Written By: Anastasis Kratsios

- The purpose of this document in unison with the spirit of this course, that is to showcase some powerful results and give some intuitive understanding of why they work.
- The results herein are not all to be found in the course's textbook and are much more far-reaching.
- Refer to the notes from my tutorial for more experience with them.

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1 General tests

These tests are the most general and widely applicable, they tend to work in general but are not always the most efficient use of your time (they may be computationally brutal; whence they present a higher chance of computational error).

Theorem 1. ***Root Test***
*Define $L := \lim_{n\to\infty} \sqrt[n]{|a_n|}$ then one of the following **must** hold:*

- *If $L > 1$ then $\sum_{i=0}^{\infty} a_n$ **does not** converge.*
- *If $L = 1$ then $\sum_{i=0}^{\infty} a_n$ may or may not converge (ie the test is inconclusive).*
- *If $L < 1$ then $\sum_{i=0}^{\infty} a_n$ converges.*

Proof. See [IR]. □

Intuition behind result 1. *The idea of this test is to approximate the general trend of $\sum_{i=0}^{\infty} a_n$ to the behavior of a **geometric series** $\sum_{i=0}^{\infty} p^i$ (which you have seen in class). Recall that such as series:*

- *diverges if $|p| \geq 1$*
- *converges if $|p| < 1$.*

*The quantity called **"L"** in the root test may be thought of as the approximation to $|p|$ of a would be geometric series.*

Remark 1. *Notice that there is a disparity between geometric series for which $|p| = 1$ and the case where the root test gives $L = 1$. That is if $L = 1$ then $\sum_{i=0}^{\infty} a_n$ **may or may not** converge, whereas $\sum_{i=0}^{\infty} p^i$ certainly diverges if $|p| = 1$, since this is the infinite sum of 1s.*

*For technical reasons the approximation of $|p|$ to L is no longer accurate **in the case where** $1 = L$, so another test should be used before jumping to any conclusions about the nature of the series $\sum_{i=0}^{\infty} a_n$.*

Remark 2. *The root test does everything the ratio test does and more, so there is no need to concern yourselves with the ratio test.*

2 Decompositional Tests

These tests break up the series $\sum_{i=0}^{\infty} a_n$ into the point-wise product of two, that is they use facts about two more manageable sequences c_n and b_n to understand any series which can be written as: $\sum_{i=0}^{\infty} c_n b_n = \sum_{i=0}^{\infty} a_n$.

Theorem 2. ***Abel's Test***
If the following hold:

- $\sum_{i=0}^{\infty} c_n$ *converges*
- b_n *is monotonically decreasing (not getting bigger but possible constant)*
- $\lim_{n\to\infty} b_n = 0$

then $\sum_{i=0}^{\infty} c_n b_n$ *converges.*

Proof. See [IR]. □

Intuition behind result 2. *Essentially the Abel test says that the series* $\sum_{i=0}^{\infty} a_n := \sum_{i=0}^{\infty} c_n b_n$*'s* $\sum_{i=0}^{\infty} b_n$ *part is "essentially negligible" since it vanishes and* $\sum_{i=0}^{\infty} c_n$ *is already tamed (since it already converges by assumption).*

Theorem 3. ***Dirichlet Test***
If the following hold:

- *There is a real number M satisfying: for every natural number n* $|\sum_{i=0}^{n} c_n| \leq M$ *(the partial sums of* $\sum_{i=0}^{\infty} c_n$ *are squeezed between $-M$ and M).*
- b_n *is monotonically decreasing (not getting bigger but possible constant)*
- $\lim_{n\to\infty} b_n = 0$

then $\sum_{i=0}^{\infty} c_n b_n$ *converges.*

Proof. See [IR]. □

Intuition behind result 3. *Essentially the Dirichlet test says that there is an "essentially constant" part of the series* $\sum_{i=0}^{\infty} a_n := \sum_{i=0}^{\infty} c_n b_n$*, the* $\sum_{i=0}^{\infty} c_n$ *part and so it is dominated in some sense by the decreasing* $\sum_{i=0}^{\infty} b_n$*. In fact the* $\sum_{i=0}^{\infty} b_n$*'s behavior is also "tamed" by the fluctuations of the* $\sum_{i=0}^{\infty} c_n$*.*

Remark 3. *Note the **"Alternating series test"** is noting but a special case of the Dirichlet test with $c_n := (-1)^n$. (So don't bother too much with that one).*

3 Monotonic Tests

These are powerful tests which only deal with series arising from monotonically decreasing sequences.

Theorem 4. ***Integral Test***

If the following hold:

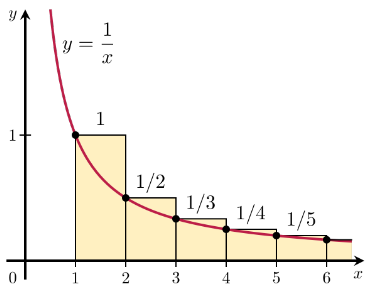
- b_n *is monotonically decreasing (not getting bigger but possible constant)*
- $\lim_{n\to\infty} b_n = 0$

then the following are ***equivalent****:*

- $\sum_{i=0}^{\infty} a_n$ *converges.*
- $\int_0^\infty f(x)dx$ *is finite (where* $f(n) = a_n$ *for* ***every*** *natural nubmer 0).*

Proof. See [IR]. □

Intuition behind result 4. *The general idea is given by this picture:*



where the blocks represent the value of the sequence $a_n = n$ *and the red curve represents the value of the real-valued function* $f(x)$.

Notice that the absence of a_0 *in this pictorial argument does not matter since* a_0 *is by definition finite, whence* $\sum_{i=0}^{\infty} a_n - a_0$ *cannot diverge unless* $\sum_{i=0}^{\infty} a_n$ *does and (visa-versa).*

4 Tricky Tests

Theorem 5. ***Comparison Test (aka: Lebesgue's Dominated convergence theorem)***

If for every real number n there $\sum_{i=0}^{n} c_n \leq \sum_{i=0}^{n} a_n \leq \sum_{i=0}^{n} b_n$*, then:*

$\sum_{i=0}^{n} a_n$ *converges if* ***both*** $\sum_{i=0}^{n} c_n$ *and* $\sum_{i=0}^{n} b_n$ *do.*

Proof. See [IR]. □

Intuition behind result 5. *In a nutshell, this result says that if the partial sums of* $\sum_{i=0}^{\infty} a_n$ *can always be contained between two convergent series* $\sum_{i=0}^{\infty} c_n$ *and* $\sum_{i=0}^{\infty} b_n$ *then* $\sum_{i=0}^{\infty} a_n$ *must also be convergent. In other words the well-behaved series* $\sum_{i=0}^{\infty} c_n$ *and* $\sum_{i=0}^{\infty} b_n$ *can be thought of as taming* $\sum_{i=0}^{\infty} a_n$.

References

[IR] Dangello, Frank, and Michael Seyfried. Introductory Real Analysis. Boston: Houghton Mifflin, 2000.

[RR] Royden, Halsey Lawrence, and Patrick Fitzpatrick. Real analysis. Vol. 32. New York: Macmillan, 1988.

[RP] Rudin, Walter. Real and complex analysis. Tata McGraw-Hill Education, 1987.

The End.