Problem Set 9 Solution

Problem 1:

a. If Y is the number arrivals in $(3,5]$, then $Y \sim Poisson(\mu = 0.5 \times 2)$. Therefore,

$$P(Y=0) = e^{-1}$$
$$= 0.37$$

b. Let Y_1, Y_2, Y_3 and Y_4 be the numbers of arrivals in the intervals $(0,1]$, $(1,2]$, $(2,3]$, and $(3,4]$. Then $Y_i \sim Poisson(0.5)$ and Y_i's are independent, so

$$\begin{aligned} P(Y_1=1, Y_2=1, Y_3=1, Y_4=1) &= P(Y_1=1) \cdot P(Y_2=1) \cdot P(Y_3=1) \cdot P(Y_4=1) \\ &= \left[0.5e^{-0.5}\right]^4 \\ &\approx 8.5 \times 10^{-3}. \end{aligned}$$

Problem 2:

(a) Let $X = N(11) - N(10.5)$, then $X \sim Poisson(10 \cdot \frac{1}{2})$, thus $P(X=0) = e^{-5}$.
(b) Let

$$X_1 = N(11) - N(10.5)$$
$$X_2 = N(12) - N(11.5)$$

Then X_1 and X_2 are two independent $Poisson(5)$ random variables so

$$\begin{aligned} P(X_1=3, X_2=7) &= P(X_1=3)P(X_2=7) \\ &= \frac{e^{-5}5^3}{3!} \cdot \frac{e^{-5}5^7}{7!} \end{aligned}$$

Problem 3:

Let

$$X_1 = N(2) - N(0)$$
$$X_2 = N(7) - N(4)$$

Then,

$$X_1 \sim Poisson(2\lambda)$$
$$X_2 \sim Poisson(3\lambda)$$

and X_1 and X_2 are independent.

$$\begin{aligned} P(X_1 = 2 \quad \text{or} \quad X_2 = 3) &= P(X_1 = 2) + P(X_2 = 3) - P(X_1 = 2, X_2 = 3) \\ &= P(X_1 = 2) + P(X_2 = 3) - P(X_1 = 2, X_2 = 3) \\ &= P(X_1 = 2) + P(X_2 = 3) - P(X_1 = 2) \cdot P(X_2 = 3) \\ &= \frac{e^{-2\lambda}(2\lambda)^2}{2!} + \frac{e^{-3\lambda}(3\lambda)^3}{3!} - \frac{e^{-5\lambda}\left(2^2 \cdot 3^3\right)(\lambda)^5}{3!2!} \end{aligned}$$

Problem 4:

a. Since $X_1 \sim Exponential(2)$, we can write

$$\begin{aligned} P(X_1 > 0.5) &= e^{-(2\times 0.5)} \\ &\approx 0.37 \end{aligned}$$

Another way to solve this is to note that

$$P(X_1 > 0.5) = P(\text{no arrivals in } (0, 0.5]) = e^{-(2\times 0.5)} \approx 0.37$$

b. We can write

$$\begin{aligned} P(X_1 > 3 | X_1 > 1) &= P(X_1 > 2) \text{ (memoryless property)} \\ &= e^{-2\times 2} \\ &\approx 0.0183 \end{aligned}$$

Another way to solve this is to note that the number of arrivals in $(1, 3]$ is independent of the arrivals before $t = 1$. Thus,

$$\begin{aligned} P(X_1 > 3 | X_1 > 1) &= P\big(\text{no arrivals in } (1, 3] \mid \text{no arrivals in } (0, 1]\big) \\ &= P\big(\text{no arrivals in } (1, 3]\big) \text{ (independent increments)} \\ &= e^{-2\times 2} \\ &\approx 0.0183 \end{aligned}$$

c. The time between the third and the fourth arrival is $X_4 \sim Exponential(2)$. Thus, the desired conditional probability is equal to

$$\begin{aligned} P(X_4 > 2 | X_1 + X_2 + X_3 = 2) &= P(X_4 > 2) \text{ (independence of the } X_i\text{'s)} \\ &= e^{-2\times 2} \\ &\approx 0.0183 \end{aligned}$$

d. When I start watching the process at time $t = 10$, I will see a Poisson process. Thus, the time of the first arrival from $t = 10$ is $Exponential(2)$. In other words, we can write

$$T = 10 + X,$$

where $X \sim Exponential(2)$. Thus,

$$\begin{aligned} ET &= 10 + EX \\ &= 10 + \frac{1}{2} = \frac{21}{2}, \\ \mathrm{Var}(T) &= \mathrm{Var}(X) \\ &= \frac{1}{4}. \end{aligned}$$

e. Arrivals before $t = 10$ are independent of arrivals after $t = 10$. Thus, knowing that the last arrival occurred at time $t = 9$ does not impact the distribution of the first arrival after $t = 10$. Thus, if A is the event that the last arrival occurred at $t = 9$, we can write

$$\begin{aligned} E[T|A] &= E[T] \\ &= \frac{21}{2}, \\ \mathrm{Var}(T|A) &= \mathrm{Var}(T) \\ &= \frac{1}{4}. \end{aligned}$$

Problem 5:

Let's assume $t_1 \geq t_2 \geq 0$. Then, by the independent increment property of the Poisson process, the two random variables $N(t_1) - N(t_2)$ and $N(t_2)$ are independent. We can write

$$\begin{aligned} C_N(t_1, t_2) &= \mathrm{Cov}\big(N(t_1), N(t_2)\big) \\ &= \mathrm{Cov}\big(N(t_1) - N(t_2) + N(t_2), N(t_2)\big) \\ &= \mathrm{Cov}\big(N(t_1) - N(t_2), N(t_2)\big) + \mathrm{Cov}\big(N(t_2), N(t_2)\big) \\ &= \mathrm{Cov}\big(N(t_2), N(t_2)\big) \\ &= \mathrm{Var}\big(N(t_2)\big) \\ &= \lambda t_2, \quad \text{since } N(t_2) \sim Poisson(\lambda t_2). \end{aligned}$$

Similarly, if $t_2 \geq t_1 \geq 0$, we conclude

$$C_N(t_1, t_2) = \lambda t_1.$$

Therefore, we can write

$$C_N(t_1, t_2) = \lambda \min(t_1, t_2), \quad \text{for } t_1, t_2 \in [0, \infty).$$

Problem 6:

$N(t)$ is a Poisson process with rate $\lambda = 1 + 2 = 3$.

a. We have

$$\begin{aligned} P(N(1)=2, N(2)=5) &= P\Big(\underline{two} \text{ arrivals in } (0,1] \text{ and } \underline{three} \text{ arrivals in } (1,2]\Big) \\ &= \left[\frac{e^{-3}3^2}{2!}\right] \cdot \left[\frac{e^{-3}3^3}{3!}\right] \\ &\approx .05 \end{aligned}$$

b.

$$\begin{aligned} P(N_1(1)=1 | N(1)=2) &= \frac{P(N_1(1)=1, N(1)=2)}{P(N(1)=2)} \\ &= \frac{P(N_1(1)=1, N_2(1)=1)}{P(N(1)=2)} \\ &= \frac{P(N_1(1)=1) \cdot P(N_2(1)=1)}{P(N(1)=2)} \\ &= \left[e^{-1} \cdot 2e^{-2}\right] / \left[\frac{e^{-3}3^2}{2!}\right] \\ &= \frac{4}{9}. \end{aligned}$$

Problem 7:

T_2~Gamma(2,λ), using this fact, we can compute the exact probability as

$$\begin{aligned} P(T_2 \leq 1) &= \int_0^1 \frac{\lambda^2}{1!} t e^{-\lambda t} \, dt \\ &= \lambda^2 \left[-\frac{e^{-\lambda t}}{\lambda^2} - \frac{t e^{-\lambda t}}{\lambda} \right]\Bigg|_0^1 \\ &= \lambda^2 \left[-\frac{e^{-\lambda}}{\lambda^2} - \frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda^2} \right] \\ &= 1 - e^{-\lambda} - \lambda e^{-\lambda} \\ &= 1 - e^{-1} - e^{-1} = 1 - 2e^{-1} = 0.2642 \end{aligned}$$

To estimate the probability, use the following code.

MATLAB CODE:

```
clear all
clc

lambda=1;%set arrival rate
T=1; %set time interval in seconds
p_hat=0;N=10000;

for n=1:N
    clear z t
    for i=1:1000
        z(i)=exprnd(1);%generate interarrival time which is exp(1)
        if i==1 %this is the first interarrival time which is the same as the
first arrival time T_1
            t(i)=z(i);
        else
            t(i)=t(i-1)+z(i);%add interarrival time to last arrival time to
get current arrival time
        end
        if(t(i)>T)%check if desired time interval has elapsed
            break
        end
    end
    M=length(t)-1; %number of arrivals in interval [0,T]
    arrivals=t(1:M);%arrival times in the interval [0,T]
    if (length(arrivals)>=2)%then this is a realization where we have at
least two arrivals in [0,T] so T_2<=T
        p_hat=p_hat+1/N;
    end

end
```

Problem 8:

We can define $N(t)$ as

$N(t) = N_c(t) + N_t(t) + N_b(t)$

where

$N_c(t)$ is the Poisson process for the number of cars with $\lambda_c = 1.2$

$N_t(t)$ is the Poisson process for the number of trucks with $\lambda_t = 0.9$

and $N_b(t)$ is the Poisson process for the number of buses with $\lambda_b = 0.7$

Since $N_c(t)$, $N_t(t)$ and $N_b(t)$ are independent, we have

$N(t)$ is a Poisson process with rate $\lambda = \lambda_c + \lambda_t + \lambda_b = 1.2 + 0.9 + 0.7$
$= 2.8$

So in a 10 minute interval

$N \sim \text{Poisson}(10 \times 2.8) = \text{Poisson}(28)$

which has PMF

$$P_N(n) = \begin{cases} \frac{28^n e^{-28}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Problem 9:

Let $N_I(t)$ be the number of internal requests at time t.

Then $N_I(t)$ is a Poisson process with rate $(0.7)(10) = 7$

Let $N_X(t)$ be the number of external requests at time t. Then $N_X(t)$ is a Poisson process with rate $(0.3)(10) = 3$

Moreover $N_X(t)$ and $N_I(t)$ are independent.

10 mins × 60 sec.

So $I \sim \text{Poisson}(600 \cdot 7) = \text{Poisson}(4200)$

and $X \sim \text{Poisson}(600 \cdot 3) = \text{Poisson}(1800)$

and I and X are independent,

so

$$P_{I,X}(i,k) = P_I(i) P_X(k) = \begin{cases} \dfrac{(4200)^i e^{-4200}}{i!} \cdot \dfrac{(1800)^k e^{-1800}}{k!} & i,k \in \{0,1,\dots\} \\ 0 & \text{otherwise} \end{cases}$$

Problem 10:

$$P[N(12)-N(7)=6]$$
$$=P[N(5)=6]$$
$$N(5) \sim Poisson(5)$$
$$\text{so } = \frac{5^6 e^{-5}}{6!} \approx 0.1462$$

$$\mathbb{E}[N(12)-N(7)] = \mathbb{E}[N(5)]$$
$$= 5. \quad (\text{since } N(5) \sim Poisson(5))$$

Problem 11:

We are interested in

$$P[N(5)=2]$$
$$N(5) \sim Poisson(\underbrace{2 \times 5}_{=10})$$
$$\text{so } P[N(5)=2] = \frac{10^2 e^{-10}}{2!} \approx 2.27 \times 10^{-3}$$

Problem 12:

$$\mathbb{E}[N(t_2)-N(t_1)]$$
$$= \mathbb{E}[N(t_2-t_1)]$$
$$N(t_2-t_1) \sim \text{Poisson}(\lambda(t_2-t_1))$$
so $\mathbb{E}[N(t_2)-N(t_1)] = \lambda(t_2-t_1)$

where λ is the rate of $N(t)$

$$\text{Var}[N(t_2)-N(t_1)] = \text{Var}[N(t_2-t_1)]$$

$$N(t_2-t_1) \sim \text{Poisson}(\lambda(t_2-t_1))$$

so $\text{Var}[N(t_2)-N(t_1)] = \lambda(t_2-t_1)$

where λ is the rate of $N(t)$

Problem 13:

$$\mathbb{E}[X(t)] = \mathbb{E}\left[\sum_{i=1}^{N(t)} U_i\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N(t)} U_i \,\middle|\, N(t)\right]\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{N(t)} \mathbb{E}[U_i \mid N(t)]\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{N(t)} \mathbb{E}[U_i]\right]$$

$$= \mathbb{E}[N(t)\,\mathbb{E}[U_i]]$$

$$= \mathbb{E}[N(t)]\,\mathbb{E}[U_i]$$

$$\mathbb{E}[U_i] = (+1)(p) + (-1)(1-p)$$

$$= p - 1 + p = 2p - 1$$

and $\mathbb{E}[N(t)] = \lambda t$

so

$$\mathbb{E}[X(t)] = \lambda t(2p-1)$$