Lecture 19

11.6 Absolute Convergence and Ration and Root Tests

Definition 1. The series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Properties of Absolutely Convergent Series

1. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, it converges.
2. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely, then for any real finite numbers α and β $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ converges absolutely.
3. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then the the series with rearranged terms converges absolutely to the same limit as $\sum_{n=1}^{\infty} a_n$.

Definition 2. If $\sum_{n=1}^{\infty} a_n$ converges, but not converges absolutely, then we say it **converges conditionally**.

The Ratio Test

Theorem 1. The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the Ratio Test is inconclusive.

The Root Test

Theorem 2. The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the Ratio Test is inconclusive.

Riemann Theorem about Rearranging the Terms in Series

Theorem 3.

(i) If the terms of absolutely convergent series are rearranged, then the rearranged series converges to the same limit as the original series.

(ii) If a series is conditionally convergent and L is any real number, then the terms can be rearranged to make a new series convergent to the limit L. The terms also can be rearranged in the way that series converges to ∞, $-\infty$ or simply diverges.