eliminated in favor of χ or u. Alternatively, we can eliminate u in favor of v by redefining χ as $\chi = v(m+S-E+V)^{-1/2}$. This different choice of χ, which we do not use in this note, results in the same physics, that is, the same wave functions (in terms of u and v), and the same energy eigenvalues. In this connection, note that Eq. (2) is invariant under the simultaneous substitutions of $u\rightarrow v$, $v\rightarrow u$, $E\rightarrow -E$, and $V\rightarrow -V$. If $V=0$, there is a symmetry between positive and negative energy eigenvalues.

[12]If $\nu<1$ and if $(g-h)(m+E)<0$, the last term of W becomes singular and attractive. Then there can be a problem for W in supporting bound states. We exclude such situations. Essentially the same remark applies to the $W^{(n)}$ of Eq. (9).

[13]F. A. B. Coutinho and Y. Nogami, ''Zero-range potential for the Dirac equation in two and three space dimensions: Elementary proof of Svenson's theorem,'' Phys. Rev. A **42**, 5716–5719 (1990).

[14]L. I. Schiff, *Quantum Mechanics* (McGraw–Hill, New York, 1968), 3rd ed., Chap. 13.

Comment on ''The falling chain and energy loss,'' by David Keiffer [Am. J. Phys. 69 (3), 385–386 (2002)]

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Recently, D. Keiffer presented a simple discrete model to elucidate the energy loss in a falling frictionless chain. Keiffer cited three references where the problem of the continuous chain is solved.[1–3] However, a reference was overlooked that already discussed the rationale for the energy loss.[4] It also includes the same discrete model discussed by Keiffer, which reproduces the continuous chain as a limit. A recent book which is not the same as Ref. 4, but which has one author in common,[5] also includes the explanation and the discrete model of the falling chain.

[1]J. B. Marion and S. T. Thornton, *Classical Dynamics of Particles and Systems* (Harcourt-Brace, Fort Worth, TX, 1995), 4th ed., p. 376.

[2]G. R. Fowles and G. L. Cassidy, *Analytical Mechanics* (Harcourt-Brace, Fort Worth, TX, 1993), 5th ed., p. 256.
[3]R. Becker, *Introduction to Theoretical Mechanics* (McGraw–Hill, New York, 1954), p. 190.
[4]E. J. Saletan and A. H. Cromer, *Theoretical Mechanics* (Wiley, New York, 1971), p. 25 and Problem 13 on p. 28.
[5]J. V. José and E. J. Saletan, *Classical Dynamics* (Cambridge U. P., Cambridge, 2002), p. 46, Problem 25.

Electrodynamics and elasticity

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We consider a Lorenz-like gauge theory with a second speed parameter that differs from the speed of light. This theory extends electrodynamics so that it has two speeds corresponding to the speeds of transverse and longitudinal waves in vacuum. The two-parameter extension of electrodynamics can be formally mapped onto the linear theory of elasticity. Because the case of equal speeds is not realized in an elastic medium, a compressible medium cannot model electromagnetism. In the Coulomb gauge electrodynamics can be formally mapped onto incompressible elasticity, which suggests that a linear-elastic incompressible medium can serve as a model of electromagnetism.

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It is well known that Maxwell arrived at his electromagnetic equations during the course of his attempts to construct a mechanical model of electromagnetic phenomena. These equations were found to be correct, but the mechanical model had fatal drawbacks and was soon rejected. One of the objections to the elastic model of electromagnetism is that a linear-elastic medium is governed by two speed parameters while electrodynamics is essentially a one-parameter theory. There also was some ambiguity in the description of an incompressible solid elastic medium.

Recently, a gauge condition was considered (see Ref. 1, and references therein) that has the form of the Lorenz gauge with a parameter c_g^2/c in place of the parameter c. The two speed electrodynamics theory thus obtained can be directly compared with a linear theory of elasticity. This comparison enables us to define the differences and similarities between the theory of elasticity and Maxwell's electrodynamics. We find, with some reservations, that the theory of a linear-elastic medium can be used as a model of electromagnetism.

Maxwell's equations in terms of electromagnetic potentials $\mathbf{A}$ and φ read as

$$\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}+\mathbf{E}+\nabla\varphi=0, \tag{1}$$

$$\frac{\partial \mathbf{E}}{\partial t}-c\nabla\times(\nabla\times\mathbf{A})+4\pi\mathbf{j}=0, \tag{2}$$

$$\nabla\cdot\mathbf{E}=4\pi\rho. \tag{3}$$

To obtain the electromagnetic potentials uniquely, Eqs. (1)–(3) should be supplemented by a gauge condition. The latter is commonly chosen in the form given by Lorenz

$$\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}=0, \tag{4}$$

or as the Coulomb gauge

$$\nabla\cdot\mathbf{A}=0. \tag{5}$$

Each of the relations (4) and (5) yields electrodynamics with a single speed parameter, in agreement with the experimental observations. Nevertheless, we will introduce into the theory another parameter c_g by considering the following generalization of the Lorenz gauge:[1]

$$\nabla\cdot\mathbf{A}+\frac{c}{c_g^2}\frac{\partial\varphi}{\partial t}=0, \tag{6}$$

where, in general, $c_g\neq c$. Equation (6) extends electrodynamics to a theory with two parameters. Both parameters can be interpreted as a wave speed: c is the speed of the wave in vacuum of the solenoidal parts of the fields $\mathbf{A}$ and $\mathbf{E}$, and c_g is the speed of the wave in vacuum of φ and of the irrotational part of $\mathbf{A}$. It is interesting to compare the two-parameter extension of electrodynamics with the linear theory of elasticity where there also are two speed parameters.

The motion of a linear-elastic homogeneous isotropic medium is governed by the Lame equation (see, for example, Ref. 2). The cases of compressible and incompressible media usually are approached in somewhat different ways. So, we will write the equation of motion in a form that covers both cases:

$$\varrho\frac{\partial^2\mathbf{s}}{\partial t^2}+\varrho c^2\nabla\times(\nabla\times\mathbf{s})+\nabla p=\varrho\mathbf{f}, \tag{7}$$

where ϱ is the density of the medium, $\mathbf{s}(\mathbf{x},t)$ is the displacement of the medium, $p(\mathbf{x},t)$ is the pressure, and $\mathbf{f}(\mathbf{x},t)$ is the field of an external force. For an incompressible medium we must combine Eq. (7) with

$$\nabla\cdot\mathbf{s}=0. \tag{8}$$

In a compressible medium we have Hooke's law for the pressure,

$$p=-\varrho c_g^2\nabla\cdot\mathbf{s}. \tag{9}$$

The parameters c and c_g are the speeds, respectively, of the transverse and longitudinal waves of the medium displacement. When $\mathbf{f}=0$, $p=\text{constant}\ (=0)$ in an incompressible medium, and the longitudinal wave is absent.

Maxwell equations (1) and (2) are isomorphic to the Lame equation (7) through the following correspondence:

$$\mathbf{A}=\kappa c\,\frac{\partial \mathbf{s}}{\partial t}, \tag{10}$$

$$\varrho\varphi=\kappa p, \tag{11}$$

$$\mathbf{E}=\kappa[-\mathbf{f}+c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s})], \tag{12}$$

$$\mathbf{j}=\frac{\kappa}{4\pi}\,\frac{\partial \mathbf{f}}{\partial t}, \tag{13}$$

where κ is a constant. Equation (2) corresponds to the relation (12) differentiated with respect to time. The Coulomb gauge (5) corresponds to the incompressibility condition (8) differentiated with respect to time. The longitudinal gauge (6) corresponds to Hooke's law for the compressibility (9) differentiated with respect to time. By using the definition (12), Eq. (3) specifies the external field $\mathbf{f}$ via a source function ρ:

$$\kappa\boldsymbol{\nabla}\cdot\mathbf{f}=-4\pi\rho. \tag{14}$$

The differentiation with respect to time in Eqs. (2), (6), and (10) indicates that we are dealing with an elastic-plastic medium (see, for example, Ref. 2, Sec. 29).

We may ignore the mechanical meaning of Eq. (7) and interpret it as merely a mathematical construction that reduces three equations, (1), (2), and (6), to one equation, (7), with Eq. (9) substituted into it.

If we consider Eq. (7) as the equation of motion of a linear-elastic medium, then we must take into account the relation between the speeds of the two waves[2]

$$c_g>c. \tag{15}$$

Equation (4) follows from the longitudinal gauge (6) with $c_g=c$. Thus, the inequality (15) implies that the Lorenz gauge (4) cannot be mapped onto an elastic medium.

We thus come to the following conclusion. The two-parameter extension of electrodynamics can be formally mapped onto the linear theory of elasticity extended to arbitrary values of the wave speeds. The mapping occurs by modeling the electrostatic field with Eq. (14) where the term $\mathbf{f}$ is an external force. The physical origin of this force is not specified in the elastic model. In the Coulomb gauge, electrodynamics can be mapped onto incompressible elasticity. This means that we can formally model electromagnetism by the dynamics of a linear-elastic incompressible medium.

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[1]J. D. Jackson, "From Lorenz to Coulomb and other explicit gauge transformations," Am. J. Phys. **70** (9), 917–928 (2002).

[2]L. D. Landau and E. M. Lifshitz, *The Theory of Elasticity* (Butterworth–Heinemann, London, 1995), 3rd ed.

The magnetic field of an infinite solenoid

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We use the Biot–Savart law for filamentary currents to show that the magnetic field produced by an infinitely long straight strip of infinitesimal width carrying a uniform transverse surface current can be written in simple geometrical terms. We use this result to calculate the magnetic field of an infinite solenoid of arbitrary but uniform cross-sectional shape.

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In 1984 Dasgupta[1] presented a simple and explicit calculation of the magnetic field produced by an infinitely long circular solenoid by representing the solenoid in terms of infinitesimal current carrying rings. The key point of his analysis is that although the angular integration associated with the magnetic field of a single ring of current cannot be solved in terms of elementary functions (but only in terms of elliptic integrals), the field of the infinite set of such rings, extending from $z=-\infty$ to $z=+\infty$ (z is the solenoid axis), can be calculated explicitly by performing the z integration *before* the angular integration. The same idea of interchanging the order of the integrations was exploited[2,3] to generalize the calculation in Ref. 1 to a solenoid of arbitrary but uniform cross-sectional shape.

The aim of this note is to present a slightly different, more geometrical way of deriving the magnetic field of the infinite solenoid of arbitrary cross-sectional shape, which we believe is more accessible to students taking an introductory first course on electromagnetism at the level of Halliday and Resnick or the equivalent.[4] The basic idea is to calculate the magnetic field due to an infinitely long straight strip (or ribbon) of *infinitesimal* width carrying a uniform transverse surface current. The result will allow us to easily calculate the field produced by any current configuration that can be considered to be a superposition of parallel strips of the kind just described, such as the solenoid and other open current configurations, in particular, an infinite slab with uniform surface current density.

Consider a point P with Cartesian coordinates $(0,y_P,0)$, $y_P>0$, and an infinitely long strip, lying in the xz plane along the z axis, of negligible thickness and infinitesimal width dw (see Fig. 1). The strip is assumed to carry a uniform transverse surface current density K. We divide the strip into elementary bars of width dz and consider the joint contribution to the magnetic field at P of two bars symmetrically situated with respect to the origin, say at $\pm z$ (shown shadowed in Fig. 1). The Biot–Savart law tells us that

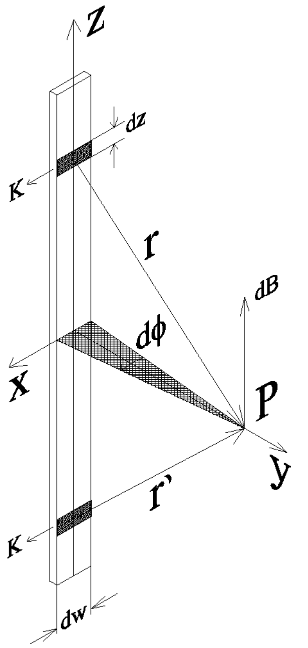

Fig. 1. The field at P due to the two symmetrically located bars (each carrying current $K\,dz$); $d\phi$ is the angle subtended by the strip as seen from P.

$$d\mathbf{B}_{\text{bars}}=\frac{\mu_0}{4\pi}(K\,dz)\,\frac{d\mathbf{l}\times(\mathbf{r}+\mathbf{r}')}{r^3}$$

$$=\frac{\mu_0 K}{2\pi}(y_P\,dw)\,\frac{dz}{(z^2+y_P^2)^{3/2}}\hat{\mathbf{z}}, \tag{1}$$

because $r=r'$, $\mathbf{r}+\mathbf{r}'=2y_P\hat{\mathbf{y}}$, and $d\mathbf{l}=dw\,\hat{\mathbf{x}}$. Notice that the field at P is found to be parallel to the strip. If we integrate over z, we find

$$d\mathbf{B}=\frac{\mu_0 K}{2\pi}y_P\,dw\,\hat{\mathbf{z}}\int_0^\infty\frac{dz}{(z^2+y_P^2)^{3/2}}=\frac{\mu_0 K}{2\pi}\,\frac{dw}{y_P}\,\hat{\mathbf{z}}=\frac{\mu_0 K}{2\pi}\,d\phi\,\hat{\mathbf{z}}. \tag{2}$$

The integral is $1/y_P^2$, and dw/y_P equals the (infinitesimal) angle $d\phi$ subtended by the strip from the observation point P.

What happens if the strip is tilted by an angle ψ with respect to the xz plane? The only change is that the $d\mathbf{l}$ appearing in Eq. (1) becomes $dw\cos\psi\,\hat{\mathbf{x}}+dw\sin\psi\,\hat{\mathbf{y}}$, so that the field $d\mathbf{B}_{\text{bars}}$ and hence also $d\mathbf{B}$ in Eq. (2) acquire an extra factor of $\cos\psi$. But the quantity $dw\cos\psi/y_P$ still equals the subtended angle $d\phi$. Therefore, we have derived the following general result: The (infinitesimal) magnetic field $d\mathbf{B}$ due to an infinitely long and infinitesimally narrow straight strip of negligible thickness carrying a transverse surface current density K is given by

$$d\mathbf{B}=\frac{\mu_0 K}{2\pi}\,d\phi\,\hat{\mathbf{u}}, \tag{3}$$

where $d\phi$ represents the (infinitesimal) angle subtended by the strip from the observation point P; $\hat{\mathbf{u}}$ is a unit vector pointing along the strip, such that if we align our thumb in the direction of $\hat{\mathbf{u}}$, then the current flows according to the right-hand rule as seen from P.

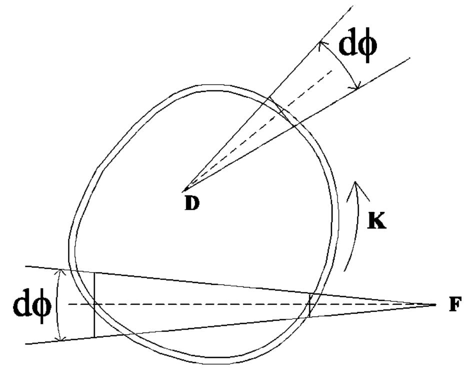

Fig. 2. Cross-sectional view of an infinite solenoid.

The result in Eq. (3) allows us to easily calculate the magnetic field produced by any surface current configuration that can be represented as the superposition of parallel infinite strips, that is, strips that are perpendicular to the same plane, carrying a uniform transverse surface current density. An immediate general result is that such a field always points everywhere parallel to the surface and perpendicular to the current.

The simplest example is perhaps an infinite slab carrying a surface current density K. From any point not on the slab, the total angle subtended by the slab is clearly π, regardless of the distance from the observation point to the slab. Hence, the magnetic field on each side of the slab is uniform, parallel to the slab and perpendicular to the direction of the surface current, with magnitude $B=\mu_0 K/2$.

But our main interest here is the solenoid (which we assume to have a uniform but otherwise arbitrary cross-sectional shape) with n turns per unit length and a current I. From any point D interior to the solenoid, the total angle subtended by the current distribution is always 2π (see Fig. 2), so that the magnetic field inside has magnitude $B=\mu_0 K=\mu_0 nI$. On the other hand, for any point F exterior to the solenoid, the two infinitesimal strips intersected by any wedge subtending an angle $d\phi$ from the observation point will produce contributions to the magnetic field at F that exactly cancel each other, due to the opposite senses of the current on the strips as seen from F. We conclude that the magnetic field exterior to the (infinite) solenoid is exactly zero.

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[1]B. B. Dasgupta, ‘‘Magnetic field due to a solenoid,’’ Am. J. Phys. **52**, 258 (1984).
[2]V. Namias, ‘‘On the magnetic field due to a solenoid of arbitrary cross section,’’ Am. J. Phys. **53**, 588 (1985).
[3]K. Fillmore, ‘‘Magnetic field of a noncircular solenoid,’’ Am. J. Phys. **53**, 782 (1985).
[4]See, for instance, D. Halliday, R. Resnick, and K. S. Krane, *Physics* (Wiley, New York, 1992), 5th ed.; E. Purcell, *Electricity and Magnetism* (Berkeley Physics Course, McGraw–Hill, New York, 1985), Vol. 2, 2nd ed.; P. A. Tipler, *Physics for Scientists and Engineers* (Worth, 1998), Vol. II, 4th ed.

The exponential of $\sqrt{\nabla^2}$ and the wave equation

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The exponentials of operators commonly encountered in vector calculus are amusing purely from a mathematical point of view and also provide us with an instructive way of looking at some familiar concepts. The exponential of the curl operator and its applications for Maxwell fields were discussed recently.[1] In this note, we define the operation of a similar operator, $\exp(\sqrt{\nabla^2})$.

Consider a scalar function $g(\mathbf{r})$ that is continuous, square integrable, and expressible by a Fourier expansion:

$$g(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} G(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r}). \tag{1}$$

If we operate with $\nabla^2 \equiv (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)$ on $g(\mathbf{r})$, we obtain

$$\nabla^2 g(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} G(\mathbf{k})(-k^2)\exp(i\mathbf{k}\cdot\mathbf{r}). \tag{2}$$

It is convenient to fractionalize the ∇^2 operator. For any real number α we define the operation $(\nabla^2)^\alpha$ as

$$(\nabla^2)^\alpha g(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} G(\mathbf{k})(-k^2)^\alpha \exp(i\mathbf{k}\cdot\mathbf{r}). \tag{3}$$

The function $(-k^2)^\alpha$ is multiple-valued. We remove this ambiguity by selecting the principal value for the argument of $(-k^2)$ as follows:

$$(-k^2)^\alpha = \exp[\alpha \log(-k^2)] = \exp[\alpha \log|k^2| + i\alpha\pi]. \tag{4}$$

From Eq. (3), it is clear that the operation $(\nabla^2)^\alpha$ on a function $g(\mathbf{r})$ involves the following steps: (a) Fourier transform the function $g(\mathbf{r})$, (b) multiply the Fourier transform by $(-k^2)^\alpha$, and (c) calculate the inverse Fourier transform of the product in step (b). It is easy to verify that for any real α and β,

$$(\nabla^2)^\alpha (\nabla^2)^\beta \equiv (\nabla^2)^{\alpha+\beta}. \tag{5}$$

The action of the fractional ∇^2 operator is thus additive in its index. The particular case $\sqrt{-\nabla^2}$ has been used before in studies of the propagation of second-order correlations of optical fields.[2]

Consider an example of use of the operator defined in Eq. (3) for the solution of the wave equation:

$$\nabla^2 \psi(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{r},t). \tag{6}$$

We can formally write the solution of Eq. (6) as

$$\psi(\mathbf{r},t) = \exp(-ct\sqrt{\nabla^2})a(\mathbf{r}) + \exp(ct\sqrt{\nabla^2})b(\mathbf{r}). \tag{7}$$

The functions $a(\mathbf{r})$ and $b(\mathbf{r})$ are time-independent. From Eqs. (5) and (7), it is clear that by differentiating $\psi(\mathbf{r},t)$ twice with respect to t we arrive at Eq. (6). All that remains now is to define the action of $\exp(\sqrt{\nabla^2})$. From Eq. (3), we obtain

$$ct\sqrt{\nabla^2} g(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}(ikct)G(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}). \tag{8}$$

We have chosen the positive sign for the square root on the right-hand side of Eq. (8) in accordance with Eq. (4). The action of $\exp(ct\sqrt{\nabla^2})$ therefore gives

$$\begin{aligned}\exp(ct\sqrt{\nabla^2})g(\mathbf{r}) &= \int \frac{d^3k}{(2\pi)^3}\left[\sum_{n=0}^{\infty}\frac{(ikct)^n}{n!}\right]G(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}) \\ &= \int \frac{d^3k}{(2\pi)^3}\exp(ikct)G(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}).\end{aligned} \tag{9}$$

From Eqs. (7) and (9), the solution $\psi(\mathbf{r},t)$ may now be written as

$$\begin{aligned}\psi(\mathbf{r},t) &= \int \frac{d^3k}{(2\pi)^3}A(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}-kct)] \\ &\quad + \int \frac{d^3k}{(2\pi)^3}B(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}+kct)].\end{aligned} \tag{10}$$

The two terms in Eq. (10) may be readily identified as the "outward" and "inward" traveling solutions, respectively.

In the study of the wave equation, one is often interested in solving the initial value problem, that is the determination of $\psi(\mathbf{r},t)$ for given $\psi(\mathbf{r},0)$ and $(\partial/\partial t)\,\psi(\mathbf{r},t)|_{t=0}$. The solution (7) expressed in terms of the $\exp(\sqrt{\nabla^2})$ operator is particularly suitable for treating such problems. From Eq. (7), we can write

$$a(\mathbf{r}) + b(\mathbf{r}) = \psi(\mathbf{r},0), \tag{11}$$

and

$$-a(\mathbf{r}) + b(\mathbf{r}) = \frac{1}{c}(\nabla^2)^{-1/2}\frac{\partial}{\partial t}\psi(\mathbf{r},t)|_{t=0}. \tag{12}$$

The simultaneous equations (11) and (12) give

$$a(\mathbf{r}) = \frac{1}{2}\left[\psi(\mathbf{r},0) - \frac{1}{c}(\nabla^2)^{-1/2}\frac{\partial}{\partial t}\psi(\mathbf{r},t)|_{t=0}\right], \tag{13}$$

$$b(\mathbf{r}) = \frac{1}{2}\left[\psi(\mathbf{r},0) + \frac{1}{c}(\nabla^2)^{-1/2}\frac{\partial}{\partial t}\psi(\mathbf{r},t)|_{t=0}\right]. \tag{14}$$

Equations (13) and (14), along with Eq. (7), thus allow us to express $\psi(\mathbf{r},t)$ in terms of the initial conditions and the fractional ∇^2 operator. The function $\psi(\mathbf{r},t)$ expressed in this way has an appearance similar to the one-dimensional case commonly treated in textbooks.[3,4] The above solution has two interesting features: the operators $\exp(\pm ct\sqrt{\nabla^2})$ govern the time evolution of $\psi(\mathbf{r},t)$, and the solution is independent of a particular choice of the coordinate system.

In summary, we have described the action of the fractional ∇^2 operator, and used the $\exp(\sqrt{\nabla^2})$ operator to illustrate the well-known fact that the wave equation has ‘‘outward’’ and ‘‘inward’’ traveling solutions. The relevance of the fractional ∇^2 operator to the initial value problem for the wave equation (6) was also described.

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[1]E. A. Power and T. Thirunamachandran, ‘‘The exponential of the curl: Applications to Maxwell fields,’’ Am. J. Phys. **70**, 1136–1142 (2002).
[2]L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995), Sec. 4.6.
[3]D. J. Griffiths, *Introduction to Electrodynamics* (Prentice–Hall, Englewood Cliffs, NJ, 1989), 2nd ed., Sec. 8.2, and the following exercises.
[4]J. W. Dettman, *Applied Complex Variables* (Dover, New York, 1984), Sec. 8.6.

Addendum to: “The one-dimensional harmonic oscillator in the presence of a dipole-like interaction” [Am. J. Phys. 71 (3), 247–249 (2003)]

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In Ref. 1 the Schrödinger equation for a one-dimensional harmonic oscillator is solved in the presence of an external dipole field. The potential for this problem is given by

$$V(x)=\frac{1}{2}m\omega^2x^2+\frac{\hbar^2}{2m}\frac{\alpha}{x^2}. \tag{1}$$

Note that the original paper had a typographical error in the second term of Eq. (2).

The solutions were found using the following Ansatz:

$$\Psi(x)=x^{\beta+1}e^{-(\lambda/2)x^2}w(x), \tag{2}$$

which defines the constant β in terms of α. If $\alpha\neq 0$, then $\beta_+=-1/2+\sqrt{1/4+\alpha}$ or $\beta_-=-1/2-\sqrt{1/4+\alpha}$. The case when $\alpha=0$ must be handled separately because of the singularity in the potential; in this case $\beta=0$ or $\beta=-1$, and the solutions for the harmonic oscillator are recovered.

Using the value β_+ along with the fact that the whole wave function has to be normalizable leads to the condition:

$$\tfrac{1}{2}(\beta_+ + \tfrac{3}{2})-\tfrac{1}{2}\mu=-n$$

and the eigenfunction given by Eq. (7) in Ref. 1. In the limit of infinitesimally small α, $\beta_+\to 0$ and the odd eigenspectrum and eigenfunctions of the harmonic oscillator are recovered. Using the value β_- leads to the eigenfunction specified by Eq. (8) in Ref. 1. Although the eigenfunctions and eigenvalues in this case appear to be distinct from those of Eq. (7) (see the former case β_+), they are in fact seen to be identical when written in terms of the same parameter α. Therefore the case $\beta=\beta_-$ provides no new information. When parameterized in terms of α the eigenvalues and eigenfunctions are

$$E_n=\hbar\omega(2n+1+\sqrt{\tfrac{1}{4}+\alpha}) \tag{3}$$

and

$$\Psi_n(x)=Bx^{1/2+\sqrt{1/4+\alpha}}e^{-(\lambda/2)x^2}{}_1F_1(-n,1+\sqrt{\tfrac{1}{4}+\alpha};\lambda x^2). \tag{4}$$

For the case $\alpha=0$, the value $\beta=0$ leads to the spatially odd eigenfunctions and associated eigenvalues of the harmonic oscillator, and the value $\beta=-1$ (which cannot be obtained from the β_+ case) leads to the even eigenfunctions. We note that the even harmonic oscillator solutions cannot be obtained as a consequence of taking a continuous limit $\alpha\to 0$ of the analytical expression of Eq. (7). This reflects the fact that the potential does not go continuously to zero as $\alpha\to 0$, because this limit does not remove the singularity in the potential at $x=0$.

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