Lecture 2
5.2 The Definite Integral
Riemann Sum

Definition 1. The set of points $P = \{x_0, x_1, \ldots, x_n\} = \{x_i\}_{i=0}^{n}$ where $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$, is called **a partition** of $[a, b]$.

Definition 2. The set of points $x_1^*, x_2^*, \ldots, x_n^*$ such that $x_k \in [x_{k-1}, x_k]$, is called **sample points**.

Definition 3. The sum $S_p = \sum_{i=1}^{n} f(x_i^*) \triangle x$ is called a **Riemann Sum** for the function f on the interval $[a, b]$.

Definition 4. Norm of the partition P is

$$\|P\| = \max_{i=\overline{1,n}} |\triangle x_i|$$

The Definite Integral as a Limit of Riemann Sums

Definition 5. Let f be a function defined on $[a, b]$ and $P = \{x_0, x_1, \dots, x_n\}$ be an arbitrary partition of $[a, b]$. The **definite integral** of f from a to b is the following limit

$$\int_a^b f(x)dx = \lim_{\|P\|=max\triangle x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\triangle x_i,$$

where $x_i^* \in [x_{i-1}, x_i]$, $\triangle x_i = x_i - x_{i-1}$, $i = \overline{1, n}$.

Integrable and Nonintegrable Functions

Theorem 1. If f is a continuous function on $[a, b]$, or if f has only a finite number of jump discontinuoties, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

Properties of the Definite Integral

1. Reversing the Limits of Integration $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. Zero Width Integral $\int_a^a f(x)dx = 0$
3. Integral of Constant $\int_a^b c dx = c(b-a)$
4. Linearity of the Definite Integral
   (a) $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
   (b) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
5. Additivity for Adjacent Intervals
   $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
6. Comparison Properties
   (a) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
   (b) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
   (c) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
   $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$