2023·秋·数理逻辑　　平时作业汇总　　大家辛苦啦！☕~~

(英文参考答案 + 不带反馈)

`hw-1 (2023/09/12)`

p3: 1-(h)　If y is an integer then z is not real, provided that x is a rational number.

Answer:

Let

$$\begin{aligned} p:&\quad y \text{ is an integer}\\ q:&\quad z \text{ is a real number}\\ r:&\quad x \text{ is a rational number}\end{aligned}$$

Therefore we have that $r \to (p \to \neg q)$ or $(r \wedge p) \to \neg q$. □

`hw-2 (2023/09/19)`

p10: (7)　Show that the statement form $(((\sim p) \to q) \to (p \to (\sim q)))$ is not a tautology. Find statement forms $\mathscr{A}$ and $\mathscr{B}$ such that $(((\sim \mathscr{A}) \to \mathscr{B}) \to (\mathscr{A} \to (\sim \mathscr{B})))$ is a contradiction.

Answer:

The following truth table shows that $(((\sim p) \to q) \to (p \to (\sim q)))$ is not a tautology.

p	q	($\neg$	p	$\to$	q)	$\to$	(p	$\to$	$\neg$	q)
T	T	F	T	T	T	F	T	F	F	T
T	F	F	T	T	F	T	T	T	T	F
F	T	T	F	T	T	T	F	T	F	T
F	F	T	F	F	F	T	F	T	T	F

When $\mathscr{A}$ and $\mathscr{B}$ are both tautologies, then $(((\sim \mathscr{A}) \to \mathscr{B}) \to (\mathscr{A} \to (\sim \mathscr{B})))$ will be a contradiction. For instance, let $\mathscr{A} = \mathscr{B} = (p \to p)$ or $\mathscr{A} = \mathscr{B} = (p \vee \neg p)$. □

除了用真值表这种比较直观的手段外，还有诸多方法。以下答案来自黄程同学，经其授权后分享给大家，感谢黄程同学👍:

Suppose that $(((\sim p) \to q) \to (p \to (\sim q)))$ is a tautology. Then the situation that $(\sim p) \to q$ be T and $p \to (\sim q)$ be F will not occur under any valuation. But considering $q = T$ and $p = T$, thus $p \to (\sim q)$ will be T. Contradiction! Therefore $(((\sim p) \to q) \to (p \to (\sim q)))$ is not a tautology.

According above answer, when $\mathscr{A}$ and $\mathscr{B}$ be T permanently, then $(((\sim \mathscr{A}) \to \mathscr{B}) \to (\mathscr{A} \to (\sim \mathscr{B})))$ will be a contradiction. In other words, $\mathscr{A}$ and $\mathscr{B}$ are both tautologies, say, $\mathscr{A} = (p \vee (\sim p))$ and $\mathscr{B} = p \to (q \to p)$. □

`hw-3 (2023/09/26)`

p15: 11-(a)　Show, using **Proposition 1.14** and **1.17**, that the statement form $((\neg(p \vee (\neg q))) \to (q \to r))$ is logically equivalent to each of the following.

(a) $((\neg(q \to p)) \to ((\neg q) \vee r))$

Recall that

- **Proposition 1.14**: If $\mathscr{B}_1$ is a statement form arising from the statement form $\mathscr{A}$ by substituting the statement form $\mathscr{B}$ for one or more occurrences of the statement form $\mathscr{A}$ in $\mathscr{A}_1$, and if $\mathscr{B}$ is logically equivalent to $\mathscr{A}$, then $\mathscr{B}_1$ is logically equivalent to $\mathscr{A}_1$.
- **Proposition 1.17 (De Morgan's Laws)**: Let $\mathscr{A}_1, \mathscr{A}_2, \cdots \mathscr{A}_n$ be any statement forms. Then:

1. $(\bigvee_{i=1}^{n}(\neg\mathscr{A}_i))$ is logically equivalent to $(\neg(\bigwedge_{i=1}^{n}\mathscr{A}_i))$.
2. $(\bigwedge_{i=1}^{n}(\neg\mathscr{A}_i))$ is logically equivalent to $(\neg(\bigvee_{i=1}^{n}\mathscr{A}_i))$.

Answer: Let $\varphi = ((\neg(p \vee (\neg q))) \to (q \to r))$ and $\chi = ((\neg(q \to p)) \to ((\neg q) \vee r))$.

It suffices to show that if $\neg(p \vee (\neg q))$ is logically equivalent to $(\neg(q \to p))$, and $(q \to r)$ is logically equivalent to $(\neg q) \vee r)$, then φ is logically equivalent to χ according to **Prop. 1.14**.

But it is easy to check, say, using truth table, that

$$\begin{aligned} \neg(p \vee (\neg q)) &\leftrightarrow (\neg(q \to p)) \qquad \text{and} \\ (q \to r) &\leftrightarrow (\neg q) \vee r) \end{aligned}$$

are tautologies, which means that $(\neg(p \vee (\neg q)))$ and $(\neg(q \to p))$, $(q \to r)$ and $(\neg q) \vee r)$ are logically equivalent, respectively. $\square$

`hw-4 (2023/10/10)`

p.19: 13-(a) Find statement forms in **conjunctive normal form** which are logically equivalent to the following:

$$(a) \qquad (((\neg p) \vee q) \to r)$$

Answer: Here we will use *three* methods to find some **conjunctive normal forms** (CNF) of $(\neg p \vee q) \to r$, the former two are from our Textbook, while the third one is new.

In the first place, let

$$\varphi = (\neg p \vee q) \to r.$$

Method-(1)

First we construct a truth table of φ's negation:

p	q	r	$\neg$ ((	$\neg$	p	$\vee$	q)	$\to$	r)
1	1	1	0	0		1		1	
<u>1</u>	<u>1</u>	<u>0</u>	1	0		1		0	
1	0	1	0	0		0		1	
1	0	0	0	0		0		1	
0	1	1	0	1		1		1	
<u>0</u>	<u>1</u>	<u>0</u>	1	1		1		0	
0	0	1	0	1		1		1	
<u>0</u>	<u>0</u>	<u>0</u>	1	1		1		0	

The combinations which give $\neg\varphi$ value 1 are 110, 010 and 000. Thus a **disjunctive normal form** of $\neg\varphi$ is

$$\chi = (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

It is clear that χ is logically equivalent to $\neg\varphi$, hence $\neg\chi$ is logically equivalent to $\neg\neg\varphi$, i.e., φ.

Then, by the **De Morgan's laws**, we have

$$\begin{aligned}
\neg\chi &= \neg[(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)] \\
&\equiv \neg(p \wedge q \wedge \neg r) \wedge \neg(\neg p \wedge q \wedge \neg r) \wedge \neg(\neg p \wedge \neg q \wedge \neg r) \\
&\equiv (\neg p \vee \neg q \vee \neg\neg r) \wedge (\neg\neg p \vee \neg q \vee \neg\neg r) \wedge (\neg\neg p \vee \neg\neg q \vee \neg\neg r) \\
&\equiv (\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)
\end{aligned}$$

Therefore, $(\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$ is a CNF of φ. □

(**NB**: here we using the symbol expression "$\alpha \equiv \beta$" to denote that formula α is logically equivalent to β)

Method-(2)

$$\begin{aligned}
\varphi &= (\neg p \vee q \to r) \\
&\equiv \neg(\neg p \vee q) \vee r && \text{(by \textbf{material implication}, cf. p.7: Example 1.4-(a))} \\
&\equiv (\neg\neg p \wedge \neg q) \vee r && \text{(by the \textbf{De Morgan's laws})} \\
&\equiv (p \wedge \neg q) \vee r \\
&\equiv (p \vee r) \wedge (\neg q \vee r) && \text{(by the \textbf{distribution} of } (\vee\text{-}\wedge)\text{, cf. p.10, Exercises-6-(b))}
\end{aligned}$$

Hence $(p \vee r) \wedge (\neg q \vee r)$ is a CNF of φ. □

Method-(3)

Similarly, we construct a truth table for φ (notice that, not for the *negation* of φ):

p	q	r	(	$\neg$	p	$\vee$	q	)	$\to$	r
1	1	1		0	1	1	1		1	1
<u>1</u>	<u>1</u>	<u>0</u>		0	1	1	1		0	0
1	0	1		0	1	0	0		1	1
1	0	0		0	1	0	0		1	0
0	1	1		1	0	1	1		1	1
<u>0</u>	<u>1</u>	<u>0</u>		1	0	1	1		0	0
0	0	1		1	0	1	0		1	1
<u>0</u>	<u>0</u>	<u>0</u>		1	0	1	0		0	0

The combinations which give φ value 0 are 110, 010 and 000. Then according to these truth combinations, we can construct three **disjunctive formulas** as follows,

$$\begin{aligned}
\varphi_1 &= (\neg p \vee \neg q \vee r) \\
\varphi_2 &= (p \vee \neg q \vee r) \\
\varphi_3 &= (p \vee q \vee r)
\end{aligned}$$

Next, we connect above three formulas in a conjunctive form, that is,

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 = (\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$$

It is easy to check that $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is a CNF of φ. And as we can see, the result in current *Method-(3)* is same as the *Method-(1)*. $\square$

p.26: 21 Suppose that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n; \therefore \mathscr{A}$ is a valid argument form. Prove that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n-1}$; $\therefore (\mathscr{A}_n \to \mathscr{A})$ is also a valid argument form.

Proof:

First, suppose that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$; $\therefore$ $\mathscr{A}$ is a valid argument form, but $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n-1}$; $\therefore$ $(\mathscr{A}_n \to \mathscr{A})$ is not.

Then there is an assignment of truth values to the statement variables such that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n-1}$ takes value T while $(\mathscr{A}_n \to \mathscr{A})$ takes value F, that is, $\mathscr{A}_n$ is T but $\mathscr{A}$ takes F. However, this contradicts to our assumption that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n; \therefore \mathscr{A}$ is a valid argument form. $\square$

`hw-5 (2023/10/17)`

p.36: 1-(c) Write out proofs in L for the following wfs.

$$(c) \qquad (p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)$$

Proof:

Method-(1)

1. $(p_1 \to (p_1 \to p_2)) \to ((p_1 \to p_1) \to (p_1 \to p_2))$ (instance of $L2$)
2. $[(p_1 \to (p_1 \to p_2)) \to ((p_1 \to p_1) \to (p_1 \to p_2))] \to$
   $[((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2))]$ (instance of $L2$)
3. $((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2))$ $(1+2, MP)$
4. $p_1 \to ((p_1 \to p_2) \to p_1)$ (instance of $L1$)
5. $[p_1 \to ((p_1 \to p_2) \to p_1)] \to [(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)]$ (instance of $L2$)
6. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)$ $(4+5, MP)$
7. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)$ $(3+6, MP)$

The proof for (c) is not unique, of course. $\square$

Method-(2)

1. $p_1 \to ((p_1 \to p_1) \to p_1)$ (instance of $L1$)

2. $(p_1 \to ((p_1 \to p_1) \to p_1)) \to ((p_1 \to (p_1 \to p_1)) \to (p_1 \to p_1))$ (instance of $L2$)

3. $(p_1 \to (p_1 \to p_1)) \to (p_1 \to p_1)$ $(1+2, MP)$

4. $p_1 \to (p_1 \to p_1)$ (instance of $L1$)

5. $(p_1 \to p_1)$ $(3+4, MP)$

6. $(p_1 \to p_1) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1))$ (instance of $L1$)

7. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)$ $(5+6, MP)$

8. $(p_1 \to (p_1 \to p_2)) \to ((p_1 \to p_1) \to (p_1 \to p_2))$ (instance of $L2$)

9. $[(p_1 \to (p_1 \to p_2)) \to ((p_1 \to p_1) \to (p_1 \to p_2))] \to$
$[((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2))]$ (instance of $L2$)

10. $((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2))$ $(8+9, MP)$

11. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)$ $(7+10, MP)$

Method-(3)

1. $\{(p_1 \to p_2) \to [((p_1 \to p_2) \to (p_1 \to p_2)) \to (p_1 \to p_2)]\} \to$
$\{[(p_1 \to p_2) \to ((p_1 \to p_2) \to (p_1 \to p_2))] \to [(p_1 \to p_2) \to (p_1 \to p_2)]\}$ (instance of $L2$)

2. $(p_1 \to p_2) \to [((p_1 \to p_2) \to (p_1 \to p_2)) \to (p_1 \to p_2)]$ (instance of $L1$)

3. $[(p_1 \to p_2) \to ((p_1 \to p_2) \to (p_1 \to p_2))] \to [(p_1 \to p_2) \to (p_1 \to p_2)]$ $(1+2, MP)$

4. $(p_1 \to p_2) \to ((p_1 \to p_2) \to (p_1 \to p_2))$ (instance of $L1$)

5. $(p_1 \to p_2) \to (p_1 \to p_2)$ $(3+4, MP)$

6. $[(p_1 \to p_2) \to (p_1 \to p_2)] \to [((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2)]$ (instance of $L2$)

7. $((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2)$ $(5+6, MP)$

8. $[((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2)] \to$
$[p_1 \to (((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2))]$ (instance of $L1$)

9. $p_1 \to (((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2))$ $(7+8, MP)$

10. $[p_1 \to (((p_1 \to p_2) \to p_1) \to ((p_1 \to p_2) \to p_2))] \to$
$[(p_1 \to ((p_1 \to p_2) \to p_1)) \to (p_1 \to ((p_1 \to p_2) \to p_2))]$ (instance of $L2$)

11. $(p_1 \to ((p_1 \to p_2) \to p_1)) \to (p_1 \to ((p_1 \to p_2) \to p_2))$ $(9+10, MP)$

12. $p_1 \to ((p_1 \to p_2) \to p_1)$ (instance of $L1$)

13. $p_1 \to ((p_1 \to p_2) \to p_2)$ $(11 + 12, MP)$

14. $[p_1 \to ((p_1 \to p_2) \to p_2)] \to [(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)]$ (instance of $L2$)

15. $(p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)$ $(13 + 14, MP)$

(ps. 上面公式中的 中括号 [ ] 和 花括号 { } 是起辅助作用的，为的是方便大家观看。但应注意的是，其本身不是命题逻辑公理系统 L 中的符号!!!)

p.37: 5 The rule HS is an example of a legitimate additional rule of deduction for L. Is the following rule legitimate in the same sense: from the wfs. $\mathscr{B}$ and $(\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))$, deduce $(\mathscr{A} \to \mathscr{C})$?

Answer:

Method-(1) (without using the **Deduction Theorem**)

1. $\mathscr{B}$ (assumption)
2. $(\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))$ (assumption)
3. $(\mathscr{A} \to (\mathscr{B} \to \mathscr{C})) \to ((\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \mathscr{C}))$ $(L2)$
4. $((\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \mathscr{C}))$ $(2 + 3, MP)$
5. $(\mathscr{B} \to (\mathscr{A} \to \mathscr{B}))$ $(L1)$
6. $(\mathscr{A} \to \mathscr{B})$ $(1 + 5, MP)$
7. $(\mathscr{A} \to \mathscr{C})$ $(6 + 4, MP)$

Hence this rule is a legitimate additional rule of deduction for L. □

Method-(2) (using the **Deduction Theorem**)

We first show that

$$\{\mathscr{B}, (\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))\} \cup \{\mathscr{A}\} \vdash_L \mathscr{C}.$$

We write out a deduction for above one as follows:

1. $\mathscr{B}$ (assumption)
2. $(\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))$ (assumption)
3. $\mathscr{A}$ (assumption)
4. $(\mathscr{B} \to \mathscr{C})$ $(2 + 3, MP)$
5. $\mathscr{C}$ $(1 + 4, MP)$

Hence by the **Deduction Theorem**, we have

$$\{\mathscr{B}, (\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))\} \vdash_L \mathscr{A} \to \mathscr{C}.$$

as required. □

hw-6 (2023/10/31) 期中作业

p.44: (8) Let $\mathscr{A}$ be a *wf.* $((\neg p_1 \to p_2) \to (p_1 \to \neg p_2))$. Show that L^+, obtained by including this $\mathscr{A}$ as a new axiom, has a larger set of theorems than L. Is L^+ a consistent extension of L? (注意：此题有两问)

Proof:

p_1	p_2	($\neg$	p_1	$\to$	p_2)	$\to$	(p_1	$\to$	$\neg$	p_2)
T	T	F	T	T	T	F	T	F	F	T
T	F	F	T	T	F	T	T	T	T	F
F	T	T	F	T	T	T	F	T	F	T
F	F	T	F	F	F	T	F	T	T	F

For the first question: Obviously $\mathscr{A} = ((\neg p_1 \to p_2) \to (p_1 \to \neg p_2))$ is not a tautology by above truth table, then by the **Soundness Theorem**, $\mathscr{A}$ is not a theorem of L, while it is a theorem of L^+, therefore L^+ has a larger set of theorems than L.

For the second question: L^+ is consistent. For suppose otherwise, then there is a formula $\mathscr{B}$ such that $\vdash_{L^+} \mathscr{B}$ and $\vdash_{L^+} \neg\mathscr{B}$. Since L^+ is obtained by including $\mathscr{A} = ((\neg p_1 \to p_2) \to (p_1 \to \neg p_2))$ as an extra axiom then L, hence we have that (note that the subscript of $\vdash$)

$$\mathscr{A} \vdash_L \mathscr{B} \quad \text{and} \quad \mathscr{A} \vdash_L \neg\mathscr{B}.$$

By the **Deduction Theorem**,

$$\vdash_L \mathscr{A} \to \mathscr{B} \quad \text{and} \quad \vdash_L \mathscr{A} \to \neg\mathscr{B},$$

which means that $(\mathscr{A} \to \mathscr{B})$ and $(\mathscr{A} \to \neg\mathscr{B})$ are tautologies according to the **Soundness Theorem**. Then by the definition, for any valuation v we have that $v(\mathscr{A} \to \mathscr{B}) = T$ and $v(\mathscr{A} \to \neg\mathscr{B}) = T$, which implies that $v(\mathscr{A}) = F$, that is, $\mathscr{A}$ is a *contradiction*. But this is impossible by the truth table of $\mathscr{A}$. Contradiction! □

p.44: (10) Let L^{++} be the extension of L obtained by including as a fourth axiom *scheme*:

$$((\neg\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \neg\mathscr{B})).$$

Show that L^{++} is inconsistent. (Hint: see Chapter 1 exercise 7 (p.10))

Proof:

Method-(1)

Let $\top = (p \to p)$ and $\varphi = (\neg\top \to \top) \to (\top \to \neg\top)$, clearly $\vdash_{L^{++}} \varphi$ (i.e., let $\mathscr{A} = \mathscr{B} = \top$). It is easy to check , say using truth table, that φ is a contradiction, hence $\neg\varphi$ is a tautology. By the **Completeness Theorem**, $\vdash_L \neg\varphi$, and thus $\vdash_{L^{++}} \neg\varphi$ since L^{++} is a extension of L.

But we have that $\vdash_{L^{++}} \varphi$ and $\vdash_{L^{++}} \neg\varphi$, by the definition, L^{++} is inconsistent as required. $\square$

Method-(2)

(下面这个证明来自 吴家儒 同学，这种证明很直接且颇具暴力美学，再次感谢家儒同学为我们带来如此精彩的证明 ♡♡♡)

Since $\vdash_L (p \to p)$ (cf. *Example 2.7-(a)* in page 31), we have that $\vdash_{L^{++}} (p \to p)$ obviously. And let $(L4)$ denotes the *fourth axiom scheme* of L^{++}, that is,

$$(L4) \qquad ((\neg\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \neg\mathscr{B})).$$

Considering the following proof sequence in L^{++}:

1. $[\neg(p \to p) \to (p \to p)] \to [(p \to p) \to \neg(p \to p)]$ (instance of $L4$)
2. $[(\neg(p \to p) \to (p \to p)) \to ((p \to p) \to \neg(p \to p))] \to$ $[((\neg(p \to p) \to (p \to p)) \to (p \to p)) \to ((\neg(p \to p) \to (p \to p)) \to \neg(p \to p))]$ (instance of $L2$)
3. $((\neg(p \to p) \to (p \to p)) \to (p \to p)) \to ((\neg(p \to p) \to (p \to p)) \to \neg(p \to p))$ $(1 + 2, MP)$
4. $(p \to p) \to [(\neg(p \to p) \to (p \to p)) \to (p \to p)]$ (instance of $L1$)
5. $(p \to p)$ ($(p \to p)$ is a theorem of L, so is for L^{++})
6. $(\neg(p \to p) \to (p \to p)) \to (p \to p)$ $(4 + 5, MP)$
7. $(\neg(p \to p) \to (p \to p)) \to \neg(p \to p)$ $(6 + 3, MP)$
8. $(p \to p) \to (\neg(p \to p) \to (p \to p))$ (instance of $L1$)
9. $\neg(p \to p) \to (p \to p)$ $(5 + 8, MP)$
10. $\neg(p \to p)$ $(9 + 7, MP)$

Hence $\vdash_{L^{++}} \neg(p \to p)$, together with previous $\vdash_{L^{++}} (p \to p)$, L^{++} is inconsistent as desired. $\square$

`hw-7 (2023/11/07)`

p.49: 2-(c) Translate each of the following statements into symbols, first using no existential quantifiers, and second using no universal quantifiers.

(c) *No mouse is heavier than any elephant.*

(注意：题目要求大家要分别用“全称量词”和“存在量词”符号化句子，因此你的翻译至少有两句)

Answer:

Let

$$
\begin{aligned}
M(x): &\quad x \text{ is a } \textit{mouse} \\
E(x): &\quad x \text{ is an } \textit{elephant} \\
H(x,y): &\quad x \text{ is } \textit{heavier than } y
\end{aligned}
$$

Using *no* existential quantifier:

1. $(\forall x)(\forall y)(M(x) \wedge E(y) \rightarrow \neg H(x,y))$, or
2. $(\forall x)(\forall y)(M(x) \rightarrow (E(y) \rightarrow \neg H(x,y)))$, or
3. $(\forall x)(M(x) \rightarrow (\forall y)(E(y) \rightarrow \neg H(x,y)))$, or
4. any other reasonable answers.

Using *no* universal quantifier:

1. $\neg(\exists x)(\exists y)(M(x) \wedge E(y) \wedge H(x,y))$, or
2. $\neg(\exists x)(M(x) \wedge (\exists y)(E(y) \wedge H(x,y)))$, or
3. any other sensible answers. □

hw-8 (2023/11/14)

p.56: 9-(d) In each case below, let $\mathscr{A}(x_1)$ be the given *wf.*, and let t be the term $f_1^2(x_1, x_3)$. Write out the wf. $\mathscr{A}(t)$ and hence decide in each case whether t is free for x_1 in the given *wf.*

$$(d) \qquad (\forall x_2)A_1^3(x_1, f_1^1(x_1), x_2) \rightarrow (\forall x_3)A_1^1(f_1^2(x_1, x_3)).$$

Recall that

- $\mathscr{A}(t)$: if x_i does occur free in $\mathscr{A}(x_1)$, then $\mathscr{A}(t)$ denotes the result of substituting term t for every free occurrence of x_i. (cf. *p.54*)

- t is free for x in a *wf.* ϕ:

定义 3.11*. (Revised defintion) 当一个项 t 可以替换 $\mathscr{A}$ 中变元 x_i 的所有自由出现，且不会使得 t 中任何变元与 $\mathscr{A}$ 的其他部分相互作用，我们就称 t 对 $\mathscr{A}$ 中 x_i 是自由的。

(注意此题有两问: 你需要 1) 写出 $\mathscr{A}(t)$，且 2) 回答 t 在 $\mathscr{A}(x_1)$ 中是否对 x_1 自由)

Answer:

Note that in

$$(d) \qquad (\forall x_2)A_1^3(x_1, f_1^1(x_1), x_2) \rightarrow (\forall x_3)A_1^1(f_1^2(x_1, x_3)).$$

x_1 has *three* occurrences are free, hence

$$\mathscr{A}(t) = (\forall x_2)A_1^3(f_1^2(x_1,x_3), f_1^1(f_1^2(x_1,x_3)), x_2) \rightarrow (\forall x_3)A_1^1(f_1^2(f_1^2(x_1,x_3), x_3))$$

And t is *not* free for x_1 in (d) of course. □

hw-9 (2023/11/21)

p.59: 11 Let $\mathscr{L}$ be the first order language which includes (besides variables, punctuation, connectives and quantifier) the individual constant a_1, the function letter f_1^2 and the predicate letter A_2^2. Let $\mathscr{A}$ denote the *wf.*

$$(\forall x_1)(\forall x_2)(A_2^2(f_1^2(x_1, x_2), a_1) \to A_2^2(x_1, x_2)).$$

Define an interpretation I of $\mathscr{A}$ as follows. D_I is $\mathbb{Z}$, $\bar{a}_1$ is 0, $\bar{f}_1^2(x, y)$ is $x - y$, $\bar{A}_2^2(x, y)$ is $x < y$. Write down the interpretation of $\mathscr{A}$ in I. Is this a true statement or a false one? Find another interpretation in which $\mathscr{A}$ is interpreted by a statement with the opposite truth value.
(注意此题有三问: 1) 用自然语言（中文/英语）写出 $\mathscr{A}$ 在 I 下的直观含义; 2) 回答在 I 下 $\mathscr{A}$ 是为真还是为假; 3) 基于你对第二问的回答，为公式 $\mathscr{A}$ 找一个新的解释，且在这个新解释中，$\mathscr{A}$ 的真值与你第二问的答案恰好相反)[所以你对第二问的回答很重要]

Answer:

(1) The formula $\mathscr{A}$ in I intuitively means that,

for any integer x_1, x_2*: if* $(x_1 - x_2) < 0$ *then* $x_1 < x_2$.

(2) This interpretation of $\mathscr{A}$ in I is *true*.

(3) Let D_I to be $\mathbb{N}$, $\bar{a}_1$ to be 0, $\bar{f}_1^2(x, y)$ is $x \times y$, and $\bar{A}_2^2(x, y)$ is $x > y$. Clearly, $\mathscr{A}$ is false in this new interpretation. (other reasonable interpretations are acceptable, of course) □

hw-10 (2023/12/13)

p.70: 22-(a) Show that none of the following *wfs.* is logically valid.

$$(a) \qquad (\forall x_1)(\exists x_2)A_1^2(x_1, x_2) \to (\exists x_2)(\forall x_1)A_1^2(x_1, x_2).$$

Proof:

It suffices to find an interpretation I such that $I \not\models (\forall x_1)(\exists x_2)A_1^2(x_1, x_2) \to (\exists x_2)(\forall x_1)A_1^2(x_1, x_2)$.

Let $D_I = \mathbb{N}$ and $\bar{A}_1^2(x, y)$ be '$x < y$'.

It is clear that the close *wf.* $(\forall x_1)(\exists x_2)A_1^2(x_1, x_2)$ is *true* in this interpretation, and the close *wf.* $(\exists x_2)(\forall x_1)A_1^2(x_1, x_2)$ is *false*. That is, every valuation satisfies the former and does not satisfy the latter. Hence no valuation satisfies the given *wf.* in clause (a). It is thus not true in this interpretation and it cannot be logically valid. □