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Assignment 02

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1. A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet six inhabitants: Fiona, Diane, Cecil, Bob, Earl and Alice. Fiona says that either Bob is a knight or Earl is a knight. Diane says, “Both Earl is a knave and Bob is a knight.” Cecil says that Diane and Earl are not the same. Bob claims that either Earl is a knave or Alice is a knight. Earl claims, “Alice and I are not the same.” Alice claims that Earl is a knave and Fiona is a knight. Can you determine who is a knight and who is a knave?

- **What has been said:**

1. Alice: "**Earl** = knave *AND* **Fiona** = knight"
2. Bob: "**Earl** = knave *OR* **Alice** = knight"
3. Cecil: "**Diane** != **Earl**"
4. Diane: "**Earl** = knave *AND* **Bob** = knight"
5. Earl: "**Alice** != **Earl**"
6. Fiona: "**Bob** = knight *OR* **Earl** = knight"

Since Alice makes an AND claim, she's saying two things must be true. Because if either of them are wrong, this makes her a knave. This will make it easier to determine what she is, so I'll begin with her.

1. First lets assume **Alice** is a knight.
    1. If **Alice** is a knight, then **Earl** is a knave.
        1. If **Earl** is a knave, then he'd be telling the truth in saying he and **Alice** are not the same.
    2. Since this would make **Earl** a knight, we know that **Alice** is lying. This is a contradiction.
2. Let's instead assume **Alice** is a knave.
    1. If **Alice** is a knave, then either **Earl** is a knight or **Fiona** is a knave.
    2. Since **Earl** says he and **Alice** aren't the same, her claim about **Earl** is a lie, thus confirming she's a knave.
    3. Since **Bob** claims **Alice** is a knight, which is false, or **Earl** is a knave, which is false, **Bob** must be a knave.
    4. Since **Dianne** says that **Earl** is a knave (false), and **Bob** is a knight (false), we know **Dianne** must be a knave.
    5. **Fionna** must be a knight, because she makes the true claim that **Earl** is a knight.
    6. **Cecil** must be a knight because **Earl** is a knight and **Dianne** is a knave.

> ∴ the six people are as follows:
> - **Alice:** Knave
> - **Bob:** Knave
> - **Cecil:** Knight
> - **Dianne:** Knave

> - **Earl:** Knight
> - **Fionna:** Knight

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2. Justify the rule of universal transitivity, which states that if $\forall x(P(x) \to Q(x))$ and $\forall x(Q(x) \to R(x))$ are true, then $\forall x(P(x) \to R(x))$ is true, where the domains of all quantifiers are the same.

> Hypotheses:
> - $\forall x(P(x) \to Q(x))$
> - $\forall x(Q(x) \to R(x))$
>
> $$\begin{aligned} P(i) \to Q(i) & \quad \text{- Universal instantiation for input } i \\ Q(i) \to R(i) & \quad \text{- Universal instantiation for input } i \\ P(i) \to R(i) & \quad \text{- Logic Chain Rule} \\ \forall i(P(i) \to R(i)) & \quad \text{- Universal generalization for input } i \\ \forall x(P(x) \to R(x)) & \quad \text{- Same rules appy for equally arbitrary input } x \end{aligned}$$

3.

A) Using equivalence transformations, prove that $((p \vee q) \wedge (p \to s) \wedge (q \to t)) \to (s \vee t) \equiv T$

> | | |
> |---|---|
> | $((p \vee q) \wedge (p \to s) \wedge (q \to t)) \to (s \vee t)$ | Original Statement |
> | $\neg((p \vee q) \wedge (\neg p \vee s) \wedge (\neg q \vee t)) \vee (s \vee t)$ | Conditional logical equivalency |
> | $(\neg(p \vee q) \vee \neg(\neg p \vee s) \vee \neg(\neg q \vee t)) \vee (s \vee t)$ | De Morgan's Law |
> | $((\neg p \wedge \neg q) \vee (\neg\neg p \wedge \neg s) \vee (\neg\neg q \wedge \neg t)) \vee (s \vee t)$ | De Morgan's Law |
> | $((\neg p \wedge \neg q) \vee (\neg p \wedge \neg s) \vee (\neg q \wedge \neg t)) \vee (s \vee t)$ | Double Negatives |
> | $(\neg p \wedge \neg q) \vee (\neg p \wedge \neg s) \vee (\neg q \wedge \neg t) \vee (s \vee t)$ | Associative Laws |
> | $(\neg p \vee (\neg q \wedge \neg s)) \vee (\neg q \wedge \neg t) \vee (s \vee t)$ | Distributive Laws |
> | $\neg p \vee (\neg q \wedge \neg s) \vee (\neg q \wedge \neg t) \vee (s \vee t)$ | Associative Laws |
> | $\neg p \vee (\neg q \vee (\neg s \wedge \neg t)) \vee (s \vee t)$ | Distributive Laws |
> | $\neg p \vee \neg q \vee (\neg s \wedge \neg t) \vee (s \vee t)$ | Associative laws |
> | $\neg p \vee \neg q \vee \neg(s \vee t) \vee (s \vee t)$ | Reverse De Morgan's Law |
> | $\neg p \vee \neg q \vee T$ | Negation Laws |
> | $\neg p \vee T$ | Domination Law |
> | T | Domination Law |

B) Prove that $(((p \vee q) \wedge (p \to s) \wedge (q \to t)) \to s \vee t)$ is a tautology, using a direct proof (with cases).

> For implication, the only way for it to be false is if the LHS is true and the RHS is false. The only way for the RHS to be false is if both s and t are false. Let's use this as our cases:
>
> $\text{Case 1: } s \text{ and } t \text{ are both false.}$
>
> | | |
> |---|---|
> | $((p \lor q) \land (p \to F) \land (q \to F)) \to (F \lor F)$ | Setting s and t to be false (F) |
> | $((p \lor q) \land (p \to F) \land (q \to F)) \to F$ | Identity Laws |
> | $\neg((p \lor q) \land (\neg p \lor F) \land (\neg q \lor F)) \lor F$ | Conditional Logical Equivalency |
> | $(\neg(p \lor q) \lor \neg(\neg p \lor F) \lor \neg(\neg q \lor F)) \lor F$ | De Morgan's Law |
> | $((\neg p \land \neg q) \lor (\neg\neg p \land \neg F) \lor (\neg\neg q \land \neg F)) \lor F$ | De Morgan's Law |
> | $((\neg p \land \neg q) \lor (p \land T) \lor (q \land T)) \lor F$ | Double Negative |
> | $(\neg p \land \neg q) \lor p \lor q \lor F$ | Identity Laws |
> | $\neg(p \lor q) \lor (p \lor q) \lor F$ | Reverse De Morgan's Law |
> | $T \lor F$ | Negation Laws |
> | T | Domination Laws |

$\therefore$ Since LHS can never be true while RHS is false, we know that this logical expression must be a tautology.

C) Prove that $((p \lor q) \land (p \to s) \land (q \to t)) \to (s \lor t)$ is a tautology by proving the contrapositive.

> $\text{Contrapositive: } \neg(s \lor t) \to \neg((p \lor q) \land (p \to s) \land (q \to t)) \equiv T$
>
> | | |
> |---|---|
> | $\neg(s \lor t) \to \neg((p \lor q) \land (p \to s) \land (q \to t))$ | Contrapositive of the original claim |
> | $(\neg s \land \neg t) \to (\neg(p \lor q) \lor \neg(p \to s) \lor \neg(q \to t))$ | De Morgan's Law twice |
> | $(\neg s \land \neg t) \to (\neg(p \lor q) \lor (p \land \neg s) \lor (q \land \neg t))$ | Conditional Logical Equivalency |
> | $\neg(\neg s \land \neg t) \lor (\neg(p \lor q) \lor (p \land \neg s) \lor (q \land \neg t))$ | Conditional Logical Equivalency |
> | $\neg(\neg s \land \neg t) \lor \neg(p \lor q) \lor (p \land \neg s) \lor (q \land \neg t)$ | Associative Laws |
> | $(\neg\neg s \lor \neg\neg t) \lor (\neg p \land \neg q) \lor (p \land \neg s) \lor (q \land \neg t)$ | De Morgan's Law |
> | $(s \lor t) \lor (\neg p \land \neg q) \lor (p \land \neg s) \lor (q \land \neg t)$ | Double Negative Law |
> | $(s \lor t) \lor (\neg p \lor (\neg q \land \neg s)) \lor (q \land \neg t)$ | Distributive Laws |
> | $(s \lor t) \lor \neg p \lor (\neg q \land \neg s) \lor (q \land \neg t)$ | Associative Laws |
> | $(s \lor t) \lor \neg p \lor (\neg q \lor (\neg s \land \neg t))$ | Distributive Laws |
> | $(s \lor t) \lor \neg p \lor \neg q \lor (\neg s \land \neg t)$ | Associative Laws |
> | $(s \lor t) \lor \neg p \lor \neg q \lor \neg(s \lor t)$ | Reverse De Morgan's Law |
> | $(s \lor t) \lor \neg(s \lor t) \lor \neg p \lor \neg q$ | Commutative Laws |
> | $T \lor \neg p \lor \neg q$ | Negation Law |
> | $T \lor \neg q$ | Domination Law |
> | T | Domination Law |

4.

A) Prove that for natural numbers n, if $n^3 + 3n + 1$ is even then n is odd, by proving the contrapositive.

> $\text{Contrapositive: if } n^3 + 3n + 1 \text{ is odd, then } n \text{ is even.}$
> We know that if you multiply anything by an even number, the result will always be even. This is because ultimately any even number is a multiple of two, and even numbers can be denoted by $2k$ where k is any integer. So we know that if n is even, and $n^3 = n * n * n$, then this must be even.
> We also know that $3n$ must be even because $3 * 2k$ (the definition of an even number) is $6k$, which will always be even.
> So if we're adding two numbers that we *know* to be even ($n^3 + 3n$), we will get a resulting even number. This can be denoted again by $2k$. Now we're simply adding 1 to $2k$, $2k + 1$, which we know by definition is an odd number.
>
> $\therefore$ if we know $n^3 + 3n + 1$ is odd whenever n is even, therefore the contrapositive must be true as well. $n^3 + 3n + 1$ must be even when n is odd.

B) Prove that if $x^3 + 3x + 3$ is irrational then x is irrational, by proving the contrapositive.

> A number x is rational iff $x = \dfrac{q_1}{q_2}$, where q_1 and q_2 are integers and $q_2 \neq 0$
>
> $$x^3 = x * x * x = \frac{q_1}{q_2} * \frac{q_1}{q_2} * \frac{q_1}{q_2} = \frac{q_1 * q_1 * q_1}{q_2 * q_2 * q_2} = \frac{(q_1)^3}{(q_2)^3} = \frac{q_3}{q_4}$$
>
> Since q_1 and q_2 are both intergers, then it follows that q_3 and q_4 must be integers as well; $\therefore$ if $x^3 = \frac{q_3}{q_4}$ we know x^3 must be rational.
>
> $$3x = 3 * \frac{q_1}{q_2} = \frac{3(q_1)}{q_2}$$
>
> Since q_1 and 3 are both integers, $3q_1$ must also be an integer. So it follows that $\frac{3q_1}{q_2}$ must be a rational number.
> So now we have $x^3 + 3x + 3 = \dfrac{q_3}{q_4} + \dfrac{3q_1}{q_2} + 3$. We know x^3 and $3x$ are both rational, and so adding them together we get another rational number, plus a rational number gives us a rational number.
> $\therefore$ we know the contrapositive of this claim to be true, and so it follows that the claim itself is true.

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5. Give a proof by contradiction to show that if the odd integers $1, 3, 5, 7, 9, 11, 13, 15, 17, 19,$ are placed randomly around a circle (without repetition), then there must exist three adjacent numbers along the circle whose sum is greater than 32.

Start by assuming that this is false, and that there *can* exist a permutation where no 3 consecutive numbers sum up to at least 32.

Let N be the set $\{x \in \mathbb{Z} | 0 <= x <= 20 \text{ and } x \mod 2 = 1\}$ - i.e The set described above.

If the average of the set is less than 32 $\left(\dfrac{\Sigma N}{3} < 32\right)$ then it would mean that it would be possible for some perumtation to exist where all sums are less than 32. However, if make a new set, N_2 that simply excludes 1 from the set N, then

$\frac{\Sigma N_2}{3} = 33$, which means that there *must* be a combination of numbers in the set that is summed up is greater than 32.

> ∴ this claim is true. No permutation of the series will have all 3 consecutive numbers less than 33, and obviously therefore 32 as well.

```python
"""
    Just for fun, the following Python code does a brute force check to see if any
    permutation of the series will yeild a result greater than 32.

    If will first generate a list of all possible permuations, then iterate through them.
    For each permuation, it will set 3 variables to be the 3 consecutive numbers to add.
    It will first set them to be the last two at the end of the array to account for
    the situation where you're counting numbers from the end and the beginning of the array.
"""

import itertools

nums = range(1,20,2)
perms = itertools.permutations(nums)

def noneLessThan32(perms):
    for perm in list(perms):
        series = [0, perm[-2], perm[-1]]
        allLess = True
        for x in perm:
            series = [series[1], series[2], x]
            if sum(series) >= 32:
                allLess = False
                break

        if allLess:
            return False
    return True

print("None are less than 32: " + str(noneLessThan32(perms)))
```

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6. Give a proof by cases that shows that $n(n^2 - 1)(n + 2)$ is a multiple of 4, for all integers n. (Hint: Only two appropriately chosen cases need to be considered.)

Case 1: n is even

> Let n be $2k$, where k is an integer. This will represent an even number.
>
> $2k(2k^2 - 1)(2k + 2)$
> $2k(8k^3 + 8k^2 - 2k - 2)$
> $16k^4 + 16k^3 - 4k^2 - 4k$
> $4(4k^4 + 4k^3 - k^2 - k)$
>
> To be a multiple of 4, we must be able to factor a 4 out of this, which we clearly can. Therefore, for n is even, this is true.

Case 2: n is odd

> If n is odd, then it can be replaced by the expression $2k+1$ where k is any integer. If we substitute n for this, we get:
> $n(n^2-1)(n+2) = (2k+1)((2k+1)^2-1)(2k+3)$
> $= (2k+1)(2k+3)(4k^2+4k+1-1)$
> $= (4k^2+8k+3)(4k^2+4k)$
> $= 16k^4+48k^3+44k^2+12k$
> $= 4(4k^4+12k^3+11k^2+3k)$
>
> Again, for this to be a multiple of 4, a 4 needs to be able to be factored out of it, which we can see here we are able to do. Therefore, for n is odd, this is true as well.

Conclusion:

> $\therefore$ since this holds up for n is any odd or even integer, it must be true.
>
> However, if we plug any of $\{-1, 0, 1\}$ into this, we get 0, since we have n^2-1, which will yeild 0 for any of these inputs. Since 0 is not a multiple of 4, the claim is false for the domain of *any* given integer. If we instead say the domain is $\{x \in \mathbb{Z} | 1 < x < -1\}$, then this claim would be true.