AMCS 241 - STAT 250

Lecture # 22

Chapter 10: Some RPs (continued)

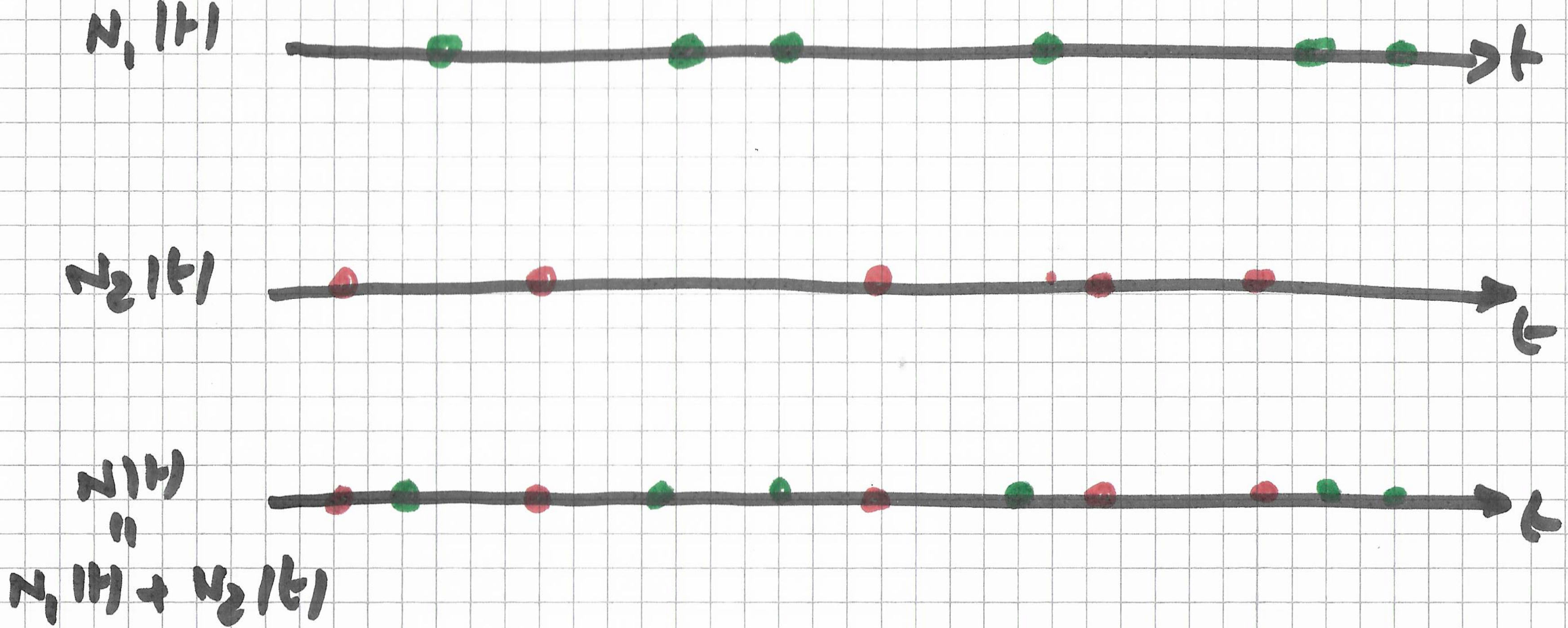
10.6.e Merging and Splitting of Poisson RPs

1. Merging

. Let $N_1(t), N_2(t), \dots, N_m(t)$ be m independent Poisson RPs with rates

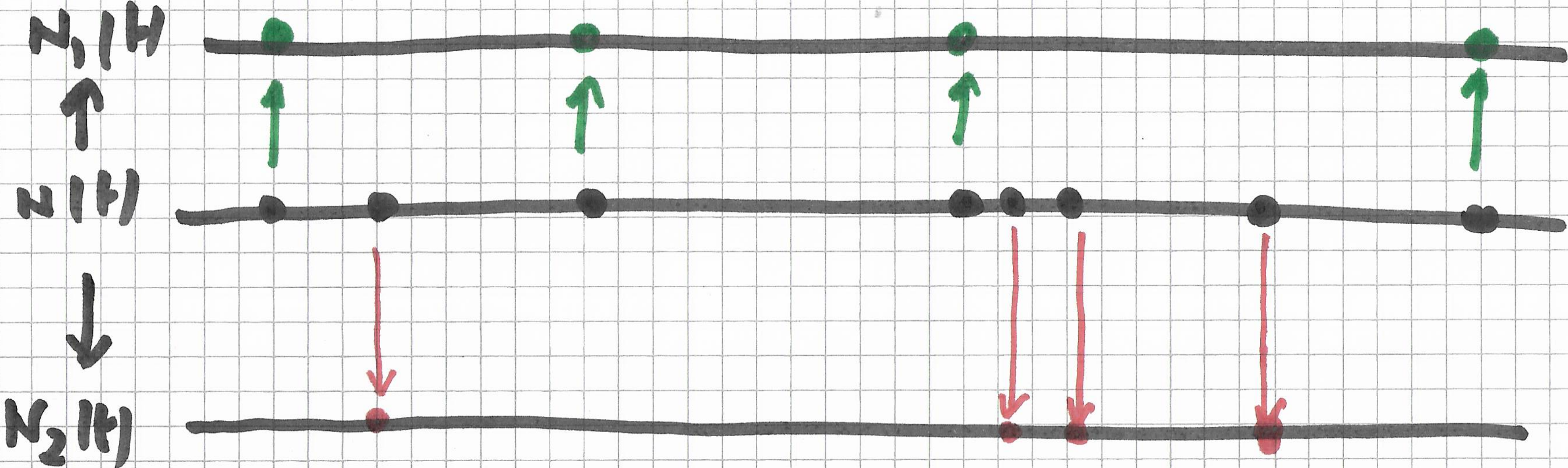
$\lambda_1, \lambda_2, \dots, \lambda_m$.

. Let also $N(t) = N_1(t) + N_2(t) + \dots + N_m(t)$ for all $t \geq 0$. Then $N(t)$ is also a Poisson RP with rate $\lambda_1 + \lambda_2 + \dots + \lambda_m$.



2- Splitting.

- Let $N(t)$ be a Poisson RP with rate λ
- Here we want to divide $N(t)$ to two RPs $N_1(t)$ and $N_2(t)$ in the following manner



- For each arrival, a coin with $\mathbb{P}[H] = p$ is tossed. For a Head, the arrival is sent to the first RP $N_1(t)$, otherwise it is sent to the second RP $N_2(t)$.

- The coin tosses are independent of each other and are independent of $N(t)$.

- Then

  1. $N_1(t)$ is a Poisson RP with rate $p\lambda$.

  2. $N_2(t)$ is a Poisson RP with rate $(1-p)\lambda$

  3. $N_1(t)$ and $N_2(t)$ are independent.

.10. 6. f Non-Homogeneous Poisson RP's

Def: Non Homogeneous Poisson RP with rate $\lambda(t) : [0, +\infty[\longrightarrow [0, +\infty[$ integrable function and

- $N(0) = 0$
- $N(t)$ has independent increments
- for any $t > 0$

$$\begin{cases} P[N(t+\Delta t) - N(t) = 0] = 1 - \lambda(t) \cdot \Delta t \\ P[N(t+\Delta t) - N(t) = 1] = \lambda(t) \cdot \Delta t \end{cases}$$

→ The resulting Non-Homogeneous Poisson RP will be characterized by

$$P[\underbrace{N(t+s) - N(t)}] \sim \text{Poisson with rate } \bar{\lambda}(s) = \int_t^{t+s} \lambda(\alpha)\, d\alpha$$

↑

Number of arrivals between t and $t+s$

↓ i.e

$$P[N(t+s) - N(t) = k] = e^{-\bar{\lambda}(s)} \cdot \frac{(\bar{\lambda}(s))^k}{k!}$$

10.6.g Compound Poisson RP

- Let $X(t) = \sum_{i=1}^{N(t)} U_i$ for $t \geq 0$

where the $\begin{cases} \bullet\ U_i \text{ are i.i.d RVs} \\ \bullet\ N(t) \text{ is a Poisson RP with rate } \lambda. \end{cases}$

- Assume that
  - $X(0) = 0$
  - U_i's and $N(t)$ are independent of each other for all time t.

Motivation

→ For example $X(t)$: Bank account balance evolution as function of time.

- $U_i > 0 \Rightarrow$ Deposit
- $U_i < 0 \Rightarrow$ withdrawal

- For a fixed $t = t_0$, what is the PDF of $X(t_0)$?

Let us use the characteristic function approach in conjunction with conditioning arguments

. For a fixed $t = t_0$, the CF of $X(t_0)$ is

$$\Phi_{X(t_0)}(\omega) = E\left[e^{j X(t_0)\omega}\right]$$

$$= E\left[e^{j\omega \sum_{i=1}^{N(t_0)} U_i}\right]$$

$$= E_{N(t_0)}\left[E_{U_1, U_2, \ldots, U_k}\left[e^{j\omega \sum_{i=1}^{N(t_0)} U_i} \Big/ N(t_0) = k\right]\right]$$

where $k = N(t_0)$.

Since the U_i's and $N(t_0)$ are independent

$$= E_{N(t_0)}\left[E_{\underset{\underset{N(t_0)}{\parallel}}{U_1 U_2 \dots U_k}} \left[e^{j w \sum_{i=1}^{k} U_i} \right] \right)$$

$$= E_{N(t_0)}\left[E_{U_1 U_2 \dots U_k} \left[\prod_{i=1}^{k} e^{j w U_i} \right] \right)$$

U_i's are independent

$$= E_{N(t_0)}\left[\prod_{i=1}^{k} E_{U_i} \left[e^{j w U_i} \right] \right)$$

$$= E_{\underset{\underset{k}{\parallel}}{N(t_0)}}\left[\prod_{i=1}^{k} \Phi_{U_i}(w) \right] \overset{\substack{U_i\text{'s are} \\ \text{identically distributed}}}{=} \longrightarrow$$

$$= E_{\underset{k}{\overset{\shortparallel}{N(t_0)}}} \left[\left(\Phi_U(\omega) \right)^k \right]$$

$$= \sum_{k=0}^{+\infty} \left(\Phi_U(\omega) \right)^k \, P\left[N(t_0) = k \right]$$

// $N(t)$ is a Poisson RP

$$e^{-\lambda t_0} \frac{(\lambda t_0)^k}{k!}$$

$$= \sum_{k=0}^{+\infty} e^{-\lambda t_0} \frac{\left(\Phi_U(\omega) \cdot \lambda t_0 \right)^k}{k!}$$

$$= e^{-\lambda t_0} \underbrace{\sum_{k=0}^{+\infty} \frac{(\Phi_U(\omega)\ \lambda t_0)^k}{k!}}$$

$$\downarrow$$

$$\exp\left(\lambda t_0\ \Phi_U(\omega)\right)$$

$$\Downarrow$$

$$\boxed{\Phi_{X(t_0)}(\omega) = \exp\left[\lambda t_0 \left(\Phi_U(\omega) - 1\right)\right]}$$

$\downarrow$ Inverse FT to get the PDF of $X(t_0)$.

Remarks

① If $U_i = 1$ for all i

$$\Phi_U(w) = E[e^{jwU}] = e^{jw}$$

$$\Downarrow$$

$$\Phi_{X(t_0)}(w) = \exp(\lambda t_0 (e^{jw} - 1))$$ which

is the CF of the Poisson Counting RP (i.e in this case $X(t_0) = N(t_0)$)

② Mean of a Compound Poisson R.V.

Recall:

$$E[X(t_0)] = \frac{1}{j} \left. \frac{d\,\Phi_{X(t_0)}(\omega)}{d\omega} \right|_{\omega=0}$$

$$= \frac{1}{j} \, \lambda t_0 \left. \frac{d\,\Phi_U(\omega)}{d\omega} \right|_{\omega=0} \cdot$$

$$\underbrace{\left. \exp\left[\lambda t_0 \left(\underbrace{\Phi_U(\omega)}_{=1} - 1 \right) \right] \right|_{\omega=0}}_{=1}$$

$$E[X(t_0)] = \frac{1}{j}\,\lambda t_0\,\frac{d\Phi_U(w)}{dw}\bigg|_{w=0}$$

but. $$\frac{1}{j}\,\frac{d\Phi_U(w)}{dw}\bigg|_{w=0} = E[U]$$

. $$\lambda t_0 = E[N(t_0)]$$

$$\Downarrow$$

$$\boxed{E[X(t_0)] = E[N(t_0)] \cdot E[U]}$$

10.7. Wiener RP and White Noise RP

10.7.a Random Walk

- We limit ourselves to one dimensional random walk.

- Let $$D_n = 2 I_n - 1$$

  with $I_n = \text{Bernoulli}(p)$

  $= 1$ with p

  $= 0$ with $1-p$

$$\begin{cases} \text{If } I_n = +1 \Rightarrow D_n = +1 \\ \text{If } I_n = 0 \Rightarrow D_n = -1 \end{cases}$$

→ D_n might represent the change of the position of a particle that moves along a straight line in jumps of ∓ 1 every time unit.

- $E(D_n) = E[2I_n - 1] = 2\,\overset{p}{E(I_n)} - 1 = 2p - 1$
- $Var(D_n) = Var(2I_n - 1) = 4\,\underset{p(1-p)}{Var(I_n)} = 4p(1-p)$

Let

$$S_n = D_1 + D_2 + \dots + D_n$$
$$= S_{n-1} + D_n$$

represents the position of the particle at time n and S_n is called a <u>one dimensional random walk</u>

10.7.b Wiener RP (Brownian Motion)

- Consider a one dimensional symmetric ($p = 1/2$) random walk

- Take steps of magnitude h every δ seconds.
- We obtain the continuous time RW

$X_\delta(t) =$ Accumulated Sum up to time t

$= h(D_1 + D_2 + \cdots + D_{t/\delta}]$

$= h\, S_n$ with $n = t/\delta$

- $E[X_\delta(t)] = h\, E[S_n]$

$$= h\, E[D_1 + D_2 + \dots + D_n]$$

$$= h \cdot 0 \implies \boxed{E[X_\delta(t)] = 0}$$

- $Var[X_\delta(t)] = h^2\, Var[S_n]$

$$= h^2\, Var[D_1 + D_2 + \dots + D_n]$$

D_i's are indep

$$= h^2 \sum_{i=1}^{n} Var(D_i)$$

identical i.i.d

$$= h^2\, n \cdot 4 \overset{\text{"}1/2\text{"}}{p} \overset{\text{"}1/2\text{"}}{(1-p)}$$

$$= h^2 n \implies \boxed{Var[X_\delta(t)] = h^2 n}$$

Let $\delta \to 0$ and $h \to 0$

but $\alpha = \dfrac{h^2}{\delta}$ = constant

$$h = \sqrt{\alpha . \delta}$$

$$\lim_{\delta \to 0} X_\delta(t) = X_0(t) \triangleq \boxed{X(t)}$$

Wiener RP

- $E[X(t)] = 0$
- $Var[X(t)] = h^2 n = \alpha \delta n$

$$= \alpha \frac{t}{n} . n = \alpha t$$

so $\boxed{Var[X(t)] = \alpha t}$

Properties of the Wiener RP

Th 1

$X(t)$ is a zero mean RP with variance αt and for a fixed $t = t_0$ $X(t_0)$ is a Gaussian RV i.e

$$f_{X(t_0)}(x) = \frac{1}{\sqrt{2\pi\alpha t_0}} e^{-\left(\frac{x^2}{2\alpha t_0}\right)}$$

$$-\infty < x < +\infty$$

Th 2

$X(t)$ has independent and stationary increments (i.e.

$x(t_1),\ x(t_2) - x(t_1),\ \ldots,\ x(t_n) - x(t_{n-1})\ ..)$

are independent Gaussian RVs

Th 3

$$R_{XX}(t_1, t_2) = E\left[X(t_1)\, X(t_2)\right]$$

$$= \alpha \min(t_1, t_2)$$

Note that $X(t)$ is **NOT** WSS.

Proof:

If $t_2 > t_1$

$$R_{xx}(t_1, t_2) = E[X(t_1)\, X(t_2)] =$$

$$E[X(t_1)\,[X(t_2) + X(t_1) - X(t_1)]] =$$

$$E[X(t_1)\,[X(t_1) + X(t_2) - X(t_1)]] =$$

$$E[(X(t_1))^2] + E(X(t_1) \cdot (X(t_2) - X(t_1)))$$

$$\| \qquad\qquad\qquad\qquad \Downarrow \text{Th}$$

$$\alpha t_1 \qquad\qquad E(X(t_1)) \cdot E(X(t_2) - X(t_1)) = 0$$

So

$$R_{xx}(t_1, t_2) = \alpha t_1$$
$$= \alpha \min(t_1, t_2)$$

If $t_2 < t_1$ .. same steps

$$R_{xx}(t_1, t_2) = \alpha t_2 = \alpha \min(t_1, t_2)$$

(4) <u>Th 4</u>

<u>$x(t)$ is a Gaussian R.P.</u>

10.7.c White Noise RP

$$N(t) = \frac{d\,X(t)}{dt} \longrightarrow \text{Wiener RP}$$

(1.) $X(t)$ is a Gaussian RP $\Longrightarrow$

$N(t)$ is a Gaussian RP

(2.) $R_{NN}(t_1, t_2) = E[N(t_1)\, N(t_2)]$

$$= E\left[\left.\frac{d\,X(t)}{dt}\right|_{t=t_1} \left.\frac{d\,X(t)}{dt}\right|_{t=t_2}\right]$$

$$= \frac{d}{dt_1}\frac{d}{dt_2}\, E[X(t_1)\, X(t_2)]$$

$$R_{NN}(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_{xx}(t_1, t_2)$$

Recall that $R_{xx}(t_1, t_2) = \alpha \min(t_1, t_2)$

$$= \begin{cases} \alpha t_1 & t_1 < t_2 \\ \alpha t_2 & t_2 < t_1 \end{cases}$$

$$\frac{d R_{xx}(t_1, t_2)}{\partial t_2} = \begin{cases} 0 & t_1 < t_2 \\ \alpha & t_2 < t_1 \end{cases}$$

$$= \alpha \; u(t_1 - t_2)$$

unit step Function

$$\frac{d^2 R_{XX}(t_1, t_2)}{dt_1\, dt_2} = \alpha\, \delta(t_1 - t_2)$$

which is function of $t_2 - t_1$

$\Downarrow$ $\qquad$ $\uparrow +$

$N(t)$ is WSS (since $E[N(t)] = 0$)

So in summary

$N(t)$ is a Gaussian WSS RP with

$$\begin{cases} E[N(t)] = 0 \\ R_{NN}(\tau) = \alpha\, \delta(\tau) \end{cases}$$

White Noise Gaussian RP