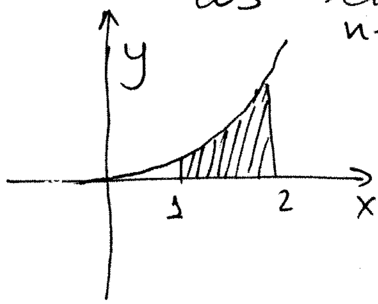
Math-205 Midterm test

23 May 2019

Solutions

Q1(a). Write the sigma notation formula for the right Riemann sum R_n of $f(x) = 3x^2$ on the interval $[1,2]$ with n subintervals, and calculate the definite integral $\int_1^2 f(x)\,dx$ as $\lim_{n\to\infty} R_n$.



$$\Delta_n = \frac{2-1}{n} = \frac{1}{n}$$

$$x_i = 1 + i\cdot\Delta_n \; ; \qquad i = 0, 1, \dots n.$$

with $x_0 = 1$ and $x_n = 2$.

$$R_n = \sum_{i=1}^{n} 3\left(1 + i\frac{1}{n}\right)^2 \frac{1}{n} =$$

$$\underline{= 3\sum_{i=1}^{n}\left(1 + 2\frac{i}{n} + \frac{i^2}{n^2}\right)\frac{1}{n}} \text{ — the righ R-sum}$$

$$\lim_{n\to\infty} R_n = 3\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} 1 + \lim_{n\to\infty}\frac{3\cdot 2}{n^2}\left(\sum_{i=1}^{n} i\right) + \lim_{n\to\infty}\frac{3}{n^3}\sum_{i=1}^{n} i^2 =$$

$$= \lim_{n\to\infty}\left(3\frac{n}{n} + \frac{6}{n^2}\,\frac{n(n+1)}{2} + \frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 3 + 3\lim_{n\to\infty}\left(1+\frac{1}{n}\right) + \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right) = 7$$

(b) Find the local extrema of $F(x) = \int_0^{x^2} \frac{t-4}{1+\cos^2(t)}\,dt$.

Solution:

$$F'(x) = \left.\frac{t-4}{1+\cos^2 t}\right|_{t=x^2} \cdot \frac{d}{dx}(x^2) = \frac{x^2-4}{1+\cos^2(x^2)} \cdot 2x$$

From $F'(x)=0 \Rightarrow (x^2-4)x = 0$ (since $1+\cos^2(x^2) \neq 0$)

Thus, the critical values are $x_1=-2,\ x_2=0,\ x_3=2$.

Sign chart for $F'(x)$:

$F'(x)$: <0 | -1 | >0 | 0 | <0 | 2 | >0

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$\Rightarrow$ x_1 and x_3 – local minima

at x_2 – local maximum.

Q 2. Find the function $F(x)$ such that $F(0)=1$ and

$$\frac{dF}{dx} = \frac{\sec^2(x)}{1+\tan(x)} + x \cdot e^{-x^2}.$$

Solution: $F(x) = \int \frac{\sec^2(x)}{1+\tan(x)}\,dx + \int x e^{-x^2}\,dx = F_1 + F_2$

where $F_1(x) = \int \frac{\sec^2(x)}{1+\tan(x)}\,dx =$ $\leftarrow$ $\boxed{\begin{array}{l} u = 1+\tan(x) \\ du = \sec^2(x)\,dx \end{array}}$

$$= \int \frac{du}{u} = \ln u + C = \ln[1+\tan(x)] + C$$

$$F_2(x) = \int x e^{-x^2}\,dx = \frac{1}{2}\int e^{-x^2}\,d(x^2) = -e^{-x^2} + C$$

$$F(x) = \ln(1+\tan(x)) - \frac{1}{2}e^{-x^2} + C; \Rightarrow$$

$$\ln[1+\tan(0)] - \frac{1}{2}e^0 + C = 1 \Rightarrow C = \frac{3}{2}$$

$$F(x) = \frac{3}{2} + \ln[1+\tan(x)] - \frac{1}{2}e^{-x^2}.$$

Q3. Calculate the following indefinite integrals. (3)

(a) $$\int x\left(x^{-1}+x^{1/2}\right)^2 dx = \int\left(x^{-1}+2x^{1/2}+x^2\right)dx$$

$$= \ln|x| + 2\frac{1}{(3/2)}x^{3/2} + \frac{1}{3}x^3 + C$$

$$= \ln|x| + \frac{4}{3}x^{3/2} + \frac{1}{3}x^3 + C.$$

(b) $$f(t) = \frac{t^2+4t}{t^2+4}\ ; \quad \int f(t)\,dt = ?$$

$$f(t) = \frac{t^2+4+4t-4}{t^2+4} = 1 + \frac{4t}{t^2+4} - \frac{4}{t^2+4}$$

$$\Rightarrow \int f(t)\,dt = t + \int\frac{4t}{t^2+4}dt - 4\int\frac{dt}{t^2+4} =$$

$$= t + 2\int\frac{d(t^2+4)}{t^2+4} - 2\arctan\left(\frac{t}{2}\right) + C$$

$$= t + 2\ln(4+t^2) - 2\arctan\left(\frac{t}{2}\right) + C.$$

(c) $$F(x) = \int\frac{e^x\,dx}{e^{2x}-9}\ ; \quad \begin{array}{l} t=e^x \\ dt=e^x dx \end{array} \Rightarrow F(x) = \int\frac{dt}{t^2-9}$$

$$\frac{1}{t^2-9} = \frac{1}{(t-3)(t+3)} = \frac{A}{t-3} + \frac{B}{t+3} \Rightarrow 1 = A(t+3)+B(t-3)$$

$$\Rightarrow 1 = A(3+3) + (3-3)\cdot B = 6A + 0;\quad A = \frac{1}{6}.$$

similarly, $B = -\frac{1}{6}$; $\quad \frac{1}{t^2-9} = \frac{1}{6}\left(\frac{1}{t-3} - \frac{1}{t+3}\right)$

$$F(x) = \int\frac{1}{6}\left(\frac{1}{t-3} - \frac{1}{t+3}\right)dt = \frac{1}{6}\left(\ln|t-3| - \ln|t+3|\right) =$$

$$= \frac{1}{6}\ln\left|\frac{e^x-3}{e^x+3}\right| + C.$$

4. Evaluate definite integrals.

(a)

$$\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx =$$

$\boxed{\begin{array}{l} u = 2x+1. \quad - \text{new variable} \\ du = 2dx, \\ x = (u-1)/2. \\ u(0) = 1; \quad u(4) = 9. \end{array}}$

$$= \frac{1}{2}\int_1^9 \frac{x\,du}{u^{1/2}} = \frac{1}{4}\int_1^9 (u-1)u^{-1/2}du =$$

$$= \frac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})du = \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]\Bigg|_1^9 =$$

$$= \frac{1}{4}\left[\frac{2}{3}9^{3/2} - 2\cdot 9^{1/2} - \frac{2}{3}\cdot 1^{3/2} + 2\cdot 1^{1/2}\right] = \frac{1}{4}\left(2\cdot 9 - 2\cdot 3 - \frac{2}{3} + 2\right)$$

$$= \frac{1}{4}\cdot\left(14 - \frac{2}{3}\right) = \frac{10}{3}.$$

(b)

$$\int_0^1 x^2\cos(\pi x)\,dx = \frac{1}{\pi}\int_0^1 x^2\,d\sin(\pi x) =$$

$$= \frac{1}{\pi}\,x^2\sin(\pi x)\Big|_0^1 - \frac{2}{\pi}\int_0^1 \sin(\pi x)\cdot x\,dx =$$

$$= \frac{1}{\pi}(0-0) + \frac{2}{\pi^2}\int x\,d\cos(\pi x) =$$

$$= \frac{2}{\pi^2}\,x\cos(\pi x)\Big|_0^1 - \frac{2}{\pi^2}\int_0^1 \cos(\pi x)\,dx =$$

$$= \frac{2}{\pi^2}(1\cdot\cos\pi - 0) - \frac{2}{\pi^3}\sin(\pi x)\Big|_0^1 = \frac{2}{\pi^2}(-1) - \frac{2}{\pi^3}(0-0)$$

$$= -\frac{2}{\pi^2}$$

Q5. Mean value of $f(x) = \sin^3(x)\cos^2(x)$ on $[0, \frac{\pi}{2}]$. (5)

$$\bar{f} = \frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx \Rightarrow$$

$u = \cos x$
$du = -\sin(x)dx$.
$u(0) = 1$.
$u(\frac{\pi}{2}) = 0$.
$\sin^2(x) = 1-u^2$

$$\bar{f} = \frac{2}{\pi}\int_1^0 -(1-u^2)u^2 du =$$

$$= \frac{2}{\pi}\int_0^1 (u^2-u^4)du = \frac{2}{\pi}\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)\Big|_0^1 =$$

$$= \frac{2}{\pi}\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{4}{15\pi}.$$

Bonus. $F(x) = \int_1^x [x + f(t)]dt = \int_1^x x\,dt + \int_1^x f(t)\,dt$

$$\Rightarrow F(x) = x\cdot\int_1^x 1\,dt + \int_1^x f(t)\,dt = x(x-1) + \int_1^x f(t)dt$$

$$\Rightarrow \underline{F'(x) = 2x-1+f(x).}$$