Problem Set 7

Sum of RVs and Convergence

Date: Monday, 1st of July, 2019.

Problem 1:
A bank teller serves customers standing in the queue one by one. Suppose that the service time X_i for customer i has mean $\mathbb{E}[X_i] = 2$ (minutes) and Var$[X_i] = 1$. We assume that service times for different bank customers are independent. Let Y be the total time the bank teller spends serving 50 customers. Find $P[90 < Y < 110]$.

Problem 2:
In a communication system each data packet consists of 1000 bits. Due to the noise, each bit may be received in error with probability 0.1. It is assumed bit errors occur independently. Find the probability that there are more than 120 errors in a certain data packet.

Problem 3:
There are 100 men on a plane. Let X_i be the weight (in pounds) of the i^{th} man on the plane. Suppose that the X_i's are i.i.d., and $\mathbb{E}[X_i] = \mu = 170$ and $\sigma_{X_i} = \sigma = 30$. Find the probability that the total weight of the men on the plane exceeds 18000 pounds.

Problem 4:
Let $X_1, X_2, \ldots, X_{25}$ be i.i.d. with the following PMF

$$P_X(k) = \begin{cases} 0.6 & k = 1 \\ 0.4 & k = -1 \\ 0 & \text{otherwise} \end{cases}$$

and let

$$Y = X_1 + X_2 + \cdots + X_n. \tag{1}$$

Using the CLT and continuity correction, estimate $P[4 \leq Y \leq 6]$.

Problem 5:
You have invited 64 guests to a party. You need to make sandwiches for the guests. You believe that a guest might need 0, 1 or 2 sandwiches with probabilities 14, 12, and 14 respectively. You assume that the number of sandwiches each guest needs is independent from other guests. How many sandwiches should you make so that you are 95% sure that there is no shortage?

Problem 6:
Let $X_1, X_2, \ldots, X_n$ be i.i.d. Exponential(λ) random variables with $\lambda = 1$. Let

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

How large should n be such that

$$P\left(0.9 \leq \overline{X} \leq 1.1\right) \geq 0.95 \text{ ?}$$

Problem 7:

A CD contains digitized samples of an acoustic waveform. In a CD palyer with a one-bit digital to analog converter, each digital sample is represented to an accuracy of ± 0.5 mV. The CD player oversamples the waveform by making eight independent measurements corresponding to each sample. The CD player obtains a waveform sample by calculating the average of the eight measurements. What is the probability that the error in the waveform is greater than 0.1 mV?

Problem 8:

A modem transmits one million bits. Each bit is 0 or 1 independently with equal probability.

(a) Estimate the probability of at least 502000 ones.
(b) Estimate the probability that there are at least 499000 ones but no more than 501000 ones.

Problem 9:

There are 100 resistors in a bin labeled 10 ohms. Due to manufacturing tolerances, however, the resistance of the resistors are somewhat different. Assume that the resistance can be modeled as a random variable with mean 10 ohms and a variance of 2 ohms. If 100 resistors are chosen from the bin and connected in series (so the resistances add together), what is the approximate probability that the total resistance will exceed 1030 ohms?

Problem 10:

A particle undergoes collisions with other particles. Each collision causes its horizontal velocity to change according to a $\mathcal{N}(0, 0.1)$ cm/sec random variable. After 100 independent collisions, what is the probability that the particle's velocity will exceed 5 cm/sec if it is initially at rest? Is this result exact or approximate?

Problem 11:

50 students live in a dormitory. The parking lot has the capacity for 30 cars. Each student has a car with probability $1/2$, independently from other students. Use the CLT (with continuity correction) to find the probability that there won't be enough parking spaces for all the cars.

Problem 12:

Let $X_2, X_3, X_4, \ldots$ be a sequence of random variable such that

$$F_{X_n}(x) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{nx} & x > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Show that X_n converges in distribution to Exponential(1).

Problem 13:

Let $X_n \sim$ Exponential(n). Show that $X_n \xrightarrow{p} 0$.

Problem 14:

Let $X_n \sim$ Uniform$(0, \frac{1}{n})$. Show that $Xn \xrightarrow{L^r} 0$, for any $r \geq 1$.

Problem 15:

Consider a sequence $\{X_n, n = 1, 2, 3, \ldots\}$ such that

$$X_n = \begin{cases} n^2 & \text{with probability } \frac{1}{n} \\ 0 & \text{with probability } 1 - \frac{1}{n} \end{cases}$$

Show that

(a) $X_n \xrightarrow{p} 0$

(b) X_n does not converge in the r^{th} mean for any $r \geq 1$.

Problem 16:

Consider a sequence $\{X_n, n = 1, 2, 3, \ldots\}$ such that

$$X_n = \begin{cases} -\frac{1}{n} & \text{with probability } \frac{1}{2} \\ \frac{1}{n} & \text{with probability } \frac{1}{2} \end{cases}$$

Show that $X_n \xrightarrow{a.s.} 0$.

Problem 17:

Let X_1, X_2, X_3,... be a sequence of i.i.d. $Uniform(0, 1)$ random variables. Define the sequence Y_n as

$$Y_n = \min(X_1, X_2, \cdots, X_n). \tag{2}$$

Prove the following convergence results independently (i.e, do not conclude the weaker convergence modes from the stronger ones).

1) $Y_n \xrightarrow{d} 0$
2) $Y_n \xrightarrow{p} 0$
3) $Y_n \xrightarrow{L^r} 0$ for all $r \geq 1$
4) $Y_n \xrightarrow{a.s.} 0$

Problem 18:

Let $X_2, X_3, X_4, \cdots$ be a sequence of random variables such that

$$F_{X_n}(x) = \begin{cases} \frac{e^{n(x-1)}}{1+e^{n(x-1)}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that X_n converges in distribution to $X = 1$.

Problem 19:

If $X_i \sim \mathcal{N}(1, 1)$ for $i = 1, 2, \ldots, N$ are i.i.d. random variables, plot a realization of the sample mean random variable against N. Should the realization converge, and if so, to what value?