Lecture 14

11.1 Sequences

Definition 1. A **sequence** (or infinite sequence) is an ordered list of infinitely many elements (or terms). Terms are always real numbers.

Definition 2. A **sequence** is a special kind of function f with domain as a set of integer numbers extending from some starting integer to infinity $a_n = f(n)$.

Ways to specify a sequence

1. A list of the first few terms following by ..., if the pattern is obvious.
2. A formula for the general term a_n as a function of n.
3. Recursive.

Algebraic Operations

1. **Sum**: $\{a_n\}_{n=1}^{\infty} + \{b_n\}_{n=1}^{\infty} = \{a_n + b_n\}_{n=1}^{\infty}$
2. **Subtraction**: $\{a_n\}_{n=1}^{\infty} - \{b_n\}_{n=1}^{\infty} = \{a_n - b_n\}_{n=1}^{\infty}$
3. **Multiplication by a number**: $k\{a_n\}_{n=1}^{\infty} = \{ka_n\}_{n=1}^{\infty}$
4. **Multiplication**: $\{a_n\}_{n=1}^{\infty} \cdot \{b_n\}_{n=1}^{\infty} = \{a_n \cdot b_n\}_{n=1}^{\infty}$
5. **Division**: $\frac{\{a_n\}_{n=1}^{\infty}}{\{b_n\}_{n=1}^{\infty}} = \left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty},\ b_n \neq 0$

Bounded and Unbounded Sequences

Definition 3. The sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded below** by a number L if $a_n \geq L$ for every $n = 1, 2, \ldots$ L is called a **lower bound**.

Definition 4. The sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded above** by a number M if $a_n \leq M$ for every $n = 1, 2, \ldots$ M is called a **upper bound**.

Definition 5. The greatest of the lower bounds is called **the most lower bound**.

Definition 6. The smallest of the upper bounds is called **the least upper bound**.

Definition 7. The sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded** if it is both bounded below and bounded above. In this case there exist a constant $K = \max\{|L|, |M|\}$ such that $|a_n| \leq K$.

Definition 8. The sequence $\{a_n\}_{n=1}^{\infty}$ is **nonnegative** if it is bounded below by 0, that is $a_n \geq 0$ for every $n = 1, 2, \ldots$.

Definition 9. The sequence $\{a_n\}_{n=1}^{\infty}$ is **nonpositive** if it is bounded above by 0, that is $a_n \leq 0$ for every $n = 1, 2, \ldots$.

Definition 10. The sequence $\{a_n\}_{n=1}^{\infty}$ is **unbounded** if for any $A > 0$ there exists a number N such that $|a_n| > A$ for $n > N$.

Monotonic Sequences

Definition 11. The sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **decreasing** if $a_{n+1} < a_n$ for every $n \geq 1$.

Definition 12. The sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **increasing** if $a_{n+1} > a_n$ for every $n \geq 1$.

Definition 13. The sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **monotonic** if it is increasing or decreasing.

Definition 14. The sequence $\{a_n\}_{n=1}^{\infty}$ is **alternating** if $a_n \cdot a_{n+1} < 0$ for every $n \geq 1$.

Theorem 1. If $a_n = f(n)$ where $f(x)$ is a differentiable function on $[1, \infty)$, then $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence if $f'(x) < 0$ on $[1, \infty)$ and an increasing sequence if $f'(x) > 0$ on $[1, \infty)$.

Convergence of a Sequence

Definition 15. We say that the sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the limit L $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there exists an integer number N such that

$$|a_n - L| < \varepsilon \quad \text{for any} \quad n \geq N.$$

Definition 16. If there doesn't exist a finite number L, the sequence $\{a_n\}_{n=1}^{\infty}$ **diverges**. If $\lim_{n\to\infty} a_n = \infty$, the sequence **diverges on** ∞, if $\lim_{n\to\infty} a_n = -\infty$, the sequence **diverges on** $-\infty$. If $\lim_{n\to\infty} a_n$ does not exist, the sequence **diverges**.

Theorem 2. If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$ then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = L.$$

Limit Rules for Sequences

If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences and k is a constant, then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(ka_n) = k \lim_{n\to\infty} a_n$
3. $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
5. $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p$ if $p > 0$ and $a_n > 0$
6. If $a_n \leq b_n$ ultimately, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$
7. If $a_n \leq b_n \leq c_n$ ultimately and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Theorem 3. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If $\lim_{n\to\infty} a_n = L$ and f is continuous at L and defined at all a_n, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$$

Theorem 5. If $\{a_n\}_{n=1}^{\infty}$ converges, then it is bounded.

Theorem 6. If $\{a_n\}_{n=1}^{\infty}$ is bounded above and (ultimately) increases, then it converges. If $\{a_n\}_{n=1}^{\infty}$ is bounded below and (ultimately) decreases, then it converges.