MATH 205

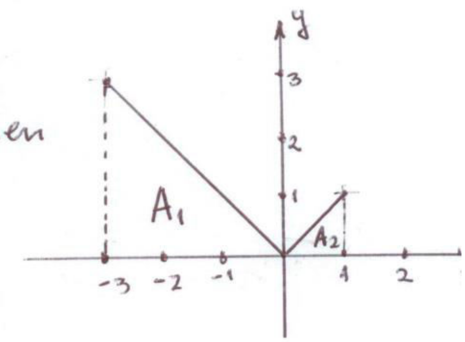
Sample Midterm Test

① (a) Evaluate the definite integral

$$\int_{-3}^{1} |x|\,dx$$

in terms of signed (net) area.

Solution. The graph of $y=|x|$, $-3 \le x \le 1$ is above the x-axis. Then



$$\int_{-3}^{1} |x|\,dx = A_1 + A_2 = \frac{3\cdot 3}{2} + \frac{1\cdot 1}{2}$$

$$= \frac{9}{2} + \frac{1}{2} = \frac{10}{2} = 5.$$

In other words

$$\int_{-3}^{1} |x|\,dx = \int_{-3}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx = A_1 + A_2 = \frac{9}{2} + \frac{1}{2} = 5.$$

(b) Find the derivative $F'(x)$ of the function

$$F(x) = \int_{x^2-1}^{0} \frac{\sin(t+1)}{t+1}\,dt$$

and by using it determine whether $F(x)$ is increasing or decreasing at $x = \sqrt{\pi/2}$.

Solution. Using Fundamental Theorem of Calculus, Part 1 we obtain

$$F(x) = -\int_{0}^{x^2-1} \frac{\sin(t+1)}{t+1}\,dt$$

$$F'(x) = -\left(\frac{\sin(t+1)}{t+1} \bigg|_{t=x^2-1} \right) \cdot (x^2-1)'$$

$$= - \frac{\sin((x^2-1)+1)}{(x^2-1)+1} \cdot (2x) = - \frac{\sin(x^2-1+1)}{x^2-1+1} \cdot (2x)$$

$$= - \frac{\sin(x^2)}{x^2} \cdot 2x = -2\frac{\sin(x^2)}{x}.$$

$$F'\left(\sqrt{\tfrac{\pi}{2}}\right) = -2 \frac{\sin\left[\left(\sqrt{\frac{\pi}{2}}\right)^2\right]}{\sqrt{\pi/2}} = -\frac{2\sqrt{2}}{\sqrt{\pi}} \sin\left(\tfrac{\pi}{2}\right) = -\frac{2\sqrt{2}}{\sqrt{\pi}} < 0$$

hence, $F(x)$ is <u>decreasing</u> at $x = \sqrt{\pi/2}$.

(2) Find the antiderivative $F(x)$ of the function

$$f(x) = x e^{-x^2}$$

such that $F(0) = 3$.

Solution. The most general antiderivative of $f(x)$ is its indefinite integral

<u>substitution</u>
$u = -x^2$
$du = -2x\,dx$
$x\,dx = -\frac{1}{2}du$

$$\int x e^{-x^2} dx = \int e^u \left(-\tfrac{1}{2} du\right)$$

$$= -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C$$

Hence, $F(x)$ must have the form

$$F(x) = -\frac{1}{2} e^{-x^2} + C.$$

$$F(0) = -\frac{1}{2} e^{-0^2} + C = -\frac{1}{2} e^0 + C = -\frac{1}{2} + C = 3 \Rightarrow C = 3 + \frac{1}{2} = \frac{7}{2}$$

$$\Rightarrow \boxed{F(x) = -\frac{1}{2} e^{-x^2} + \frac{7}{2}.}$$

③ Find the following indefinite integrals

(a) $\int \frac{(\sqrt{2x}-1)^2}{x}\,dx$.

Solution.

$$\int \frac{(\sqrt{2x}-1)^2}{x}\,dx = \int \frac{(\sqrt{2x})^2 - 2\sqrt{2x}+1}{x}\,dx$$

$$= \int \frac{2x - 2\sqrt{2}\sqrt{x}+1}{x}\,dx = \int \left(2 - 2\sqrt{2}\,x^{-1/2} + x^{-1}\right)dx$$

$$= 2x - 2\sqrt{2}\,\frac{x^{1/2}}{1/2} + \ln|x| + C = \boxed{2x - 4\sqrt{2}\sqrt{x} + \ln|x| + C.}$$

(b) $\int 4t^2 \ln(t)\,dt$.

Solution. Integration by parts; $u = \ln(t)$, $v = \frac{t^3}{3}$

$$\int 4t^2 \ln(t)\,dt = 4\int \ln(t)\,d\frac{t^3}{3} = 4\left[\frac{t^3}{3}\ln(t) - \int \frac{t^3}{3}\,d\ln(t)\right]$$

$$= 4\left[\frac{t^3}{3}\ln(t) - \frac{1}{3}\int t^3 \cdot \frac{1}{t}\,dt\right] = 4\left[\frac{t^3}{3}\ln(t) - \frac{1}{3}\int t^2 dt\right]$$

$$= \frac{4}{3}t^3\ln(t) - \frac{4}{3}\cdot\frac{t^3}{3} + C \Rightarrow$$

$$\boxed{\int 4t^2\ln(t)\,dt = \frac{4}{3}t^3\ln(t) - \frac{4}{9}t^3 + C.}$$

(c) $\displaystyle\int \frac{x-1}{x^2-7x+12}\,dx$

Solutoon. Using partoal fractoons.

First, $\deg(x-1) = 1 < 2 = \deg(x^2-7x+12)$.

Next, $x^2-7x+12 = (x-3)(x-4)$. Then

$$\frac{x-1}{x^2-7x+12} = \frac{x-1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4} \quad \text{or}$$

$x-1 = A(x-4) + B(x-3)$ for all x.

if $x=3$ we obtain $3-1 = A(3-4) + B(3-3)$

$\Rightarrow 2 = -A \Rightarrow A = -2.$

if $x=4$ we obtain $4-1 = A(4-4) + B(4-3)$

$\Rightarrow 3 = B \Rightarrow B = 3.$

Then,

$$\int \frac{x-1}{x^2-7x+12}\,dx = \int\left(-\frac{2}{x-3} + \frac{3}{x-4}\right)dx$$

$$= -2\int \frac{1}{x-3}\,dx + 3\int \frac{1}{x-4}\,dx = -2\ln|x-3| + 3\ln|x-4| + C$$

$$\Rightarrow \boxed{\int \frac{x-1}{x^2-7x+12}\,dx = 3\ln|x-4| - 2\ln|x-3| + C}$$

<u>Remark.</u> $\displaystyle\int \frac{1}{x+k}\,dx = \ln|x+k| + C.$

(4) Evaluate the following definite integrals

(a) $$\int_0^3 \frac{1+\arctan(x/3)}{9+x^2}\,dx$$

Solution 1. Substitution $u = \frac{x}{3}$, $du = \frac{dx}{3}$

$x = 3u$, $dx = 3\,du$;

$x = 0 \Rightarrow u = 0$

$x = 3 \Rightarrow u = 1$. Then

$$\int_0^3 \frac{1+\arctan(x/3)}{9+x^2}\,dx = \int_0^1 \frac{1+\arctan(u)}{9+(3u)^2}\,(3\,du)$$

$$= 3\int_0^1 \frac{1+\arctan(u)}{9+9u^2}\,du = 3\int_0^1 \frac{1+\arctan(u)}{9(1+u^2)}\,du$$

$$= \frac{1}{3}\int_0^1 \frac{1+\arctan(u)}{1+u^2}\,du$$

Substitution:

$v = \arctan(u)$

$dv = \frac{1}{1+u^2}\,du$

$u = 0 \Rightarrow v = 0$

$u = 1 \Rightarrow v = \frac{\pi}{4}$

$$= \frac{1}{3}\int_0^{\pi/4} (1+v)\,dv$$

$$= \frac{1}{3}\left[\left(v + \frac{v^2}{2}\right)\Big|_0^{\pi/4}\right]$$

$$= \frac{1}{3}\left[\left(\frac{\pi}{4} + \frac{\pi^2}{2\cdot 16}\right) - 0\right] = \frac{3\pi}{4} + \frac{\pi^2}{96}$$

$$\Rightarrow \boxed{\int_0^3 \frac{1+\arctan(x/3)}{9+x^2}\,dx = \frac{3\pi}{4} + \frac{\pi^2}{96}.}$$

Solution 2. Substitution $u = \arctan(x/3)$

$$du = \frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \frac{1}{3}\, dx$$

$$\int_0^3 \frac{1+\arctan(x/3)}{9+x^2}\, dx$$

$$du = \frac{3}{9+x^2}\, dx$$

$$= \int_0^{\pi/4} (1+u)\left(\frac{1}{3}\, du\right)$$

$$\frac{1}{3}\, du = \frac{1}{9+x^2}\, dx$$

$$= \frac{1}{3}\int_0^{\pi/4}(1+u)\, du$$

$x = 0 \Rightarrow u = 0$

$x = 3 \Rightarrow u = \frac{\pi}{4}$

$$= \frac{1}{3}\left[\left(u + \frac{u^2}{2}\right)\Big|_0^{\pi/4}\right]$$

$$= \frac{1}{3}\left[\left(\frac{\pi}{4} + \frac{1}{2}\cdot\left(\frac{\pi}{4}\right)^2\right) - \left(0 + \frac{0^2}{2}\right)\right] = \frac{1}{3}\left(\frac{\pi}{4} + \frac{\pi^2}{32}\right)$$

$$= \frac{\pi}{12} + \frac{\pi^2}{96}.$$

Solution 3. Substitution $u = 1 + \arctan(x/3)$

$$du = \frac{3}{9+x^2}\, dx$$

$$\int_0^3 \frac{1+\arctan(x/3)}{9+x^2}\, dx$$

$$\frac{1}{3}\, du = \frac{1}{9+x^2}\, dx$$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 1 + \frac{\pi}{4}$

$$= \frac{1}{3}\int_1^{1+\pi/4} u\, du = \frac{1}{3}\left[\frac{u^2}{2}\Big|_1^{1+\pi/4}\right]$$

$$= \frac{1}{6}\left[\left(1+\frac{\pi}{4}\right)^2 - 1^2\right] = \frac{1}{6}\left[\cancel{1} + 2\cdot\frac{\pi}{4} + \frac{\pi^2}{16} - \cancel{1}\right] = \frac{2\cdot\pi}{6\cdot 4} + \frac{\pi^2}{6\cdot 16}$$

$$= \frac{\pi}{12} + \frac{\pi^2}{96}.$$

(b) $\int_0^1 x e^{-x}\,dx$.

Solution. Integration by parts $u = x,\ v = -e^{-x}$

$$\int_0^1 x e^{-x}\,dx = \int_0^1 x\, d\left(\frac{e^{-x}}{-1}\right) = \int_0^1 \underset{u}{x}\, d\underbrace{(-e^{-x})}_{v}$$

$$= \left(x\cdot(-e^{-x})\right)\Big|_0^1 - \int_0^1 -e^{-x}\,dx$$

$$= \left(-xe^{-x}\Big|_0^1\right) + \int_0^1 e^{-x}\,dx$$

$$= \left[-1\cdot e^{-1} - (-0\cdot e^{-0})\right] + \left(\frac{e^{-x}}{-1}\Big|_0^1\right)$$

$$= -e^{-1} + \left[\frac{e^{-1}}{-1} - \frac{e^{-0}}{-1}\right] = -e^{-1} - e^{-1} + 1$$

$$= 1 - 2e^{-1} = 1 - \frac{2}{e} = \frac{e-2}{e}.$$

(5) Find the average value of $f(x) = 1 + \sin^2(x)$ on the interval $[0, \pi]$.

Solution.

$$f_{ave} = \frac{1}{\pi - 0}\int_0^\pi \left(1 + \sin^2(x)\right)dx = \frac{1}{\pi}\int_0^\pi\left(1 + \frac{1 - \cos(2x)}{2}\right)dx$$

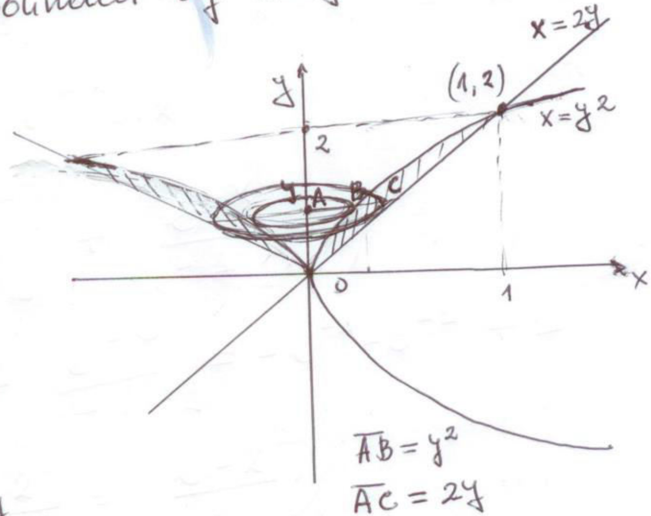
$$= \frac{1}{\pi}\int_0^\pi \left(\frac{3}{2} - \frac{\cos(2x)}{2}\right)dx = \frac{1}{2\pi}\int_0^\pi \left(3 - \cos(2x)\right)dx$$

$$= \frac{1}{2\pi}\left[\left(3x - \frac{\sin(2x)}{2}\right)\Big|_0^\pi\right] = \frac{1}{2\pi}\left[\left(3\pi - \frac{\sin(2\cdot\pi)}{2}\right) - \left(3\cdot 0 - \frac{\sin(2\cdot 0)}{2}\right.\right.$$

$$= \frac{1}{2\pi}\left[(3\pi - 0) - 0\right] = \frac{3}{2}.$$

$$\boxed{f_{ave} = \frac{3}{2}.}$$

(6) Find the volume of the solid obtained by rotating the region bounded by $x = y^2$ and $x = 2y$ about the y-axis.



Solution.

$$V = V_2 - V_1$$

$$V_2 = \int_0^2 \pi (2y)^2\,dy$$

$$V_1 = \int_0^2 \pi (y^2)^2\,dy$$

$$V = V_2 - V_1 = \int_0^2 \pi\, 4y^2\,dy - \int_0^2 \pi y^4\,dy$$

$$= \pi \int_0^2 (4y^2 - y^4)\,dy = \pi\left[\left(4\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_0^2\right]$$

$$= \pi\left[\left(4\cdot\frac{2^3}{3} - \frac{2^5}{5}\right) - 0\right] = \pi\, 2^5\left(\frac{1}{3} - \frac{1}{5}\right) = \pi\,\frac{2^6}{15} = \pi\cdot\frac{64}{15}$$

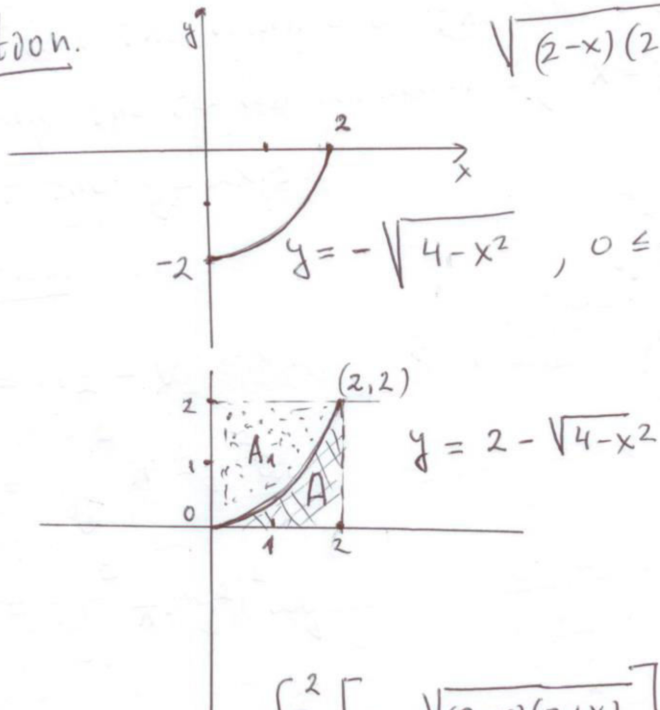
<u>Bonus question.</u> Calculate the definite integral

$$\int_0^2 \left[2 - \sqrt{(2-x)(2+x)}\right] dx$$

in terms of area.

<u>Solution.</u>

$$\sqrt{(2-x)(2+x)} = \sqrt{4-x^2}$$



Hence, $\displaystyle\int_0^2 \left[2 - \sqrt{(2-x)(2+x)}\right] dx$

$$= A = 2\cdot 2 - A_1 = 4 - \frac{\pi \cdot 2^2}{4} = 4-\pi .$$