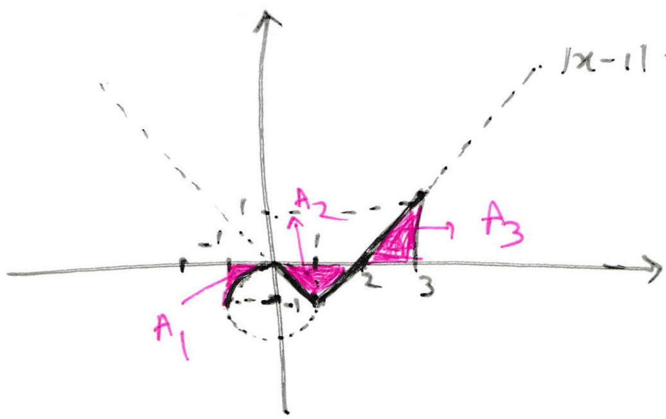
Midterm March 2019 Solution

* Problem1

$$f(x) = \begin{cases} \sqrt{1-x^2}-1 & -1 \leq x \leq 0 \\ |x-1|-1 & \text{if } x>0 \end{cases} \quad \text{on } [-1,3]$$

(a) Calculate $\int_{-1}^{3} f(x)\,dx$ using area



$$\int_{-1}^{3} f(x)\,dx = -A_1 - A_2 + A_3$$

$$= -\left(1 - \frac{1}{4}\cdot\pi\cdot 1^2\right) - \frac{1}{2}\cdot 2\cdot 1 + \frac{1}{2}\cdot 1$$

$$= -1 + \frac{\pi}{4} - 1 + \frac{1}{2}$$

$$\boxed{\int_{-1}^{3} f(x)\,dx = -\frac{3}{2} + \frac{\pi}{4}}$$

(b) Use fundemental theorem to find $f(x)$ and A

$$\int_{x}^{2} f(t)\,dt + A = x^2 + x$$

$$\left(\int_{x}^{2} f(t)\,dt + A\right)' = \left(x^2+x\right)'$$

$$-\left(\int_{2}^{x} f(t)\,dt\right)' = 2x+1$$

$$f(x) = -2x-1$$

Find A

let $x = 2$

$$\int_x^2 f(t)\,dt + A = x^2 + x$$

$$\int_2^2 f(t)\,dt + A = 4 + 2$$

$$0 + A = 6$$

$$\boxed{A = 6}$$

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* Problem 2 Find antiderivative of $f(x)$

$$f(x) = \frac{\cos^5(x) + 1}{\cos^2(x)}$$

$$F(x) = \int \left[\cos^3(x) + \frac{1}{\cos^2(x)}\right] dx$$

$$= \int \left[\left(1 - \sin^2(x)\right)\cos(x) + \sec^2(x)\right] dx$$

$$= \int \left[\cos x - \sin^2(x)\cos(x) + \sec^2(x)\right] dx$$

$$= \sin(x) - \frac{1}{3}\sin^3(x) + \tan(x) + C$$

$F(0) = 0 \Rightarrow \sin(0) - \frac{1}{3}\sin(0) + \tan(0) + C = 0$

$\Rightarrow C = 0$

So $F(x) = \sin(x) - \frac{1}{3}\sin^3(x) + \tan(x)$

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* Problem 3 : Calculate the indefinite integrals

(a) $I = \int \frac{x^2-2}{x^2-x-2}\,dx$

$I = \int \frac{x^2-x-2+x}{x^2-x-2}\,dx = \int 1 + \frac{x}{x^2-x-2}\,dx$

$I = x + \int \frac{x}{x^2-x-2}\,dx$

$\frac{x}{x^2-x-2} = \frac{x}{(x-2)(x+1)} = \frac{A}{x+1} + \frac{B}{x-2}$

$\Rightarrow \quad x = A(x-2) + B(x+1)$

$x = (A+B)x + (-2A+B)$

$\Rightarrow \begin{cases} A+B=1 \\ -2A+B=0 \end{cases} \Rightarrow A = \frac{1}{3},\ B = \frac{2}{3}$

$I = x + \int \left[\frac{1}{3(x+1)} + \frac{2}{3(x-2)}\right] dx$

$I = x + \frac{1}{3}\ln|x+1| + \frac{2}{3}\ln|x-2| + C$

(b) $I = \int e^{2x}\sqrt{e^x+1}\,dx = \int e^x\sqrt{e^x+1}\,e^x dx$

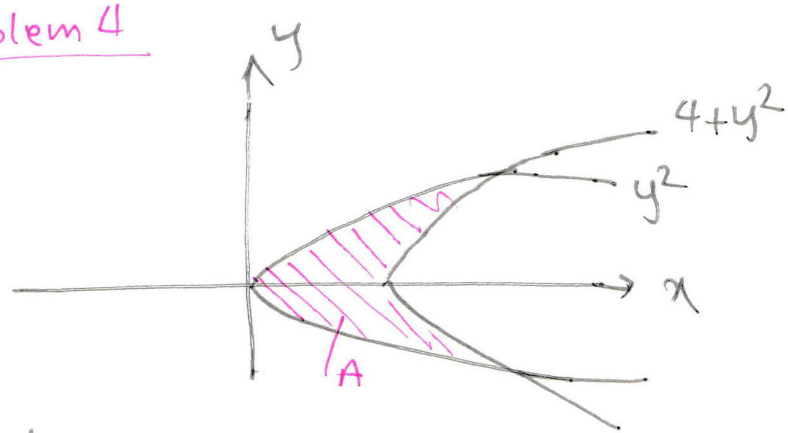
$u = e^x+1 \Rightarrow du = e^x dx$ and $e^x = u-1$

$I = \int (u-1)\sqrt{u}\,du$

$I = \int (u^{3/2} - u^{\frac{1}{2}})\,du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$I = \frac{2}{5}(e^x+1)^{5/2} - \frac{2}{3}(e^x+1)^{3/2} + C$

* Problem 4



Find the area A

$$A = \int_{-2}^{2} \left[(4+y^2) - 2y^2\right] dy \quad \text{or } A = 2\int_0^2 \left[(4+y^2) - 2y^2\right] dy$$

$$= 2\int_0^2 [4-y^2]\, dy = 2 \cdot 4y - \frac{y^3}{2} \Big]_0^2$$

$$A = \frac{32}{3}$$

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* Problem 5

(a) $I = \int_{\sqrt{2}}^{2} \sqrt{4-x^2}\, dx$

Technique 1 : Using trig. substitution

let $x = 2\sin(t) \Rightarrow dx = 2\cos(t)dt$

$$\int \sqrt{4-x^2}\, dx = \int \sqrt{4-4\sin^2 t} \cdot 2\cos(t)dt$$

$$= 4\int \cos^2(t)dt = 4\int \frac{2\cos(2t)+1}{2} dt$$

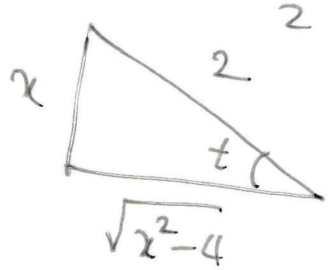
$$= 2\int \left[\cos(2t) + \frac{1}{2}\right] dt = 2\sin(2t) + 2t$$

$$= 2\sin(t)\cos(t) + 2t$$

Finding $\cos(t)$ in terms of x

$x = 2\sin(t)$

$\Rightarrow \sin t = \frac{x}{2}$



$\cos(t) = \frac{\sqrt{x^2-4}}{2}$

So $I = 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{x^2-4}}{2} + 2\sin^{-1}\left(\frac{x}{2}\right) \Big]_{\sqrt{2}}^{2}$

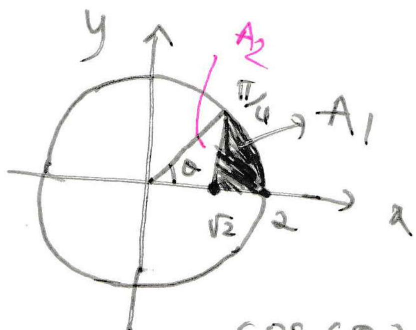
$I = \frac{\pi}{2} - 1$

Technique 2 : using the area

$I = \int_{\sqrt{2}}^{2} \sqrt{4-x^2}\, dx$

$y = \sqrt{4-x^2}$

$y^2 + x^2 = 4$ circle with center $(0,0)$ radius $r=2$



So $I = A_1$ area under the curve from $\sqrt{2}$ to 2

$\cos(\theta) = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$

$\Rightarrow A_2 = \frac{1}{2}\sqrt{2}\cdot\sqrt{2} = 1$

So $I = A_1 = \frac{1}{8} \cdot \pi \cdot r^2 - A_2 = \frac{1}{8}\pi \cdot 4 - 1$

$I = \frac{\pi}{2} - 1$

5b $I = \int_0^{\pi/4} \tan^{-1}(x)\,dx$

use integral by parts

$u = \tan^{-1}(x) \Rightarrow du = \frac{1}{x^2+1}dx$

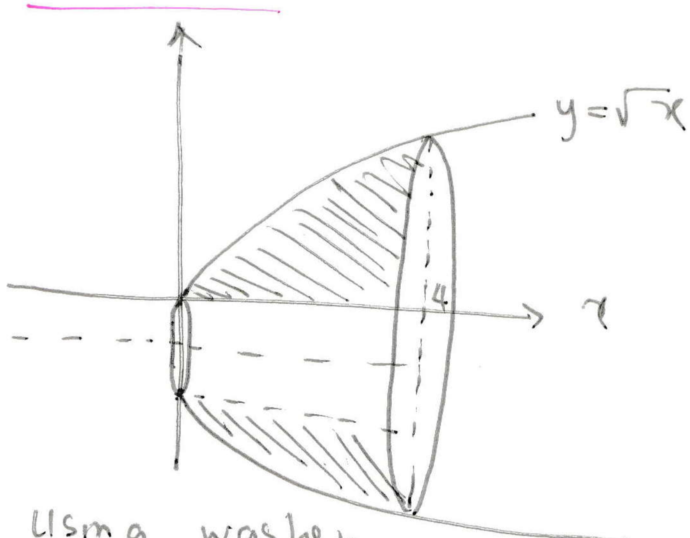
$dv = 1 \Rightarrow v = x$

$I = x\tan^{-1}(x)\Big]_0^{\pi/4} - \int_0^{\pi/4} \frac{x}{x^2+1}\,dx$

$= \frac{\pi}{4}\tan^{-1}\left(\frac{\pi}{4}\right) - \frac{1}{2}\ln|x^2+1|\Big]_0^{\pi/4}$

$I = \pi/4 - \frac{1}{2}\ln\left(\frac{\pi^2}{16}+1\right)$

* Problem 6



using washer

$R = 1+\sqrt{x}$

$r = 1$

$\Rightarrow V = \pi\int_0^4 (1+\sqrt{x})^2 - 1^2\,dx = \pi\int_0^4 \cancel{1}+2\sqrt{x}+x-\cancel{1}\,dx$

$V = \pi\int_0^4 2\sqrt{x}+x\,dx = \pi\cdot\left[2\cdot\frac{2}{3}x^{3/2}+\frac{1}{2}x^2\right]_0^4$

$V = \frac{56\pi}{3}$

* Bonus Problem

$$\int_0^1 e^{-x}[f(x)-f'(x)] = e \quad , f(0)=0$$

Find $f(1)$

$$\int_0^1 e^{-x}[f(x)-f'(x)]\,dx = \int_0^1 [e^{-x}f(x) - e^{-x}f'(x)]\,dx$$

$$= \int_0^1 e^{-x}f(x)\,dx - \int e^{-x}f'(x)\,dx$$

let $u = e^{-x}$ $\quad du = -e^{-x}dx$

$v = f'(x)$ $\quad v = f(x)$

Integral by parts of u, v

$$\int_0^1 e^{-x}f'(x)\,dx = f(x)e^{-x}\Big]_0^1 - \int_0^1 -e^{-x}f(x)\,dx$$

$$\int_0^1 e^{-x}f'(x)\,dx = f(x)e^{-x}\Big]_0^1 + \int_0^1 e^{-x}f(x)\,dx$$

$$\Rightarrow \int_0^1 e^{-x}f'(x)\,dx - \int_0^1 e^{-x}f(x)\,dx = -f(x)e^{-x}\Big]_0^1$$

$$\int_0^1 e^{-x}[f(x)-f'(x)]\,dx = -(f(1)e^{-1} - f(0)\cdot e^{-0})$$

$$e = -\frac{f(1)}{e} - 0$$

$$\Rightarrow f(1) = -e^2$$

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