1 HMMT POTW

The first thing we notice is that if the probablity to get heads is p then the probablity to get tails is $1-p$. So let's test the waters a little...

What's the probability of Adele winning on her first turn? That's just p. Since all she needs to do is get a head and she wins! Well, what about on her second turn? On her first turn she'll need to get tails instead, because we don't want her to win quite yet. Beyonce will need to get heads on her first turn because we don't want her to win either! Then, we want Adele to win on her second turn so she needs to get heads. Putting this together gives us $p^2 \cdot (1-p)$

Let's try one more situation! What's the probability of Adele winning on her third turn? On Adele's first turn she needs to get tails so that's $(1-p)$, and Beyonce needs to get heads so that's p. Adele then needs to get tails, then Beyonce gets heads, then Adele wins with heads. We get $(1-p)^2 \cdot p^3$.

Similarly, if we continue this pattern...

Adele winning on her n^{th} turn.

n	probablity
1	p
2	$(1-p)(p)^2$
3	$(1-p)^2(p)^3$
4	$(1-p)^3(p)^4$
5	$(1-p)^4(p)^5$
6	$(1-p)^5(p)^6$

Here is where we come across a prediciment. They could play the game forever! How would we count it then? Well, if we look closely, the probabilities resemble a infinite geo series! The common ratio would be $p(1-p)$. We know to add the probabilities because we can think of these situations as cases... and a whole lot of them! With casework, we add. So we have...

$$p + (1-p)(p)^2 + (1-p)^2(p)^3 + (1-p)^3(p)^4 ....$$

The formuala we use is.. $\frac{a}{1-r}$ with a being the first term and r being the ratio. We can use this fomrula only if $-1 < r < 1$, which is fine because p is a fraction!

$\frac{p}{1-p(1-p)} = \frac{1}{2021}$. After simplifying, we get $p^2 - 2022p + 1 = 0$. A quadratic! The roots are $1011 - 2\sqrt{255530}$ and $1011 + 2\sqrt{255530}$. We obviously take $1011 - 2\sqrt{255530}$ because it's a decimal, and probabilities can't be more than one! Thus, our answer is.. $p = 1011 - 2\sqrt{255530} \approx \boxed{0.000495}$