Lecture 14

Chapter 7 (Continued)

7.4 Normal (Gaussian) Random Vectors

A- General Case

$\boxed{* \text{ Def}}$

A random vector $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$

is said to be Normal or Gaussian if the RVs $X_1, X_2, \ldots, X_n$ are jointly Gaussian i.e the joint PDF of $\{X_i\}_{i=1}^{n}$ is given by

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det[C]}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{m})^{T}[C]^{-1}(\underline{x}-\underline{m})\right]$$

<u>where</u> $\underline{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}$ with $m_i = E[X_i]$

$$[C] = \begin{bmatrix} \mathrm{Cov}[X_1, X_1] & \mathrm{Cov}[X_1, X_2] & \cdots & \mathrm{Cov}[X_1, X_n] \\ \vdots & \ddots & & \\ & & \ddots & \\ \mathrm{Cov}[X_1, X_n] & \cdots & \cdots & \mathrm{Cov}[X_n, X_n] \end{bmatrix}$$

* Properties

$$\text{P 1}\quad \text{Uncorrelation} \;\overset{\text{Always True}}{\Longleftarrow}\; \underset{\text{Gaussianity}}{\Longrightarrow}\; \text{Independence}$$

Proof: If $x_1, x_2, \dots, x_n$ are uncorrelated

$$C_{ij} = Cov[x_i, x_j] = 0 \quad \text{for all } i \neq j$$

Thus

$$[C] = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix}_{n\times n}$$

So $\det[C] = \prod_{i=1}^{n} \sigma_i^2$ and $[C]^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & & 0 \\ & \frac{1}{\sigma_2^2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sigma_n^2} \end{bmatrix}$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\prod_{i=1}^{n} \sigma_i^2}} \cdot \exp\left[-\frac{1}{2} \sum_{i=1}^{n} \frac{(x_i - m_i)^2}{\sigma_i^2}\right]$$

$$\prod_{i=1}^{n} \exp\left(-\frac{1}{2} \frac{(x_i - m_i)^2}{\sigma_i^2}\right)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \sigma_i^2}} \exp\left[-\frac{1}{2} \frac{(x_i - m_i)^2}{\sigma_i^2}\right]$$

$$= \prod_{i=1}^{n} f_{X_i}(x_i)$$

where $X_i \sim N(m_i, \sigma_i^2)$

QED

$\boxed{P2}$ Linear Transformation preserves "Gaussianity"

Th: If • $\underline{X} = [X_1 \; X_2 \; .. \; X_n]^T$ is a Normal vector $\underline{X} \sim \mathcal{N}(\underline{m}, [C])$

• $[A]$ is an $m \times n$ fixed matrix

• $\underline{b}$ is an m-dimensional fixed vector

Then

$$\underline{Y} = [A]\,\underline{X} + \underline{b} \sim \mathcal{N}\left([A]\,\underline{m} + \underline{b} \,;\, [A][C][A]^T\right)$$

Proof: See Textbook

Application (or Implication) of $\boxed{P2}$

If $X_1, X_2, \ldots, X_n$ are correlated multi-variate Gaussian RVs (i.e $[C]_{\underline{X}}$ is not diagonal), one can choose $[A]$ such as

$$[C]_{\underline{Y}} = [A][C]_X [A]^T \text{ is diagonal}$$

$$\Downarrow$$

$Y_1, Y_2, \ldots, Y_n$ are uncorrelated and because they are jointly Gaussian RVs, then they are <u>independent</u>.

P3 Joint CF

$$\Phi_{\underline{x}}(\underline{w}) \triangleq E\left[e^{j\underline{w}^T\underline{x}}\right]$$

$$= E\left[e^{jw_1x_1 + jw_2x_2 + \cdots + jw_nx_n}\right]$$

$\downarrow$ jointly Gaussian RVs

$$= \exp\left[-\frac{1}{2}\underline{w}^T[C]\,\underline{w} + j\,\underline{m}^T\underline{w}\right]$$

$$= \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} Cov[X_j X_k]\, w_j w_k + j\sum_{k=1}^{n} m_k w_k\right]$$

(24) Joint Moments

Th: Let $X_1, X_2, \ldots, X_n$ be n-zero mean jointly Gaussian RVs then

$$E[X_1 X_2 \cdots X_n] = \begin{cases} 0 & n \text{ odd} \\ \sum \prod_{j \neq k}^{n} E[X_j X_k] & n \text{ even} \end{cases}$$

where the sum is taken over all distinct pairs

<u>Illustration for $n=4$</u>

$$E[X_1 X_2 X_3 X_4] = E[X_1 X_2] E[X_3 X_4] + E[X_1 X_3] E[X_2 X_4] + E[X_1 X_4] E[X_2 X_3]$$

as a special case we can write:

$$\left\{\begin{array}{l} X_1 = X_2 = X \\ X_3 = X_4 = Y \end{array}\right\} \text{ jointly Gaussian}$$

$$E[X^2 Y^2] = E[X^2]\, E[Y^2] + E[XY]\, E(XY) + E(XY)\, E(XY)$$

$$\Downarrow$$

$$\boxed{E[X^2 Y^2) = E(X^2)\, E(Y^2) + 2\,(E[XY])^2}$$

B- Special Case $n=2$

a- Definition

Two RVs X and Y are said to have a bivariate normal distribution with parameters $\mu_x, \sigma_x^2, \mu_y, \sigma_y^2$, and ρ if their joint PDF is given by :

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot$$

$$\exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right]$$

with

$$\mu_x = E[x] \in \mathbb{R}$$

$$\mu_y = E[y] \in \mathbb{R}$$

$$\sigma_x = \sqrt{\operatorname{var}(x)} > 0$$

$$\sigma_y = \sqrt{\operatorname{var}(y)} > 0$$

$$\rho = \frac{\operatorname{Cov}[x, y]}{\sigma_x \sigma_y} \quad \text{and} \quad -1 \leq \rho \leq 1$$

<u>Proof</u> $n = 2$

$$[C] = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$\det[C] = \sigma_1^2 \sigma_2^2 - \rho^2 \sigma_1^2 \sigma_2^2 = (1 - \rho^2) \sigma_1^2 \sigma_2^2$$

For $\rho \neq \mp 1$ we can write

$$[C]^{-1} = \frac{1}{(1-\rho^2)\,\sigma_1^2\sigma_2^2} \cdot \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

$$= \frac{1}{(1-\rho^2)} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

So in the argument of the exponential we have

$$-\frac{1}{2}\left(\underset{1\times 2}{\begin{bmatrix} x_1 - m_1 & x_2 - m_2 \end{bmatrix}} \frac{1}{(1-\rho^2)} \underset{2\times 2}{\begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 - m_1 \\ x_2 - m_2 \end{bmatrix}} \right)$$

$\searrow$ ....

Once we do the matrix multiplications involved in the argument of the exponential we get the desired result.

b- Properties

$\boxed{P1}$ If X and Y are bivariate Normal and uncorrelated then they are independent

$\boxed{P2}$ If X and Y are bivariate Normal then

$$X \sim \mathcal{N}(E[X], Var[X])$$

$$Y \sim \mathcal{N}(E[Y], Var[Y])$$

$\boxed{P3}$ Let X and Y be two bivariate normal RVs. Then there exist <u>independent</u> <u>standard</u> normal random variables Z_1 and Z_2 such that

$$\begin{cases} X = \sigma_x Z_1 + m_x \\ Y = \sigma_y \left(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2 \right) + m_y \end{cases}$$

with $\left.\begin{array}{l} Z_1 \sim \mathcal{N}(0,1) \\ Z_2 \sim \mathcal{N}(0,1) \end{array}\right\}$ independent

$\boxed{P4}$ <u>Theorem</u>

. Suppose X and Y are jointly Normal RVs with parameters m_x, σ_x^2, m_y, σ_y^2, and ρ

Then Given $X = x$, Y is normally distributed with

$$\begin{cases} E[Y/X=x] = m_y + \rho\,\sigma_y \dfrac{x - m_x}{\sigma_x} \\ Var[Y/X=x] = (1-\rho^2)\,\sigma_y^2 \end{cases}$$

(P5) For $n=2$, the joint characteristic Function is

$$\Phi_{X_1 X_2}(\omega_1, \omega_2) = E\left[e^{j\omega_1 X_1 + j\omega_2 X_2}\right]$$

$$= e^{j m_1 \omega_1 + j m_2 \omega_2} \cdot$$

$$\exp\left[-\frac{1}{2}\left(\sigma_1^2 \omega_1^2 + 2\,\omega_1 \omega_2 \rho_{12} \sigma_1 \sigma_2 + \sigma_2^2 \omega_2^2\right)\right]$$