期中作业

hw-6 (2023/10/31)　　姓名:　　　　学号:

p.44: (8) Let $\mathscr{A}$ be a *wf.* $((\neg p_1 \to p_2) \to (p_1 \to \neg p_2))$. Show that L^+, obtained by including this $\mathscr{A}$ as a new axiom, has a larger set of theorems than L. Is L^+ a consistent extension of L? (10 points)

(注意：此题有两问)

Your proof:

p_1	p_2	($\neg$	p_1	$\to$	p_2)	$\to$	(p_1	$\to$	$\neg$	p_2)
T	T	F	T	T	T	F	T	F	F	T
T	F	F	T	T	F	T	T	T	T	F
F	T	T	F	T	T	T	F	T	F	T
F	F	T	F	F	F	T	F	T	T	F

For the first question: Obviously $\mathscr{A} = ((\neg p_1 \to p_2) \to (p_1 \to \neg p_2))$ is not a tautology by above truth table, then by the **Soundness Theorem**, $\mathscr{A}$ is not a theorem of L, while it is a theorem of L^+, therefore L^+ has a larger set of theorems than L.

For the second one: L^+ is consistent. For suppose otherwise, then there is a formula $\mathscr{B}$ such that

$$\vdash_{L^+} \mathscr{B} \quad \text{and} \quad \vdash_{L^+} \neg\mathscr{B}.$$

Since L^+ is obtained by including $\mathscr{A} = ((\neg p_1 \to p_2) \to (p_1 \to \neg p_2))$ as an extra axiom then L, hence we have that (note that the subscript of $\vdash$)

$$\mathscr{A} \vdash_L \mathscr{B} \quad \text{and} \quad \mathscr{A} \vdash_L \neg\mathscr{B}.$$

By the **Deduction Theorem**,

$$\vdash_L \mathscr{A} \to \mathscr{B} \quad \text{and} \quad \vdash_L \mathscr{A} \to \neg\mathscr{B},$$

which means that $(\mathscr{A} \to \mathscr{B})$ and $(\mathscr{A} \to \neg\mathscr{B})$ are tautologies according to the **Soundness Theorem**. Then by the definition, for any valuation v we have that $v(\mathscr{A} \to \mathscr{B}) = T$ and $v(\mathscr{A} \to \neg\mathscr{B}) = T$, which implies that $v(\mathscr{A}) = F$, that is, $\mathscr{A}$ is a *contradiction*. But this is impossible by the truth table of $\mathscr{A}$. Contradiction! □

p.44: (10) Let L^{++} be the extension of L obtained by including as a fourth axiom *scheme*:

$$((\neg\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \neg\mathscr{B})).$$

Show that L^{++} is inconsistent. (Hint: see Chapter 1 exercise 7 (p.10)) (10 points)

Your proof:

method-(1)

Let

$$\top = (p \to p) \quad \text{and} \quad \varphi = (\neg\top \to \top) \to (\top \to \neg\top),$$

clearly $\vdash_{L^{++}} \varphi$ (i.e., let $\mathscr{A} = \mathscr{B} = \top$). It is easy to check , say using truth table, that φ is a contradiction, hence $\neg\varphi$ is a tautology. By the **Completeness Theorem**, $\vdash_L \neg\varphi$, and then $\vdash_{L^{++}} \neg\varphi$ since L^{++} is a extension of L.

But we have that $\vdash_{L^{++}} \varphi$ and $\vdash_{L^{++}} \neg\varphi$, by the definition, L^{++} is inconsistent as required. □

(*continue on next page*)

method-(2) (下面这个证明来自 吴家儒 同学，这种证明很直接且颇具暴力美学，再次感谢吴家儒同学为大家带来如此精彩的证明 ♡♡♡)[1]

Since $\vdash_L (p \to p)$ (cf. *Example 2.7-(a)* in page 31), we have that $\vdash_{L^{++}} (p \to p)$ obviously. And let $(L4)$ denotes the *fourth axiom scheme* of L^{++}, that is,

$$(L4) \qquad ((\neg\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \neg\mathscr{B})).$$

Considering the following proof sequence in L^{++}:

1. $[\neg(p \to p) \to (p \to p)] \to [(p \to p) \to \neg(p \to p)]$ (instance of $L4$)
2. $[(\neg(p \to p) \to (p \to p)) \to ((p \to p) \to \neg(p \to p))] \to$ $[((\neg(p \to p) \to (p \to p)) \to (p \to p)) \to ((\neg(p \to p) \to (p \to p)) \to \neg(p \to p))]$ (instance of $L2$)
3. $((\neg(p \to p) \to (p \to p)) \to (p \to p)) \to ((\neg(p \to p) \to (p \to p)) \to \neg(p \to p))$ $(1+2, MP)$
4. $(p \to p) \to [(\neg(p \to p) \to (p \to p)) \to (p \to p)]$ (instance of $L1$)
5. $(p \to p)$ ($(p \to p)$ is a theorem of L, so is for L^{++})
6. $(\neg(p \to p) \to (p \to p)) \to (p \to p)$ $(4+5, MP)$
7. $(\neg(p \to p) \to (p \to p)) \to \neg(p \to p)$ $(6+3, MP)$
8. $(p \to p) \to (\neg(p \to p) \to (p \to p))$ (instance of $L1$)
9. $\neg(p \to p) \to (p \to p)$ $(5+8, MP)$
10. $\neg(p \to p)$ $(9+7, MP)$

Hence $\vdash_{L^{++}} \neg(p \to p)$, together with previous $\vdash_{L^{++}} (p \to p)$, L^{++} is inconsistent as desired. □

....作业反馈..

- 大部分人还是没有区分「元语言」和「对象语言」，所以严格来说很多人的回答都是不合法的甚至是错误的。不过改作业的时候已经采取十分宽容的态度了，还希望大家一定要重视这点，这对后续的逻辑学习是十分重要的。
- 依旧强烈建议不要使用「简化真值表」，这并不是说「简化真值表」是什么洪水猛兽大家碰不得，只不过照现在的作业来看，同学一画「简化真值表」就容易画错。
- 虽然很多同学借鉴了教材 *p.205* 的提示，但这种提示往往省略了超多细节，这些细节应该要补充完整的，直接抄书行不得！一个证明首先要说服自己才能说服别人！
- 建议以后用**黑笔**作答，期末考试时也是一样的。
- 很多同学都误用了 $(L3) : (\neg\mathscr{A} \to \neg\mathscr{B}) \to (\mathscr{B} \to \mathscr{A})$ 公理，如下的公式并不是 $(L3)$ 公理的一个实例：

$$(p \to q) \to (\neg q \to \neg p) \qquad \text{或} \qquad (p \to \neg q) \to (q \to \neg p)$$

单单只使用公理模式 $(L3)$ 得不到上述公式是 L 的定理的，注意否定符号的位置。

[1]虽然下面陈述的和家儒同学的原始版本有亿点点不同... 再赞叹一次，家儒同学的证明真的很有暴力美学，爱了爱了😘

- 同样容易误用的是 **Proposition 2.19**:

  > Let L^* be a consistent extension of L and let φ be a formula which is not a theorem of L^*. Then L^{**} is also consistent, where L^{**} is the extension of L obtained from L^* by including $(\neg\varphi)$ as an additional axiom. (p. 40)

  显然 L 是其本身的一个一致扩张，并且很多人做第 8 题第二问的时候，确实证明了 $\nvdash_L \neg\mathscr{A}$，然后直接运用 **Prop. 2.19** 就说 L^+ 是 L 的一致扩张，这中间其实还有一个 gap 要补充的。

  根据 **Prop. 2.19** 和 $\nvdash_L \neg\mathscr{A}$ 我们只能得到 $L \cup \{\neg\neg\mathscr{A}\}$ 是一致的（注意否定的个数），而题目中的是 $L^+ = L \cup \{\mathscr{A}\}$。虽然语义直观上 $\mathscr{A}$ 和 $\neg\neg\mathscr{A}$ 是一个意思，但是仅仅作为字符串来说二者是完全不同的东西。因此，如果硬是要用 **Prop. 2.19** 的话，我们就必须还得论证：$L \cup \{\neg\neg\mathscr{A}\}$ 和 $L \cup \{\mathscr{A}\}$ 是同一个系统。然而这在教材中是没有明确说明的。[2]

  希望大家可以从这点看出逻辑证明的严格性。

- 抄作业的情况有点严重呀！虽然鼓励同学们相互讨论，但写作业的时候也别直接抄呀，都做对就还好啦，错都错一样的话就很难说过去了 *:(*

[2]在其他逻辑著作中，这往往涉及到一种称之为「置换」的操作，要注意「置换」和「代入」不同。