Notes on Category & Coalgebra

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1 Basic Category Theory

1.1 Categories

Definition 1.1 (Categories) A **category** C consists of a class $\mathsf{Ob}(\mathsf{C})$ of **objects**, and for each pair of objects A, B, a family $\mathsf{C}(A, B)$ of **arrows**. If f is a arrow in $\mathsf{C}(A, B)$, we write $f\colon A \to B$ or $A \xrightarrow{f} B$, and call A the **domain** and B the **codomain** of the arrow. The collection of arrows is endowed with some algebraic structure:

1. Identity Arrows: for every object A of C there is an arrow $id_A\colon A \to A$ (or $1_A\colon A \to A$).
2. Composition: every pair of arrows $f\colon A \to B$, $g\colon B \to C$ can be uniquely composed to an arrow $g \circ f\colon A \to C$.
3. Associativity: The operations are supposed to satisfy the *associative law* for composition, while the appropriate identity arrows are left and right neutral elements.

An arrow $f\colon A \to B$ is an **iso** if it has an **inverse**, that is, an arrow $g\colon B \to A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$. $\dashv$

Example 1.2 (Some categories) 1. For every algebraic similarity Σ, the class $\mathsf{Alg}(\Sigma)$ of Σ-algebras with homomorphisms as arrows.

$\dashv$

Example 1.3 The **formulas** and **proofs** in logic form a category. Formulas are objects, and for any formulas φ, ψ, the proofs for implication $\varphi \to \psi$, we write $\vdash \varphi \to \psi$, as arrows.

For formula φ, id_φ is $\vdash \varphi \to \varphi$. The composition of proof is just HS (hypothetical syllogism) rule: $\vdash \varphi \to \psi \quad + \quad \vdash \psi \to \chi \quad \Rightarrow \quad \vdash \varphi \to \chi$. $\dashv$

Example 1.4 (The category of sets, Set) （集合范畴）

Set is the category with (i) objects: all sets, and (ii) arrows: for any sets X, Y, every (total) function $f\colon X \to Y$ is an arrow.

Remarks on the category Set:

(1) The identity function on any set is the identity arrow.

(2) The set–function $f\colon A \to B$ and $g\colon B \to C$ always compose.

(3) The arrows in Set, like any category arrows, must come with determinate targets and codomains. Hence, we can't simply identify an arrow in Set with a function's graph (i.e. the set $\hat{f} = \{\langle x, y\rangle \mid f(x) = y\}$).

We can define a set–function $f\colon A \to B$ as a triple $(A, \hat{f}, B)$.

(4) Empty set:

- There is an identity arrow for $\emptyset$ in Set. Vacuously, for any set Y there is exactly one function $f\colon \emptyset \to Y$, that is, the one whose graph is the empty set. Hence in particular there is a function $1_\emptyset\colon \emptyset \to \emptyset$.
- In Set, the empty set is the one and only set s.t. there is exactly one arrow *from* it to any other set.

  (This tells us how we can characterize a significant object in a category by what arrows it has to and from other objects. For example, we can define *singletons* in Set by relying on the observation that there is exactly one arrow from any set *to* a singleton.)

$\dashv$

The categories whose objects are sets, perhaps equipped with some structure (e.g. groups, monoids, etc.), and whose arrows are structure–respecting set–functions, are often called *concrete categories*. （具体范畴）

Example 1.5 Some examples of concrete categories:

- FinSet: the category whose objects are finite sets and whose arrows are the set–functions between such objects.

- Rel: the category of relations.

  objects: all sets

  arrows: any relation R between A and B. We can let a relation R be a triple $R = (A, \hat{R}, B)$ where $\hat{R} \subseteq A \times B$ is $R's$ extension. We allow the case where $\hat{R}$ is empty and the arrow of the form $(A, \emptyset, B)$.

  The identity arrow on A is the diagonal relation whose extension is $\{(a, a) \mid a \in A\}$.

$\dashv$

1.2 Commutative diagrams

[Arbib and Manes, 1975, p. 2]: "*commutare* is the Latin for *exchange*, and we say that a diagram commutes if we can exchange paths, between two given points, with impunity."

a diagram commutes if each minimal polygon in the diagram commutes.

（「交换」意指箭头的合成殊途同归）

1.3 So many arrows: monomorphisms, epimorphisms and isomorphisms

Definition 1.6 (Opposite category) The **opposite category** C^{op} of a given category C has the same objects as C, while $\mathsf{C}^{op}(A, B) = \mathsf{C}(B, A)$ for all objects A, B from C, and the operations on arrows are defined in the obvious way. $\dashv$

Definition 1.7 单态射、满态射、 $\dashv$

Definition 1.8 (Initial and final objects) An object X is **initial** in a category C if for every object A in C there is a unique arrow $\alpha\colon X \to A$, and **final** if for all A there is a unique $\alpha\colon A \to X$. $\dashv$

***Note* 1.9** The initial and finial objects in a given category may not unique. $\dashv$

In Set, the empty set $\emptyset$ is initial, ans the final objects are precisely the singletons.

Definition 1.10 (Products & Coproducts) A **product** of two objects A_0 and A_1 in a category $\mathcal{C}$ consists of a triple $(A, A \xrightarrow{\alpha_0} A_0, A \xrightarrow{\alpha_1} A_1)$, such that for every triple $(A', A' \xrightarrow{\alpha'_0} A_0, A' \xrightarrow{\alpha'_1} A_1)$ there is a unique arrow $f\colon A' \to A$ such that $\alpha_i \circ f = \alpha'_i$ for both i.

$$\begin{tikzcd}
 & A \arrow[dl, "\alpha_0"'] \arrow[dr, "\alpha_1"] & \\
A_0 & & A_1 \\
 & A' \arrow[ul, "\alpha'_0"] \arrow[ur, "\alpha'_1"'] \arrow[uu, "f"] &
\end{tikzcd}$$

Figure 1: product of objects

Coproducts of A_0 and A_1 are defined dually as triples $(A, A_0 \xrightarrow{\alpha_0} A, A_1 \xrightarrow{\alpha_1} A)$, such that for every triple $(A', A_0 \xrightarrow{\alpha'_0} A', A_1 \xrightarrow{\alpha'_1} A')$ there is a unique arrow $f\colon A \to A'$

such that $f \circ \alpha_i = \alpha'_i$ for each i.

$$\begin{tikzcd}
& A \arrow[dd, "f"] & \\
A_0 \arrow[ur, "\alpha_0"] \arrow[dr, "\alpha'_0"'] & & A_1 \arrow[ul, "\alpha_1"'] \arrow[dl, "\alpha'_1"] \\
& A' &
\end{tikzcd}$$

Figure 2: Coproducts

⊣

Example 1.11 The category Set has both products and coproducts — that is, every pair (S_0, S_1) of sets has both a product — the cartesian product $S_0 \times S_1$ together with the two projection functions $\pi_i \colon S_0 \times S_1 \to S_i$.

$$\begin{tikzcd}
& S_0 \times S_1 \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
S_0 & & S_1
\end{tikzcd}$$

Figure 3: product of sets

And a coproduct — the disjoint union $S_0 \uplus S_1 = S_0 \times \{0\} \cup S_1 \times \{1\}$ together with the coproduct maps κ_0 and κ_1 given by $\kappa_i(s) = (s, i)$.

$$\begin{tikzcd}
& S_0 \uplus S_1 & \\
S_0 \arrow[ur, "\kappa_0"] & & S_1 \arrow[ul, "\kappa_1"']
\end{tikzcd}$$

Figure 4: coproduct of sets

⊣

1.4 Functors

Definition 1.12 (Functors) A **functor** $\Omega \colon \mathsf{C} \to \mathsf{D}$ from a category C to a category D consists of an operation mapping objects and arrows of C to objects and arrows of D, respectively. in such a way that for all objects and arrows involved

(1) $\Omega f \colon \Omega A \to \Omega B$ if $f \colon A \to B$.

(2) $\Omega(id_A) = id_{\Omega A}$.

(3) $\Omega(g \circ f) = (\Omega g) \circ (\Omega f)$

A functor $\Omega\colon \mathsf{C} \to \mathsf{D}^{op}$ is sometimes called a **contravariant functor** from C to D.

An **endofunctor** on C is a functor $\Omega\colon \mathsf{C} \to \mathsf{C}$. ⊣

Example 1.13 (Set functors) We consider the following endofunctor on Set.

(1) For a fixed set C, the **constant functor** mapping all sets to C and all arrows to id_C; this functor is denoted as C.

(2) The **power set functor** $\mathcal{P}$, which maps amy set S to its powerset $\mathcal{P}S$, and any map $f\colon S \to S'$ to the map $\mathcal{P}f\colon \mathcal{P}S \to \mathcal{P}S'$ given by $\mathcal{P}f\colon X \mapsto f[X]$, where $f[X] = \{fx \in S' \mid x \in X \subseteq S\}$.

(3) For every cardinal κ, the variant $\mathcal{P}_\kappa$ of the powerset functor, which maps any set S to the collection $\mathcal{P}_\kappa := \{X \subseteq S \mid |X| < \kappa\} \subseteq \mathcal{P}S$, and agrees with $\mathcal{P}$ on the arrows for which is defined.

⊣

Definition 1.14 (Product of functors) Given two functors Ω_0 and Ω_1, their **product functor** $\Omega_0 \times \Omega_1$ is given on objects by

$$(\Omega_0 \times \Omega_1)S := \Omega_0 S \times \Omega_1 S,$$

while for $f\colon S \to S'$ the map $(\Omega_0 \times \Omega_1)f$ is given as

$$((\Omega_0 \times \Omega_1)f)(\sigma_0, \sigma_1) := ((\Omega_0 f)(\sigma_0), (\Omega_1 f)(\sigma_1)).$$

The **coproduct functor** is defined similarly. ⊣

Definition 1.15 (Identity functor) Every category C admits the **identity functor** $\mathcal{I}_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{C}$ which is the identity on both objects and arrows of C. ⊣

Definition 1.16 (Natural transformation) Let C and D be two categories, and let Ω and Ψ be two functors from C to D. A **natural transformation** τ from Ω to Ψ, notation $\tau\colon \Omega \Rightarrow \Psi$, consists of D-arrows $\tau_A\colon \Omega A \to \Psi A$ such that

$$\tau_B \circ \Omega f = \Psi f \circ \tau_A$$

for each $f\colon A \to B$ in C.

$$\begin{tikzcd}
 & A \arrow[d, "f"'] & \arrow[r, dashed, bend left, "\Omega"] & {} & \Omega A \arrow[r, "\Omega f"] \arrow[d, "\tau_A"'] \arrow[dr] & \Omega B \arrow[d, "\tau_B"] & \\
\mathsf{C} & B & \arrow[r, dashed, bend right, "\Psi"'] & {} & \Psi A \arrow[r, "\Psi f"'] & \Psi B & \mathsf{D}
\end{tikzcd}$$

Figure 5: $\tau\colon \Omega \Rightarrow \Psi$

⊣

???? equivalent , isomorphic, dual/ dually equivalent

2 Coalgebra

Coalgebra can be conceived as a general and uniform theory of *dynamic systems*, taken in a broad sense. Many structures in mathematics and theoretical computer science can naturally be represented as coalgebras.

Probably the first example was provided by Aczel [2], who models *transition systems* and *non–well–founded sets* as coalgebras.

For modal logicians, it will be Kripke frames and models that provide the prime examples of coalgebras, this link goes back to at least Abramsky [1].

In fact, the modal model theory is coalgebraic in nature, so modal logicians entering the field will have much the same experience as group theorists learning about universal algebra, in that they will recognize many familiar notions and results, lifted to a higher level of generality and abstraction.

- algebraic operations are ways to construct complex objects out of simple ones, coalgebraic operations, going out of the carrier set, should be seen as ways to unfold or observe objects.

- coalgebras over the base category C are dual to algebras over the opposite category C^{op}. This explains not only the name "coalgebra".

- the link between modal logics and coalgebra is so tight, that one may even claim that modal logic is the natural logic for coalgebras — just like equational logic is that for algebra.

Definition 2.1 (Coalgebras) （余代数）Given an endofunctor (see Def. **??**) Ω on a category C, an Ω–**coalgebra** is a pair $\mathbb{A} = (A, \alpha)$, also denoted as $\mathbb{A} = (A, \alpha\colon A \to \Omega A)$ or $\mathbb{A} = (A, A \xrightarrow{\alpha} \Omega A)$, where A is an object of C called the **carrier** of $\mathbb{A}$, and $\alpha\colon A \to \Omega A$ is an arrow in C, called the **transition map** of $\mathbb{A}$.

In that case that Ω is an endofunctor on Set, Ω–coalgebras may also be called Ω–**systems**. A **pointed Ω-system** is a triple (A, α, a) where (A, α) is an Ω-system and $a \in A$. $\dashv$

An Ω–coalgebra $\mathbb{A} = (A, \sigma)$ can be pictured by

$$A \xrightarrow{\alpha} \Omega A$$

Figure 6: Ω–Coalgebra $\mathbb{A} = (A, \sigma)$

Example 2.2 (Automata) Deterministic automata are usually modeled as quintuples

$$\mathbb{A} = (Q, a_0, \Sigma, \delta, F)$$

such that Q is the state space of the automaton, $a_0 \in Q$ is its *initial state*, Σ its *alphabet*, $\delta\colon Q \times \Sigma \to Q$ its *transition function* and finally, $F \subseteq Q$ its *accepting states*.

We can represent F by by its characteristic map $C_F\colon Q \to 2$ (where 2 denoting the set $\{0,1\}$) such that $C_F(a) = 1$ if $a \in F$ and $C_F(a) = 0$ otherwise. Furthermore, we can view δ as a map from $Q \to Q^\Sigma$:

$$\delta\colon Q \times \Sigma \to Q \qquad \rightsquigarrow \qquad \delta\colon Q \to (\Sigma \to Q) \qquad \rightsquigarrow \qquad \delta\colon Q \to Q^\Sigma$$

where Q^Σ denotes the collection of maps from Σ to Q.

Thus, we may represent a deterministic automaton over the alphabet Σ as a *pointed system* over the functor $\Omega\colon S \mapsto 2 \times S^\Sigma$ for any set S. $\dashv$

Example 2.3 (Kripke Frams & Models) We now see that Kripke frames and models are in fact coalgebras in disguise.

(1) **Frame**: Considering the frame $\mathfrak{F} = (W,R)$ (for the basic modal similarity type). The crucial observation is that the binary relation R on W can be represented as the function
$$R[\cdot]\colon W \to \mathcal{P}W$$
mapping a point w to its R–successors $R[w] = \{u \in W \mid Rwu\}$. Thus frames $\mathfrak{F} = (W,R)$ correspond to coalgebras over the powerset functor $\mathcal{P}$. Therefore, a frame $\mathfrak{F} = (W,R)$ is a $\mathcal{P}$–coalgebra or $\mathcal{P}$–system $(W, R\colon W \to \mathcal{P}W)$. **Pointed frame** $(\mathfrak{F}, w)$ is just the pointed $\mathcal{P}$–system $(W, W \xrightarrow{R} \mathcal{P}W, w)$.

(Note that: the powerset functor $\mathcal{P}$ maps any set S to its powerset $\mathcal{P}(S)$ and a function $f\colon S \to S'$ to the image map $\mathcal{P}f$ given by $(\mathcal{P}f)(X) \coloneqq f[X] = \{f(x) \mid x \in X\}$.)

(2) **Image finite frames**, that is, frames in which $R[w]$ is finite for all points w, correspond to coalgebras over the finitary powerset functor $\mathcal{P}_\omega$.

(3) **Ternary frames**: W with a ternary relation $T \subseteq W^3$ forms a ternary frame (W,T) (a frame for temporal logic with since S or until U). Similarly, T can be represented as
$$T[\cdot]\colon W \to \mathcal{P}(W^2)$$
s.t. $T[w] = \{(w_1,w_2) \in W^2 \mid (w,w_1,w_2) \in T\}$. Thus, a ternary frame (W,T) is a coalgebra $(W, T\colon W \to \mathcal{P}(W^2))$ under the functor Ω which $\Omega\colon S \mapsto \mathcal{P}(S^2)$ for any set S, and for function $f\colon S \to S'$, Ωf is given by $(\Omega f)(R) \coloneqq \{(f(x_1), f(x_2) \mid (x_1,x_2) \in R)\}$ where R is a (binary) relation in $\mathcal{P}(S^2)$.

(4) **Models**: Now concerning models with the form $\mathfrak{M} = (W,R,V)$. It is easy to see that a valuation $V\colon \mathsf{PROP} \to \mathcal{P}(W)$ could equivalently have been defined as a $\mathcal{P}(\mathsf{PROP})$–coloring of W, that is, mapping a state w to the collection
$$V^{-1}[s] = \{p \in \mathsf{PROP} \mid s \in V(p)\}$$
of proposition letters holding at w. Thus models can be identified with coalgebras of the functor Ω given by
$$\Omega\colon W \mapsto \mathcal{P}(\mathsf{PROP}) \times \mathcal{P}(W)$$

for any set W, and for function $f\colon S \to S'$, Ωf is given by $(\Omega f)(X', X'') \coloneqq (f[X], f[X''])$ where $X \subseteq S$ and $X'' \subseteq S''$.

(5) Recap:

For frame $\mathfrak{F} = (W, R)$ and model $\mathfrak{M} = (W, R, V)$, the corresponding coalgebras are

$$\begin{array}{lll} \mathfrak{F} = (W, R) : & (W, W \xrightarrow{R} \mathcal{P}W) & \text{where } R(w) = \{u \in W \mid Rwu\} \\ \mathfrak{M} = (W, R, V) : & (W, W \xrightarrow{\alpha} \mathcal{P}(\mathsf{PROP}) \times \mathcal{P}W) & \text{where } \alpha(w) = \langle \{p \mid w \Vdash p\}, R(w) \rangle \end{array}$$

$\dashv$

Example 2.4 (Neighborhood frames) **contravariant powerset functor** $\breve{\mathcal{P}}$ $\dashv$

References

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