MATH - 205
Midterm Solutions
July 22, 2015

1 (a). $f(x) = 1+x^2$, Right Riemann sum on $[-1, 1]$

$$\Delta_n = \frac{2}{n}; \quad x_i = -1 + \frac{2}{n} i, \quad 1 \le i \le n$$

$$R_n = \frac{2}{n} \sum_{i=1}^{n} \left(1 + \left(-1 + \frac{2}{n} i\right)^2\right) = \frac{2}{n} \sum_{i=1}^{n} \left(2 + \frac{4}{n^2} i^2 - \frac{4}{n} i\right)$$

$$\int_{-1}^{1} f(x)\,dx = \lim_{n \to \infty} R_n$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[2 \cdot n + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{4}{n} \frac{n(n+1)}{2}\right]$$

$$= 4 + \lim_{n\to\infty}\left[\frac{4}{3} \frac{n(1+1/n)\cdot 2n(1+1/2n)}{n^2} - 4 \frac{n+1}{n}\right]$$

$$= 4 + \frac{8}{3} - 4 = \frac{8}{3};$$

Check: $\int_{-1}^{1} (1+x^2)\,dx = \left(x + \frac{x^3}{3}\right)\Big|_{-1}^{1} = 1 + \frac{1}{3} - \left(-1 - \frac{1}{3}\right) = \frac{8}{3}$

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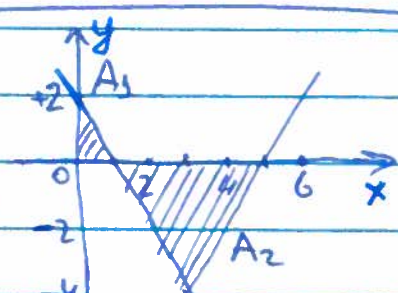
1 b; $f(x) = |2x-6| - 4$



$$\int_{0}^{5} f(x)\,dx = A_1 - A_2 = \frac{1}{2} 2 \cdot 1 - \frac{1}{2} 4 \cdot 4$$

$$= 1 - 8 = -7$$

N2 Find the antiderivative:

$$f(x) = (1-\sqrt{x})^2 + x e^{1-x^2}$$
$$= 1 - 2\sqrt{x} + x + x e^{1-x^2}$$

$$F(x) = \int f(x)\,dx = \int (1-2x^{1/2}+x)\,dx + \int x e^{1-x^2}\,dx \qquad \left| \begin{array}{l} 1-x^2 = t \\ dt = -2x\,dx \\ dx = -\frac{1}{2x}\,dt \end{array} \right.$$

$$= x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 - \frac{1}{2}\int e^t\,dt$$

$$= x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 - \frac{e^{1-x^2}}{2} + C.$$

$$F(0) = 0, \Longrightarrow 0 - \frac{1}{2}e^{1-0} + C = 0 \Rightarrow C = \frac{e}{2}$$

$$\Rightarrow F(x) = x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 - \frac{1}{2}e^{1-x^2} + \frac{e}{2}$$

N3

(a). $$\int \cos\sqrt{x}\,dx = \int 2u\cos u\,du \qquad \boxed{\begin{array}{l} u = \sqrt{x},\ x = u^2 \\ dx = 2u\,du \end{array}}$$

$$= 2\int u\,d\sin u = 2u\sin u - 2\int \sin u\,du$$

$$= 2u\sin u + 2\cos u + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

(b) $$\int \frac{x+4}{x^2-x-2}\,dx$$

$$f(x) = \frac{x+4}{x^2-x-2} = \frac{x+4}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$\Rightarrow (x+4) = A(x+1) + B(x-2) \Longrightarrow$$

$$\left.\begin{array}{ll} \text{at } x=2: & 2+4 = A(2+1)+0, \Rightarrow A = 2 \\ \text{at } x=-1 & -1+4 = 0 + B(-1-2) \Rightarrow B = -1 \end{array}\right\}$$

$$\Rightarrow \int f(x)\,dx = \int\left(\frac{2}{x-2} - \frac{1}{x+1}\right)dx$$

$$= 2\ln|x-2| - \ln|x+1|\,dx + C$$

(c)

$$\int \frac{\ln^2 x}{x^2}dx = \int \ln^2 x\, d\left(-\frac{1}{x}\right) =$$

$$= -\frac{1}{x}\ln^2 x + \int \frac{1}{x} 2\ln x \cdot \frac{1}{x}dx$$

$$= -\frac{1}{x}\ln^2 x - 2\int \ln x\, d\left(\frac{1}{x}\right)$$

$$= -\frac{1}{x}\ln^2 x - \frac{2}{x}\ln x + \int \frac{1}{x}\,\frac{1}{x}dx$$

$$= -\frac{1}{x}\left(\ln^2 x + 2\ln x + 1\right) + C$$

N4 Definite integrals:

(a) $\int_0^2 x^2\sqrt{4-x^2}\,dx$

$x = 2\sin t$
$dx = 2\cos t\,dt$
at $x=0 \Rightarrow t=0$
at $x=2 \Rightarrow t=\frac{\pi}{2}$
$\sqrt{4-x^2} = 2\cos t$

$$= \int_0^{\pi/2} 4\sin^2 t \cdot 2\cos t \cdot 2\cos t\,dt$$

$$= 16\int_0^{\pi/2} \sin^2 t\cos^2 t\,dt$$

$$= 4\int_0^{\pi/2}(1-\cos 2t)(1+\cos 2t)\,dt = 4t\Big|_0^{\pi/2} - 4\int_0^{\pi/2}\cos^2(2t)\,dt$$

$$= 4\cdot\left(\frac{\pi}{2}-0\right) - 2\int_0^{\pi/2}(1+\cos 4t)\,dt = 2\pi - \pi - \frac{1}{2}\sin(4t)\Big|_0^{\pi/2}$$

$$= \pi - 0 = \pi$$

4b

$$\int_0^{\pi/3} \tan^3(x)\sec^3(x)\,dx$$

$$\Rightarrow \quad = \int_1^2 (t^2-1)t^2\,dt$$

$$= \left(\frac{t^5}{5} - \frac{t^3}{3}\right)\Big|_1^2 = \left(\frac{32}{5} - \frac{8}{3} - \frac{1}{5} + \frac{1}{3}\right)$$

$$= \frac{31}{5} - \frac{7}{3} = \frac{31\cdot 3 - 5\cdot 7}{15} = \underline{\underline{\frac{58}{15}}}$$

$t = \sec(x)$.

$dt = \sec(x)\tan(x)\,dx$

$\left(= \frac{\sin(x)}{\cos^2(x)}\,dx\right)$

$\tan^2(x) = t^2 - 1$

$\sec(0) = 1$.

$\sec(\frac{\pi}{3}) = 2$

N5

$$f(x) = \frac{x^2}{\sqrt{x+1}} \;;\quad \text{on } [0,3]$$

$$\bar{f} = \frac{1}{3}\int_0^3 \frac{x^2\,dx}{\sqrt{x+1}}\;;$$

$$= \frac{1}{3}\int_1^2 \frac{(t^2-1)^2}{t}\,2t\,dt$$

$$= \frac{2}{3}\int_1^2 (t^4 - 2t^2 + 1)\,dt$$

$$= \frac{2}{3}\left(\frac{t^5}{5} - \frac{2}{3}t^3 + t\right)\Big|_1^2 = \frac{2}{3}\left(\frac{32-1}{5} - \frac{2}{3}(8-1) + (2-1)\right)$$

$$= \frac{2}{3}\left(\frac{31}{5} - \frac{14}{3} + 1\right)\left(= \frac{76}{45}\right)$$

$t = \sqrt{x+1}$.

$t(0) = 1$

$t(3) = 2$.

$x = t^2 - 1$

$dx = 2t\,dt$.

Bonus:

$$F(x) = \int_0^x \left(\int_{\cos t}^1 \sqrt{1+z^2}\, dz \right) dt$$

$$F'(x) = \int_{\cos(x)}^1 \sqrt{1+z^2}\, dz \Rightarrow$$

$$F''(x) = -\sqrt{1+z^2}\,\Big|_{z=\cos(x)} \cdot (\cos x)'$$

$$= + \sqrt{1+\cos^2 x} \cdot \sin(x)$$