Lecture 20

11.8 Power Series

Definition 1. The series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is called a **power series** in $x-a$ or a **power series centered at** a or a **power series about** a. Constants c_n are called **coefficients of the series**.

Theorem 1. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all real x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. In this case the series may or may not converge at either of two endpoints $x = a - R$ and $x = a + R$.

Definition 2. We call the interval, where $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges, the **interval of convergence** of the series.

It must be one of the form:

(i) The isolated point $x = a$.

(ii) The entire real line $(-\infty, \infty)$.

(iii) A finite interval centered at $x = a$:

$$(a-R, a+R), [a-R, a+R), (a-R, a+R], [a-R, a+R]$$

Definition 3. The number R is called the **radius of convergence** of the series.

There are only three possibilities:

(i) $R = 0$.

(ii) $R = \infty$).

(iii) R is finite positive number.

Algebraic Operations on Power Series

Theorem 2. Let $\sum_{n=0}^{\infty} a_n(x-a)^n$ and $\sum_{n=0}^{\infty} b_n(x-a)^n$ be two series with the radius of convergence R_a and R_b, respectively. Let c be a real constant. Then

1. $\sum_{n=0}^{\infty} a_n(x-a)^n \pm \sum_{n=0}^{\infty} b_n(x-a)^n = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-a)^n$, $R = \min\{R_a, R_b\}$
2. $c\sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty}(ca_n)(x-a)^n$, $R = R_a$
3. $\left(\sum_{n=0}^{\infty} a_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) = \sum_{n=0}^{\infty} c_n(x-a)^n$ where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$

Definition 4. The sum $\sum_{n=0}^{\infty} c_n(x-a)^n$ is called the **Cauchy product of series** $\sum_{n=0}^{\infty} a_n(x-a)^n$ and $\sum_{n=0}^{\infty} b_n(x-a)^n$.