Temporal Analysis of Synchronization of Classical Coupled Pendulums based on Varying String Lengths

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Abstract

Christiaan Huygens discovered the synchronization of coupled pendulums considering a specific closed system. This paper considers the effects of synchronization time based on changes in string length of simple pendulums on a coupled system designed on a moving platform. We use two simple pendulums which are connected by being placed on a wooden board which is then placed on cylindrical cans. String length and synchronization time seem to display an inverse relationship based on trends of raw data. Explanations for other behaviors, such as short oscillator death and anti phase versus in phase synchronization are explained using Laws of Classical Mechanics. The effect of synchronization arises from the media of the pendulums and the dampening of the system, observed through modeling in Wolfram Alpha Mathematica and qualitative observations of man made apparatus. The model in Mathematica is identical to the man made apparatus, with a few modifications: the media that connects the two pendulums is a very stiff spring to account for swaying. It's expected to find in phase synchronization but we find anti-phase synchronization for all trials. The findings presented can be used to speed up natural synchronization processes such as synching end to end encrypted systems where the distance between the signals is analogous to string length. The equations of motion/energy are modeled with Lagrange techniques finishing with a conclusion of the paper.

1 Overview

Synchrony is everywhere in our world, some notable examples being how our own circadian rhythm is a group of cells that act as a chronometer to keep us in synchrony with our world. Or even the less known synchrony of social phenomena, when you might synchronize your motion with someone when talking to them in a conversation. [4] Synchrony in the natural world is not a new topic, where Christiaan Huygens discovered that when setting up two pendulum apparatuses and releasing them at different times, they eventually synchronize their motion;

experiencing continous synchronization or oscillator death. But many questions remain unanswered, more than 350 years after Huygens' discovery. What, for example, are the "minimal" requirements for self-synchronization? This paper will be delving deeper into the topic of synchrony in order to answer Hyugen's ambitions. We will have the pendulums arranged in close proximity, and change the string length of both simple pendulums (rotational intertia is that of a point particle) such that they are approximately the same eigneperiod and find the differences in synchronization times, generalizing into a relationship. Shown in previous research there is a transition period for damping coefficient ρ in [0.06, 0.07] there is quasiperiodic motion for in phase [1]; where Huygens only experienced anti-phase motion. This paper utilizes wood that has a high enough density to fall within that range. For pure experimental purposes, Pantaleone [2] created an experimental platform with two metronomes with the phase difference close to 0. Increasing the dampening effect of the moving platform reported anti-phase motion; supported by Kuramoto model involving Adler's equation.

Specifically, we are considering forced oscillators where the amplitude of the oscillation is not fixed and stable. In autonomous oscillators, the phase of the oscillations is also free due to the time - shift invariance. With a small periodic force $\epsilon sin(\omega * t)$, where Adler's equation is $\frac{d\varphi}{dt} = \triangle(\omega) + \epsilon sin(\varphi)$ where φ is defined as the phase difference and $\triangle(\omega)$ is the damping. Noticing that when $\triangle(\omega) < \epsilon$, then can we only acheive a stable state at some point during the synchronicity; a asynchronous quasiperiodic motion. We define stable states as anti-phase motion [10] and unstable states as in-phase motion.

Talking more about the non identical pendulums, there is a function for the $\triangle$ or the detuning variable [6] which is given as...

$$\Delta = 4\pi\mu(\Gamma + \Upsilon)$$

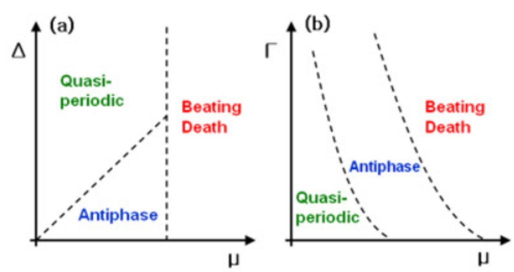

Figure 1: a) A Δ vs. μ graph with Γ fixed, b) a Γ vs. μ graph with Δ fixed, Image from Sae Hong [9]

μ is the damping and Γ and Υ are the periods of the respective pendulums. We see that the general motion of the pendulum system should be first in quasi periodic motion or approaching synchronization. Then it should approach anti phase motion where the pendulums are synchronzing with opposite symmetry. Lasting, it should have beating death where one pendulum is at rest and the other moves with periodic motion.

What this paper accomlishes is uncovering more about the mystery of synchrony in nature, and the physical properties behind it. The first discovery of inanimate synchronization led to the invention of the laser, power grids, and computer clocks. Previous research suggests that there needs to be a connection medium between the two pendulums in order for there to be synchronization [5]. This paper's main result is the fact that with two identical coupled oscillators that there is a inverse relationship between string length and synchronization time. We further analyze the motion using Langrangian equations and laws of Classical mechanics.

2 Method

A large part of the experiment is making the apparatus. The essential materials that were used are Balsa wood, Eisco pendulum bobs, Poplar wood boards (4 inch x 18 inch), metal staples, and extra slabs of poplar wood (1 inch x 5 inch).

Builiding First, the schematic shown in Fig. 2 below has the necessary measurements for a side of a single pendulum tower. Two of them were made and connected with balsa wood pieces to create the tower. We duplicated this process to create another pendulum tower.

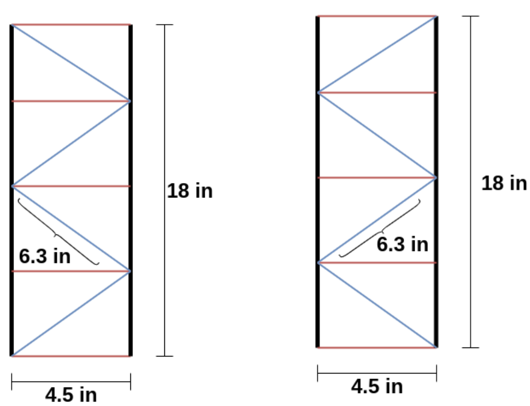

Figure 2: Diagram of apparatus frames, model made using BioRender software

Fig. 2 shows a 4.5 in x 18 in rectangle that has rigid cross supports. The sides of the rectangle show balsa wood sticks that are double stacked meaning that

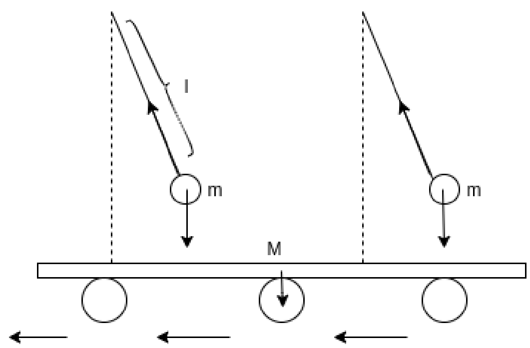

Figure 3: Diagram of the setup, included essential internal and external forces; gravitational, tension, and friction. The masses of the cans are considered negligible and the masses of the rest are shown above, l = string length; made using BioRender.

two balsa wood sticks are fastened together to create a stronger stick. The glue between the wood will increase the wood bondage surface area and strengthen the frame as a whole.

When gluing the horizantal and diagonal supports it's best to avoid using lap joints as those will reduce the gluing surface. Cut the edges of the diagonal support to fit on top of the horizantal support. The diagonal supports of the two seperate towers should also be pointing in opposite directions as shown in Fig. 2. Now when two rectangular frames are created, they can be connected using a singular 4 inch long balsa stick on each side. At the end, it should have the shape of a rectangular prisim. Duplicate this process for a second tower. After building it is critical to recognize that the pendulum stand must be flat. This can be ensured by using fine grain sandpaper to sand away excess balsa.

Now, the mechanism for changing the string length was made after. It doesn't quite matter how this is constructed as long as it performs efficently. A feasible method is placing a thin wooden bar at the top of the tower. In that wooden bar was hammered two metal staple hooks and they would be where the string was tied. It's sufficent to just tie one string to the staple and tape the other end to the top of the wooden bar. This conserves string and makes the process flow easier. Lastly, we attach the pendulum bob onto the string. This is made as if the string is not rigid, so we operate two ends of the string rather than just one end to ensure the pendulum moves in a singular plane of motion.

Then the wooden bar should be taped or secured somehow onto the top of the tower. Tape allows for the easy removal of the top. Duplicate this process to create a second tower.

With the towers complete, they were be placed onto a sturdy wooden board. Look at Fig. 3 for reference. The wooden board is to rest on three congruent soda cans. The soda cans that were used were about 2 inches in radius. Using two soda cans might damage the cans (from all the weight), thus affecting the results. The idea of a moving wooden board was put forward by Kortweg, 1906 [3]. Make markings for where the cans should be placed so the placement is relatively constant.

Sampling strategy The data to be taken here is very straighforward. First adjust the string length to the length that is needed for that specific trial. This paper varied string lengths from 14.5 inches to 8 inches with 0.5 increments. One pendulum bob will be at rest while the other will be pulled back and when released it will push the other to move, eventually synchronizing. Mark on one of the pendulum stands to where the pendulum bob will be released. The same pendulum should be used each time to ensure the consistencies in the data. When recording the data, one should look at when the pendulums first synchronize. This happens when the platform stops moving completely. A motion sensor could be used to sense this movement to be as precise as possible by connecting an arduino. The time taking method can be using a stop watch; starting the time as soon as pendulum is released and stopping when the platform ceases to move. Additionally, it's good to measure when the first and second pendulums stop respectively. Lapping the timer at those points works.

The amount of trials can vary, but for each string length an average about 40 - 50 trials is good to ensure utmost accuracy. This includes redos with string lengths if something went wrong.

Quality control Sometimes while doing the trials, irregularities will pop up. In forms such as swinging for unordinally long amounts of time or not synchronizing totally. In these cases, the root cause is usually misplacement or too much movement of the cans. Once that is fixed, the platform should move as normal. In such cases, restart the timer but still record down the time.

3 Results

Additional Observations Some key facts to note were that when in the process of synchronizing, one pendulum would stop completely and then restart while the other pendulum would do the same. After both pendulums have done that, then they would synchronize. The pendulums also only swung in anti-phase which may be due to the initial conditions that were imposed. Sometimes, the string length may be off by a little bit, and the board would not stop completely.

The density of the wooden board has an effect on the synchronization time more than the periods of each pendulum. Replacing the dense poplar wooden board with a hollow wooden board led to the pendulums having in phase synchronization; albeit with an additional phase difference. Placing a dense wood board resulted in oscillator death which was a curious result.

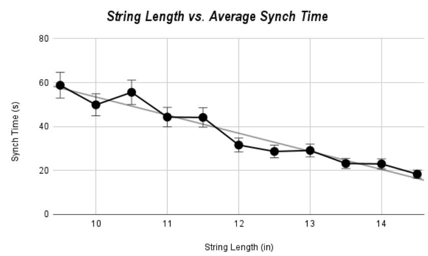

Figure 4: The pendulum system is changed by string length, shown above with appropriate error bars and string lengths. The relationship seems to be inverse; synch time goes down as string length goes down.

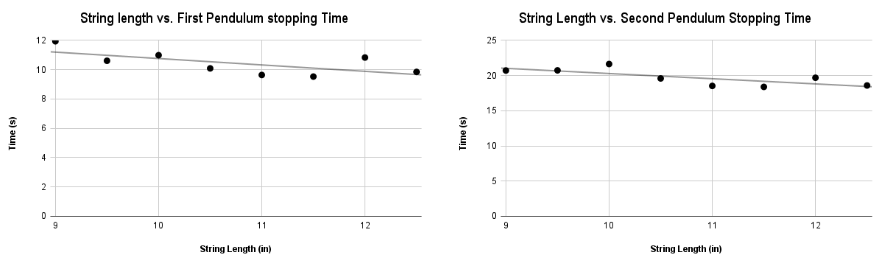

Figure 5: The pendulum that is released is the "first" pendulum that stops, and the other pendulum is the "second" pendulum. They both display slow and shallow decreasing times as string length gets longer. In other words, these are the graphs of the beating deaths.

4 Discussion

We notice that the data from Fig. 4 suggests a general inverse relationship relating the string length and the synchronization time. As string length goes up in magnitude, then the synchronization time tends to go down. The period of a simple pendulum is dependent on the string length or generally the arm length if the rod may be rigid. It's given as $2\pi \times \sqrt{\frac{L}{g}}$, with L being the string length and g universally defined as $9.8\ \frac{m}{s^2}$. [7] We can consider our apparatus as a simple pendulum, meaning that it only moves in singular planar direction; no chaotic movement. As string length increases, the period also increases which would mean that it takes longer for the pendulum to oscillate. It follows that the restoring force is also less with more of a period.

The decreasing synch time due to string length may be due to the fact that since the period is longer it's easier for the pendulums to synchronize due to the ease of finding an intersection time, where as if the pendulums are going faster it's as if they have their own rhythm in simple terms. When the two pendulums are moving slowly, it's easier to find an intersection point at which all the forces on that system cancel. Opposed to if they move chaotically, it'll be harder for them to find a synchronization point.

There is also the interesting fact that when the pendulums first synchronize, they stay synchronized. Due to the fact these are classical pendulums and not metronomes, the pendulums do not contain their own internal rhythm. This influences the fact that they only appeared in anti-phase movement. We recall that anti-phase movement is when ...

$$\lim_{t_n \to \infty} -\theta_1(t_n) = \lim_{t_n \to \infty} \theta_2(t_n)$$

holds true. We see the pendulums go in anti-phase, when the platform stops moving. Refer to Fig 3 for force body diagram. At this point, the system won't move again unless there's an external force acting on it. If the platform is placed on a smooth floor (friction is considered negligible) and no external noises, then it would not be forced to move. It synchronizes in anti-phase so the forces are all equal and opposite to each other and cancel out each other per Newton's 1st Law if we consider the pendulums as our system.

Also note the fact during the synchronization process that the pendulum that is released stops its motion for a brief moment then continues swinging. The pendulum that is not released also stops for a brief moment some time after the first one. This is due to complete energy transfer from one pendulum to another. The medium that the pendulums are placed lets the energy from the first pendulum go into the board and then the board transfers the energy to the second pendulum. Once the pendulums are synchronized, they both slow down simultaneously due to internal friction from the swinging mechanism.

In the results above, we can see that the stopping time of the first pendulum slowly decreases as string length goes up. This might be a general consequence of the fact that the synchronization time as a whole decreases. Although what's strange is that the time hovers around the [8,11] second range even when you go from 8 inches to 14.5 inches in length. This might suggest that the function is logistic for the stopping time of the first pendulum. The same thing occured for the stopping time of the second pendulum except that time hovered around [18,22] seconds. As a note, we notice that the period of the pendulums lies in a range of [1,3] seconds which is about $\frac{1}{3}$ or $\frac{1}{4}$ of the stopping time. The period is how long it takes for the pendulum to swing for one oscillation. When the pendulum fully stops, all the of energy is transferred to the other pendulum by energy conservation.

In general, it's not that hard to see that the connecting medium, the wood board in this case, is one of the main factors that causes the synchronization. Although friction also plays a big part. We can see this with some modeling in Wolfram demonstrations. Intuitively we can see that the phase difference between our two pendulums should drift in a periodic way with the absence of any dissipative effects. In otherwords, we should have in phase synchronization. The arrangement for the setup of Huygen's pendulums is strictly non linear hence why he saw anti - phase motion. Using Langrage techniques we can derive equations of motion for the synchronizing pendulum.

Let's recall that we pull the first pendulum bob to an angle of θ_1 and that position vector has dimensions $r_1 = (x + sin(\theta_1), cos(\theta_1))$ with x being our position parameter.

We know that kinetic energy is defined as $\frac{1}{2} * (mv^2)$, so we can find the magintude and diffrentiate our vector to find the kinetic energy of our pendulum.

We acheive $v_1 = (\dot{x} + cos(\theta_1), -sin(\theta_1))$ and so the kinetic energy of the first pendulum bob is

$$2T = M(\dot{x}^2 + 2\dot{x}\dot{\theta_1}cos(\theta_1) + (\theta_1)^2)$$

.

where T is kinetic energy in Lagrange notation. Notice that we also have to take into account the kinetic energy of the moving platform which would be $2T = M(\dot{x}^2)$. The second pendulum bob will have the same kinetic energy just with θ_2 instead of θ_1. The potential energy is a little more tricky to figure out. We know that by Hooke's law we can consider the restoring force energy to be of $\frac{1}{2}kx^2$, so we can include that along with gravity acting in terms of cosine, which is the component of gravity that it opposite that of the tension force, so we can have...

$$V = \frac{1}{2}kx^2 - Mglcos(\theta_1) - Mglcos(\theta_2)$$

with the weights of the respective bobs being the same. Our coordinate system is defined as negative downwards and leftwards.

We can write down the Lagrange form of L = T - V, and apply it twice with θ and x which is quite nice instead of working with 3d coordinates. As we recall the Euler-Lagrange equations are a second order differential equation where if we have the three parameters (x, θ_1, θ_2) then [8]...

$$\frac{\partial L}{\partial x} = (\frac{d}{d\theta_1})\frac{\partial L}{\partial \theta_2}$$

Accounting for the string length, l, in our langrangian we get...

$$L = \frac{1}{2}(M + 2m)(\dot{x}^2) + m\dot{x}l(\theta_1 cos(\theta_1)$$

$$+\theta_2 cos(\theta_2)) + \ ml^2(\theta_1^2 + \theta_2^2)$$

$$+mgl(cos(\theta_1) + cos(\theta_2)) - \frac{1}{2}kx^2$$

The equation of motion itself is motivation for this research, as string length is a factor in the Langragian equation. Using Mathematica we can use the function EulerEquations[L, θ_1, θ_2, x] to plug in the software. Studying sample graphs as the one below, we notice that the damping coefficents is what gives synchronization. For context, the damping coefficent is usually an extra term that is proportional to v. Through this experiment, a lot of it depended on the damping coefficent. As stated in the Overview, there is a certain damping coefficent from where the motion is quasi-periodic. It follows that there would be a damping coefficent to where beating death could happen. When placing the coupled pendulums on the denser (than poplar) wooden board, beating death occured and this must be due to the density of the wood.

This adds to the already understood fact that anti-phase synchronization can occur due to more damping of a platform [3].

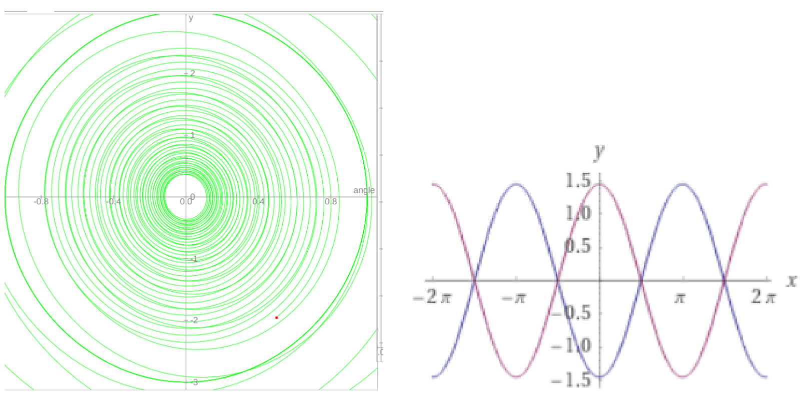

Figure 6: Left: Graphed using myphysics lab, the singular classical pendulum graphed as a function of the angle; damping coefficent of 0.18 and under the differential method of Euler, Right: A sample graph of the anti-phase motion using the Euler - Langrange equations using Wolfram

With more cans that are used, there is more resitive forces due to friction in turn can affect how fast the pendulums can synchronize in anti-phase. The damped string simulates the interaction effects that propagate through the elastic and resistive media.

5 Conclusion

In general, we've shown that synchronization time can depend on string length, which may also be due to the initial conditions that we've imposed on the system. Using the laws of Classical Mechanics we may also explain certain phenomena that happen with synchronization in general which tend to be curious results. The relationship that this paper uncovered may incite other mathematical models of not just Synchronization time, but Synchronization in general. During my experiments, there were multiple attempts to reduce the amount of possible errors, such as making sure everything except the string length was constant. Although, since the apparatus was man-made it will not be error-free. This might've incited a damping constant within the build's mass itself which made the synchronization as anti-phase isntead of the expected in-phase. The findings of this experiment may show that natural synchronization processes can be sped up. An example being end to end encrypted systems where if the distance between the signals (analogous to string length) is changed, then the time is takes for the information to travel can change as well. This was done along with making sure the room in which the experiments were done was quiet as the sound waves can affect the synchronization. Speaking of sound waves, an interesting extension to this problem would be testing what frequency of sounds create certain types of motion on the pendulums (oscillator death, in phase, anti-phase, etc). It would also be interesting to see what happens when you change the medium of the pendulums. There could also be more analysis of how the density affects the synchronization states and times mathematically since that seemed to be a big factor. Synchronization has a lot of applications in other fields such as optics, fluids, mechanics, and much more! Thus, this is a problem that will always have extension problems making it an interesting investigation.

Acknowledgements

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