Sample midterm test - Math 205

MARKS #

9 **1.** Let $A = \int\limits_1^4 \left(5 - x^2\right) dx$.

(a) Approximate A by a right Riemann sum $\mathbf{R}_3$ using 3 approximating rectangles of equal width.

(b) Approximate A by a left Riemann sum $\mathbf{L}_3$ using 3 approximating rectangles of equal width.

(c) Calculate the exact value of A and explain why $\frac{\mathbf{R}_3 + \mathbf{L}_3}{2}$ is expected closer to A than either $\mathbf{R}_3$ or $\mathbf{L}_3$.

5 **2.** Use the Fundamental Theorem of Integral Calculus to evaluate $F'(x)$ if $F(x) = \int\limits_{\cos x}^{3} \sqrt{1 + t^3} dt$.

6 **3.** If $f'(x) = \frac{10t}{\sqrt[3]{t - 2}}$ and $f(8) = -20$ calculate $f(x)$.

12 **4.** Calculate the following indefinite integrals:

(a) $\int \frac{x^3}{\sqrt{16 - x^2}} dx$

(b) $\int \frac{3x^2 + 4x + 4}{x^3 + x} dx$.

12 **5.** Calculate the following definite integrals (*do not approximate*):

(a) $\int\limits_0^{\pi} \cos^4 x \tan^2 x dx$

(b) $\int\limits_1^{e} x^2 \ln x dx$

6 **6.** Calculate the area of the region enclosed by $x = |y|$ and $x = y^2 - 2$.

3 **Bonus** Given that $\int\limits_0^{\pi} [f(x) + f''(x)] \sin x dx = 2$ and $f(\pi) = 1$, calculate $f(0)$.

Solutions are on the next page

Solutions:

1. Let $A = \int\limits_1^4 \left(5 - x^2\right) dx$. Let $f(x) = 5 - x^2$, then

   (a) Approximate A by a right Riemann sum $\mathbf{R}_3$ using 3 approximating rectangles of equal width h:
   $h = \dfrac{4-1}{3} = 1$ the right points are $1+1 = 2$, $1+2 = 3$, and $1+3 = 4 \to$ Let $f(x) = 5 - x^2$, then $\mathbf{R}_3 = f(2) + f(3) + f(4) = \mathbf{-14}$

   (b) Approximate A by a left Riemann sum $\mathbf{L}_3$ using 3 approximating rectangles of equal width. Then the left points are $1+0 = 1$, $1+1 = 2$, and $1+2 = 3 \to$ Then $\mathbf{L}_3 = f(1) + f(2) + f(3) = \mathbf{1}$

   (c) Calculate the exact value of $A = \int\limits_1^4 \left(5 - x^2\right) dx = \mathbf{-6}$ and explain why $\dfrac{\mathbf{R}_3 + \mathbf{L}_3}{2}$ is expected closer to A than either $\mathbf{R}_3$ or $\mathbf{L}_3$. As $f(x)$ is a decreasing function on $[1, 4]$ then $\mathbf{R}_3 < A < \mathbf{L}_3$ and therefore $\mathbf{R}_3 < \dfrac{\mathbf{R}_3 + \mathbf{L}_3}{2} < \mathbf{L}_3$ and is expected to be closer than either $\mathbf{R}_3$ or $\mathbf{L}_3$

2. Use the Fundamental Theorem of Integral Calculus to evaluate $F'(x)$ if $F(x) = \int\limits_{\cos x}^{3} \sqrt{1+t^3}dt$. As $\int \sqrt{1+t^3}dt = G(t) + C$ then (Fundamental Theorem of Integral Calculus) $G'(t) = \sqrt{1+t^3}$. Then $F(x) = \int\limits_{\cos x}^{3} \sqrt{1+t^3}dt = G(3) - G(\cos x)$ and $F'(x) = -G'(\cos x)(\cos x)' = -\sqrt{1+\cos^3 x}\,(-\sin x) = (\sin x)\sqrt{\cos^3 x + 1}$.

3. If $f'(x) = \dfrac{10t}{\sqrt[3]{t-2}}$ and $f(8) = -20$ calculate $f(x)$. Let $u = t - 2 \to dx = du$. Then: $f(x) = -20 + \int\limits_8^x \dfrac{10t}{\sqrt[3]{t-2}}dt = -20 + \int\limits_6^{x-2} \dfrac{10(u+2)}{\sqrt[3]{u}}du = -20 + 10\int\limits_6^{x-2} \left(u^{2/3} + 2u^{-1/3}\right) du = 30\sqrt[3]{(x-2)^2} + 6(x-2)^{\frac{5}{3}} - 66\sqrt[3]{3}\sqrt[3]{12} - 20$

4. Calculate the following indefinite integrals:

(a) $\int \frac{x^3}{\sqrt{16-x^2}}dx = \left| \begin{array}{ll} t=\sqrt{16-x^2} & 2xdx=-2tdt \\ x^2=16-t^2 & xdx=-tdt \end{array} \right| = \int \frac{\left(16-t^2\right)tdt}{t} = \frac{t\left(48-t^2\right)}{3}+C = \frac{\sqrt{16-x^2}\left(48-\left(16-x^2\right)\right)}{3}+C = \frac{\left(x^2+32\right)\sqrt{16-x^2}}{3}+C$

(b) $\int \frac{3x^2+4x+4}{x^3+x}dx = \int \left(\frac{A}{x}+\frac{B}{x^2+1}+\frac{2Cx}{x^2+1}\right)dx \to$

$3x^2+4x+4 = A\left(x^2+1\right)+x\left(2Cx+B\right) \to x=0 : 4=A$

and compare coefficients of: $x^2 : 3 = 4+2C \to C=-\frac{1}{2}$; $x : 4 = B$

$\int \frac{3x^2+4x+4}{x^3+x}dx = \int \left(\frac{4}{x}+\frac{4}{x^2+1}+\frac{(-1/2)\,2x}{x^2+1}\right)dx = 4\ln x - \frac{1}{2}\ln\left(x^2+1\right)+4\arctan x+C$

5. Calculate the following definite integrals (*do not approximate*): $\int\limits_1^e x^2\ln xdx = \frac{2e^3}{9}+\frac{1}{9}$

(a) $\int\limits_0^\pi \cos^4 x\tan^2 xdx = \frac{1}{4}\int\limits_0^\pi 4\cos^2 x\sin^2 xdx = \frac{1}{4}\int\limits_0^\pi \sin^2 2xdx = \frac{1}{8}\int\limits_0^\pi (1-\cos 4x)\,dx = \frac{\pi}{8}$

(b) $\int\limits_1^e x^2\ln xdx = \left| \begin{array}{ll} u=\ln x & u'=\frac{1}{x} \\ v'=x^2 & v=\frac{x^3}{3} \end{array} \right| = \frac{x^3\ln x}{3}]_1^e - \int\limits_1^e \frac{x^2}{3}dx = \frac{2e^3}{9}+\frac{1}{9}$

6. Calculate the area A of the region enclosed by $x=|y|$ and $x=y^2-2$. The intersections are $|y|=y^2-2 \to y=\pm 2$

$A = 2\left|\int\limits_0^2 \left(y^2-2-y\right)dy\right| = \frac{20}{3}.$

Bonus Given that $=2$ and $f(\pi)=1$, calculate $f(0)$.

$2 = \int\limits_0^\pi \left[f(x)+f''(x)\right]\sin xdx = \int\limits_0^\pi f(x)\sin xdx + \int\limits_0^\pi f''(x)\sin xdx$

(a) $\int\limits_0^\pi f(x)\sin xdx = \left| \begin{array}{ll} u=f(x) & v'=\sin x \\ u'=f'(x) & v=-\cos x \end{array} \right| = -f(x)\cos x]_0^\pi + \int\limits_0^\pi f'(x)\cos xdx = f(\pi)+f(0)+\int\limits_0^\pi f'(x)\cos xdx;$

(b) $\int\limits_0^{\pi} f''(x)\sin x dx = \left| \begin{array}{cc} u' = f''(x) & v = \sin x \\ u = f'(x) & v' = \cos x \end{array} \right| = f'(x)\sin x]_0^{\pi} - \int\limits_0^{\pi} f'(x)\cos x dx =$
$- \int\limits_0^{\pi} f'(x)\cos x dx.$

Therefore, $2 = \int\limits_0^{\pi} [f(x) + f''(x)]\sin x dx = f(\pi) + f(0) \rightarrow f(0) = 1.$