1. Payoff Matrix Nash Equilibria

Consider the following payoff matrix where the rows correspond to player A's strategies and the columns correspond to player B's strategies. The first entry in each box is player A's payoff, and the second entry is player B's payoff:

$$\begin{pmatrix} (3,3) & (1,2) \\ (2,1) & (3,0) \end{pmatrix}$$

(a) **Find all pure strategy Nash equilibria of this game.**

When examining the best responses for each player, for Player A's best responses, if B plays **L**, A's payoffs are 3 for U and 2 for D. So A would prefer U. If B plays **R**, A's payoffs are 1 for U and 3 for D. So A would prefer D. For Player B's best responses, if A plays **U**, B's payoffs are 3 for L and 2 for R. So B prefers L. If A plays **D**, B's payoffs are 1 for L and 0 for R. So B prefers L. The only strategy profile where both players are best responding to each other is **(U, L)**. So the only pure strategy Nash equilibrium is **(U, L)**.

(b) **Notice from the payoff matrix above that player A's payoff from the pair of strategies (U, L) is 3. Can you change player A's payoff from this pair of strategies to some non-negative number such that the resulting game has no pure-strategy Nash equilibrium? Provide a brief (1-3 sentence) explanation. Note: Only change Player A's payoff for this pair of strategies (U, L), leaving the rest of the game unchanged, including player B's payoff for (U, L).**

No, because changing A's payoff in (U, L) doesn't alter the best responses enough to eliminate all pure-strategy Nash equilibria. This is because B has L as its dominant strategy. So either (U, L) or (D, L) will still be a Nash equilibrium since B would choose L since it gives the most payoff.

(c) **Now return to the original payoff matrix. Can you change player B's payoff from the pair of strategies (U, L) to some non-negative number such that the resulting game has no pure-strategy Nash equilibrium? Provide a brief (1-3 sentence) explanation. Note: Only change Player B's payoff for the pair (U, L), keeping the rest of the game unchanged.**

Yes, by reducing B's payoff in (U, L) to a number less than 2, B's best response to A playing **U** changes from **L** to **R**. This adjustment disrupts the mutual best responses, resulting in no pure-strategy Nash equilibrium in the modified game.

2. Nash Equilibria for a Different Payoff Matrix

Consider the following payoff matrix where the rows correspond to player A's strategies and the columns correspond to player B's strategies. The first entry in each box is player A's payoff, and the second entry is player B's payoff:

$$\begin{pmatrix} (1,1) & (4,2) \\ (3,3) & (2,2) \end{pmatrix}$$

Find all Nash equilibria (both pure and mixed strategy) for the game described by this payoff matrix.

We should examine the best responses for each player. For Player A's best responses, if B plays **L**, A's payoffs are 1 for U and 3 for D. So A would prefer D. If B plays **R**, A's payoffs are 4 for U and 2 for D. So A would prefer U. For Player B's best responses, if A plays **U**, B's payoffs are 1 for L and 2 for R. So A would prefer R. If A plays **D**, B's payoffs are 3 for L and 2 for R. So A would prefer L. From this, we can gather that the two Pure Nash Equilibria are **(U,R)** and **(D,L)**.

To find the Mixed-strategy Nash equilibrium let:

- $p =$ Probability that Player A plays U
- $1 - p =$ Probability that Player A plays D
- $q =$ Probability that Player B plays L
- $1 - q =$ Probability that Player B plays R

Such that we find p and q such that both players are indifferent between their strategies.

For Player A: Expected Payoff of Playing U: $E_A(U) = q \cdot 1 + (1-q) \cdot 4 = 1q + 4(1-q)$ which simplifies to $E_A(U) = 1q + 4 - 4q = 4 - 3q$

Expected Payoff of Playing D: $E_A(D) = q \cdot 3 + (1-q) \cdot 2 = 3q + 2(1-q)$ which simplifies to $E_A(D) = 3q + 2 - 2q = 2 + q$

Set Equal for Indifference: $E_A(U) = E_A(D) \implies 4 - 3q = 2 + q$
Solve for q: $4 - 3q = 2 + q \implies 4 - 2 = -3q - q \implies 2 = -4q \implies q = \frac{1}{2}$

For Player B:
Expected Payoff of Playing L: $E_B(L) = p \cdot 1 + (1-p) \cdot 3 = 1p + 3(1-p)$ which simplifies to $E_B(L) = p + 3 - 3p = 3 - 2p$

Expected Payoff of Playing R: $E_B(R) = p \cdot 2 + (1-p) \cdot 2 = 2p + 2(1-p)$ which simplifies to $E_B(R) = 2p + 2 - 2p = 2$

Set Equal for Indifference: $E_B(L) = E_B(R) \implies 3 - 2p = 2$

Solve for q: $3 - 2p = 2 \implies 3 - 2 = 2p \implies 1 = 2p \implies p = \frac{1}{2}$
Thus we get the Mixed-strategy Nash Equilibira that

- **Player A:** Plays U with probability $p = \frac{1}{2}$, and D with probability $1 - p = \frac{1}{2}$.
- **Player B:** Plays L with probability $q = \frac{1}{2}$, and R with probability $1 - q = \frac{1}{2}$.

The game has **two pure-strategy Nash equilibria**: (U, R) and (D, L), and **one mixed-strategy Nash equilibrium**, where both players randomize equally between their two strategies, playing each with probability $\frac{1}{2}$.

3. Guessing Game

In a class guessing game, each student guesses a number between 0 and 100, aiming to make their guess as close as possible to two-thirds of the average guess of all other students. What is the Nash equilibrium of this game?

The unique Nash equilibrium is for all players to guess 0. There is no strictly dominant strategy, the only unique pure strategy Nash equilibrium could be found through the iterated elimination of weakly dominated strategies. Through this iterative reasoning, any number above 0 is not sustainable because players will continually adjust their guesses downward to be two-thirds of the average, ultimately converging to zero.

4. Auction Scenario

A seller runs a second-price, sealed-bid auction with two bidders, a and b. Each bidder has independent private values, v_i, which are either 0 or 1, with equal probability. However, bidder b occasionally misjudges his value. Half of the time, b's value is 1, and he knows it; the other half, b's value is 0, but he mistakenly believes it is 1 with probability 1/2. Bidder a never makes mistakes about his value but knows about b's mistakes.

(a) **Is bidding his true value still a dominant strategy for bidder a? Briefly explain.**

Yes, bidding his true value remains a dominant strategy for bidder **a**. In a second-price auction, a bidder's bid affects only whether they win, not the price they pay (which is the second-highest bid). Despite bidder **b**'s potential misjudgment, bidder **a** maximizes his expected payoff by bidding his true value to ensure he wins when his value is higher.

(b) **What is the seller's expected revenue? Briefly explain.**

The seller's expected revenue is **0.375**. This is calculated based on the probability that both bidders bid 1, which occurs when bidder **a**'s value is 1 (probability 0.5) and bidder **b** believes his value is 1 (probability 0.75), resulting in a combined probability of 0.375. In this case, the seller receives a revenue of 1; in all other cases, the revenue is 0.