Problem Set 8

The Poisson Process and Branching Processes

Date: Tuesday, 2^{nd} of July, 2019.

Problem 1:

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate $\lambda = 0.5$.

(a) Find the probability of no arrivals in $(3, 5]$

(b) Find the probability that there is exactly one arrival in each of the following time intervals: $(0, 1]$, $(1, 2]$, $(2, 3]$, and $(3, 4]$.

Problem 2:

The number of orders arriving at a service facility can be modeled by a Poisson process with intensity $\lambda = 10$ orders per hour.

(a) Find the probability that there are no orders between $10:30$ and 11.

(b) Find the probability that there are 3 orders between $10:30$ and 11 and 7 orders between $11:30$ and 12

Problem 3:

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ. Find the probability that there are two arrivals in $(0, 2]$ or three arrivals in $(4, 7]$.

Problem 4:

Let $N(t)$ be a Poisson process with rate $\lambda = 2$, and let $X_1, X_2, \ldots$ be the corresponding interarrival times.

(a) Find the probability that the first arrival occurs after $t = 0.5$, i.e. $P[X_1 > 0.5]$.

(b) Given that we have had no arrivals before $t = 1$, find $P[X_1 > 3]$.

(c) Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrival occurs after $t = 4$.

(d) I start watching the process at $t = 10$. Let T be the time of the first arrival that I see. In other words, T is the first arrival after $t = 10$. Find $\mathbb{E}[T]$ and $\text{Var}[T]$.

(e) I start watching the process at time $t = 10$. Let T be the time of the first arrival I see. Find the conditional expectation and the conditional variance of T given that I am informed that the last arrival occurred at time $t = 9$.

Problem 5:

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson Process with rate λ. Find its covariance function

$$C_N(t_1, t_2) = \text{Cov}\big(N(t_1), N(t_2)\big), \quad \text{for } t_1, t_2 \in [0, \infty)$$

Problem 6:

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Let $N(t)$ be the merged process $N(t) = N_1(t) + N_2(t)$.

(a) Find the probability that $N(1) = 2$ and $N(2) = 5$.

(b) Given that $N(1) = 2$, find the probability that $N_1(1) = 1$.

Problem 7:

Use a computer simulation to generate multiple realizations of a Poisson process with $\lambda = 1$. Use the simulation to estimate $P[T_2 \leq 1]$. Compare your result to the true value.

Problem 8:

A branching process has family size C such that $C \sim$ Bernoulli $\left(\frac{1}{3}\right)$.

(a) What is the expected number of nomads in the n^{th} generation, $\mathbb{E}[Z_n]$?

(b) What is the variance of the number of nomads in the n^{th} generation, Var$[Z_n]$?

Problem 9:

If each family-size of a branching process has $C \sim$ Binomial$(2, p)$, show that the probability of ultimate extinction is

$$e = \begin{cases} 1 & \text{if } 0 \leq p \leq \frac{1}{2}, \\ \left(\frac{1-p}{p}\right)^2 & \text{if } \frac{1}{2} \leq p \leq 1, \end{cases}$$

Problem 10:

Most bacteria, including Salmonella, reproduce by a process known as binary fission in which the bacterium splits into two halves, producing two bacteria. Now assume that a food supply has been contaminated by a single Salmonella bacterium. We wish to model this outbreak as a branching process.

(a) Specify a PMF to model each family-size of the branching process, assuming the bacteria reproduce independently of each other and that the probability of binary fission of any individual bacterium is p. Also assume that if the bacterium does not undergo binary fission, it dies. (5 points)

(b) What is the probability of ultimate extinction of this branching process? (5 points)

Problem 11:

Cars, trucks, and buses arrive at a toll booth as independent Poisson processes with rates $\lambda_c = 1.2 =$ cars/minute, $\lambda_t = 0.9$ trucks/minute, and $\lambda_b = 0.7$ buses/minute. In a 10-minute interval, what is the PMF of N, the number of vehicles (cars, trucks, or buses) that arrive?

Problem 12:

A corporate Web server records hits (requests for HTML documents) as a Poisson process at a rate of 10 hits per second. Each page is either an internal request (with probability 0.7) or an external request (with probability 0.3) from the Internet. Over a 10-minute interval, what is the joint PMF of I, the number of internal requests, and X, the number of external requests?

Problem 13:

Find the probability of 6 arrivals of a Poisson random process in the time interval $[7, 12]$ if $\lambda = 1$. Next determine the average number of arrivals for the same time interval.

Problem 14:

For a Poisson random process with an arrival rate of 2 arrivals per second, find the probability of exactly 2 arrivals in 5 successive time intervals of length 1 second each.

Problem 15:
If $N(t)$ is a Poisson counting random process, determine $\mathbb{E}[N(t_2) - N(t_1)]$ and $\text{Var}[N(t_2) - N(t_1)]$.

Problem 16:
A compound Poisson random process $X(t)$ is composed of random variables U_i that can take on the values ± 1 with $P[U_i = 1] = p$. What is the expected value of $X(t)$?