# Problem Set 6 Solution

**Problem 1:**

1.

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dxdydz \\
&= \int_0^1\int_0^1\int_0^1 c(x+2y+3z)\ \ dxdydz \\
&= \int_0^1\int_0^1 c\left(\frac{1}{2}+2y+3z\right)\ \ dydz \\
&= \int_0^1 c\left(\frac{3}{2}+3z\right)\ \ dz \\
&= 3c.
\end{aligned}
$$

Thus, $c = \frac{1}{3}$.

2. To find the marginal PDF of $X$, we note that $R_X = [0,1]$. For $0 \le x \le 1$, we can write

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dydz \\
&= \int_0^1\int_0^1 \frac{1}{3}(x+2y+3z)\ \ dydz \\
&= \int_0^1 \frac{1}{3}(x+1+3z)\ \ dz \\
&= \frac{1}{3}\left(x+\frac{5}{2}\right).
\end{aligned}
$$

Thus,

$$
f_X(x) = \begin{cases} \frac{1}{3}\left(x+\frac{5}{2}\right) & 0 \le x \le 1 \\ \\ 0 & \text{otherwise} \end{cases}
$$

**Problem 2:**

We have

$$
\begin{aligned}
\mathbf{C_X} &= \mathbf{E}[(\mathbf{X}-\mathbf{EX})(\mathbf{X}-\mathbf{EX})^{\mathbf{T}}] \\
&= \mathbf{E}[(\mathbf{X}-\mathbf{EX})(\mathbf{X}^{\mathbf{T}}-\mathbf{EX}^{\mathbf{T}})] \\
&= \mathbf{E}[\mathbf{XX}^{\mathbf{T}}]-\mathbf{EXEX}^{\mathbf{T}}-\mathbf{EXEX}^{\mathbf{T}}+\mathbf{EXEX}^{\mathbf{T}} \quad \text{(by linearity of expectation)} \\
&= \mathbf{R_X}-\mathbf{EXEX}^{\mathbf{T}}.
\end{aligned}
$$

## Problem 3:

Note that by linearity of expectation, we have

$$\mathbf{EY} = \mathbf{AEX} + \mathbf{b}.$$

By definition, we have

$$\begin{aligned}\mathbf{C_Y} &= \mathbf{E}[(\mathbf{Y}-\mathbf{EY})(\mathbf{Y}-\mathbf{EY})^{\mathbf{T}}] \\ &= \mathbf{E}[(\mathbf{AX}+\mathbf{b}-\mathbf{AEX}-\mathbf{b})(\mathbf{AX}+\mathbf{b}-\mathbf{AEX}-\mathbf{b})^{\mathbf{T}}] \\ &= E[\mathbf{A}(\mathbf{X}-\mathbf{EX})(\mathbf{X}-\mathbf{EX})^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}] \\ &= \mathbf{AE}[(\mathbf{X}-\mathbf{EX})(\mathbf{X}-\mathbf{EX})^{\mathbf{T}}]\mathbf{A}^{\mathbf{T}} && \text{(by linearity of expectation)} \\ &= \mathbf{AC_X}\mathbf{A}^{\mathbf{T}}.\end{aligned}$$

## Problem 4:

We first obtain the marginal PDFs of $X$ and $Y$. Note that $R_X = R_Y = (0,1)$. We have for $x \in R_X$

$$\begin{aligned}f_X(x) &= \int_0^1 \frac{3}{2}x^2 + y \; dy \\ &= \frac{3}{2}x^2 + \frac{1}{2}, \quad \text{for } 0 < x < 1.\end{aligned}$$

Similarly, for $y \in R_Y$, we have

$$\begin{aligned}f_Y(y) &= \int_0^1 \frac{3}{2}x^2 + y \; dx \\ &= y + \frac{1}{2}, \quad \text{for } 0 < y < 1.\end{aligned}$$

From these, we obtain $EX = \frac{5}{8}$, $EX^2 = \frac{7}{15}$, $EY = \frac{7}{12}$, and $EY^2 = \frac{5}{12}$. We also need $EXY$. By LOTUS, we can write

$$\begin{aligned}EXY &= \int_0^1 \int_0^1 xy\left(\frac{3}{2}x^2 + y\right) dx dy \\ &= \int_0^1 \frac{3}{8}y + \frac{1}{2}y^2 dy \\ &= \frac{17}{48}.\end{aligned}$$

From this, we also obtain

$$\begin{aligned}\text{Cov}(X,Y) &= EXY - EXEY \\ &= \frac{17}{48} - \frac{5}{8}.\frac{7}{12} \\ &= -\frac{1}{96}.\end{aligned}$$

The correlation matrix $R_U$ is given by

$$\mathbf{R_U} = \mathbf{E}[\mathbf{UU}^{\mathbf{T}}] = \begin{bmatrix} EX^2 & EXY \\ EYX & EY^2 \end{bmatrix} = \begin{bmatrix} \frac{7}{15} & \frac{17}{48} \\ \frac{17}{48} & \frac{5}{12} \end{bmatrix}.$$

The covariance matrix $\mathbf{C_U}$ is given by

$$\mathbf{C_U} = \begin{bmatrix} \text{Var}(X) & \text{Cov}(X,Y) \\ \text{Cov}(Y,X) & \text{Var}(Y) \end{bmatrix} = \begin{bmatrix} \frac{73}{960} & -\frac{1}{96} \\ -\frac{1}{96} & \frac{11}{144} \end{bmatrix}.$$

**Problem 5:**

$$f_{XYZ}(x,y,z) = \begin{cases} x+y & 0 \le x,y,z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) $f_{XY}(x,y) = \int_0^1 f_{XYZ}(x,y,z)\,dz$

$$= \int_0^1 (x+y)\,dz = (x+y)\Big[z\Big]_0^1$$

$$= x+y \quad 0 \le x,y \le 1$$

so $f_{XY}(x,y) = \begin{cases} x+y & 0 \le x,y \le 1 \\ 0 & \text{otherwise} \end{cases}$

(b) $f_X(x) = \int_0^1 f_{XY}(x,y)\,dy = \int_0^1 (x+y)\,dy$

$$= \Big[xy + y^2/2\Big]_0^1$$

$$= x + 1/2, \quad 0 \le x \le 1$$



so

$$f_X(x) = \begin{cases} x + 1/2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le z \le 1$

(c) $$f_Z(z) = \int_0^1 \int_0^1 f_{XYZ}(x,y,z)\,dx\,dy$$

$$= \int_0^1 \int_0^1 (x+y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{1}{2} + y\right) dy = \left[\frac{1}{2}y + \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

so $$f_Z(z) = \begin{cases} 1 & 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

so

$$f_{XY|Z}(x,y|z) = \frac{f_{XYZ}(x,y,z)}{f_Z(z)}$$

$$= \begin{cases} x+y & 0 \le x,y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le z \le 1$

(d) $\cancel{f_{XY|Z}(x,y|z) = f_{X|Z}(x,y) f_Z(z)}$

$$f_{XYZ}(x,y,z) = f_{XY|Z}(x,y|z)$$

therefore X and Y are independent of Z. ...

**Problem 6:**

$$f_X(x) = \begin{cases} 6e^{-x_1-2x_2-3x_3}, & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x_1,x_2,x_3) = \int_0^{x_3}\int_0^{x_2}\int_0^{x_1} 6e^{-x_1'-2x_2'-3x_3'}\,dx_1'\,dx_2'\,dx_3'$$

$$= 6\int_0^{x_3} e^{-3x_3'}\int_0^{x_2} e^{-2x_2'}\int_0^{x_1} e^{-x_1'}\,dx_1'\,dx_2'\,dx_3'$$

$$= 6\int_0^{x_3} e^{-3x_3'}\int_0^{x_2} e^{-2x_2'}\left[-e^{-x_1'}\right]_0^{x_1} dx_2'\,dx_3'$$

$$= 6\int_0^{x_3} e^{-3x_3'}\int_0^{x_2} e^{-2x_2'}\left(1-e^{-x_1}\right) dx_2'\,dx_3'$$

$$= 6(1-e^{-x_1}) \int_0^{x_3} e^{-3x_3'} \left[ -\frac{1}{2} e^{-2x_2'} \right]_0^{x_2} dx_3'$$

$$= 3(1-e^{-x_1})(1-e^{-2x_2}) \int_0^{x_3} e^{-3x_3'} dx_3'$$

$$= 3(1-e^{-x_1})(1-e^{-2x_2}) \left[ -\tfrac{1}{3} e^{-3x_3'} \right]_0^{x_3}$$

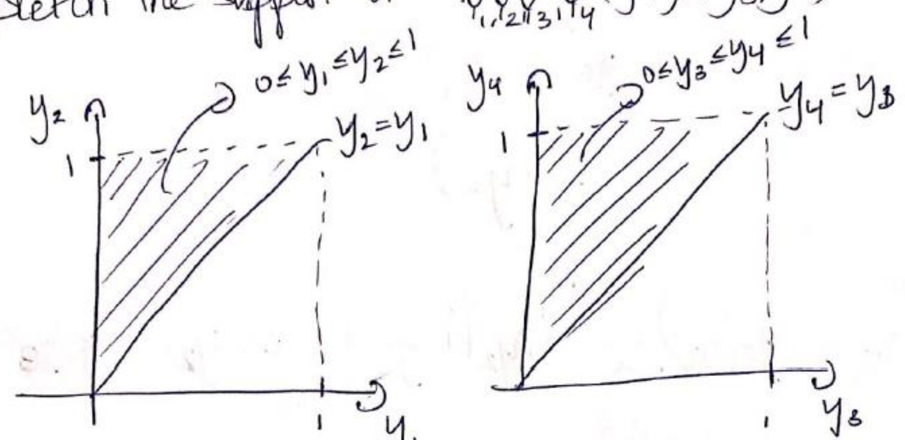
$$= (1-e^{-x_1})(1-e^{-2x_2})(1-e^{-3x_3}) \quad \text{for} \quad x_1, x_2, x_3 \geq 0$$

so

$$F_X(x) = \begin{cases} (1-e^{-x_1})(1-e^{-2x_2})(1-e^{-3x_3}) & x_1, x_2, x_3 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 7:**

Sketch the support of $f_{Y_1,Y_2,Y_3,Y_4}(y_1, y_2, y_3, y_4)$



• $f_{Y_1,Y_4}(y_1,y_4) = \int_0^{y_4}\int_{y_1}^{1} f_{Y_1,Y_2,Y_3,Y_4}(y_1,y_2,y_3,y_4)\, dy_2\, dy_3$

$$= \int_0^{y_4}\int_{y_1}^{1} 4\, dy_2\, dy_3$$

$$= 4(1-y_1)y_4 \quad 0 \le y_1 \le 1,\ 0 \le y_4 \le 1$$

so $f_{Y_1,Y_4}(y_1,y_4) = \begin{cases} 4y_4(1-y_1) & 0 \le y_1 \le 1,\ 0 \le y_4 \le 1 \\ 0 & \text{otherwise} \end{cases}$

---

• $f_{Y_2,Y_3}(y_2,y_3) = \int_{y_3}^{1}\int_0^{y_2} 4\, dy_1\, dy_4$

$$= 4y_2(1-y_3)$$

so $f_{Y_2,Y_3}(y_2,y_3) = \begin{cases} 4y_2(1-y_3) & 0 \le y_2 \le 1,\ 0 \le y_3 \le \\ 0 & \text{otherwise} \end{cases}$

$$\bullet\ f_{Y_3}(y_3) = \int_0^1 f_{Y_2,Y_3}(y_2,y_3)\,dy_2$$

$$= \left[2y_2^2(1-y_3)\right]_0^1 = 2(1-y_3) \quad 0 \le y_3 \le 1$$

so $f_{Y_3}(y_3) = \begin{cases} 2(1-y_3) & 0 \le y_3 \le 1 \\ 0 & \text{otherwise} \end{cases}$

**Problem 8:**

For independence, we need

$$f_{Y_1,Y_2,Y_3,Y_4}(y_1,y_2,y_3,y_4) = f_{Y_1}(y_1) f_{Y_2}(y_2) f_{Y_3}(y_3) f_{Y_4}(y_4)$$

From Problem 7:

$$f_{Y_1,Y_4}(y_1,y_4) = \begin{cases} 4y_4(1-y_1) & 0 \le y_1 \le 1,\ 0 \le y_4 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

so

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_4}(y_1,y_4)\,dy_4$$

$$= \int_0^1 4y_4(1-y_1)\,dy_4 = \left[2y_4^2(1-y_1)\right]_0^1$$

$$= 2(1-y_1) \quad 0 \le y_1 \le 1$$

also

$$f_{Y_4}(y_4) = \int_0^1 4y_4(1-y_1)\,dy_1 = 4y_4\left[-\frac{(1-y_1)^2}{2}\right]_0^1 =$$

$$= 4y_4\left(\tfrac{1}{2}\right) = 2y_4 \quad 0 \le y_4 \le 1$$

from Problem 7

$$f_{Y_3}(y_3) = 2(1-y_3) \quad 0 \le y_3 \le 1$$

and

$$f_{Y_2,Y_3}(y_2,y_3) = 4y_2(1-y_3) \quad 0 \le y_2 \le 1,\ 0 \le y_3 \le 1$$

so

$$f_{Y_2}(y_2) = \int_0^1 4y_2(1-y_3)\,dy_3$$

$$= 4y_2\left[-\tfrac{1}{2}(1-y_3)^2\right]_0^1$$

$$= 2y_2 \quad 0 \le y_2 \le 1$$

Finally,

$$f_{Y_1}(y_1)f_{Y_2}(y_2)f_{Y_3}(y_3)f_{Y_4}(y_4)$$

$$= \begin{cases} 16(1-y_1)y_2(1-y_3)y_4 & 0 \le y_1, y_2, y_3, y_4 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

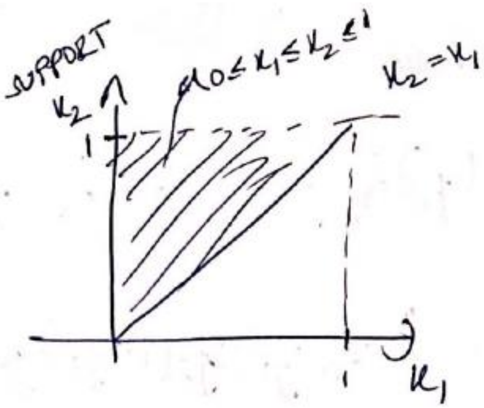
$$\neq f_{Y_1,Y_2,Y_3,Y_4}(y_1, y_2, y_3, y_4)$$

Therefore $Y_1, Y_2, Y_3, Y_4$ are NOT independent random variables.

**Problem 9:**

Problem 9

$$\bullet\ \mathbb{E}[X] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \end{bmatrix}$$



$$\mathbb{E}[X_1] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f_X(x)\, dx_1\, dx_2$$

$$= \int_0^1 \int_0^{x_2} 2x_1\, dx_1\, dx_2 = \int_0^1 2\left[\tfrac{1}{2}x_1^2\right]_0^{x_2} dx_2$$

$$= \int_0^1 x_2^2\, dx_2$$

$$= \left[\frac{x_2^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\mathbb{E}[X_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 f_X(x)\, dx_1\, dx_2 = \int_0^1\int_0^{x_2} 2x_2\, dx_1\, dx_2$$

$$= \int_0^1 2x_2^2\, dx_2 = 2\left[\frac{x_2^3}{3}\right]_0^1 = \frac{2}{3}$$

so $\mathbb{E}[X] = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

• $R_X = \begin{bmatrix} \mathbb{E}[X_1^2] & \mathbb{E}[X_1 X_2] \\ \mathbb{E}[X_1 X_2] & \mathbb{E}[X_2^2] \end{bmatrix}$

$$\mathbb{E}[X_1^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 f_X(x)\, dx_1 dx_2$$

$$= \int_0^1\int_0^{x_2} 2x_1^2\, dx_1 dx_2 = 2\int_0^1 \left[\frac{x_1^3}{3}\right]_0^{x_2} dx_2$$

$$= \frac{2}{3}\int_0^1 x_2^3\, dx_2$$

$$= \frac{2}{3}\left[\frac{x_2^4}{4}\right]_0^1 = \frac{1}{6}$$

$$\mathbb{E}[X_2^2] = \int_0^1\int_0^{x_2} 2x_2^2\, dx_1 dx_2$$

$$= \int_0^1 2x_2^3\, dx_2 = \frac{2}{4}\left[x_2^4\right]_0^1 = \frac{1}{2}$$

$$\mathbb{E}[X_1X_2] = \int_0^1\int_0^{x_2} 2x_1x_2\, dx_1 dx_2$$

$$= \int_0^1 2x_2\left[\frac{x_1^2}{2}\right]_0^{x_2} dx_2$$

$$= \int_0^1 x_2^3\, dx_2 = \left[\frac{x_2^4}{4}\right]_0^1 = \frac{1}{4}$$

so $R_X = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$

$$C_X = R_X - \mathbb{E}[X]\mathbb{E}[X^T]$$

$$\mathbb{E}[X]\mathbb{E}[X^T] = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix}$$

$$\text{so } C_X = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix}$$

$$= \begin{bmatrix} 1/18 & 1/36 \\ 1/36 & 1/18 \end{bmatrix}$$

**Problem 10:**

$$\mu_Y = A\mu_X + b$$

$$= \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2 \\ 3 \end{bmatrix}$$

$R_Y = ?$ $\quad R_X = \mathbb{E}[XX^T]$

$$\text{so } R_Y = \mathbb{E}[YY^T] = \mathbb{E}[(AX+b)(AX+b)^T]$$

$$= \mathbb{E}[AXX^TA^T + AXb^T + bX^TA^T + bb^T]$$

$$= AR_XA^T + A\mu_Xb^T + b\mu_X^TA^T + bb^T$$

$$= AR_XA^T + (A\mu_X)b^T + b(A\mu_X)^T + bb^T$$

$$A\mu_X = \begin{bmatrix} 1/3 \\ 4 \\ 5 \end{bmatrix}$$

$$AR_x A^T = \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & 1/4 \\ 7/4 & 3 \\ 2 & 15/4 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & 7/4 & 2 \\ 7/4 & 39/2 & 93/4 \\ 2 & 93/4 & 57/2 \end{bmatrix}$$

$$bb^T = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

$$A m_x b^T = \begin{bmatrix} 1/3 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -2/3 & -2/3 \\ 0 & -8 & -8 \\ 0 & -10 & -10 \end{bmatrix}$$

so

$$R_y = \begin{bmatrix} 1/6 & 7/4 & 2 \\ 7/4 & 39/2 & 93/4 \\ 2 & 93/4 & 57/2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -2/3 & -2/3 \\ 0 & -8 & -8 \\ 0 & -10 & -10 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -2/3 & -8 & -10 \\ -2/3 & -8 & -10 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & 13/12 & 4/3 \\ 13/12 & 15/2 & 37/4 \\ 4/3 & 37/4 & 25/2 \end{bmatrix}$$

$$C_Y = A C_X A^T$$

$$= \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1/18 & 1/36 \\ 1/36 & 1/18 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1/18 & 1/36 \\ 5/12 & 1/3 \\ 1/3 & 5/12 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1/18 & 5/12 & 1/3 \\ 5/12 & 7/2 & 13/4 \\ 1/3 & 13/4 & 7/2 \end{bmatrix}$$

**Problem 11:**

(a)

$$R_{XY} = R_X A^T + \mu_X b^T$$

$$= \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 0 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & 7/4 & 2 \\ 1/4 & 3 & 15/4 \end{bmatrix} + \begin{bmatrix} 0 & -2/3 & -2/3 \\ 0 & -4/3 & -4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & 13/12 & 4/3 \\ 1/4 & 5/3 & 29/12 \end{bmatrix}$$

$$C_{XY} = C_X A^T = \begin{bmatrix} 1/18 & 1/36 \\ 1/36 & 1/18 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1/18 & 5/12 & 1/3 \\ 1/36 & 1/3 & 5/12 \end{bmatrix}$$

(b)

$$\rho_{Y_1 Y_3} = \frac{Cov[Y_1, Y_3]}{\sqrt{Var[Y_1]\, Var[Y_3]}}$$

→ read from $C_Y$

→ read from $C_Y$

$$= \frac{1/3}{\sqrt{(1/18)(7/2)}} \approx 0.756$$

→ read from $C_{XY}$

$$\rho_{X_2, Y_1} = \frac{Cov[X_2, Y_1]}{\sqrt{Var[X_2]\, Var[Y_1]}}$$

→ read from $C_X$ and $C_Y$

$$= \frac{1/36}{\sqrt{(1/18)(1/18)}} = 1/2$$

**Problem 12:**

$$\begin{aligned} f_{XY}(x,y) &= \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dz \\ &= \int_0^1 \frac{1}{3}(x+2y+3z)dz \\ &= \frac{1}{3}\left[(x+2y)z + \frac{3}{2}z^2\right]_0^1 \\ &= \frac{1}{3}\left(x+2y+\frac{3}{2}\right), \qquad \text{for} \quad 0 \leq x, y \leq 1. \end{aligned}$$

Thus,

$$f_{XY}(x,y) = \begin{cases} \frac{1}{3}\left(x+2y+\frac{3}{2}\right) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 13:**

Since $X_1$ and $X_2$ are jointly Gaussian, they have the bivariate Gaussian PDF of the form

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho_{X_1X_2}^2}} \exp\left(\frac{-1}{2(1-\rho_{X_1X_2}^2)} \cdot \left(\left(\frac{x_1-\mu_{X_1}}{\sigma_{X_1}}\right)^2 - \frac{2\rho_{X_1X_2}(x_1-\mu_{X_1})(x_2-\mu_{X_2})}{\sigma_{X_1}\sigma_{X_2}} + \left(\frac{x_2-\mu_{X_2}}{\sigma_{X_2}}\right)^2\right)\right)$$

From the given information,

$\mu_{X_1} = 50$, $\mu_{X_2} = 62$

$\sigma_{X_1} = \sqrt{16} = 4$ , $\sigma_{X_2} = \sqrt{16} = 4$

From $C_X$, $\rho_{X_1X_2} = \frac{\mathrm{Cov}[X_1,X_2]}{\sqrt{(16)(16)}} = \frac{12.8}{16} = 0.8$

so

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{(19.2)\pi} \exp\left(-\frac{1}{0.72}\left(\frac{(x_1-50)^2}{16} - \frac{1.6(x_1-50)(x_2-62)}{16} + \frac{(x_2-62)^2}{16}\right)\right)$$

$$= \frac{1}{19.2\pi} \exp\left(-\frac{1}{11.52}\left((x_1-50^2) - 1.6(x_1-50)(x_2-62) + (x_2-62)^2\right)\right)$$

**Problem 14:**

$Y = AX + b$ where

$$A = \begin{bmatrix} 5/9 & 0 & 0 \\ 0 & 5/9 & 0 \\ 0 & 0 & 5/9 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -160/9 \\ -160/9 \\ -160/9 \end{bmatrix}$$

(a) $$\mu_Y = A\mu_X + b = \begin{bmatrix} 5/9 & 0 & 0 \\ 0 & 5/9 & 0 \\ 0 & 0 & 5/9 \end{bmatrix} \begin{bmatrix} 50 \\ 62 \\ 58 \end{bmatrix} + \begin{bmatrix} -160/9 \\ -160/9 \\ -160/9 \end{bmatrix}$$

$$= \begin{bmatrix} 250/9 \\ 310/9 \\ 290/9 \end{bmatrix} + \begin{bmatrix} -160/9 \\ -160/9 \\ -160/9 \end{bmatrix} = \begin{bmatrix} 10 \\ 50/3 \\ 130/9 \end{bmatrix}$$

(b) $C_Y = AC_XA^T = (5/9)^2 C_X$

(c) $$f_Y(y) = \frac{1}{(2\pi)^{3/2}\sqrt{|C_Y|}} \exp\left(-\frac{1}{2}(y-\mu_Y)^T C_Y^{-1}(y-\mu_Y)\right)$$

**Problem 15:**

$$\begin{aligned} x &= \frac{w - \mu_W}{\sigma_W} \\ y &= \frac{z - \mu_Z}{\sigma_Z}. \end{aligned}$$

The inverse Jacobian matrix becomes

$$\frac{\partial(x, y)}{\partial(w, z)} = \begin{bmatrix} 1/\sigma_W & 0 \\ 0 & 1/\sigma_Z \end{bmatrix}$$

and therefore, since

$$p_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right]$$

$$p_{W,Z}(w,z) = \frac{1}{2\pi\sqrt{1-\rho^2}}$$
$$\cdot \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{w-\mu_W}{\sigma_W}\right)^2 - 2\rho\left(\frac{w-\mu_W}{\sigma_W}\right)\left(\frac{z-\mu_Z}{\sigma_Z}\right) + \left(\frac{z-\mu_Z}{\sigma_Z}\right)^2\right)\right]\frac{1}{\sigma_W\sigma_Z}$$

or finally

$$p_{W,Z}(w,z) = \frac{1}{2\pi\sqrt{(1-\rho^2)\sigma_W^2\sigma_Z^2}}$$
$$\cdot \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{w-\mu_W}{\sigma_W}\right)^2 - 2\rho\left(\frac{w-\mu_W}{\sigma_W}\right)\left(\frac{z-\mu_Z}{\sigma_Z}\right) + \left(\frac{z-\mu_Z}{\sigma_Z}\right)^2\right)\right]. \tag{12.24}$$

**Problem 16:**

$$X, Y, \text{ and } Z \text{ } are \text{ independent } \Rightarrow \begin{cases} EX^2 \cdot EY + EX \cdot EY \cdot EZ = 13 \\ EX \cdot EY^2 + EZ \cdot EX^2 = 14 \end{cases}$$

Since $Y, Z \sim Uniform(0,2)$, we conclude

$$EY = EZ = 1; \text{Var}(Y) = \text{Var}(Z) = \frac{(2-0)^2}{12} = \frac{1}{3}.$$

Therefore,

$$EY^2 = \frac{1}{3} + 1 = \frac{4}{3}.$$

Thus,

$$\begin{cases} EX^2 + EX = 13 \\ \frac{4}{3}EX + EX^2 = 14 \end{cases}$$

We conclude $EX = 3$, $EX^2 = 10$. Therefore,

$$\begin{cases} \mu = 3 \\ \mu^2 + \sigma^2 = 10 \end{cases}$$

So, we obtain $\mu = 3, \sigma = 1$.

**Problem 17:**

(a) From $m$ and $c$ we have $X_2 \sim N(1, 2)$. Thus

$$\begin{aligned} P(0 \leq X_2 \leq 1) &= \Phi\left(\frac{1-1}{\sqrt{2}}\right) - \Phi\left(\frac{0-1}{\sqrt{2}}\right) \\ &= \Phi(0) - \Phi\left(\frac{-1}{\sqrt{2}}\right) = 0.2602 \end{aligned}$$

(b)

$$\begin{aligned} m_Y &= EY = AEX + b \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}. \end{aligned}$$

(c)

$$\begin{aligned} C_Y &= AC_X A^T \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \end{aligned}$$

(d) From $m_Y$ and $c_Y$ we have $Y_3 \sim N(3, 1)$, thus

$$P(Y_3 \leq 4) = \Phi\left(\frac{4-3}{1}\right) = \Phi(1) = 0.8413$$

**Problem 18:**

Given $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$, $n = 1, 2, \cdots, N$

(1) — Show $Y = \sum_{n=1}^{N} a_n X_n$ is a Gaussian RV

Let $y_n = a_n x_n$ $\quad y_n \sim \mathcal{N}(\underbrace{a_n \mu_n}_{\bar{y}_n}, \underbrace{a_n^2 \sigma_n^2}_{\sigma_{y_n}^2})$ We can write:

$Y = \sum_{n=1}^{N} y_n$

Since $y_n$ is Gaussian. We have $\phi_{y_n}(\omega) = E_{y_n}[e^{j\omega y_n}] = e^{j\omega \bar{y}_n - \frac{1}{2}\omega^2 \sigma_{y_n}^2}$

Hence $\phi_Y(\omega) = E_Y[e^{j\omega Y}] = \prod_{n=1}^{N} E_{y_n}[e^{j\omega y_n}]$ by independence of $y_n$

$$= \prod_{n=1}^{N} e^{j\omega \bar{y}_n - \frac{1}{2}\omega^2 \sigma_{y_n}^2}$$

$$= e^{j\omega \underbrace{\left(\sum_{n=1}^{N} \bar{y}_n\right)}_{\bar{y}} - \frac{1}{2}\omega^2 \underbrace{\left(\sum_{n=1}^{N} \sigma_{y_n}^2\right)}_{\sigma_y^2}}$$ Which is a CF of Gaussian RV

$\Rightarrow$ Y is a Gaussian Random Variable with mean $\bar{Y} = \sum_{n=1}^{N} \bar{y}_n$ and Variance $\sigma_y^2 = \sum_{n=1}^{N} \sigma_{y_n}^2$

(2) — Find mean & Variance (Double Check)

$$\bar{y} = E[Y] = E\left[\sum_{n=1}^{N} a_n x_n\right] = \sum_{n=1}^{N} a_n E[x_n] = \sum_{n=1}^{N} a_n \mu_n$$

$$\sigma_y^2 = E[Y^2] - (E[Y])^2 = E\left[\left(\sum_{n=1}^{N} a_n x_n\right)^2\right] - \left(\sum_{n=1}^{N} a_n \mu_n\right)^2$$

$$= E\left[\sum_{n=1}^{N} a_n^2 x_n^2 + 2\sum_{n \neq m} a_n a_m x_n x_m\right] - \left(\sum_{n=1}^{N} a_n \mu_n\right)^2$$

$$= \sum_{n=1}^{N} a_n^2 E[x_n^2] + 2\sum_{n \neq m} a_n a_m \mu_n \mu_m - \underbrace{\left(\sum_{n=1}^{N} a_n \mu_n\right)^2}_{= \sum_{n=1}^{N} a_n^2 \mu_n^2 + 2\sum_{n \neq m} a_n a_m \mu_n \mu_m}$$

$$= \sum_{n=1}^{N} a_n^2 E[x_n^2] - \sum_{n=1}^{N} a_n^2 \mu_n^2$$

$$= \sum_{n=1}^{N} a_n^2 (E[x_n^2] - \mu_n^2) = \sum_{n=1}^{N} a_n^2 \sigma_n^2$$

(3) $\quad P_r[Y \leq 10] = 1 - Q\left(\frac{10 - \mu_y}{\sigma_y}\right)$

$$= 1 - Q\left[\frac{10 - \sum_{n=1}^{N} a_n \mu_n}{\sqrt{\sum_{n=1}^{N} a_n^2 \sigma_n^2}}\right].$$

**Problem 19:**

MATLAB CODE:

```
clear all
clc

%Generate M realizations of X and Y
M=1e7;x=zeros(M,1);y=zeros(M,1);
for m=1:M
    u=rand;
    if(u<=1/8)
        x(m)=0;y(m)=0;
    elseif(u>1/8&&u<=1/4)
        x(m)=0;y(m)=1;
    elseif(u>1/4&&u<=1/2)
        x(m)=1;y(m)=0;
    else
        x(m)=1;y(m)=1;
    end
end

%Estimate p(0,0)
p00_hat=sum(x+y==0)/M;
```