Midterm Oct 2017

* Problem 1:

(1a) Write the sigma notation formula for L_n (left Riemann Sum) of $f(x)$ on n subinterval

given

$$f(x) = (1+x)^2 \text{ on } [-1, 2]$$

$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$x_i = x_0 + \Delta x . i = -1 + \frac{3}{n} i$$

$$f(x_i) = (1+x_i)^2 = (1-1+\frac{3}{n}i)^2 = \frac{9i^2}{n^2}$$

So $L_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{n} \frac{9i^2}{n^2} \cdot \frac{3}{n} = \sum_{i=1}^{n} \frac{27}{n^3} i^2 = \frac{27}{n^3} \sum_{i=1}^{n} i^2$$

But we have $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

So

$$L_n = \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{9\,n(n+1)(2n+1)}{2\,n^3}$$

Evaluate $\int_{-1}^{2} f(x)\,dx$

$$\int_{-1}^{2} f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{9\,n(n+1)(2n+1)}{2\,n^3} = \frac{9}{2} \cdot 2 = 9$$

$$\int_{-1}^{2} f(x)\,dx = 9$$

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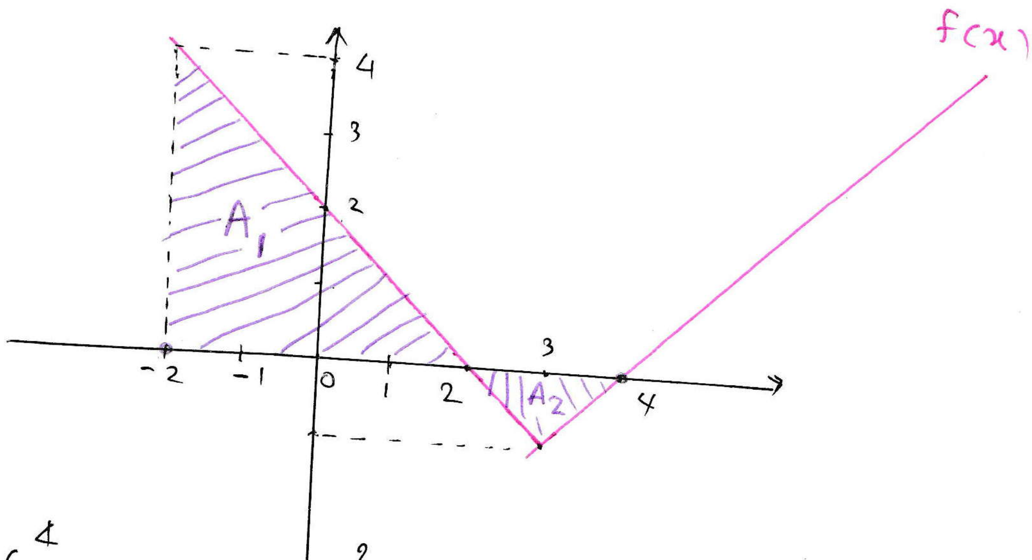
16 Calculate the integral in terms of area

$$I = \int_{-2}^{4} f(x)\,dx \quad \text{where } f(x) = |x-3| - 1$$

sketch the graph



$$I = \int_{-2}^{4} f(x)\,dx = \int_{-2}^{2} \left[|x-3|-1\right] dx + \int_{2}^{4} \left[|x-3|-1\right] dx$$

$$= A_1 - A_2 = \frac{1}{2}\cdot 4\cdot 4 - \frac{1}{2}\cdot 2\cdot 1 = 8 - 1 = 7$$

$$\boxed{I = 7}$$

Verify: Not interm of Area $\quad |x-3| = \begin{cases} x-3 & \text{if } x \geq 3 \\ -x+3 & \text{if } x < 3 \end{cases}$

$$I = \int_{-2}^{4} f(x)\,dx = \int_{-2}^{3} \left[|x-3|-1\right] dx + \int_{3}^{4} \left[|x-3|-1\right] dx$$

$$= \int_{-2}^{3} (-x+3-1)\,dx + \int_{3}^{4} (x-3-1)\,dx$$

$$= \int_{-2}^{3} (-x+2)\,dx + \int_{3}^{4} (x-4)\,dx =$$

$$= -\frac{1}{2}x^2 + 2x \Big]_{-2}^{3} + \frac{1}{2}x^2 - 4x \Big]_{3}^{4} = \boxed{7}$$

* **Problem 2** :

Use fundemental thm of Calculus, evaluate $F'(x)$

given: $F(x) = x^2 + \int_{-2x}^{2x} [3+\sin(t^2)]\,dt$

So $F'(x) = \left[x^2 + \int_{-2x}^{2x} [3+\sin(t^2)]\,dt\right]'$

$= 2x + \left[\int_{-2x}^{2x} [3+\sin(t^2)]\,dt\right]'$

$= 2x + \left[\int_{-2x}^{0} [3+\sin(t^2)]\,dt + \int_{0}^{2x} [3+\sin^2(t)]\,dt\right]'$

$= 2x + \underbrace{\left[-\int_{0}^{-2x} [3+\sin(t^2)]\,dt\right]'}_{y_1} + \underbrace{\left[\int_{0}^{2x} [(3+\sin(t^2)]\,dt\right]'}_{y_2}$

let $[y_1]' = \dfrac{dy_1}{dx} = \left[-\int_0^{-2x} [3+\sin(t^2)]\,dt\right]'$

let

$u = -2x \Rightarrow \dfrac{du}{dx} = -2$

$\dfrac{dy_1}{dx} = \dfrac{dy_1}{du}\cdot\dfrac{du}{dx} = -2\,\dfrac{dy_1}{du}$

$y_1 = \int_0^{-2x} [3+\sin(t^2)]\,dt$

replace $-2x$ by u

$\Rightarrow y_1 = \int_0^{u} [3+\sin(t^2)]\,dt$

$\Rightarrow \dfrac{dy_1}{du} = \left[\int_0^{u} [(3+\sin(t^2)]\,dt\right]'$

By Fundement of Calculus, derivative undo the integral

$\Rightarrow \dfrac{dy_1}{du} = +[3+\sin(t^2)] = +[3+\sin(-2x)^2] = +[3+\sin(4x^2)]$

$$\frac{dy_1}{du} = 3+\sin(4x^2)$$

$$\Rightarrow \frac{dy_1}{dx} = -\frac{dy_1}{du} \cdot \frac{du}{dx} = -(3+\sin(4x^2)) \cdot (-2)$$

$$\frac{dy_1}{dx} = 2[3+\sin(4x^2)]$$

Similarly

$$\frac{dy_2}{dx} = [3+\sin(4x^2)] \cdot 2$$

So $\frac{dy_1}{dx} = \frac{dy_2}{dx}$

Then

$$F'(x) = 2x + 4[3+\sin(4x^2)]$$

$$F'(x) = 2x + 12 + 4\sin(4x^2)$$

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* Problem 3 Calculate the indefinite integrals

3a $$I = \int \frac{x}{x^2-3x+2}\,dx$$

consider

$$\frac{x}{x^2-3x+2} = \frac{x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)}$$

$$\frac{x}{x^2-3x+2} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)}$$

$$\Rightarrow \quad x = A(x-1) + B(x-2)$$

$$= (A+B)\,x - (A+2B)$$

$$\Rightarrow \quad \begin{cases} A+B=1 \\ -(A+2B)=0 \end{cases} \Rightarrow \quad A = 2, \; B = -1$$

So $$I = \int \frac{x}{x^2-3x+2}\,dx = \int \left(\frac{2}{x-2} - \frac{1}{x-1} \right) dx$$

$$\boxed{I = 2\ln|x-2| - \ln|x-1| + C}$$

3b $$I = \int \cos^4(x)\,dx = \int \cos^2(x) \cdot \cos^2(x)\,dx$$

We have $\cos^2(x) = \frac{1}{2}\left[\cos(2x)+1\right]$

$$I = \frac{1}{4}\int \left[\cos(2x)+1\right]\left[\cos(2x)+1\right] dx$$

$$= \frac{1}{4}\int \left[\cos(2x)+1\right]^2 dx$$

$$= \frac{1}{4}\int \left[\cos^2(2x) + 2\cos(2x) + 1\right] dx$$

$$\cos^2(2x) = \frac{1}{2}\left[\cos(4x)+1\right]$$

$$I = \frac{1}{4}\left[\int \frac{1}{2}\left[\cos(4x)+1\right]dx + 2\int \cos(2x)\,dx + x\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}\int \cos(4x)\,dx + \frac{1}{2}x + 2\cdot\frac{1}{2}\sin(2x) + x\right]$$

$$= \frac{1}{4}\left[\frac{1}{8}\sin(4x) + \frac{1}{2}x + \sin(2x) + x\right]$$

$$= \frac{1}{4}\left[\frac{1}{8}\sin(4x) + \frac{3x}{2} + \sin(2x)\right]$$

$$I = \frac{1}{32}\sin(4x) + \frac{3}{8}x + \frac{\sin(2x)}{4} + C$$

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* Problem 5 : Evaluate

5a $I = \int_0^{\ln(4)} \frac{e^x}{e^{2x}+16}\, dx = \int_0^{\ln 4} \frac{e^x}{(e^x)^2+16}\, dx$

consider infinite integral

$I_1 = \int \frac{e^x}{(e^x)^2+16}\, dx = \int \frac{e^x}{16\left[\left(\frac{e^x}{4}\right)^2+1\right]}\, dx = \frac{1}{16}\int \frac{e^x}{\left(\frac{e^x}{4}\right)^2+1}\, dx$

let $t = \frac{e^x}{4} \Rightarrow dt = \frac{e^x dx}{4} \Rightarrow$

$I_1 = \frac{1}{16} \int \frac{4\,dt}{t^2+1} = \frac{1}{4}\int \frac{dt}{t^2+1} = \frac{1}{4}\tan^{-1}(t) + C$

$I_1 = \frac{1}{4}\tan^{-1}\left(\frac{e^x}{4}\right) + C$

$I = \int_0^{\ln 4} \frac{e^x}{e^{2x}+16}\, dx = \frac{1}{4}\tan^{-1}\left(\frac{e^x}{4}\right)\Big]_0^{\ln(4)}$

$I = \frac{1}{4}\left[\tan^{-1}\left(\frac{e^{\ln 4}}{4}\right) - \tan^{-1}\left(\frac{e^0}{4}\right)\right.$

$I = \frac{1}{4}\tan^{-1}(1) - \tan^{-1}\left(\frac{1}{4}\right)$

5b $I = \int_0^3 x^2 \sqrt{1+x}\, dx$

Using substitution

let $t = 1+x \Rightarrow dt = dx$

and $x = t-1$

when $x = 0$ then $t = 1+x = 1+0 = 1$

when $x = 3$ then $t = 1+3 = 4$

$$I = \int_1^4 (t-1)^2 \sqrt{t}\, dt = \int_1^4 (t^2-2t+1) \cdot t^{\frac{1}{2}} dt$$

$$= \int_{-1}^4 \left(t^{5/2} - 2t^{3/2} + t^{\frac{1}{2}}\right) dt$$

$$= \frac{2}{7} t^{7/2} - 2 \cdot \frac{2}{5} t^{5/2} + \frac{2}{3} t^{3/2} \Big]_{-1}^{4}$$

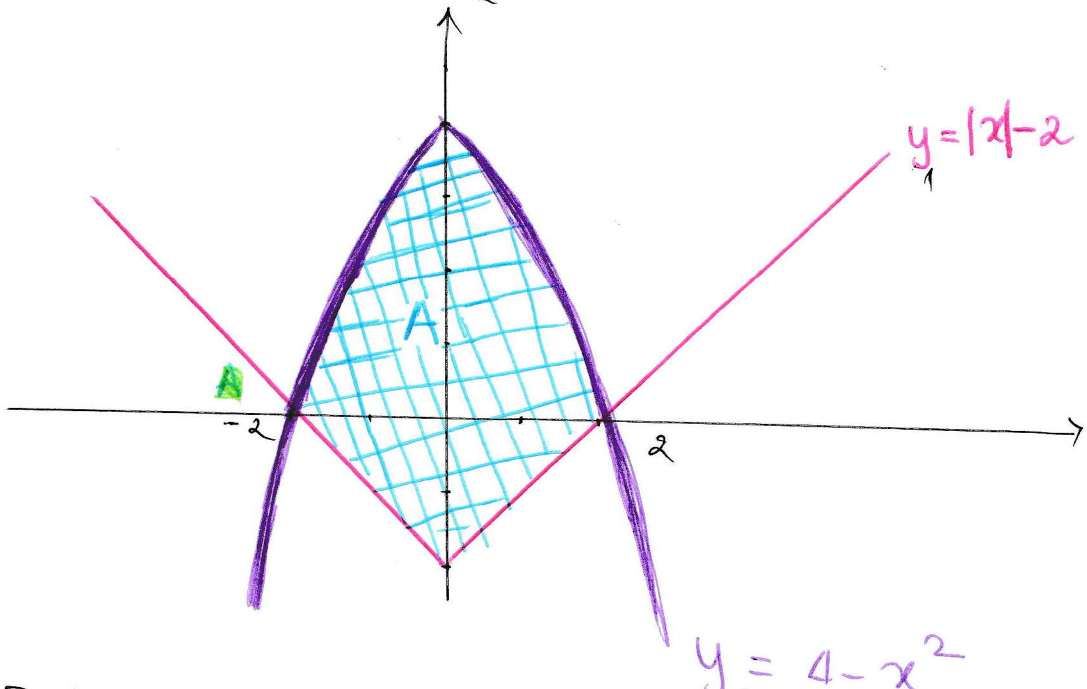
$$= \frac{2}{7} t^{7/2} - \frac{4}{5} t^{5/2} + \frac{2}{3} t^{3/2} \Big]_{-1}^{4}$$

$$I = \frac{2}{7}(4)^{7/2} - \frac{4}{5}(4)^{5/2} + \frac{2}{3}(4)^{3/2} - \left(\frac{2}{7}(-1)^{\frac{7}{2}} - \frac{4}{5}(-1)^{5/2} + \frac{2}{3}(-1)^{3/2}\right]$$

*** Problem 6 :**

* sketch the graph $y_1 = |x| - 2$

$y_2 = 4 - x^2$



* Find the area enclosed by the graph

Points of intersection by the graph $(x = -2 , y = 0)$
$(x = 2 , y = 0)$

$$A = \int_{-2}^{2} (y_2 - y_1)\, dx$$

$$= \int_{-2}^{2} \left[(4 - x^2) - (|x| - 2) \right] dx$$

we have $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$

$$A = \int_{-2}^{0} \left[(4 - x^2) - (-x - 2) \right] dx + \int_{0}^{2} (4 - x^2) - (x - 2) \Big] dx$$

$$A = \int_{-2}^{0} (-x^2 + x + 6)\, dx + \int_{0}^{2} (-x^2 - x + 6) \cdot dx = 24$$

Bonus Question

$$I = \int_{-\pi}^{\pi} \frac{\sin(x)}{1+x^2}\,dx$$

$\sin(x)$ is odd because $\sin(-x) = -\sin(x)$

$1+x^2$ is even

$\Rightarrow \dfrac{\sin(x)}{1+x^2}$ is odd

$$I = \int_{-\pi}^{\pi} \frac{\sin(x)}{1+x^2}\,dx = 0$$

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