** Midterm March 2018

Problem 1

(1a) sketch the graph of $f(x)$

$$f(x) = \begin{cases} \sqrt{9-x^2}-1 & -3 \le x \le 0 \\ |2x-3|-1 & 0 < x \le 2 \end{cases}$$

Case $-3 \le x \le 0$

$y+1 = \sqrt{9-x^2}$

$(y+1)^2 = 9-x^2$

$x^2+(y+1)^2 = 9$

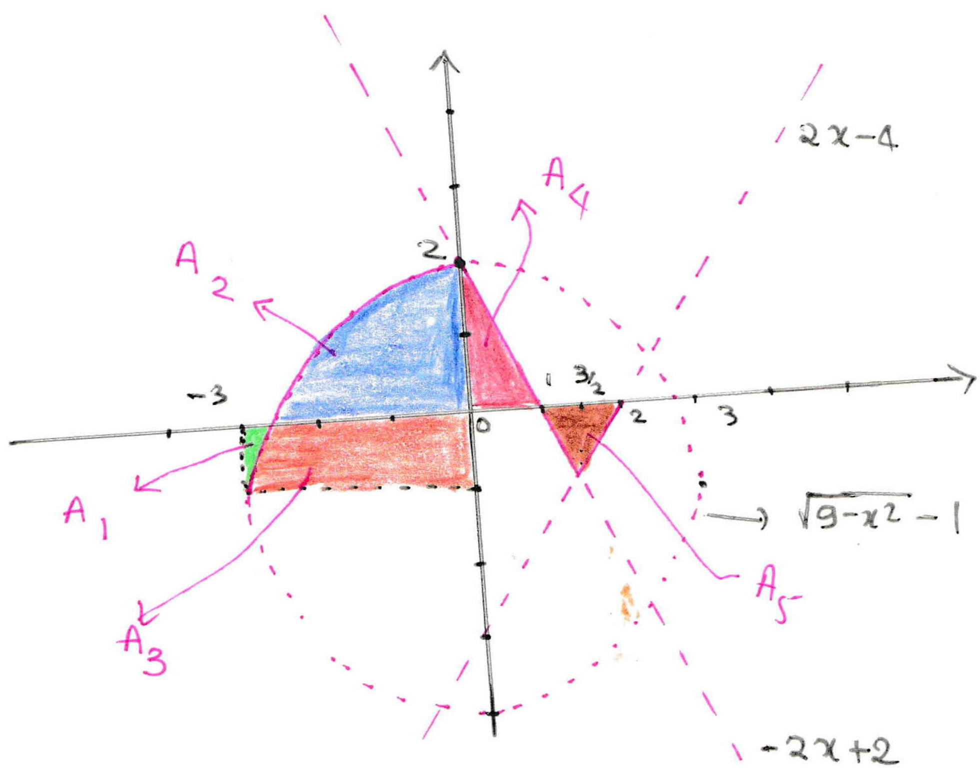
It is a circle with radius 3 and center at $(0,-1)$

Case $0 < x \le 2$

$f(x) = |2x-3|-1$

So

$$f(x) = \begin{cases} 2x-4 & \text{if } \frac{3}{2} \le x \le 2 \\ -2x+2 & \text{if } 0 < x \le \frac{3}{2} \end{cases}$$



* Find the definite integral $\int_{-3}^{2} f(x)dx$

$$I = \int_{-3}^{2} f(x)dx = \int_{-3}^{0} f(x)dx + \int_{0}^{2} f(x)dx$$

① Consider $\int_{-3}^{0} f(x)dx$

$\int_{-3}^{0} f(x)\,dx$ is the difference between the area A_1 and A_2. Since the area A_2 is above x-axis A_2 takes positive sign(+) A_1 is below x-axis then A_1 take (−)

So

$$\int_{-3}^{0} f(x)dx = A_2 - A_1$$

Based on the graph $A_2 = \underbrace{\frac{1}{4}\pi \cdot 3^2}_{\frac{1}{4}\text{ of the area of the circle radius 3}} - A_3 = \frac{9}{4}\pi - A_3$

Then

$$\int_{-3}^{0} f(x)dx = \frac{9}{4}\pi - A_3 - A_1 = \frac{9}{4}\pi - (A_1 + A_3)$$

$A_1 + A_3$ = area of the rectangle width 1 and length 3

$$A_1 + A_3 = 1 \cdot 3 = 3$$

So

$$\int_{-3}^{0} f(x)dx = \frac{9}{4}\pi - 3$$

2. consider $\int_0^2 f(x)\,dx$

$$\int_0^2 f(x)\,dx = A_4 - A_5$$

$$A_4 = \frac{1}{2} \cdot 2 \cdot 1 = 1$$

$$A_5 = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

$$\int_0^2 f(x)\,dx = 1 - \frac{1}{2} = \frac{1}{2}$$

Then

$$\int_{-2}^3 f(x)\,dx = \frac{9}{4}\pi - 3 + \frac{1}{2} = \frac{9}{4}\pi - \frac{5}{2}$$

$$\int_{-2}^3 f(x)\,dx = \frac{9}{4}\pi - 5/2$$

#

(6) Use fundemental thm of Calcullus to calculate $F'(x)$ (4)

given: $F(x) = \int_{-x^2}^{0} (t-1)\cos^4(t+1)\,dt$

$$F'(x) = \left[\int_{-x^2}^{0} (t-1)\cos^4(t+1)\,dt\right]'$$

$$= \left[-\int_{0}^{-x^2} (t-1)\cos^4(t+1)\,dt\right]'$$

$$F'(x) = -(-2x).(-x^2-1)\cos^4(-x^2+1)$$

$$F'(x) = -2x.(x^2+1)\cos^4(-x^2+1)$$

#

* Problem 3 : Evaluate the indefinite integral

3a $\quad I = \int \ln(\sqrt{x})\,dx$

let $t = \sqrt{x} \Rightarrow dt = \frac{1}{2\sqrt{x}}dx$

$\Rightarrow dx = 2\sqrt{x}\,dt = 2t\,dt$

$\Rightarrow I = \int \ln(t) \cdot 2t\,dt = 2 \cdot \int t\ln(t)\,dt$

let $u = \ln(t) \Rightarrow du = \frac{dt}{t}$

$dv = t \Rightarrow v = \frac{1}{2}t^2$

$I = 2\left[\frac{1}{2}t^2\ln(t) - \int \frac{1}{2}t^2 \cdot \frac{dt}{t}\right]$

$= \cancel{2}\cdot\frac{1}{\cancel{2}}\left[t^2\ln(t) - \int t\,dt\right]$

$= t^2\ln(t) - \frac{1}{2}t^2 + C$

$I = (\sqrt{x})^2\ln(\sqrt{x}) - \frac{1}{2}(\sqrt{x})^2 + C$

$I = x\ln(\sqrt{x}) - \frac{1}{2}x + C$

3b $$I = \int \frac{x+1}{x^3+x}\,dx$$

$$\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)}$$

let $$\frac{x+1}{x^3+x} = \frac{Ax+B}{x^2+1} + \frac{C}{x}$$

$$\Rightarrow \quad x+1 = (Ax+B)\cdot x + C(x^2+1)$$

$$x+1 = Ax^2 + Bx + Cx^2 + C$$

$$x+1 = (A+C)x^2 + Bx + C$$

$$\Rightarrow \begin{cases} A+C=0 \\ B=1 \\ C=1 \end{cases} \quad \Rightarrow \quad \begin{array}{l} A=-1 \\ B=C=1 \end{array}$$

So $$\frac{x+1}{x^3+x} = \frac{-x+1}{x^2+1} + \frac{1}{x}$$

$$I = \int \left(\frac{-x+1}{x^2+1} + \frac{1}{x} \right) dx = \int \left(\frac{-x}{x^2+1} + \frac{1}{x^2+1} + \frac{1}{x} \right) dx$$

$$I = -\frac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + \ln(x) + c$$

#

* Problem 4 : Evaluate

4a $\quad I = \int_0^a x\sqrt{a-x}\,dx$

let $u = a - x \Rightarrow du = -dx$ and $x = a - u$

There are 2 ways of solving this problem

Approach I

$$\int x\sqrt{a-x}\,dx = \int (a-u)\sqrt{u}(-du)$$

$$= -\int (a\sqrt{u} - u^{3/2})\,du$$

$$= -\left[\frac{2}{3}a\,u^{3/2} - \frac{2}{5}u^{5/2}\right]$$

$$= -\frac{2}{3}a.(a-x)^{3/2} + \frac{2}{5}(a-x)^{5/2}$$

$$\Rightarrow I = -\frac{2}{3}a(a-x)^{3/2} + \frac{2}{5}(a-x)^{5/2}\Big]_0^a$$

$$= 0 + 0 - \left(-\frac{2}{3}a(a-0)^{3/2} + \frac{2}{5}(a-0)^{5/2}\right]$$

$$= \frac{2}{3}a.a^{3/2} - \frac{2}{5}a^{5/2}$$

$$= \frac{2}{3}a^{5/2} - \frac{2}{5}a^{5/2}$$

$$I = \frac{4}{15}a^{5/2}$$

Approach II

$u = a - x$

when $x = 0 \Rightarrow u_1 = a$

$x = a \Rightarrow u_2 = 0$

So

$$I = \int_{u_1}^{u_2} (a-u)\sqrt{u}\,(-du)$$

$$= -\int_a^a (a u^{\frac{1}{2}} - u^{3/2})\,du$$

$$= \int_0^a (a u^{\frac{1}{2}} - u^{3/2})\,du$$

$$= a.\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\Big]_0^a$$

$$= a.\frac{2}{3}.a^{3/2} - \frac{2}{5}a^{5/2}$$

$$= \frac{2}{3}a^{5/2} - \frac{2}{5}a^{5/2}$$

$$I = \frac{4}{15}a^{5/2}$$

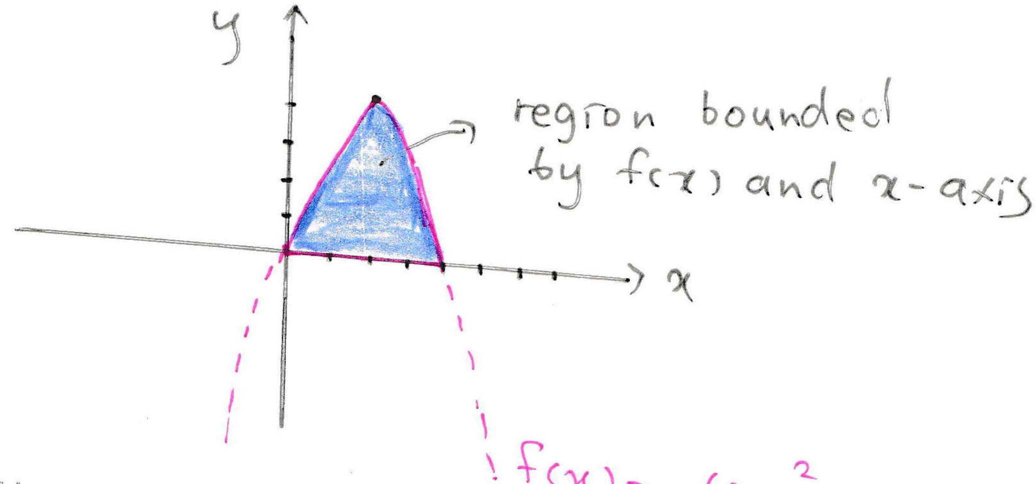
So either way, the answer is the same.

* Problem 5

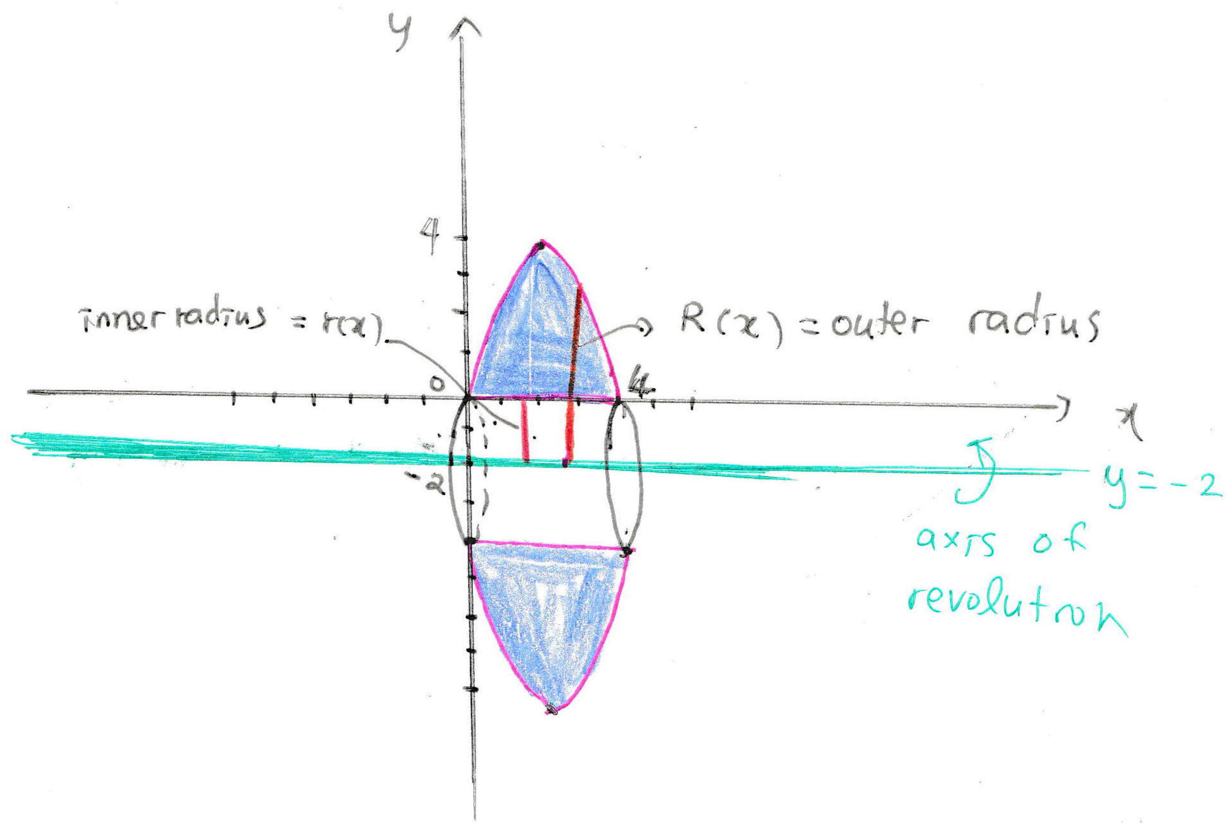
* Sketch the region bounded by $f(x) = 4x - x^2$ and x-axis

$$f(x) = 4x - x^2 = -(x^2 - 4x) = -(\underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4)$$

$$f(x) = -(x-2)^2 + 4$$



* Rotate the region about $y = -2$ and find volume of the solid obtained



Using washer to find the volume of the solid obtained since there is a hole in the shape of cylinder for the solid obtained from the revolution.

$$V = \pi \int_a^b [R^2(x) - r^2(x)]\, dx = \pi \int_0^4 [R^2(x) - r^2(x)]\, dx$$

* Note:

$R(x)$ outer radius : the distance from the axis of revolution to the curve bordering the solid of revolution

From the graph,

$$R(x) = 2 + f(x) = 2 + 4x - x^2$$

* Note:

$r(x)$ inner radius : the distance from the axis of revolution to the inne curve bording the solid of revol.

From the graph $r(x) = 2$

$$V = \pi \int_0^4 [(2+4x-x^2)^2 - 2^2]\, dx$$

$$= \pi \int_0^4 (\cancel{4} + 16x^2 + x^4 + 16x - 4x^2 - 8x^3 - \cancel{4})\, dx$$

Here we are using $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$

$$= \pi \int_0^4 (12x^2 + x^4 + 16x - 8x^3)\, dx = \frac{4^5}{5}\pi$$

check the calculation!

*** Problem 6 :**

Find the ave. valu of $f(x)=(x-3)^2$ on $[2,5]$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3}\int_2^5 (x-3)^2\,dx$$

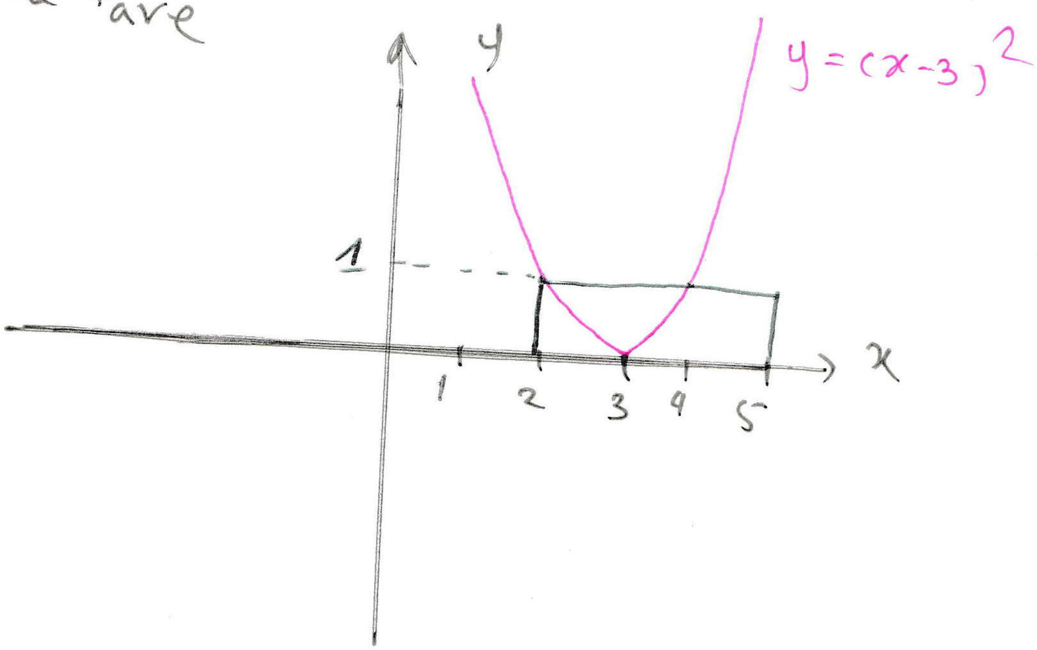
$$= \frac{1}{3}\int_2^5 x^2-6x+9\,dx$$

$$= \frac{1}{3}\cdot \left.\frac{x^3}{3}-3x^2+9x\right]_2^5$$

$$f_{ave} = \frac{1}{3}\left[\frac{5^3}{3}-3.25+9.5-\left(\frac{2^3}{3}-3.2^2+9.2\right)\right]$$

$$f_{ave} = 1$$

sketch the graph and draw a rectangle with the base of interval [2,5] and height of the fare



Bonus:

Evaluate the limit using the def of Riemann sum

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{i^4}{n^5}+\frac{i}{n^2}\right)\right) \quad \text{on } [0,1]$$

def of Riemann sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

In this problem

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = a + i\cdot\Delta x = 0 + i\cdot\frac{1}{n} = \frac{i}{n}$$

we have $\sum_{i=1}^{n}\left(\frac{i^4}{n^5}+\frac{i}{n^2}\right) = \sum_{i=1}^{n}\left(\frac{i^4}{n^4}+\frac{i}{n}\right)\cdot\frac{1}{n}$

So we have our $f(x_i) = \frac{i^4}{n^4}+\frac{i}{n}$

since $x_i = \frac{i}{n}$

$\Rightarrow f(x_i) = x_i^4 + x_i$

So

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_0^1 f(x)\,dx = \int_0^1 x^4 + x\,dx$$

$$= \frac{1}{5}x^5 + \frac{1}{2}x^2\Big]_0^1 = \boxed{\frac{1}{5}+\frac{1}{2}=\frac{7}{10}}$$