2023·秋·数理逻辑　　平时作业汇总　　大家辛苦啦！~~

（英文参考答案 + 作业反馈）

hw-1 (2023/09/12)

p3: 1-(h)　If y is an integer then z is not real, provided that x is a rational number.

Answer:

Let

$$\begin{aligned} p: &\quad y \text{ is an integer} \\ q: &\quad z \text{ is a real number} \\ r: &\quad x \text{ is a rational number} \end{aligned}$$

Therefore we have that $r \to (p \to \neg q)$ or $(r \wedge p) \to \neg q$. □

.... hw-1: feedback ....

1. 没有把“z is not real”中的**否定联结词**提取出，进而翻译为公式时缺少否定符号 $\neg$。
2. 对英语的语序产生了错误的判断，将“provided that x is a rational number”这个句子的成分放置到了错误的地方。
3. 一些同学额外做了教材 *p.3* 第一题中的 $(a)-(h)$，但没有注意到 $(c),(e),(g)$ 中的“either ... or ...”表达的是 **不兼容析取**，从而对句子产生了不当的翻译。

hw-2 (2023/09/19)

p10: (7)　Show that the statement form $(((\sim p) \to q) \to (p \to (\sim q)))$ is not a tautology. Find statement forms $\mathscr{A}$ and $\mathscr{B}$ such that $(((\sim \mathscr{A}) \to \mathscr{B}) \to (\mathscr{A} \to (\sim \mathscr{B})))$ is a contradiction.

Answer:

The following truth table shows that $(((\sim p) \to q) \to (p \to (\sim q)))$ is not a tautology.

p	q	($\neg$	p	$\to$	q)	$\to$	(p	$\to$	$\neg$	q)
T	T	F	T	T	T	F	T	F	F	T
T	F	F	T	T	F	T	T	T	T	F
F	T	T	F	T	T	T	F	T	F	T
F	F	T	F	F	F	T	F	T	T	F

When $\mathscr{A}$ and $\mathscr{B}$ are both tautologies, then $(((\sim \mathscr{A}) \to \mathscr{B}) \to (\mathscr{A} \to (\sim \mathscr{B})))$ will be a contradiction. For instance, let $\mathscr{A} = \mathscr{B} = (p \to p)$ or $\mathscr{A} = \mathscr{B} = (p \vee \neg p)$. □

除了用真值表这种比较直观的手段外，还有诸多方法。以下答案来自黄程同学，经其授权后分享给大家，感谢黄程同学:

Suppose that $(((\sim p) \to q) \to (p \to (\sim q)))$ is a tautology. Then the situation that $(\sim p) \to q$ be T and $p \to (\sim q)$ be F will not occur under any valuation. But considering $q = T$ and $p = T$, thus $p \to (\sim q)$ will be T. Contradiction! Therefore $(((\sim p) \to q) \to (p \to (\sim q)))$ is not a tautology.

According above answer, when $\mathscr{A}$ and $\mathscr{B}$ be T permanently, then $(((\sim\mathscr{A}) \to \mathscr{B}) \to (\mathscr{A} \to (\sim\mathscr{B})))$ will be a contradiction. In other words, $\mathscr{A}$ and $\mathscr{B}$ are both tautologies, say, $\mathscr{A} = (p \vee (\sim p))$ and $\mathscr{B} = p \to (q \to p)$. □

.... hw-2: feedback ....

1. 本次作业一共有两问，但存在同学只回答第一问的情况，请大家以后细心。
2. 用 0 和 1 来替代 F 和 T 是可以的，有时这样会更为简洁。
3. 第一问有同学用一种「简化真值表」来回答，如

(	$\neg$	p	$\to$	q	)	$\to$	(	p	$\to$	$\neg$	q	)
	F	T	T	T		F		T	F	F	T	
	F	T	T	F		T		T	T	T	F	
	T	F	T	T		T		F	T	F	T	
	T	F	F	F		T		F	T	T	F	

   这是可行且正确的。不过建议还是把 p 和 q 的真值单独列在表前，这样在画真值表找「析取范式」的时候不容易眼花，不过这不是强制性的。

4. 第二问要求大家确实为 $\mathscr{A}$ 和 $\mathscr{B}$ 找到某种「命题形式」，很多同学只是声明其为重言式而没有找出具体的「命题形式」，严格来说这是不够的，不过默认大家都掌握了。

hw-3 (2023/09/26)

p15: 11-(a) Show, using **Proposition 1.14** and **1.17**, that the statement form $((\neg(p \vee (\neg q))) \to (q \to r))$ is logically equivalent to each of the following.

(a) $((\neg(q \to p)) \to ((\neg q) \vee r))$

Recall that

- **Proposition 1.14**: If $\mathscr{B}_1$ is a statement form arising from the statement form $\mathscr{A}$ by substituting the statement form $\mathscr{B}$ for one or more occurrences of the statement form $\mathscr{A}$ in $\mathscr{A}_1$, and if $\mathscr{B}$ is logically equivalent to $\mathscr{A}$, then $\mathscr{B}_1$ is logically equivalent to $\mathscr{A}_1$.
- **Proposition 1.17 (De Morgan's Laws)**: Let $\mathscr{A}_1, \mathscr{A}_2, \cdots \mathscr{A}_n$ be any statement forms. Then:

  1. $(\bigvee_{i=1}^{n}(\neg\mathscr{A}_i))$ is logically equivalent to $(\neg(\bigwedge_{i=1}^{n}\mathscr{A}_i))$.
  2. $(\bigwedge_{i=1}^{n}(\neg\mathscr{A}_i))$ is logically equivalent to $(\neg(\bigvee_{i=1}^{n}\mathscr{A}_i))$.

Answer: Let $\varphi = ((\neg(p \vee (\neg q))) \to (q \to r))$ and $\chi = ((\neg(q \to p)) \to ((\neg q) \vee r))$.

It suffices to show that if $\neg(p \vee (\neg q))$ is logically equivalent to $(\neg(q \to p))$, and $(q \to r)$ is logically equivalent to $(\neg q) \vee r)$, then φ is logically equivalent to χ according to **Prop. 1.14**.

But it is easy to check, say, using truth table, that

$$\neg(p \vee (\neg q)) \leftrightarrow (\neg(q \to p)) \qquad \text{and}$$
$$(q \to r) \leftrightarrow (\neg q) \vee r)$$

are tautologies, which means that $(\neg(p \vee (\neg q)))$ and $(\neg(q \to p))$, $(q \to r)$ and $(\neg q) \vee r)$ are logically equivalent, respectively. $\square$

. . . . no feedback for hw-3 . . . .

`hw-4 (2023/10/10)`

p.19: 13-(a) Find statement forms in **conjunctive normal form** which are logically equivalent to the following:

$$(a) \qquad (((\neg p) \vee q) \to r)$$

Answer: Here we will use *three* methods to find some **conjunctive normal forms** (CNF) of $(\neg p \vee q) \to r$, the former two are from our Textbook, while the third one is new.

In the first place, let

$$\varphi = (\neg p \vee q) \to r.$$

Method-(1)

First we construct a truth table of φ's negation:

p	q	r	$\neg$	((	$\neg$	p	$\vee$	q	)	$\to$	r	)
1	1	1	0		0		1			1		
$\underline{1}$	$\underline{1}$	$\underline{0}$	1		0		1			0		
1	0	1	0		0		0			1		
1	0	0	0		0		0			1		
0	1	1	0		1		1			1		
$\underline{0}$	$\underline{1}$	$\underline{0}$	1		1		1			0		
0	0	1	0		1		1			1		
$\underline{0}$	$\underline{0}$	$\underline{0}$	1		1		1			0		

The combinations which give $\neg\varphi$ value 1 are 110, 010 and 000. Thus a **disjunctive normal form** of $\neg\varphi$ is

$$\chi = (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

It is clear that χ is logically equivalent to $\neg\varphi$, hence $\neg\chi$ is logically equivalent to $\neg\neg\varphi$, i.e., φ.

Then, by the **De Morgan's laws**, we have

$$\begin{aligned}
\neg\chi &= \neg[(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)] \\
&\equiv \neg(p \wedge q \wedge \neg r) \wedge \neg(\neg p \wedge q \wedge \neg r) \wedge \neg(\neg p \wedge \neg q \wedge \neg r) \\
&\equiv (\neg p \vee \neg q \vee \neg\neg r) \wedge (\neg\neg p \vee \neg q \vee \neg\neg r) \wedge (\neg\neg p \vee \neg\neg q \vee \neg\neg r) \\
&\equiv (\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)
\end{aligned}$$

Therefore, $(\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$ is a CNF of φ. $\square$

(**NB**: here we using the symbol expression "$\alpha \equiv \beta$" to denote that formula α is logically equivalent to β)

Method-(2)

$$\begin{aligned}
\varphi &= (\neg p \vee q \to r) \\
&\equiv \neg(\neg p \vee q) \vee r && \text{(by } \textbf{material implication}\text{, cf. p.7: Example 1.4-(a))} \\
&\equiv (\neg\neg p \wedge \neg q) \vee r && \text{(by the } \textbf{De Morgan's laws}\text{)} \\
&\equiv (p \wedge \neg q) \vee r \\
&\equiv (p \vee r) \wedge (\neg q \vee r) && \text{(by the } \textbf{distribution} \text{ of } (\vee\text{-}\wedge)\text{, cf. p.10, Exercises-6-(b))}
\end{aligned}$$

Hence $(p \vee r) \wedge (\neg q \vee r)$ is a CNF of φ. $\square$

Method-(3)

Similarly, we construct a truth table for φ (notice that, not for the *negation* of φ):

p	q	r	(	$\neg$	p	$\vee$	q	)	$\to$	r
1	1	1		0	1	1	1		1	1
$\underline{1}$	$\underline{1}$	$\underline{0}$		0	1	1	1		0	0
1	0	1		0	1	0	0		1	1
1	0	0		0	1	0	0		1	0
0	1	1		1	0	1	1		1	1
$\underline{0}$	$\underline{1}$	$\underline{0}$		1	0	1	1		0	0
0	0	1		1	0	1	0		1	1
$\underline{0}$	$\underline{0}$	$\underline{0}$		1	0	1	0		0	0

The combinations which give φ value 0 are 110, 010 and 000. Then according to these truth combinations, we can construct three **disjunctive formulas** as follows,

$$\begin{aligned}
\varphi_1 &= (\neg p \vee \neg q \vee r) \\
\varphi_2 &= (p \vee \neg q \vee r) \\
\varphi_3 &= (p \vee q \vee r)
\end{aligned}$$

Next, we connect above three formulas in a conjunctive form, that is,

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 = (\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$$

It is easy to check that $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is a CNF of φ. And as we can see, the result in current *Method-(3)* is same as the *Method-(1)*. $\square$

p.26: 21 Suppose that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n; \therefore \mathscr{A}$ is a valid argument form. Prove that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n-1}; \therefore (\mathscr{A}_n \to \mathscr{A})$ is also a valid argument form.

Proof:

First, suppose that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n; \therefore \mathscr{A}$ is a valid argument form, but $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n-1}; \therefore (\mathscr{A}_n \to \mathscr{A})$ is not.

Then there is an assignment of truth values to the statement variables such that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n-1}$ takes value T while $(\mathscr{A}_n \to \mathscr{A})$ takes value F, that is, $\mathscr{A}_n$ is T but $\mathscr{A}$ takes F. However, this contradicts to our assumption that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n; \therefore \mathscr{A}$ is a valid argument form. $\square$

......................................hw-4: feedback ....

1. 还是有同学少写题目呀，题目少写的话想给你们找分都很难了。考试的时候也差不多，尽量不要空题不做呀 ☹

2. 还有很多同学写证明的时候，一句话中往往不写「定语」和「状语」，比如会出现如下情况：

   所以 φ ....

   所以 φ 什么呢？ φ 是重言式？ φ 是矛盾式？这些都是需要额外加以说明的。

`hw-5 (2023/10/17)`

p.36: 1-(c) Write out proofs in L for the following wfs.

$$(c) \qquad (p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)$$

Proof:

Method-(1)

1. $(p_1 \to (p_1 \to p_2)) \to ((p_1 \to p_1) \to (p_1 \to p_2))$ (instance of $L2$)
2. $[(p_1 \to (p_1 \to p_2)) \to ((p_1 \to p_1) \to (p_1 \to p_2))] \to$
   $[((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2))]$ (instance of $L2$)
3. $((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2))$ $(1+2, MP)$
4. $p_1 \to ((p_1 \to p_2) \to p_1)$ (instance of $L1$)
5. $[p_1 \to ((p_1 \to p_2) \to p_1)] \to [(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)]$ (instance of $L2$)
6. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)$ $(4+5, MP)$
7. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)$ $(3+6, MP)$

The proof for (c) is not unique, of course. $\square$

Method-(2)

1. $p_1 \to ((p_1 \to p_1) \to p_1)$ (instance of $L1$)
2. $(p_1 \to ((p_1 \to p_1) \to p_1)) \to ((p_1 \to (p_1 \to p_1)) \to (p_1 \to p_1))$ (instance of $L2$)
3. $(p_1 \to (p_1 \to p_1)) \to (p_1 \to p_1)$ $(1+2, MP)$
4. $p_1 \to (p_1 \to p_1)$ (instance of $L1$)

5. $(p_1 \to p_1)$ $(3+4, MP)$

6. $(p_1 \to p_1) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1))$ (instance of $L1$)

7. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)$ $(5+6, MP)$

8. $(p_1 \to (p_1 \to p_2)) \to ((p_1 \to p_1) \to (p_1 \to p_2))$ (instance of $L2$)

9. $[(p_1 \to (p_1 \to p_2)) \to ((p_1 \to p_1) \to (p_1 \to p_2))] \to$
$[((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2))]$ (instance of $L2$)

10. $((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2))$ $(8+9, MP)$

11. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)$ $(7+10, MP)$

Method-(3)

1. $\{(p_1 \to p_2) \to [((p_1 \to p_2) \to (p_1 \to p_2)) \to (p_1 \to p_2)]\} \to$
$\{[(p_1 \to p_2) \to ((p_1 \to p_2) \to (p_1 \to p_2))] \to [(p_1 \to p_2) \to (p_1 \to p_2)]\}$ (instance of $L2$)

2. $(p_1 \to p_2) \to [((p_1 \to p_2) \to (p_1 \to p_2)) \to (p_1 \to p_2)]$ (instance of $L1$)

3. $[(p_1 \to p_2) \to ((p_1 \to p_2) \to (p_1 \to p_2))] \to [(p_1 \to p_2) \to (p_1 \to p_2)]$ $(1+2, MP)$

4. $(p_1 \to p_2) \to ((p_1 \to p_2) \to (p_1 \to p_2))$ (instance of $L1$)

5. $(p_1 \to p_2) \to (p_1 \to p_2)$ $(3+4, MP)$

6. $[(p_1 \to p_2) \to (p_1 \to p_2)] \to [((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2)]$ (instance of $L2$)

7. $((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2)$ $(5+6, MP)$

8. $[((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2)] \to$
$[p_1 \to (((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2))]$ (instance of $L1$)

9. $p_1 \to (((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2))$ $(7+8, MP)$

10. $[p_1 \to (((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2))] \to$
$[(p_1 \to ((p_1 \to p_2) \to p_1)) \to (p_1 \to ((p_1 \to p_2) \to p_2))]$ (instance of $L2$)

11. $(p_1 \to ((p_1 \to p_2) \to p_1)) \to (p_1 \to ((p_1 \to p_2) \to p_2))$ $(9+10, MP)$

12. $p_1 \to ((p_1 \to p_2) \to p_1)$ (instance of $L1$)

13. $p_1 \to ((p_1 \to p_2) \to p_2)$ $(11+12, MP)$

14. $[p_1 \to ((p_1 \to p_2) \to p_2)] \to [(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)]$ (instance of $L2$)

15. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)$ $(13+14, MP)$

(ps. 上面公式中的 中括号 [ ] 和 花括号 { } 是起辅助作用的，为的是方便大家观看。但应注意的是，其本身不是命题逻辑公理系统 L 中的符号!!!)

p.37: 5 The rule HS is an example of a legitimate additional rule of deduction for L. Is the following rule legitimate in the same sense: from the wfs. $\mathscr{B}$ and $(\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))$, deduce $(\mathscr{A} \to \mathscr{C})$?

Answer:

Method-(1) (without using the **Deduction Theorem**)

1. $\mathscr{B}$ (assumption)
2. $(\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))$ (assumption)
3. $(\mathscr{A} \to (\mathscr{B} \to \mathscr{C})) \to ((\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \mathscr{C}))$ $(L2)$
4. $((\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \mathscr{C}))$ $(2+3, MP)$
5. $(\mathscr{B} \to (\mathscr{A} \to \mathscr{B}))$ $(L1)$
6. $(\mathscr{A} \to \mathscr{B})$ $(1+5, MP)$
7. $(\mathscr{A} \to \mathscr{C})$ $(6+4, MP)$

Hence this rule is a legitimate additional rule of deduction for L. □

Method-(2) (using the **Deduction Theorem**)

We first show that

$$\{\mathscr{B}, (\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))\} \cup \{\mathscr{A}\} \vdash_L \mathscr{C}.$$

We write out a deduction for above one as follows:

1. $\mathscr{B}$ (assumption)
2. $(\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))$ (assumption)
3. $\mathscr{A}$ (assumption)
4. $(\mathscr{B} \to \mathscr{C})$ $(2+3, MP)$
5. $\mathscr{C}$ $(1+4, MP)$

Hence by the **Deduction Theorem**, we have

$$\{\mathscr{B}, (\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))\} \vdash_L \mathscr{A} \to \mathscr{C}.$$

as required. □

.... hw-5: feedback ....

- 很多同学都误解了什么是一个「L 中的证明」，在其中，是不能出现"假设"、"因为-所以"这样的字眼的。因此 *p.36* 1-(c) 的证明也不能用「演绎定理」，这个是内定理证明，证明的序列中出现的只能是公理或者由前面的公式使用 MP 得到。还请大家特别要注意这点！

hw-6 (2023/10/31) 期中作业

p.44: (8) Let $\mathscr{A}$ be a *wf.* $((\neg p_1 \to p_2) \to (p_1 \to \neg p_2))$. Show that L^+, obtained by including this $\mathscr{A}$ as a new axiom, has a larger set of theorems than L. Is L^+ a consistent extension of L? (注意：此题有两问)

Proof:

p_1	p_2	(	$\neg$	p_1	$\to$	p_2	) $\to$ (	p_1	$\to$	$\neg$	p_2)
T	T		F	T	T	T	F	T	F	F	T
T	F		F	T	T	F	T	T	T	T	F
F	T		T	F	T	T	T	F	T	F	T
F	F		T	F	F	F	T	F	T	T	F

For the first question: Obviously $\mathscr{A} = ((\neg p_1 \to p_2) \to (p_1 \to \neg p_2))$ is not a tautology by above truth table, then by the **Soundness Theorem**, $\mathscr{A}$ is not a theorem of L, while it is a theorem of L^+, therefore L^+ has a larger set of theorems than L.

For the second question: L^+ is consistent. For suppose otherwise, then there is a formula $\mathscr{B}$ such that $\vdash_{L^+} \mathscr{B}$ and $\vdash_{L^+} \neg\mathscr{B}$. Since L^+ is obtained by including $\mathscr{A} = ((\neg p_1 \to p_2) \to (p_1 \to \neg p_2))$ as an extra axiom then L, hence we have that (note that the subscript of $\vdash$)

$$\mathscr{A} \vdash_L \mathscr{B} \quad \text{and} \quad \mathscr{A} \vdash_L \neg\mathscr{B}.$$

By the **Deduction Theorem**,

$$\vdash_L \mathscr{A} \to \mathscr{B} \quad \text{and} \quad \vdash_L \mathscr{A} \to \neg\mathscr{B},$$

which means that $(\mathscr{A} \to \mathscr{B})$ and $(\mathscr{A} \to \neg\mathscr{B})$ are tautologies according to the **Soundness Theorem**. Then by the definition, for any valuation v we have that $v(\mathscr{A} \to \mathscr{B}) = T$ and $v(\mathscr{A} \to \neg\mathscr{B}) = T$, which implies that $v(\mathscr{A}) = F$, that is, $\mathscr{A}$ is a *contradiction*. But this is impossible by the truth table of $\mathscr{A}$. Contradiction! $\square$

p.44: (10) Let L^{++} be the extension of L obtained by including as a fourth axiom *scheme*:

$$((\neg\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \neg\mathscr{B})).$$

Show that L^{++} is inconsistent. (Hint: see Chapter 1 exercise 7 (p.10))

Proof:

Method-(1)

Let $\top = (p \to p)$ and $\varphi = (\neg\top \to \top) \to (\top \to \neg\top)$, clearly $\vdash_{L^{++}} \varphi$ (i.e., let $\mathscr{A} = \mathscr{B} = \top$). It is easy to check , say using truth table, that φ is a contradiction, hence $\neg\varphi$ is a tautology. By the **Completeness Theorem**, $\vdash_L \neg\varphi$, and thus $\vdash_{L^{++}} \neg\varphi$ since L^{++} is a extension of L.

But we have that $\vdash_{L^{++}} \varphi$ and $\vdash_{L^{++}} \neg\varphi$, by the definition, L^{++} is inconsistent as required. □

Method-(2)

(下面这个证明来自 吴家儒 同学，这种证明很直接且颇具暴力美学，再次感谢家儒同学为我们带来如此精彩的证明 ♡♡♡)

Since $\vdash_L (p \to p)$ (cf. *Example 2.7-(a)* in page 31), we have that $\vdash_{L^{++}} (p \to p)$ obviously. And let $(L4)$ denotes the *fourth axiom scheme* of L^{++}, that is,

$$(L4) \qquad ((\neg\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \neg\mathscr{B})).$$

Considering the following proof sequence in L^{++}:

1. $[\neg(p \to p) \to (p \to p)] \to [(p \to p) \to \neg(p \to p)]$ (instance of $L4$)
2. $[(\neg(p \to p) \to (p \to p)) \to ((p \to p) \to \neg(p \to p))] \to$
   $[((\neg(p \to p) \to (p \to p)) \to (p \to p)) \to ((\neg(p \to p) \to (p \to p)) \to \neg(p \to p))]$ (instance of $L2$)
3. $((\neg(p \to p) \to (p \to p)) \to (p \to p)) \to ((\neg(p \to p) \to (p \to p)) \to \neg(p \to p))$ $(1+2, MP)$
4. $(p \to p) \to [(\neg(p \to p) \to (p \to p)) \to (p \to p)]$ (instance of $L1$)
5. $(p \to p)$ ($(p \to p)$ is a theorem of L, so is for L^{++})
6. $(\neg(p \to p) \to (p \to p)) \to (p \to p)$ $(4+5, MP)$
7. $(\neg(p \to p) \to (p \to p)) \to \neg(p \to p)$ $(6+3, MP)$
8. $(p \to p) \to (\neg(p \to p) \to (p \to p))$ (instance of $L1$)
9. $\neg(p \to p) \to (p \to p)$ $(5+8, MP)$
10. $\neg(p \to p)$ $(9+7, MP)$

Hence $\vdash_{L^{++}} \neg(p \to p)$, together with previous $\vdash_{L^{++}} (p \to p)$, L^{++} is inconsistent as desired. □

.... hw-6: feedback ....

1. 大部分人还是没有区分「元语言」和「对象语言」，所以严格来说很多人的回答都是不合法的甚至是错误的。不过改作业的时候已经采取十分宽容的态度了，还希望大家一定要重视这点，这对后续的逻辑学习是十分重要的。
2. 依旧强烈建议不要使用「简化真值表」，这并不是说「简化真值表」是什么洪水猛兽大家碰不得，只不过照现在的作业来看，一画「简化真值表」就容易画错。

3. 虽然很多同学借鉴了教材 *p.205* 的提示，但这种提示往往省略了超多细节，这些细节应该要补充完整的，直接抄书行不得！一个证明首先要说服自己才能说服别人！

4. 建议用**黑笔**! **黑笔**! **黑笔**! 作答，期末考试时也是一样的。

5. 很多同学都误用了 $(L3): (\neg\mathscr{A} \to \neg\mathscr{B}) \to (\mathscr{B} \to \mathscr{A})$ 公理，如下的公式并不是 $(L3)$ 公理的一个实例：
$$(p \to q) \to (\neg q \to \neg p) \qquad \text{或} \qquad (p \to \neg q) \to (q \to \neg p)$$
单单只使用公理模式 $(L3)$ 得不到上述公式是 L 的定理的，注意否定符号的位置。

6. 同样容易误用的是 **Proposition 2.19**:

   > Let L^* be a consistent extension of L and let φ be a formula which is not a theorem of L^*. Then L^{**} is also consistent, where L^{**} is the extension of L obtained from L^* by including $(\neg\varphi)$ as an additional axiom. (p. 40)

   显然 L 是其本身的一个一致扩张，并且很多人做第 8 题第二问的时候，确实证明了 $\nvdash_L \neg\mathscr{A}$，然后直接运用 **Prop. 2.19** 就说 L^+ 是 L 的一致扩张，这中间其实还有一个 gap 要补充的。

   根据 **Prop. 2.19** 和 $\nvdash_L \neg\mathscr{A}$ 我们只能得到 $L \cup \{\neg\neg\mathscr{A}\}$ 是一致的（注意否定的个数），而题目中的是 $L^+ = L \cup \{\mathscr{A}\}$。虽然语义直观上 $\mathscr{A}$ 和 $\neg\neg\mathscr{A}$ 是一个意思，但是仅仅作为字符串来说二者是完全不同的东西。因此，如果硬是要用 **Prop. 2.19** 的话，我们就必须还得论证：$L \cup \{\neg\neg\mathscr{A}\}$ 和 $L \cup \{\mathscr{A}\}$ 是同一个系统。然而这在教材中是没有明确说明的。

7. 抄作业的情况有点严重呀！虽然鼓励同学们相互讨论，但写作业的时候也别直接抄呀，都做对就还好啦，错都错一样的话就很难说过去了 *:(*

`hw-7 (2023/11/07)`

p.49: 2-(c) Translate each of the following statements into symbols, first using no existential quantifiers, and second using no universal quantifiers.

(c) *No mouse is heavier than any elephant.*

（注意：题目要求大家要分别用“全称量词”和“存在量词”符号化句子，因此你的翻译至少有两句）

Answer:

Let
$$\begin{aligned} M(x):&\quad x \text{ is a } \textit{mouse}\\ E(x):&\quad x \text{ is an } \textit{elephant}\\ H(x,y):&\quad x \text{ is } \textit{heavier than } y \end{aligned}$$

Using *no* existential quantifier:

1. $(\forall x)(\forall y)(M(x) \wedge E(y) \to \neg H(x,y))$, or

2. $(\forall x)(\forall y)(M(x) \to (E(y) \to \neg H(x,y)))$, or

3. $(\forall x)(M(x) \to (\forall y)(E(y) \to \neg H(x,y)))$, or

4. any other reasonable answers.

Using *no* universal quantifier:

1. $\neg(\exists x)(\exists y)(M(x) \wedge E(y) \wedge H(x,y))$, or

2. $\neg(\exists x)(M(x) \wedge (\exists y)(E(y) \wedge H(x,y)))$, or

3. any other sensible answers. $\square$

.... hw-7: feedback ....

这次作业大部分人都写得很好，但有两点还请大家要尤其注意下：

1. 对谓词的拆解不完全。有人的用诸如 $D(x)$ 这样的符号来表示谓词“x 比老鼠重”，便会有如下的翻译（$E(x)$ 表示“x 是大象”）：

$$(\forall x)(E(x) \to D(x))$$

   这种翻译就没有把谓词“... 比 ... 重”符号化。

2. 如果用 $H(x,y)$ 表示 “x 比 y 重”，有些同学会把 $H(y,x)$ 理解为 $H(x,y)$ 的否定，即认同 $H(y,x) = \neg H(x,y)$，进而有如下的翻译：

$$(\forall x)(\forall y)(M(x) \wedge E(y) \to H(y,x)) \qquad (\divideontimes)$$

   这种翻译直观上好像可以，但仔细想想，如果我们令 $x = y$，就会产生下面的问题

$$H(x,x) \qquad = \qquad \neg H(x,x)$$

   采用这种翻译的同学其实在脑海中预设了 $H(x,y)$ 是一个严格偏序关系（即“反自反 + 传递”），但这就需要额外的一阶公式来说明 H 是一个严格偏序关系，因此严格来说上面的翻译 $(\divideontimes)$ 是不符合题意的。不过改作业还是采取了宽容的态度，但这并不说明这种答案可行，请特别注意这点！

`hw-8 (2023/11/14)`

p.56: 9-(d) In each case below, let $\mathscr{A}(x_1)$ be the given *wf.*, and let t be the term $f_1^2(x_1, x_3)$. Write out the wf. $\mathscr{A}(t)$ and hence decide in each case whether t is free for x_1 in the given *wf.*

$$(d) \qquad (\forall x_2)A_1^3(x_1, f_1^1(x_1), x_2) \to (\forall x_3)A_1^1(f_1^2(x_1, x_3)).$$

Recall that

- $\mathscr{A}(t)$: if x_i does occur free in $\mathscr{A}(x_1)$, then $\mathscr{A}(t)$ denotes the result of substituting term t for every free occurrence of x_i. (cf. *p.54*)
- t is free for x in a *wf.* ϕ:

定义 3.11*. (Revised defintion) 当一个项 t 可以替换 $\mathscr{A}$ 中变元 x_i 的所有自由出现，且不会使得 t 中任何变元与 $\mathscr{A}$ 的其他部分相互作用，我们就称 t 对 $\mathscr{A}$ 中 x_i 是自由的。

（注意此题有两问：你需要 1) 写出 $\mathscr{A}(t)$，且 2) 回答 t 在 $\mathscr{A}(x_1)$ 中是否对 x_1 自由）

Answer:

Note that in

$$(d) \qquad (\forall x_2)A_1^3(x_1, f_1^1(x_1), x_2) \to (\forall x_3)A_1^1(f_1^2(x_1, x_3)).$$

x_1 has *three* occurrences are free, hence

$$\mathscr{A}(t) = (\forall x_2)A_1^3(f_1^2(x_1, x_3), f_1^1(f_1^2(x_1, x_3)), x_2) \to (\forall x_3)A_1^1(f_1^2(f_1^2(x_1, x_3), x_3))$$

And t is *not* free for x_1 in (d) of course. □

.... hw-8: feedback ....

1. 关于代入后的结果。对 x_1 的自由出现代入 t 后，一定得在所得的公式中把 t 展开了，仅仅写成

   $$(\forall x_2)A_1^3(t, f_1^1(t), x_2) \to (\forall x_3)A_1^1(f_1^2(t, x_3))$$

   这个样子是不可行滴，且就定义而言，上面这个符号串也不是一个合式公式（因为一阶语言的字母表中并没有 t 这样的符号，t 只是元语言中的符号）。

2. 关于符号的写法。对于全称量词或存在量词，可以采取书上的写法，即 $\forall x_i$ 和 $\exists x_i$ 外面有对括号：$(\forall x_i)\varphi$ 、$(\exists x_i)\varphi$。比较现代的记法一般省略会这对括号，直接写作：$\forall x_i\varphi, \exists x_i\varphi$。但有些同学会在把变元用括号括起来，从而有形如

   $$\forall(x_i)\varphi \qquad \exists(x_i)\varphi$$

   这样的写法。不过这种写法既不太美观也不通用，有时还会让人看得比较困惑，所以还是建议不要自创记法为好。

3. 关于代入自由。一个项 t 对于某个公式 φ 中的变元 x 是自由的，一定是相对于整个公式 φ 来说的，当 φ 是一个蕴含式（或者其他复合公式）时，没有「t 对 φ 的前件代入自由」或者「t 对 φ 的后件不是代入自由」这类说法。

`hw-9 (2023/11/21)`

p.59: 11 Let $\mathscr{L}$ be the first order language which includes (besides variables, punctuation, connectives and quantifier) the individual constant a_1, the function letter f_1^2 and the predicate letter A_2^2. Let $\mathscr{A}$ denote the *wf.*

$$(\forall x_1)(\forall x_2)(A_2^2(f_1^2(x_1, x_2), a_1) \to A_2^2(x_1, x_2)).$$

Define an interpretation I of $\mathscr{A}$ as follows. D_I is $\mathbb{Z}$, $\bar{a}_1$ is 0, $\bar{f}_1^2(x,y)$ is $x-y$, $\bar{A}_2^2(x,y)$ is $x<y$. Write down the interpretation of $\mathscr{A}$ in I. Is this a true statement or a false one? Find another interpretation in which $\mathscr{A}$ is interpreted by a statement with the opposite truth value.
(注意此题有三问: 1) 用自然语言（中文/英语）写出 $\mathscr{A}$ 在 I 下的直观含义; 2) 回答在 I 下 $\mathscr{A}$ 是为真还是为假; 3) 基于你对第二问的回答，为公式 $\mathscr{A}$ 找一个新的解释，且在这个新解释中，$\mathscr{A}$ 的真值与你第二问的答案恰好相反)[所以你对第二问的回答很重要]
Answer:

(1) The formula $\mathscr{A}$ in I intuitively means that,

for any integer x_1, x_2: if $(x_1 - x_2) < 0$ then $x_1 < x_2$.

(2) This interpretation of $\mathscr{A}$ in I is *true*.

(3) Let D_I to be $\mathbb{N}$, $\bar{a}_1$ to be 0, $\bar{f}_1^2(x,y)$ is $x \times y$, and $\bar{A}_2^2(x,y)$ is $x > y$. Clearly, $\mathscr{A}$ is false in this new interpretation. (other reasonable interpretations are acceptable, of course) □

.... hw-9: feedback ....

1. 对于第三问，当规定了论域 D_I ，一定要小心对常元 a_1 和函数符号 $f_1^2(x_1,x_2)$ 的解释是否对论域 D_I **封闭**！比如：若我们规定 D_I 为所有**正整数**，那么就不能让 $\bar{a}_1 = 0$，因为 0 不是正整数！同理此时不能把 $f_1^2(x_1,x_2)$ 解释为 $x_1 - x_2$，因为正整数不对（通常意义上的）减法封闭！我们当然可以重新定义那种对正整数封闭的“减法运算”，不过这就得额外给出明确的形式定义。因此，当考虑为一个一阶语言中的公式寻找解释的时候，一定要注意对非逻辑符号的解释是否**对论域封闭**的问题。

`hw-10 (2023/12/13)`

p.70: 22-(a) Show that none of the following *wfs.* is logically valid.

$$(a) \qquad (\forall x_1)(\exists x_2)A_1^2(x_1,x_2) \rightarrow (\exists x_2)(\forall x_1)A_1^2(x_1,x_2).$$

Proof:

It suffices to find an interpretation I such that $I \not\models (\forall x_1)(\exists x_2)A_1^2(x_1,x_2) \rightarrow (\exists x_2)(\forall x_1)A_1^2(x_1,x_2)$. Let $D_I = \mathbb{N}$ and $\bar{A}_1^2(x,y)$ be '$x<y$'.

It is clear that the close *wf.* $(\forall x_1)(\exists x_2)A_1^2(x_1,x_2)$ is *true* in this interpretation, and the close *wf.* $(\exists x_2)(\forall x_1)A_1^2(x_1,x_2)$ is *false*. That is, every valuation satisfies the former and does not satisfy the latter. Hence no valuation satisfies the given *wf.* in clause (a). It is thus not true in this interpretation and it cannot be logically valid. □

.... hw-10: feedback ....

1. 在找反例的时候，建议大家多找找数学上的例子。日常生活中的很多现象，比如“朋友关系”等都有很大的模糊性，不同人对这些观念的理解可能相差甚大。