ogy (Freeman, San Francisco, 1974), p. 18.
[4]H. M. França, G. C. Marques, and A. J. DaSilva, Nuovo Cimento A **48**, 65 (1978). They call the Schott energy term $m\tau\mathbf{a}\cdot\mathbf{v}$, while G. N. Plass, Ref. 15, calls it the *Schott acceleration energy*. Some authors such as F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965), call Schott's term $m\tau\dot{\mathbf{a}}$.
[5]J. L. Jiménez and I. Campos, Am. J. Phys. **55**, 1017 (1987).
[6]C. Teitelboim, Phys. Rev. D **1**, 1572 (1970); **3**, 297 (1971).
[7]P. A. M. Dirac, Proc. R. Soc. London Ser. A **176**, 148 (1938).
[8]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), p. 582.
[9]M. Planck, Ann. Phys. **63**, 419 (1897).
[10]R. Bellman, *Perturbation Techniques in Mathematics, Engineering and Physics* (Dover, New York, 1972), pp. 50–51.
[11]Reference 8, p. 191.
[12]W. L. Burke, Phys. Rev. A **2**, 1501 (1970).
[13]J. Thompson, Philos. Mag. **11**, 229 (1881). The concept of electromagnetic mass was developed some years later by, among others, O. Heaviside, Philos. Mag. **27**, 324 (1884).
[14]D. Bohm and M. Weinstein, Phys. Rev. **74**, 1789 (1948).
[15]G. N. Plass, Rev. Mod. Phys. **33**, 37 (1961).
[16]R. Becker, *Electromagnetic Fields and Interactions, Vol. 2* (Blaisdell, New York, 1964), p. 23.
[17]L. de la Peña, J. L. Jiménez, and R. Montemayor, Nuovo Cimento B **69**, 71 (1982).
[18]W. Pauli, *Theory of Relativity* (Pergamon, New York, 1958), p. 186.
[19]A. Sommerfeld, *Electrodynamics* (Academic, New York, 1967), p. 236.

# On the magnetic field generated by a short segment of current

Arthur Hovey
*Amity Regional High School, Woodbridge, Connecticut 06525*

(Received 12 November 1987; accepted for publication 17 August 1988)

This article describes a series of simple experiments with which students can discover the well-known formula for the magnetic field generated by an infinitesimal current segment. No such procedure seems to have been described previously in the literature.

## I. BACKGROUND

One of the most basic relations in electricity and magnetism is the inverse-square formula describing the magnetic field $\mathbf{dB}$ generated by a short segment of current-carrying wire,

$$\mathbf{dB} = kI\,\mathbf{ds}\times\hat{\mathbf{r}}/r^2,$$

where $I$ represents the current, $\mathbf{ds}$ represents the length of the current segment, $\mathbf{r}$ represents the position of the observation point relative to the segment, $\hat{\mathbf{r}}$ represents a unit vector in the $\mathbf{r}$ direction, and $k$ represents a proportionality constant.

Some authors, including Sears and Zemansky[1] and Fowler and Meyer[2] call it Biot's law and carefully distinguish it from the Biot–Savart law describing the field generated by a long, straight current. Others[3–7] call it the Biot–Savart law and give no special name at all to the long straight current formula that Biot and Savart actually discovered. In older texts, we find it called "Ampere's theorem,"[8] "an alternate form of Ampere's law,"[9,10] or "Ampere's formula."[11]

Probably most of us accept the formula because we have seen it used to make predictions, and we have seen those predictions tested in the laboratory. A first-year physics student does not yet have the luxury of that experience, however, and is generally expected to accept our word for it. Parasnis[12] seems to reinforce that dangerously trusting attitude when he claims that the formula "cannot be deduced by any logical procedure from the results of experiments."

The only author I have been able to find who actually describes the work of Biot and Savart is Magie.[13] The experiment he describes is similar to experiment IV-8 in the original PSSC lab manual.[14] Deflection of a magnetic compass needle is used to measure the strength of the wire's field at several distances from the wire. Anyone who has tried this experiment can verify that it produces results that are not so good when examined quantitatively unless very complicated measures are taken to compensate for the stray fields generated by the rest of the circuit.

Neither Jeans[15] nor Maxwell[16] describes anything remotely resembling a procedure for discovering the current-segment law, nor do they mention Biot and Savart. They both describe an experiment that demonstrates the straight-wire law of Biot and Savart, but gives no clue about making the transition to the current-segment law. Some authors claim that Biot made that leap, some claim that both Biot and Savart did it, and some claim that it was Ampere. None, however, support their claims with direct quotes. Maxwell devotes an entire chapter in his treatise to the work of Ampere. After many pages filled with double integrals and vector potentials, he leaves us with this fascinating remark:

> The whole theory and experiment seems as if it had leaped full grown and full armed from the brain of the "Newton of electricity".... Though cast into an inductive form, [the method of Ampere] does not allow us to trace the formation of the ideas which guided it.... We are led to suspect, indeed he tells us himself, that he discovered the law by some process which he has not shewn us, and

that when he had afterwards built up a perfect demonstration he removed all the traces of the scaffolding by which he had raised it.[17]

## II. PREREQUISITES

The appropriate time for this investigation is just after students have used magnetic compasses and current-carrying wires (as in the original PSSC lab manual) to make the following discoveries.

(1) Electric currents generate magnetic fields. (Students place a magnetic compass near a wire and then turn on the current.)

(2) The magnetic field lines formed by a long, straight, current-carrying wire form concentric circles around the wire in planes perpendicular to the wire. (Students observe this with iron filings and with compasses.)

(3) At any point in the plane of a current-carrying loop with any shape, the generated magnetic field is perpendicular to the plane of the loop. (Students find a way to orient the loop or coil so that turning the current off and on has no effect on a compass placed in the plane. In addition to investigating the field at the center of a loop, it is useful to investigate points near the corner and inside of a triangular loop, and outside the corner of a square loop.)

(4) Reversing the current in the coil causes the coil's magnetic field to be reversed.

(5) The strength of the coil's field is proportional to the product of the coil current and the number of turns in the coil. (Orient the coil to make the generated field point eastward. The magnitude of the generated field is then equal to the magnitude of the Earth's magnetic field multiplied by the tangent of the compass needle deflection angle. Students make graphs of generated field versus current and/or generated field versus number of turns.)

(6) The magnetic field at a given location produced by two or more current-carrying wires or coils in *any* configuration is the vector sum of the fields generated individually by those current-carrying wires. (Measure fields by the "tangent galvanometer" method described in No. 5.)

## III. HOW DO WE KNOW THAT dB IS PROPORTIONAL TO $I$ ds?

Imagine a circular current loop marked off into a large number of identical segments. Symmetry demands that each of those segments must contribute equally to the generated field at any point on the axis of the loop. Let $\mathbf{B}$ represent the generated field at the center of the loop, and let $\mathbf{dB}$ represent the field contributed there by one such segment. Let $\mathbf{ds}$ represent the length of the segment; let $I$ represent the current in the segment.

The experiments described in Nos. 5 and 6 of Sec. II show that magnetic fields generated by current-carrying wires follow the rules of vector addition. Therefore, the field generated by two short current segments placed end to end must be the sum of the fields contributed individually by the two segments. If the two segments are identical, then the field at any location far from the segments must then be twice as strong as the field generated by one segment acting alone. Since placing two segments end to end is equivalent to doubling $\mathbf{ds}$, we must conclude that $\mathbf{dB}$ is proportional to the length of the current segment.

In Sec. II, No. 5 shows that the coil's field is proportional to the coil current, indicating that $\mathbf{dB}$ must also be proportional to $I$. To reinforce that evidence, imagine placing two identical current segments side by side to produce a new segment with twice the original current. The magnitude of $\mathbf{dB}$ must be doubled, indicating again that $\mathbf{dB}$ is proportional to the current.

## IV. HOW DO WE KNOW THAT dB IS PROPORTIONAL TO $r^{-2}$?

Imagine moving every segment of a circular loop closer to the center, forming a two-turn coil with half the original radius but the same overall length of wire. If $R$ represents the old radius and $R'$ represents the new radius, then $R' = R/2$. The distributive law tells us that this modification must increase the strength of both $\mathbf{B}$ and $\mathbf{dB}$ by the same factor. Let $F$ represent that factor, so that $\mathbf{dB}' = F\,\mathbf{dB}$.

If we can find the value of $F$ by experiment, then we can determine the unknown exponent from that $F$ value. For example, if we find that $F = 2$, then we must conclude that $\mathbf{dB}$ is proportional to $1/r$. If we find that $F = 4$, we must conclude that $\mathbf{dB}$ is proportional to the reciprocal of the square of the radius. In general, if $\mathbf{dB}$ is proportional to $r^{-x}$, then

$$\frac{\mathbf{dB}'}{\mathbf{dB}} = \left(\frac{R'}{R}\right)^{-x} = F.$$

In the special case described above, the radius ratio $R'/R = \frac{1}{2}$. Therefore, $F = 2^x$.

To find the value of $F$, we proceed in two stages: First, we determine its approximate value by a "brute force" experiment that is simple but not very accurate. Then we guess its exact value and perform a sensitive experiment to test that guess.

First, construct two circular coils, giving the second coil half the radius and twice as many turns as the first. (Both coils must be large compared to the compass needle. I chose to make my first coil have 8 turns with a 40-cm radius.) Set up the first coil with its axis east–west, as in the tangent galvanometer experiment. Place a magnetic compass at its center, and adjust the coil current until the compass needle has a moderate deflection, i.e., 10 to 20 deg from north. After recording the current, replace the large coil with the smaller one. Adjust the current until it is the same as before, and record the new needle deflection. From the two needle deflections, we find that the magnetic field generated by the second coil is roughly four times stronger than the field generated by the first coil, so that $F$ is approximately 4. This tells us that the unknown exponent is approximately $-2$.

To find out if that conclusion is accurate, make a third circular coil with the same radius as the second but $\frac{1}{4}$ as many turns. Mount the first and third coils concentrically on a sheet of plywood with a hole cut at the center to accommodate a magnetic compass. Connect the two coils in series so that the current is clockwise in one and counterclockwise in the other. The two generated fields at the center will then tend to cancel. Twist the coil leads together in pairs to minimize stray magnetic fields and mount the pair of coils in a vertical plane.

Even with a current greater than in the previous experiment, we find no deflection of the compass needle. If we add one turn or delete one turn from either coil, we do see some needle deflection. This shows with great accuracy that $F = 4$. We must conclude that the exponent in the

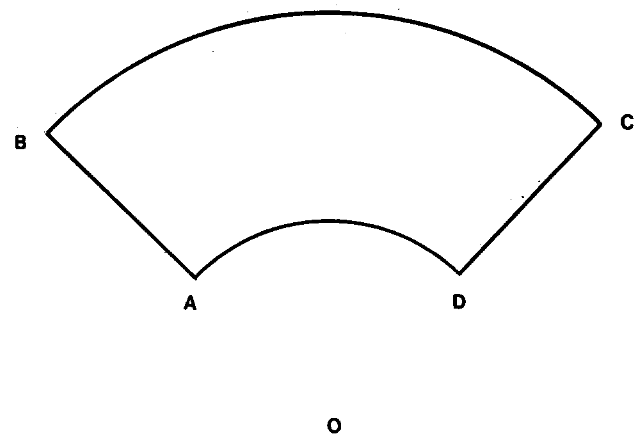


Fig. 1. Shape of a special coil for demonstrating that **dB** vanishes at points collinear with a current segment.

current segment formula is $-2$. (Incidentally, this procedure is not limited to coils that are circular in shape. It works just as well for any other configuration of current-carrying wire.)

## V. HOW DO WE KNOW THAT dB IS ZERO AT POINTS IN LINE WITH THE CURRENT SEGMENT?

Assume for the moment that this is not the case, i.e., that a current segment or moving charged particle aimed directly toward or away from an observer does contribute to the magnetic field at the observer's location. In what directions could that contribution point? Any direction not parallel or antiparallel to the current segment would be a violation of symmetry. Therefore, we design an experiment to detect a hypothetical generated field that is collinear with the segment.

Construct a coil with two sides in the form of concentric circular arcs and two radial sides forming a 90-deg angle, as in Fig. 1. The fields generated at the center by the currents in the two arcs BC and DA will be perpendicular to the plane of the coil, as we know from the tangent galvanometer experiments. As we learned in the previous paragraph, the hypothetical fields generated at point O by the currents in AB and CD must be collinear with those segments. Since there must be an angle of 90 deg between them and their magnitudes must be equal, their sum must be either horizontal or vertical when the coil is in the orientation shown in Fig. 1, if they exist at all.

Set up the coil in a vertical plane with its axis directed toward magnetic north. If the compass at point O indicates any direction other than north or south when the current is turned on, that will indicate that the hypothetical collinear magnetic field exists. The experiment must be repeated with the coil rotated 90 deg about its north–south axis to look for both of the possible sums mentioned in the previous paragraph. When we find no eastward or westward needle deflection in either case, we must conclude that $\mathbf{dB} = 0$ at any point collinear with the current segment.

## VI. HOW DO WE KNOW THAT dB IS PROPORTIONAL TO sin θ?

Let $\theta$ represent the angle between $I\,\mathbf{ds}$ and $\hat{\mathbf{r}}$. Resolve $I\,\mathbf{ds}$ into components parallel and perpendicular to $\mathbf{r}$. The parallel component is then $I\,ds\cos\theta$, and the perpendicular component is $I\,ds\sin\theta$. In Sec. V we found that the parallel component contributes no magnetic field. Only the perpendicular component generates a magnetic field. (Some students wonder if **dB** exists at all at locations where $\theta$ is not 90 deg. The field outside the corner of a square coil demonstrates that it does.)

## VII. HOW DO WE KNOW THE DIRECTION OF dB?

Experiment 3 in Sec. II suggests that **dB** is perpendicular to the plane of $I\,\mathbf{ds}$ and $\mathbf{r}$. If that is not true everywhere, then **dB** will have components in that plane, parallel and/or perpendicular to $I\,\mathbf{ds}$. (Observations made at the center of a circular or polygonal coil do not preclude the existence of such hypothetical components because symmetry requires such components to cancel there.) We saw in Sec. V that all components of **dB** must be zero when $\theta$ is 0 or 180°. Therefore, if such components do exist, they must depend on $\theta$. This leads us to ask if the hypothetical components might be proportional to $\sin\theta$, to $\sin 2\theta$, or to something similar.

If **dB** had a "forward" component (parallel to $I\,\mathbf{ds}$) and that component were proportional to $\sin\theta$, then the magnetic field lines observed in preliminary experiment 2 would have been helices rather than circles. A perpendicular component proportional to $\sin\theta$ would cause them to be spirals instead of circles. If there were forward or perpendicular components proportional to $\sin 2\theta$, they would not be noticed in experiment 2 but they would become evident in experiment 3. The field generated by a triangular current loop at a point in the plane of the loop near a corner would not be perpendicular to that plane.

There appears to be no way such components can exist without contradicting one or more of the observations made in Sec. II. We conclude that the hypothetical parallel and perpendicular components do not exist, so that **dB** must be perpendicular to the plane of $I\,\mathbf{ds}$ and $\mathbf{r}$.

## VIII. SUMMARY AND CONCLUSIONS

The experimental evidence indicates that:

(1) The magnetic field generated at any location by any static configuration of current-carrying wires can be predicted by summing the fields generated by the individual segments of current.

(2) The field **dB** contributed by one such segment is proportional to $I\,\mathbf{ds}$, to $\sin\theta$, and to $r^{-2}$.

(3) The direction of **dB** is perpendicular to the plane of $I\,\mathbf{ds}$ and $\mathbf{r}$.

(4) Combining those conclusions, we obtain the formula attributed to Biot and Savart,

$$\mathbf{dB} = kI\,\mathbf{ds}\times\hat{\mathbf{r}}/r^2.$$

[1]F. W. Sears and M. W. Zemansky, *University Physics* (Addison-Wesley, Reading, MA, 1964), 3rd ed., p. 708.

[2]R. G. Fowler and D. I. Meyer, *Physics for Engineers and Scientists* (Allyn & Bacon, Boston, 1958), p. 273.

[3]D. Halliday and R. Resnick, *Physics for Students of Science and Engineering* (Wiley, New York, 1960), pp. 768–769.

[4]R. A. Serway, *Physics for Scientists and Engineers* (Saunders College Publishing, Philadelphia, 1982), p. 574.

[5]D. S. Parasnis, *Magnetism* (Harper & Brothers Science Today Series, New York, 1961), p. 15.

[6]J. C. Slater and N. H. Frank, *Electromagnetism* (McGraw-Hill, New York, 1947).
[7]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[8]Harvey E. White, *Modern College Physics* (Van Nostrand, New York, 1948), 2nd ed., p. 535.
[9]Leigh Page and N. I. Adams, *Principles of Electricity* (Van Nostrand, Princeton, NJ, 1931), 3rd ed., pp. 204 and 210.
[10]Sir James Jeans, *The Mathematical Theory of Electricity and Magnetism* (Cambridge U.P., New York, 1908), 5th ed., pp. 438–439.
[11]Reference 10, p. 514.
[12]Reference 5.
[13]W. F. Magie, *A Source Book in Physics* (McGraw-Hill, New York, 1935).
[14]Physical Science Study Committee, *Laboratory Guide for Physics* (Heath, Boston, 1960), Experiment IV-7.
[15]Reference 10.
[16]James Clerk Maxwell, *A Treatise on Electricity and Magnetism* (Constable & Company, London, 1891), 3rd ed.
[17]Reference 16.

# Self-similarity and long-tailed distributions in the generation of thermal light

Edward B. Rockower
*Department of Operations Research, Naval Postgraduate School, Monterey, California 93943*

(Received 7 July 1987; accepted for publication 2 August 1988)

Two counterintuitive phenomena are studied. (1) It is well known that a thermal electromagnetic field has a Bose–Einstein (geometric) distribution of photons within a coherence volume. This arises because of the photon clumping characteristic of a thermal Boson field. On the other hand, the distribution of the number of atoms emitting photons through spontaneous emission must be Poisson if emissions are truly independent. (2) The average time between atomic decays is finite, being just the inverse of the total decay rate of the atoms. However, it is shown that in a coherence volume or in a single mode of the resulting Gaussian electromagnetic field, the average photon interarrival time is infinite. Hence, on average, an infinite length of time must pass before $\langle N \rangle$ photons arrive in the field. These apparent paradoxes are discussed, showing how both arise from random interference of Boson fields. The infinite waiting time is seen to be one manifestation of a long-tailed distribution. Such distributions are increasingly important by virtue of their relation to self-similarity and fractals, e.g., strange attractors in the description of deterministic chaos; therefore, it is of interest to understand their counterintuitive properties and see how they arise naturally even in more traditional analyses.

## I. INTRODUCTION

Long-tailed distributions with infinite moments beyond a given order occasionally arise in physical models. One well-known example of such a distribution is the Lorentz (Cauchy) distribution. Students sometimes feel it somewhat paradoxical that a distribution with infinite mean or variance can have any physical significance. We show that a long-tailed distribution arises as the probability density function (pdf) for the time of arrival of photons in a coherence volume or in a single mode of a linearly growing Gaussian electromagnetic field. Such a field has a Bose–Einstein (geometric) photon number distribution and may be generated by the spontaneous emission of excited atoms or laser scattering from a rotating ground glass.[1] The distribution of photon interarrival times does not, in fact, possess even a finite mean. On the other hand, the average number of photons in the field is nonzero. In spontaneous emission, $\langle N \rangle$ equals the average number of decayed atoms. In laser light scattered from a rotating ground glass, $\langle N \rangle$ is proportional to the integrated laser field intensity. In either case $\langle N \rangle$ can be written as $At$.

The apparent paradox relating to infinite photon interarrival times is related to another puzzling fact. If spontaneous emissions by atoms are really independent events, then the distribution of the number of decayed atoms must be Poisson. How can this be reconciled with the well-known Bose–Einstein photon distribution of thermal light? We answer these questions by showing how both of these phenomena can be understood physically as arising from the random interference of boson fields.

In Sec. II, we present a method for finding the distribution of the time until the first photon arrives and apply it to a Poisson (independent emissions) process. Section III applies the method to the linearly growing Bose–Einstein distribution within a coherence volume of the field and shows generally that the mean time between photon arrivals is infinite. In Sec. IV, this apparent paradox is shown to be simply a manifestation of a distribution having a very slowly decreasing tail. By analyzing a simple, idealized distribution, the situation is clarified and the mathematical nature of the apparent paradox is explained. A simple physical model is presented in Sec. V, showing how random interference of boson fields leads to the long-tailed distributions of photon interarrival times in a coherence volume.

The distribution derived in Sec. III is of a type known as

  