Lecture 11
7.4 Integration of Rational Functions by Partial Fractions
Rational Functions

Definition 1. The function

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where n is integer nonnegative number, is called a **polynomial of degree** n.

Definition 2. A number b is called a **root** of polynomial $P_n(x)$ if $P_n(b) = 0$. If b is a root of $P_n(x)$, then $P_n(x) = (x - b)P_{n-1}(x)$.

Definition 3. A number b is said to be a **root** of polynomial $P_n(x)$ of **multiplicity** m if $P_n(x) = (x - b)^m P_{n-m}(x)$.

Definition 4. If $m = 1$ the root b is said to be a **simple root**.

Statement 1. Any polynomial can be written as a product of linear $P_1(x) = ax + b$ and quadratic $P_2(x) = ax^2 + bx + c$ polynomials taking into account their multiplicity.

Definition 5. A function which is a ratio of polynomials $\frac{P_n(x)}{Q_m(x)}$ is called a **rational function**.

Definition 6. If $n < m$, the fraction $\frac{P_n(x)}{Q_m(x)}$ is called **proper**. If $n \geq m$, the fraction $\frac{P_n(x)}{Q_m(x)}$ is called **improper**.

$$\frac{P_n(x)}{Q_m(x)} = P_{n-m}(x) + \frac{P_{m-1}(x)}{Q_m(x)}, \quad n \geq m$$

Expression a Proper Fractions as a Sum of Partial Fractions

Statement 2. A proper rational function $\frac{P_n(x)}{Q_m(x)}$ $(m > n)$ can be expressed as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^i}$$

Case I Denominator has only simple real roots, that is

$$Q_m(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_mx + b_m)$$

Then

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_m}{a_mx + b_m}$$

Unknown constants A_i $(i = \overline{1, m})$ can be determined by the method of undetermined coefficients.

Case II Denominator has multiple real roots, that is

$$Q_m(x) = (ax+b)^r P_{m-r}(x)$$

Then instead of single term $\frac{A}{ax+b}$, we use r terms

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_r}{(ax+b)^r}$$

Unknown constants A_i ($i = \overline{1, r}$) can be determined by the method of undetermined coefficients.

Case III Denominator contains quadratic irreducible factors, none of which is repeated, that is

$$Q_m(x) = (ax^2 + bx + c)P_{m-2}(x)$$

where $D = b^2 - 4ac < 0$.

Then expression for $\frac{P_n(x)}{Q_m(x)}$ will have a term of the form $\frac{Ax+B}{ax^2+bx+c}$.

Case IV Denominator contains repeated quadratic factors, that is

$$Q_m(x) = (ax^2 + bx + c)^r P_{m-2r}(x)$$

where $D = b^2 - 4ac < 0$.

Then instead of single fraction $\frac{Ax+B}{ax^2+bx+c}$, the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fractions of $\frac{P_n(x)}{Q_m(x)}$.