Test #4 will cover WebWork assignments 9, 10 and the class notes for Cramer's Rule (pages 74 to 76) and Section 6 – Vectors (pages 86 to 111), along with the problems done in class for those sections. A formula sheet will be provided.

The breakdown for Test #4 is as follows (out of 35 points):

7 pts – Vector arithmetic (3 questions)

12 pts – Applications of Vectors (3 questions)

6 pts – Proof with vectors

10 pts – Solve a system of linear equations using Cramer's rule

Good luck!

Formula sheet

$\cos\theta = \dfrac{\vec{u} \cdot \vec{w}}{\|\vec{u}\|\|\vec{w}\|}$

Area of parallelogram: $\|\vec{v} \times \vec{w}\|$

Volume of parallelepiped: $|\vec{u} \cdot (\vec{v} \times \vec{w})|$

VECTOR OPERATIONS

Operations: Addition, Subtraction, Scalar Multiplication, Norm, Dot Product, Cross Product, Scalar Triple Product

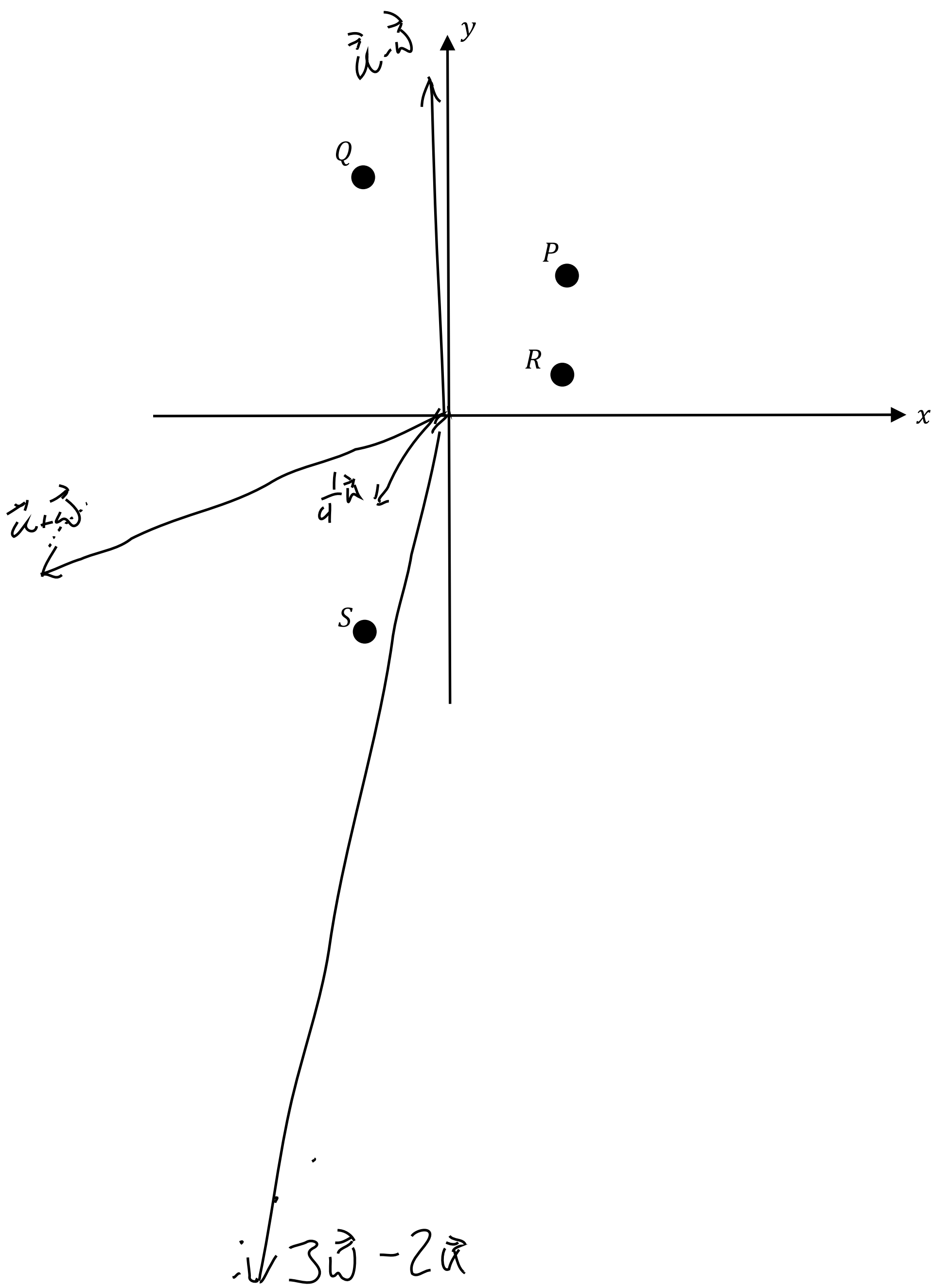
Ex 1) Given the points P, Q, R and S (shown below), construct vectors $\vec{u} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{RS}$ and then perform the following operations :

a. $\cancel{\vec{u} + \vec{w}}$

b. $\vec{u} - \vec{w}$

c. $\cancel{\frac{1}{4}\vec{w}}$

d. $\cancel{3\vec{w} - 2\vec{u}}$



Ex 2) Consider the vectors $\vec{a} = (1,0,2)$, $\vec{b} = (-1,-3,3)$ and $\vec{c} = (0,4,5)$. Find the following, if possible. If it is not possible, explain why not.

a. $3\vec{b} - 2\vec{c}$

$3(-1,-3,3) - 2(0,4,5)$

$(-3,-9,9) - (0,8,10)$

$(-3,-17,-19)$

b. $||4\vec{a}||$

$||4(1,0,2)||$

$= ||(4,0,8)||$

$= \sqrt{4^2 + 8^2}$

$= 4\sqrt{5}$

c. $4(\vec{a} \cdot \vec{c})$

$= 4((1,0,2)\cdot(0,4,5))$

$= 4(0 + 0 + 10)$

$= 4(10)$

$= 40$

d. $\vec{c} \times \vec{a}$

$= (0, 4, 5) \times (1, 0, 2)$

$= \left(\begin{vmatrix} 4 & 5 \\ 0 & 2 \end{vmatrix}, -\begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} \right)$

$= (8-0, -(0-5), 0-4)$

$= (8, 5, -4)$

e. $\vec{a} \cdot \vec{c} + 3\vec{b}$

Can't add a vector and a scalar

f. $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$

Can't take the dot product of a vector and a scalar

g. $2\vec{a} \cdot 5\vec{c} + \|\vec{a}\|\vec{b}$

Can't add a vector and a scalar

h. $(\vec{a} \cdot \vec{b}) \times \vec{c}$

Can't find the cross product between a vector and a scalar

i. $\vec{a} \cdot (\vec{b} \times \vec{c})$

$= (1, 0, 2) \cdot ((-1, -3, 3) \times (0, 4, 5))$

$= (1, 0, 2) \cdot \left(\begin{vmatrix} -3 & 3 \\ 4 & 5 \end{vmatrix}, - \begin{vmatrix} -1 & 3 \\ 0 & 5 \end{vmatrix}, \begin{vmatrix} -1 & -3 \\ 0 & 4 \end{vmatrix} \right)$

$= (1, 0, 2) \cdot (-27, 5, -4)$

$= -27 - 8$

$= -35$

ANGLE BETWEEN VECTORS $\dfrac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\,\|\vec{w}\|}$

Ex 1) Let $\vec{u} = (1, -5, 4)$ and $\vec{w} = (-2, 3, 2)$. Find the angle between $\vec{u}$ and $\vec{w}$

$$= \frac{(1,-5,4)\cdot(-2,3,2)}{\sqrt{1^2+5^2+4^2}\,\sqrt{2^2+3^2+2^2}}$$

$$= \frac{-2-15+8}{\sqrt{42}\,\sqrt{17}}$$

$$= \frac{-9}{\sqrt{42}\,\sqrt{17}}$$

$$= 1.914 \text{ rad}$$

Ex 2) Find the angle at B determined by the directed line segments $\overrightarrow{BA}$ and $\overrightarrow{BC}$, where $A(3,0,2)$, $B(4,3,0)$ and $C(8,1,-1)$.

$$\overrightarrow{BA} = (3,0,2)-(4,3,0)$$
$$= (-1,-3,2)$$
$$\overrightarrow{BC} = (8,1,-1)-(4,3,0)$$
$$= (4,-2,-1)$$

$$= \frac{(-1,-3,2)\cdot(4,-2,1)}{\sqrt{1^2+3^2+2^2}\sqrt{4^2+2^2+1^2}}$$

$$= \frac{(-1\cdot 4 + -3\cdot -2 + -2\cdot 1)}{\sqrt{14}\sqrt{21}}$$

$$= \frac{0}{\sqrt{14}\sqrt{21}}$$

$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2}$$

APPLICATION OF VECTORS

Types: Area, Volume

Ex 1) Find the area of the triangle with vertices $P(0,-2,-2)$, $Q(5,-1,3)$ and $R(7,0,6)$.

$$\vec{u} = (0,-2,-2)-(5,-1,3)$$
$$= (-5,-1,-5)$$

$$\vec{v} = (7,0,6)-(5,-1,3)$$
$$= (2,1,3)$$

$$= \|(-5,-1,-5) \times (2,1,3)\| \cdot \frac{1}{2}$$

$$= \left\| \left(\begin{vmatrix} -1 & -5 \\ 1 & 3 \end{vmatrix}, -\begin{vmatrix} -5 & -5 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} -5 & -1 \\ 2 & 1 \end{vmatrix} \right) \right\| \cdot \frac{1}{2}$$

$$= \|(2,5,-3)\| \cdot \frac{1}{2}$$

$$= \sqrt{2^2+5^2+3^2} \cdot \frac{1}{2}$$

$$= \sqrt{4+25+9} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{38}}{2}$$

Ex 2) Find the area of the parallelogram with vertices $P(5,-1,3)$, $Q(8,1,4)$, $R(13,2,9)$ and $S(16,4,10)$.

$\vec{u} = (5,-1,3)-(8,1,4)$
$= (-3,-2,-1)$

$\vec{v} = (13,2,9)-(8,1,4)$
$= (5,1,5)$

$= \|(-3,-2,-1) \times (5,1,5)\|$

$= \left\| \left(\begin{vmatrix} -2 & -1 \\ 1 & 5 \end{vmatrix}, -\begin{vmatrix} -3 & 1 \\ 5 & 5 \end{vmatrix}, \begin{vmatrix} -3 & -2 \\ 5 & 1 \end{vmatrix} \right) \right\|$

$= \|(-9, 10, 7)\|$

$= \sqrt{9^2 + 10^2 + 7^2}$

$= \sqrt{81 + 100 + 49}$

$= \sqrt{230}$

Ex 3) Given $\vec{u} = (5,1,-2)$, $\vec{v} = (-3,-1,-3)$ and $\vec{w} = (2,-3,-3)$

a. Find the volume of the parallelepiped whose adjacent edges are defined by the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$

$$\left|(5,1,-2)\cdot\left((-3,-1,-3)\times(2,-3,-3)\right)\right|$$

$$\left|(5,1,-2)\cdot\left(\begin{vmatrix}-1 & -3\\ -3 & -3\end{vmatrix}, -\begin{vmatrix}-3 & -3\\ 2 & -3\end{vmatrix}, \begin{vmatrix}-3 & -1\\ 2 & -3\end{vmatrix}\right)\right|$$

$$\left|(5,1,-2)\cdot(-6,-15,11)\right|$$

$$\left|(5\cdot -6 + -15 + -22)\right|$$

$$\left|(-30-15-22)\right|$$

$$|-67|$$

$$=67$$

b. Are the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar? Explain.

No, they are not, because $\vec{u}\cdot(\vec{v}\times\vec{w})\neq 0$

Ex 4) Given the points $A(1,0,2)$, $B(3,-2,3)$, $C(4,3,4)$ and $D(2,-5,-1)$

a. Find the volume of the parallelepiped having adjacent edges defined by the points.

$\vec{u} = (1,0,2)-(2,-5,-1)$
$= (-1,5,3)$

$\vec{v} = (3,-2,3)-(2,-5,-1)$
$= (3,-2,4)$

$\vec{w} = (4,3,4)-(2,-5,-1)$
$= (6,8,5)$

$(-1,5,3)\cdot((5,-2,4) \times (6,8,5))$

$(-1,5,3)\cdot\left(\begin{vmatrix} -2 & 4 \\ 8 & 5 \end{vmatrix}, -\begin{vmatrix} 5 & 4 \\ 6 & 5 \end{vmatrix}, \begin{vmatrix} 5 & -2 \\ 6 & 8 \end{vmatrix}\right)$

$(-1,5,3)\cdot(-42,-1,52)$

$(42-5+156)$

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b. Are the points A, B, C and D coplanar? Explain.

VECTOR PROOFS

Ex 1) Let $\vec{u}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ and k be a scalar. Prove that $k(\vec{u}+\vec{w}) = k\vec{u} + k\vec{w}$

$$k(\vec{u}+\vec{w}) = k\vec{u} + k\vec{w}$$

$$= k(u_1, u_2) + k(w_1, w_2)$$

$$= (ku_1, ku_2) + (kw_1, kw_2)$$

$$= (ku_1 + kw_1, ku_2 + kw_2)$$

$$= k(u_1 + w_1, u_2 + w_2)$$

$$= k((u_1, u_2) + (w_1, w_2))$$

$$= k(\vec{u} + \vec{w})$$

Ex 2) Let $\vec{u}$ be a vector in $\mathbb{R}^2$ and k be a scalar. Prove that $\|k\vec{u}\| = |k|\,\|\vec{u}\|$

$\|k\vec{u}\| = |k|\,\|\vec{u}\|$

$= |k|\;\|(u_1, u_2)\|$

$= \sqrt{k^2}\,\sqrt{u_1^2 + u_2^2}$

$= \sqrt{k^2 u_1^2 + k^2 u_2^2}$

$= k\sqrt{u_1^2 + u_2^2}$

$= \|k(u_1, u_2)\|$

$= \|k\vec{u}\|$

Ex 3) Let $\vec{u}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ and k be a scalar. Prove that $(k\vec{u}) \cdot \vec{w} = k(\vec{u} \cdot \vec{w})$

$$(k\vec{u}) \cdot \vec{w} = k(\vec{u} \cdot \vec{w})$$
$$= k((u_1, u_2) \cdot (w_1, w_2))$$
$$= k(u_1 w_1 + u_2 w_2)$$
$$= k(u_1 w_1) + k(u_2 w_2)$$
$$= (ku_1)w_1 + (ku_2)w_2$$
$$= (ku_1, ku_2) \cdot (w_1, w_2)$$
$$= k(u_1, u_2) \cdot (w_1, w_2)$$
$$= (k\vec{u}) \cdot \vec{w}$$

Ex 4) Let $\vec{u}$ be a vector in $\mathbb{R}^2$ and c and d be scalars. Prove that $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

$$(c+d)\vec{u} = c\vec{u} + d\vec{u}$$
$$= c(u_1, u_2) + d(u_1, u_2)$$
$$= (cu_1, cu_2) + (du_1, du_2)$$
$$= (cu_1 + du_1, cu_2 + du_2)$$
$$= ((c+d)u_1, (c+d)u_2)$$
$$= (c+d)(u_1, u_2)$$
$$= (c+d)\vec{u}$$

Ex 5) Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$. Prove that $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$$

$$= (u_1, u_2)(v_1, v_2) + (u_1, u_2)\cdot(w_1, w_2)$$

$$= (u_1 v_1 + u_2 v_2) + (u_1 w_1 + u_2 w_2)$$

$$= (u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2)$$

$$= (u_1(v_1 + w_1) + u_2(v_2 w_2))$$

$$= (u_1, u_2)\cdot((v_1 + w_1), (v_2 w_2))$$

$$= \vec{u}\cdot((v_1, v_2) + (w_1, w_2))$$

$$= \vec{u}\cdot(\vec{v} + \vec{w})$$

CRAMER'S RULE

Ex 1) Find the general solution of the following system of linear equations by using Cramer's rule.

$$2x-4y+z=1$$
$$-x+y-z=2$$
$$x-2y=1$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 1 \\ -1 & 1 & -1 & 2 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

$$0+4+2-1-4-0=1$$

$$D=\begin{vmatrix} 2 & -4 & 1 \\ -1 & 1 & -1 \\ 1 & -2 & 0 \end{vmatrix} \Rightarrow \begin{array}{ccccc} 2 & -4 & 1 & 2 & -4 \\ -1 & 1 & -1 & -1 & 1 \\ 1 & -2 & 0 & 1 & -2 \end{array}$$

$$0+4+-4-1-2-0=-3$$

$$D_x=\begin{vmatrix} 1 & -4 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 0 \end{vmatrix} \Rightarrow \begin{array}{ccccc} 1 & -4 & 1 & 1 & -4 \\ 2 & 1 & -1 & 2 & 1 \\ 1 & -2 & 0 & 1 & -2 \end{array}$$

$$0+-1+-1-2-(-2)-0=-2$$

$$D_y=\begin{vmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & 0 \end{vmatrix} \Rightarrow \begin{array}{ccccc} 2 & 1 & 1 & 2 & 1 \\ -1 & 2 & -1 & -1 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{array}$$

$$2+-8+2-1-(-8)-4=-1$$

$$D_z=\begin{vmatrix} 2 & -4 & 1 \\ -1 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix} \Rightarrow \begin{array}{ccccc} 2 & -4 & 1 & 2 & -4 \\ -1 & 1 & 2 & -1 & 1 \\ 1 & -2 & 1 & 1 & -2 \end{array}$$

$x=-3$

$y=-2$

$z=-1$

Ex 2) Find the general solution of the following system of linear equations by using Cramer's rule.

$$x - 2y + 6z = 0$$
$$-4x + 9y - 23z = -3$$
$$2x - 4y + 10z = 1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & 0 \\ -4 & 9 & -23 & -3 \\ 2 & -4 & 10 & 1 \end{array}\right]$$

$$90 + 92 + 96 - 108 - 92 - 80 = -2$$

$$D = \begin{vmatrix} 1 & -2 & 6 \\ -4 & 9 & -23 \\ 2 & -4 & 10 \end{vmatrix} \quad \begin{matrix} 1 & -2 & 6 & 1 & -2 \\ -4 & 9 & -23 & -4 & 9 \\ 2 & -4 & 10 & 2 & -4 \end{matrix}$$

$$0 + 46 + 72 - 54 - 0 - 60 = 4$$

$$D_x = \begin{vmatrix} 0 & -2 & 6 \\ -3 & 9 & -23 \\ 1 & -4 & 10 \end{vmatrix} \quad \begin{matrix} 0 & -2 & 6 & 0 & -2 \\ -3 & 9 & -23 & -3 & 9 \\ 1 & -4 & 10 & 1 & -4 \end{matrix}$$

$$-30 + 0 + (-24) - (-36) - (-23) - 0 = 5$$

$$D_y = \begin{vmatrix} 1 & 0 & 6 \\ -4 & -3 & -23 \\ 2 & 1 & 10 \end{vmatrix} \quad \begin{matrix} 1 & 0 & 6 & 1 & 0 \\ -4 & -3 & -23 & -4 & -3 \\ 2 & 1 & 10 & 2 & 1 \end{matrix}$$

$$9 + (-12) + 0 - 0 - (-12) - 8 = 1$$

$$D_z = \begin{vmatrix} 1 & -2 & 0 \\ -4 & 9 & -3 \\ 2 & -4 & 1 \end{vmatrix} \quad \begin{matrix} 1 & -2 & 0 & 1 & -2 \\ -4 & 9 & 3 & -4 & 9 \\ 2 & -4 & 1 & 2 & 4 \end{matrix}$$

$$x = -2$$

$$y = -5/2$$

$$z = -1/2$$