Problem 1

ON → '1'

OFF → '0'

$p_{00} + p_{01} = 1$

$p_{00} + p = 1$

$p_{00} = 1 - p$

(a)

$p_{11} + p_{10} = 1$

(b) $P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$

$1-p + p = 1$

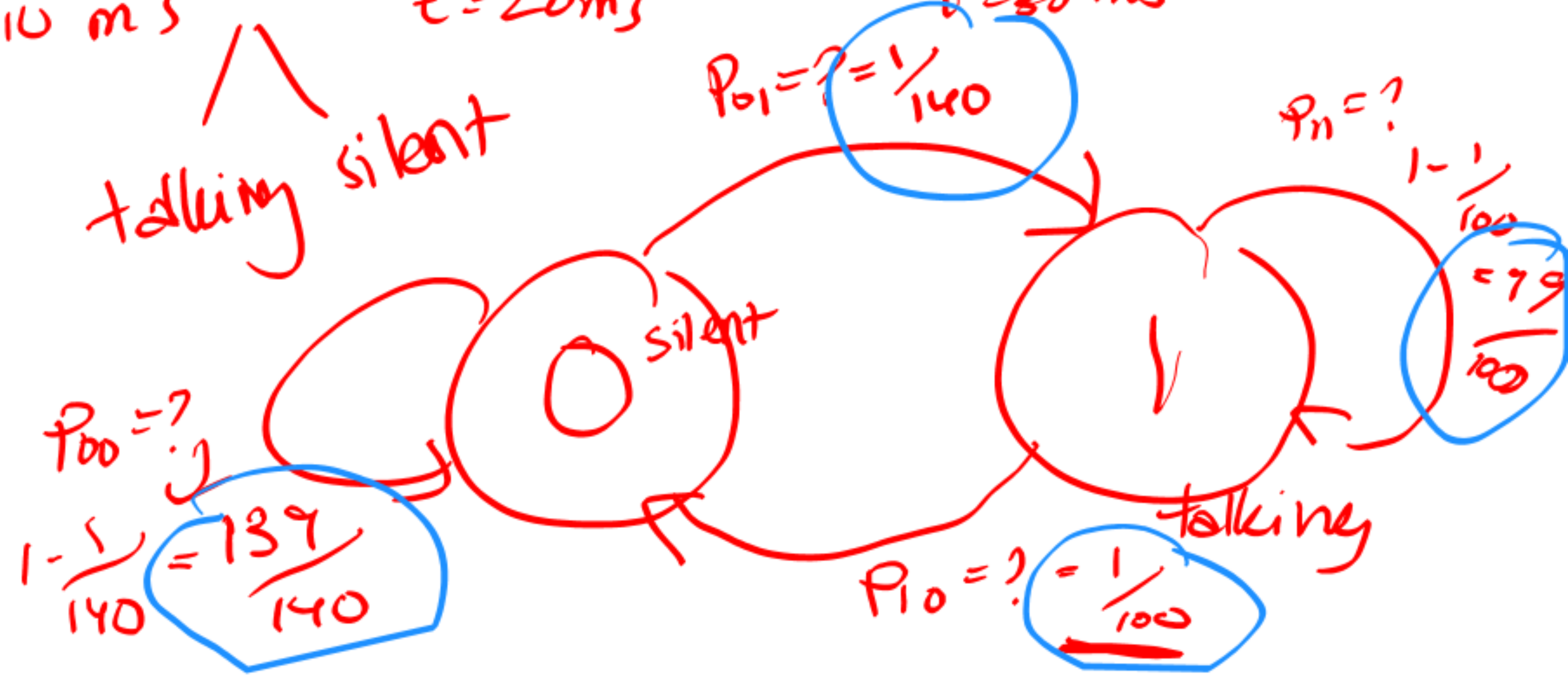
$q + 1-q = 1$

(c) $P^{(2)} = \begin{bmatrix} p_{00}^{(2)} & p_{01}^{(2)} \\ p_{10}^{(2)} & p_{11}^{(2)} \end{bmatrix} = P^2 = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$

$$= \boxed{\begin{bmatrix} (1-p)^2 + pq & p^2 + (1-q)^2 \\ q(1-p) + (1-q)q & pq + (1-q)^2 \end{bmatrix}}$$

$P^{(n)} = P^n$

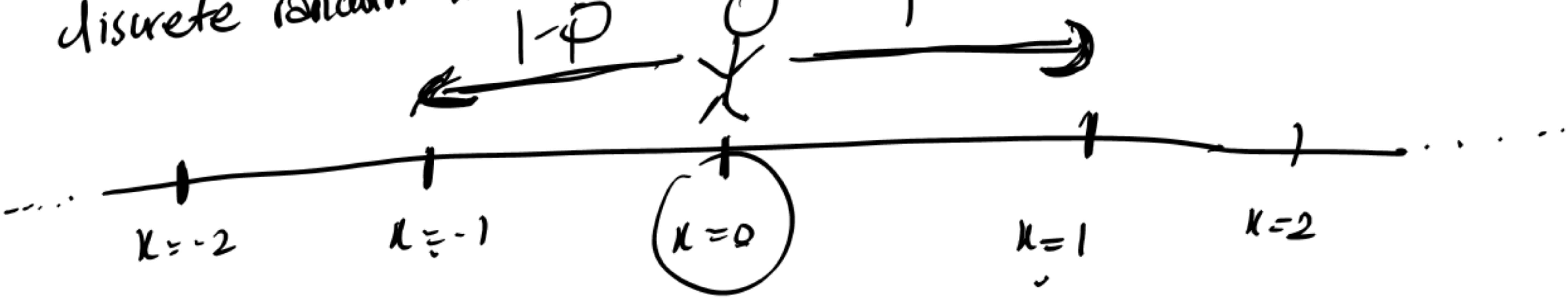
Problem 2



$$P=\begin{bmatrix} \frac{139}{140} & \frac{1}{140} \\ \frac{1}{100} & \frac{99}{100} \end{bmatrix}$$

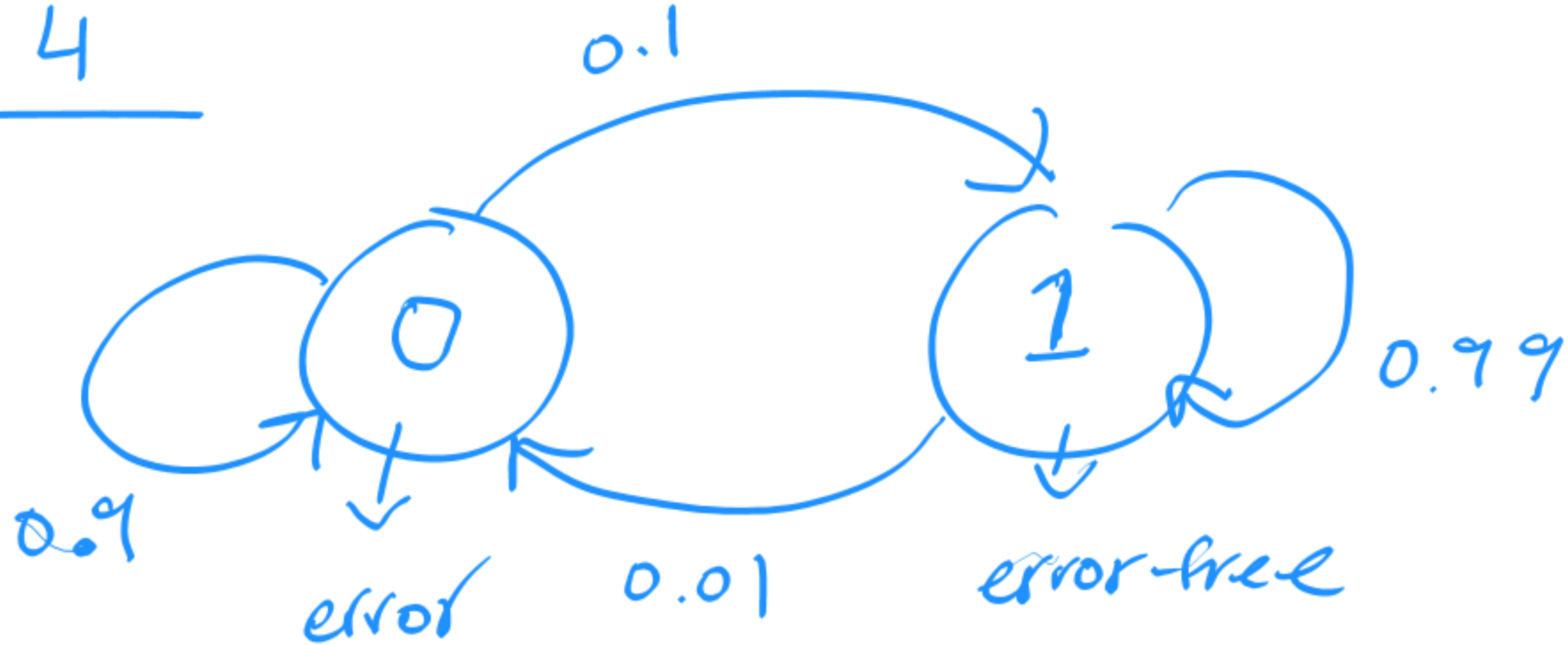
Problem 3

discrete random walk → a kind of Markov chain



$$S = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$$

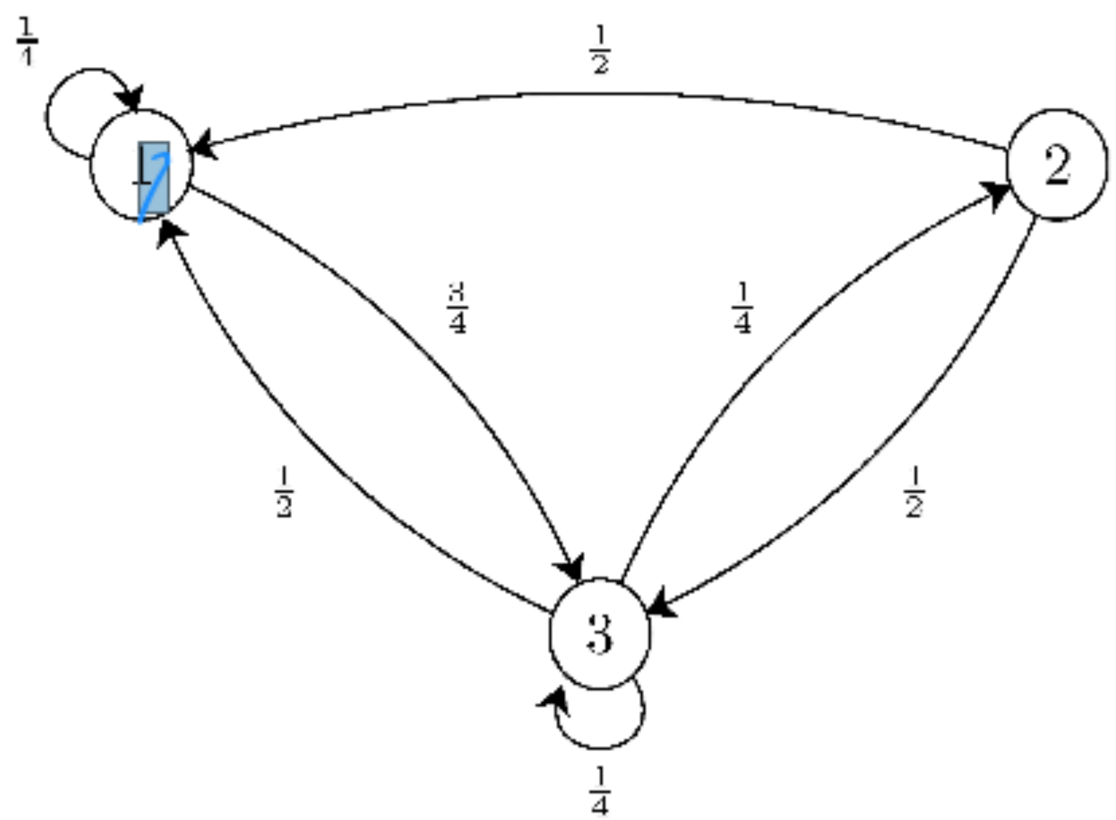
Problem 4



$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.01 & 0.99 \end{bmatrix} \begin{matrix} =1 \\ =1 \end{matrix}$$

Problem 5:

Consider the Markov chain with three states $S = 1, 2, 3$, that has the following state transition diagram Suppose $P[X_1 = 1] = 1/2$ and $P[X_1 = 2] = 1/4$.



(a) Find the state transition matrix for this chain.
(b) Find $P[X_1 = 3, X_2 = 2, X_3 = 1]$.
(c) Find $P(X_1 = 3, X_3 = 1)$.

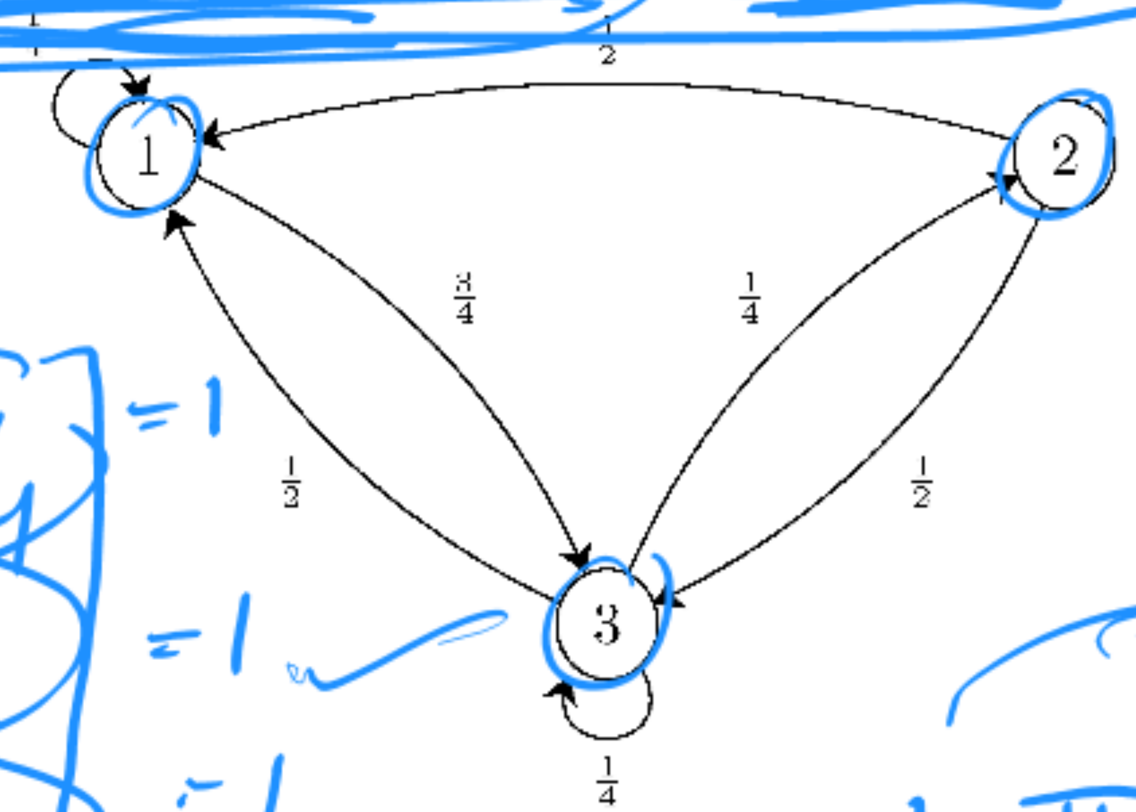
Problem 6:

Problem 5:

Consider the Markov chain with three states $S = 1, 2, 3$, that has the following state transition diagram. Suppose $P[X_1 = 1] = 1/2$ and $P[X_1 = 2] = 1/4$.

$P[X_1 = 3] = ?$

$\frac{1}{2} - \frac{1}{4}$



$= \frac{1}{2} - \frac{1}{4}$

$P[X_2 = 3]$

$P[X_3 = 3]$

$$P = \begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{matrix} = 1 \\ = 1 \\ = 1 \end{matrix}$$

$\frac{1}{4}$

π^{10}

$$\pi^{(n+1)} = \pi^{(n)} P$$

(a) Find the state transition matrix for this chain.

(b) Find $P[X_1 = 3, X_2 = 2, X_3 = 1]$.

(c) Find $P(X_1 = 3, X_3 = 1)$.

$$\pi^{(n)} = [P[X_n = 1] \; P[X_n = 2] \; P[X_n = 3]]$$

Problem 6:

$$P = \begin{bmatrix} 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

$C = \{X_1 = 3\}$

$B = \{X_2 = 2\}$

$A = \{X_3 = 1\}$

$A = \{X_1 = 3\}$

$B = \{X_2 = 2\}$

$C = \{X_3 = 1\}$

$P[X_2 = 2 \mid X_1 = 3]$

$P[X_1 = 3, X_2 = 2] \over P[X_2 = 2]$

$\hookrightarrow P[X_1 = 3 \mid X_2 = 2, X_3 = 1] \cdot P[X_2 = 2 \mid X_3 = 1] \cdot P[X_3 = 1]$

$= P[X_1 = 3 \mid X_2 = 2] \cdot P[X_2 = 2 \mid X_3 = 1] \cdot P[X_3 = 1]$

(b) $P[X_1 = 3, X_2 = 2, X_3 = 1] = P[X_3 = 1 \mid X_2 = 2, \cancel{X_1 = 3}] \cdot P[X_2 = 2 \mid X_1 = 3] \cdot$

$P[A, B] = P[A \mid B] P[B]$

$P[A, B, C] = P[A \mid B, C] P[B, C]$

$= P[A \mid B, C] P[B \mid C] P[C]$

$= P[X_3 = 1 \mid X_2 = 2] P[X_2 = 2 \mid X_1 = 3] P[X_1 = 3]$

$= P_{21} \cdot P_{32} \cdot \frac{1}{4} = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$

$= \frac{1}{32}$

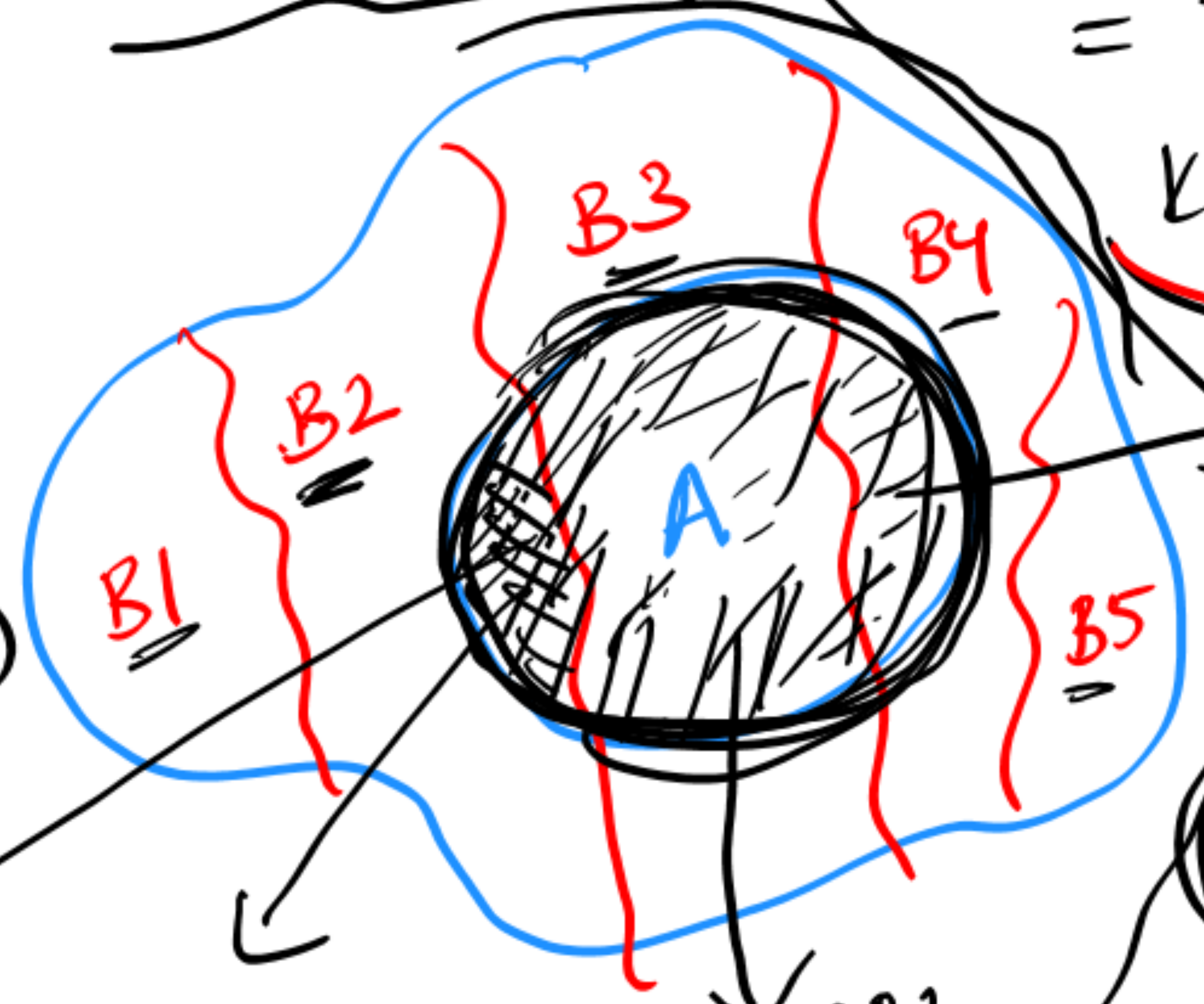
(c) $P[X_1=3, X_3=1] = \sum_{k=1}^{3} P[X_1=3, X_2=k, X_3=1]$

in 2 steps

$= \sum_{k=1}^{3} P[X_1=3]\, p_{3k}\, p_{k1}$

X_2

Law of total probability



$A \cap B4$

$A \cap B3$

$A \cap B2$

$P[A] = P[A \cap B2] + P[A \cap B3] + P[A \cap B4]$

(b)

$P[X_1=3, X_2=\cancel{2}\,k, X_3=1]$

$= P[X_1=3]\, p_{3\cancel{2}k}\, p_{\cancel{2}k1}$

$$P[X_3=j \mid X_1=i] = \sum_{k=1}^{3} p_{ik}\, p_{kj}$$

$n=1$
$m=1$

$$P[X_3=j,\ X_1=i]$$

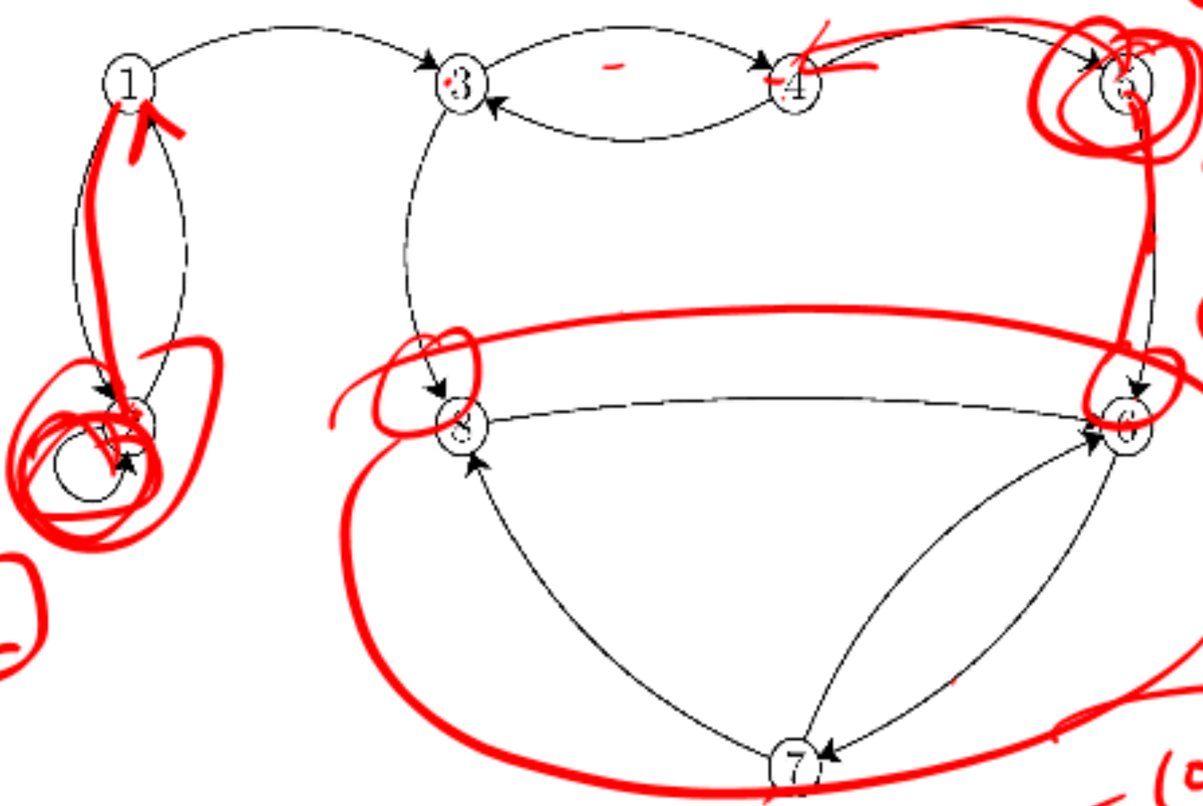
$$= P[X_1=i]\ P[X_3=j \mid X_1=i]$$

$$= P[X_1=i] \sum_{k=1}^{3} p_{ik}\, p_{kj}$$

Problem 6

Problem 6:

Consider the Markov chain shown in the figure below. It is assumed that when there is an arrow from state i to state j, then $p_{ij} > 0$. Find the communicating classes (also called equivalence classes) for this Markov chain.



$i \leftrightarrow j$ $\quad i \leftrightarrow i$

$i \to j$

$i \leftarrow j$

$1 \leftarrow 2$

class 1 = {1, 2}

class 2 = {3, 4}

class 3 = {6, 7, 8}

class 4 = {5}

$P_{ii}^{(0)} = 1$

Problem 7:

Consider the Markov chain in Problem 6.

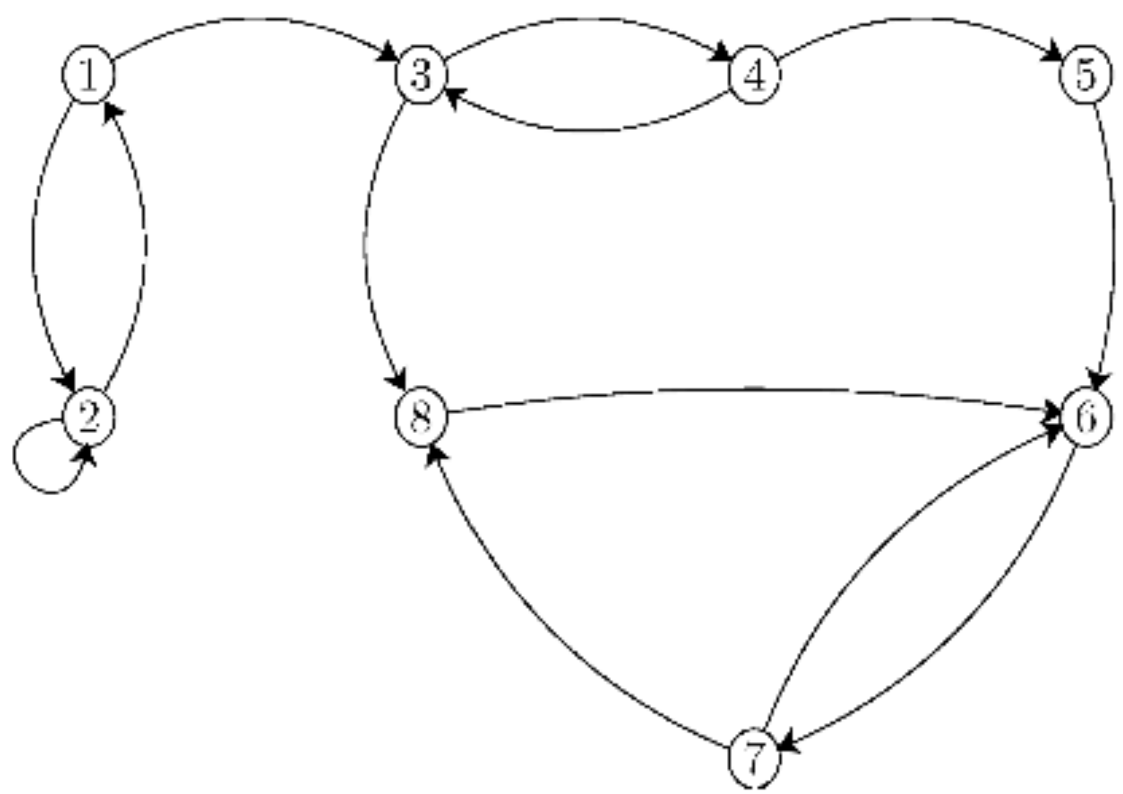
(a) Is Class 1 = {state 1, state 2} aperiodic?

(b) Is Class 2 = {state 3, state 4} aperiodic?

(c) Find $P(X_1 = 3, X_3 = 1)$.

Problem 6:

Consider the Markov chain shown in the figure below. It is assumed that when there is an arrow from state i to state j, then $p_{ij} > 0$. Find the communicating classes (also called equivalence classes) for this Markov chain.



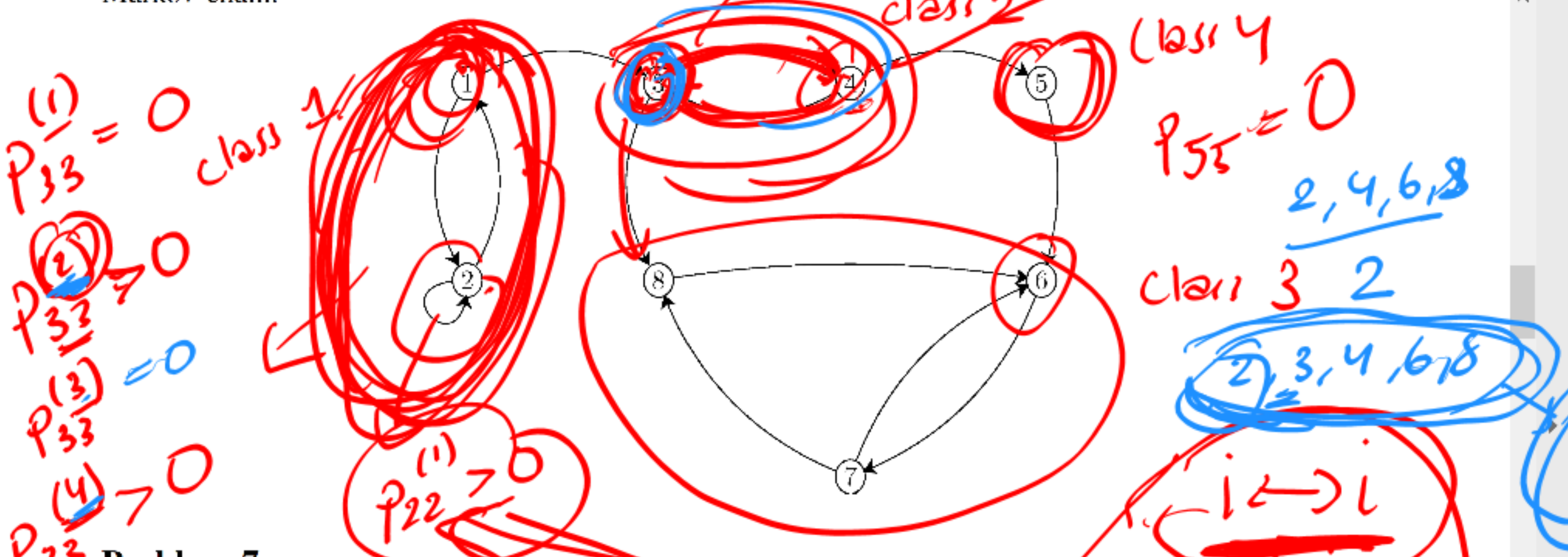
Problem 7:

Consider the Markov chain in Problem 6.

(a) Is Class 1 = {state 1, state 2} aperiodic?

(b) Is Class 2 = {state 3, state 4} aperiodic?

Markov chain.



Problem 7:

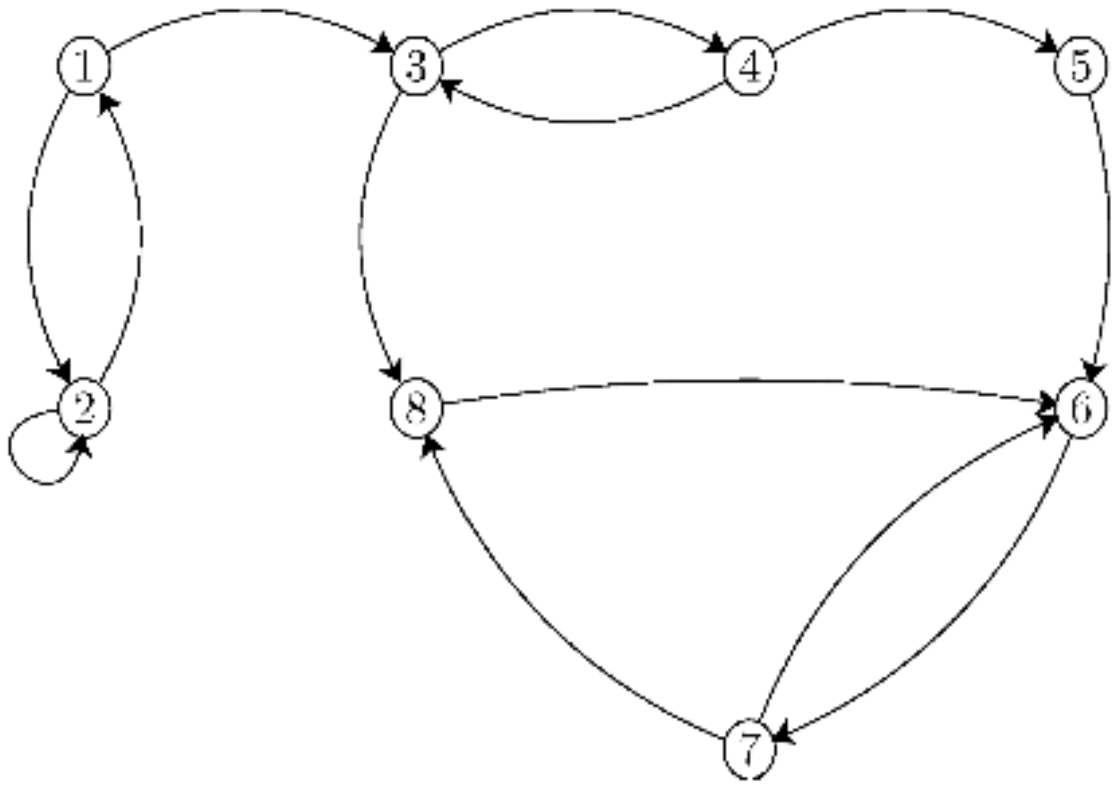
Consider the Markov chain in Problem 6.

(a) Is Class 1 = {state 1, state 2} aperiodic? aperiodic

(b) Is Class 2 = {state 3, state 4} aperiodic? periodic

g

Markov chain.



Problem 7:

Consider the Markov chain in Problem 6.

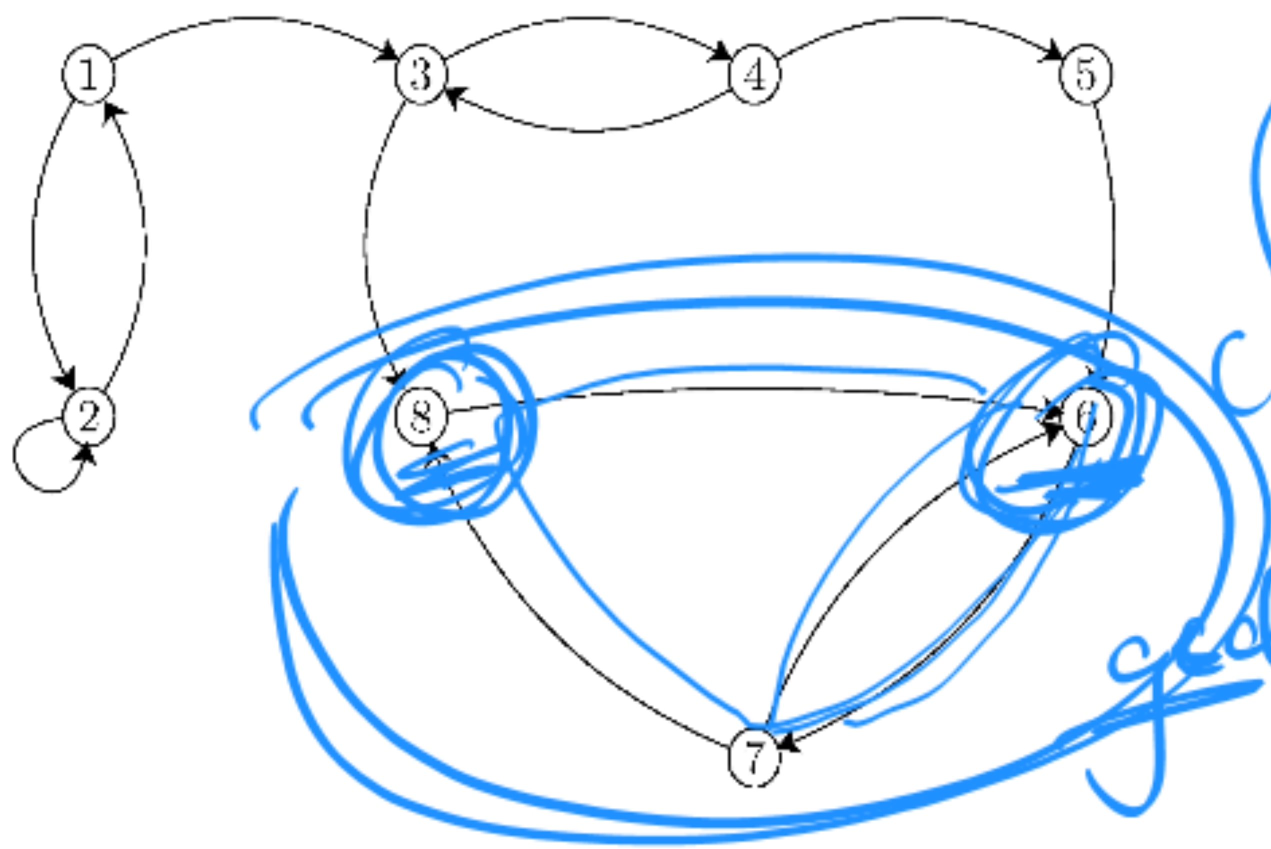
(a) Is Class 1 = {state 1, state 2} aperiodic?

(b) Is Class 2 = {state 3, state 4} aperiodic?

Problem 6:

Consider the Markov chain shown in the figure below. It is assumed that when there is an arrow from state i to state j, then $p_{ij} > 0$. Find the communicating classes (also called equivalence classes) for this Markov chain.

$P_{88}^{(1)} = 0$
$P_{88}^{(2)} = 0$
$P_{88}^{(3)} > 0$
$P_{88}^{(4)} = 0$
$P_{88}^{(5)} = 0$
$P_{88}^{(6)} > 0$



$P_{66}^{(1)} = 0$
$P_{66}^{(2)} > 0$
$P_{66}^{(3)} > 0$
Class 4
$\gcd\{2, 3\} = 1$

Problem 7:

Consider the Markov chain in Problem 6.

(a) Is Class 1 = {state 1, state 2} aperiodic?

(b) Is Class 2 = {state 3, state 4} aperiodic?

Problem 8

$S = \{0, 1\}$ $\quad P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$

where a and b are in $[0,1]$

such that $0 < a+b < 2$ $\rightarrow$ $a \neq 0, b \neq 0$ $\quad a \neq 1, b \neq 1$

$$\pi^{(0)} = \begin{bmatrix} P[X_0 = 0] & P[X_0 = 1] \end{bmatrix} = \begin{bmatrix} \alpha & 1-\alpha \end{bmatrix}$$

(a)

$$\begin{tikzcd} 0 \arrow[loop left, "1-a"] \arrow[r, bend left, "a"] & 1 \arrow[l, bend left, "b"] \arrow[loop right, "1-b"] \end{tikzcd}$$

class $= \{0, 1\}$

recurrent

aperiodic

irreducible

To have a limiting distribution for a finite markov chain we need $\rightarrow$ irreducible (= 1 class that is recurrent)

$\rightarrow$ aperiodic $\rightarrow$ $P^{(n)} = P^{(n)}$ $\rightarrow$ 1 step

(b) $P^{(n)} = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$

$\begin{bmatrix} P_{00}^{(n)} & P_{01}^{(n)} \\ P_{10}^{(n)} & P_{11}^{(n)} \end{bmatrix}$

$\lim_{n\to\infty} P^{(n)}$

$\lim_{n\to\infty} \underline{\pi}^{(n)} = \pi^{(0)} \lim_{n\to\infty} P^{(n)}$

$$\pi^{(n)} = \underline{\pi^{(0)}\, P^{(n)}}$$

$$\left[\alpha \quad 1-\alpha\right] \frac{1}{a+b}\left[\alpha \quad 1-\alpha\right]\begin{bmatrix} b & a \\ b & a \end{bmatrix}$$

$$\left[P[X_n = 0] \quad P[X_n = 1]\right]$$

$$= \frac{1}{a+b}\left[b \quad a\right]$$

$$\underline{\lim_{n\to\infty} \pi^{(n)}} = \lim_{n\to\infty} \underline{\pi}^{(0)} P^{(n)} = \pi^{(0)} \lim_{n\to\infty} P^{(n)}$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e., $\pi^{(0)} = [P(X_0 = 0)\ P(X_0 = 1)] = [\alpha\ 1 - \alpha]$, where $\alpha \in [0, 1]$.

(a) Sketch the Markov chain. How many classes are in the chain, are they recurrent/transient, and are they periodic/aperiodic?

(b) Given that

$a+b<2$ $\quad |1-(a+b)| \leq 1$

$$\lim_{n\to\infty} \quad P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \cancel{\frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}}. \quad =$$

show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

$\lim_{n\to\infty} 0.5^n =$

and that the Markov chain has a limiting distribution given by

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

$\lim_{n\to\infty} \frac{1}{2^n} \to 0$

(c) Find the stationary distribution for this chain by solving the balance equations. Is this also the limiting distribution for this chain? Why or why not?

(d) Consider the case when $a = b = 1$. Sketch the Markov chain and comment on the existence of a limiting distribution for this Markov chain.

(e) Consider the case when $a = b = 0$. Sketch the Markov chain and comment on the existence of a limiting distribution for this Markov chain.

(c)

$$\underline{\pi} = \underline{\pi} P$$

$$\sum_{j=0}^{1} \underline{\pi_j} = 1$$

$$\underline{\pi}^{(n)} = \pi^{(0)} P^n$$

$n+1$

$$\pi = \pi P$$

$$\underline{\pi} = [\pi_0 \quad \pi_1]$$

$$\lim_{n\to\infty} P[X_{\underline{n}} = 0] \qquad \lim_{n\to\infty} P[X_n = 1]$$

(c) Is Class 4 = {state 6, state 7, state 8} aperiodic?

Problem 8:

Consider a Markov chain with two possible states, $S = \{0, 1\}$. In particular, suppose that the transition matrix is given by

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

$a \neq 1, b \neq 1$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e., $\pi^{(0)} = [P(X_0 = 0)\ P(X_0 = 1)] = [\alpha\ 1-\alpha]$, where $\alpha \in [0, 1]$.

(a) Sketch the Markov chain. How many classes are in the chain, are they recurrent/transient, and are they periodic/aperiodic?

(b) Given that

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

$$\pi = \pi P \Rightarrow \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Final answer: $\pi = \begin{bmatrix} \dfrac{b}{a+b} & \dfrac{a}{a+b} \end{bmatrix}$

$$\pi_0 + \pi_1 = 1$$

$$\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} = \begin{bmatrix} (1-a)\pi_0 + b\pi_1 & a\pi_0 + (1-b)\pi_1 \end{bmatrix}$$

1

$$\pi_0 = (1-a)\pi_0 + b\pi_1$$

$$\cancel{\pi_1 = a\pi_0 + (1-b)\pi_1}$$

$$\pi_0 + \pi_1 = 1$$

2

From 2

$$\pi_0 = 1 - \pi_1 \Rightarrow \pi_1 = 1 - \pi_0$$

Plug in to 1

$$\pi_0 = (1-a)\pi_0 + b(1-\pi_0)$$

$$a\pi_0 = b - b\pi_0 \Rightarrow \pi_0 = \frac{b}{a+b}$$

$$\pi_1 = \frac{a}{a+b}$$

$$\pi = \pi P$$

$$\sum_j \pi_j = 1$$

→ find stationary dist.

because my chain is irreducible and aperiodic

This is the limiting dist

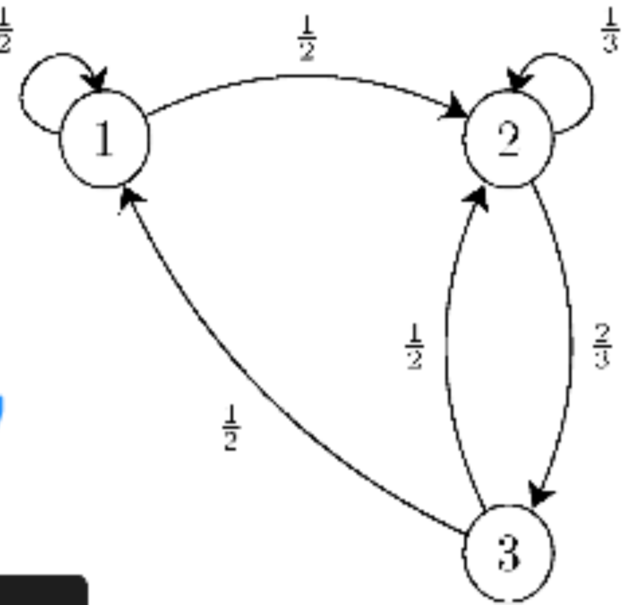
$$\lim_{n \to \infty} \pi^{(n)}$$

(d) Is the stationary distribution a limiting distribution for the chain?

Problem 10:

Consider the following Markov chain

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{matrix} =1 \\ =1 \\ =1 \end{matrix}$$



$$\pi = \pi P$$
$$\sum \pi_j = 1$$

π is stationary

$\Downarrow$

limiting.

(a) Is this chain irreducible? ✓

(b) Is this chain aperiodic?

(c) Find the stationary distribution for this chain. ✓

(d) Is the stationary distribution a limiting distribution for the chain?

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\pi = \pi P, \quad \sum_i \pi_i = 1$$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 & \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 & \frac{2}{3}\pi_2 \end{bmatrix}$$

$$\pi_1 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 \qquad \pi_2 = \pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 \qquad \pi_3 = \frac{2}{3}\pi_2$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

Problem 8:

Consider a Markov chain with two possible states, $S = \{0, 1\}$. In particular, suppose that the transition matrix is given by

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e., $\pi^{(0)} = [P(X_0 = 0) \; P(X_0 = 1)] = [\alpha \; 1-\alpha]$, where $\alpha \in [0, 1]$.

(a) Sketch the Markov chain. How many classes are in the chain, are they recurrent/transient, and are they periodic/aperiodic?

(b) Given that

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

and that the Markov chain has a limiting distribution given by

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

(c) Find the stationary distribution for this chain by solving the balance equations. Is this also the limiting distribution for this chain? Why or why not?