Lecture 13

7.8 Improper Integrals

Type I Improper Integrals

Definition 1. (a) if $\int_a^t f(x)dx$ exists for every $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists.

(b) if $\int_t^b f(x)dx$ exists for every $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided this limit exists.

Integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

Type II Improper Integrals

Definition 2. (a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

provided this limit exists.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

provided this limit exists.

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where ($a < c < b$), and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Tests for Convergence/Divergence of Improper Integrals

Comparison Test

Theorem 1. Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for all $x \geq a$.

(a) If $\int_a^\infty f(x)dx$ is convergent then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent then $\int_a^\infty f(x)dx$ is divergent.

Limit Comparison Test

Theorem 2. If $f(x) \geq 0$ and $g(x) \geq 0$ and continuous on $[a, \infty)$ and if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, where $0 < L < \infty$, then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or both diverge.