AMCS 241 - STAT 250

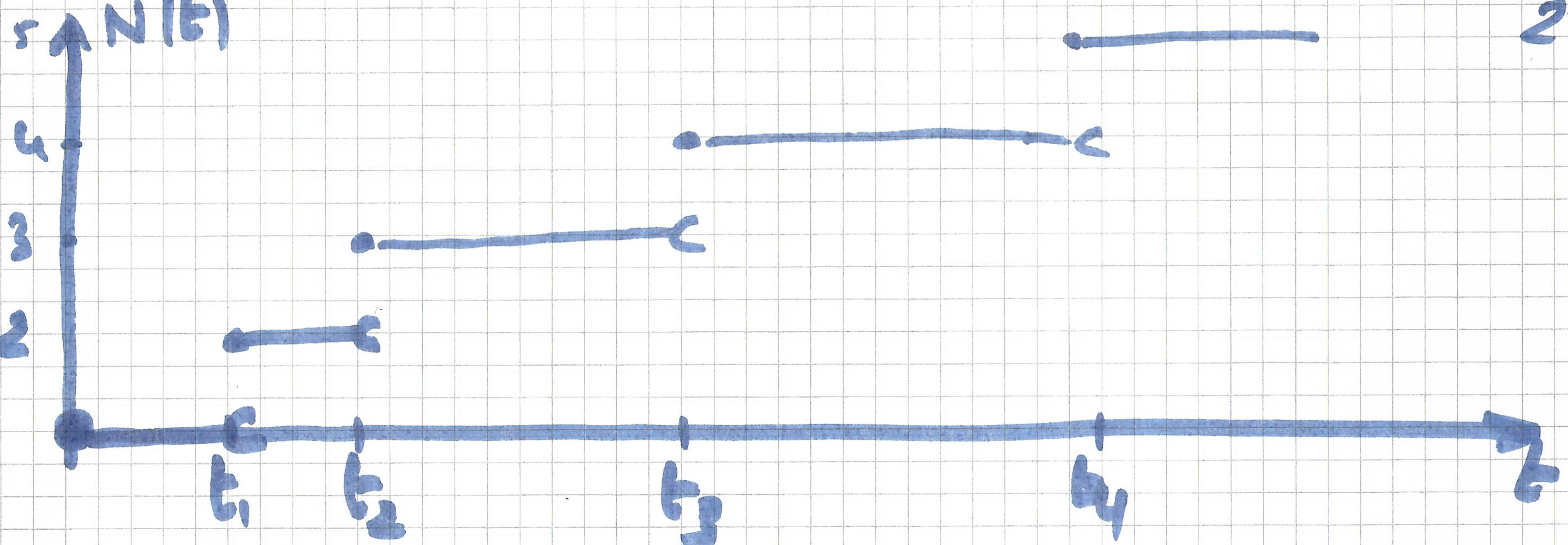
Lecture #10

. Chapter 10 (Some RPs)

10.6 Poisson RP

10.6.a Introduction

A typical illustration of the Poisson RP is :



- The RP whose realizations is a set of times $(t_1, t_2, t_3, \ldots)$ is called the <u>Poisson RP</u>
- The RP that counts the number of events in the time interval $[0, t]$ and which is denoted by $N(t)$ is called the <u>Poisson Counting RP</u>

- N(t) is a discrete valued continuous time RP and is used to model a wide range of physical or man-made phenomena such as
  - Arrival of customers at a cashier
  - Requests of service in a computer network
  - ....

10.6.b Derivation of the Poisson Counting RP

Our starting Point is 3 Axioms:

<u>Axiom 1</u>

. $N(0) = 0$ (since we assume that the RP starts at $t = 0$)

<u>Axiom 2</u>

. $N(t)$ has independent and stationary increments

→ For two increments we assume that the RVs:

$$I_1 : N(t_2) - N(t_1)$$

$$I_2 : N(t_4) - N(t_3)$$

* I_1 and I_2 are independent if

$$t_4 \geq t_3 \geq t_2 \geq t_1$$

* I_1 and I_2 have the same PDF if $t_2 - t_1 = t_4 - t_3$

Axiom 3

$$P[N(t+\Delta t) - N(t) = k] = \begin{cases} 1 - \lambda \Delta t & \text{for } k = 0 \\ \lambda \Delta t & \text{for } k = 1 \end{cases}$$

→ We are now ready to derive the Poisson Counting RP.

→ Consider first $P[N(t)=0]$
for an arbitrary $t>0$

So to have no arrivals in $[0,t)$,
there must be no arrivals in $[0, t-\Delta t]$
and no arrivals in $[t-\Delta t, t)$

Thus

$$P[N(t)=0) = P(N(t-\Delta t)=0 \text{ and }$$

$$N(t) - \Delta(t-\Delta t) = 0)$$

↓ independence
Axiom 2

$$= P[N(t-\Delta t)=0] \cdot P[N(t)-N(t-\Delta t)=0]$$

$$\downarrow \text{Axiom 3}$$

$$= P[N(t-\Delta t)=0] \cdot (1-\lambda\,\Delta t)$$

Now let $P[N(t)=0] \triangleq P_0(t)$

So we can write

$$P_0(t) = P_0(t-\Delta t) \cdot (1-\lambda\,\Delta t)$$

$$\frac{P_0(t) - P_0(t-\Delta t)}{\Delta t} = -\lambda\, P_0(t-\Delta t)$$

$$\downarrow \Delta t \to 0$$

$$\frac{d P_0(t)}{dt} = -\lambda P_0(t)$$

$$\downarrow$$

$$P_0(t) = K e^{-\lambda t}$$

Using Axiom 1

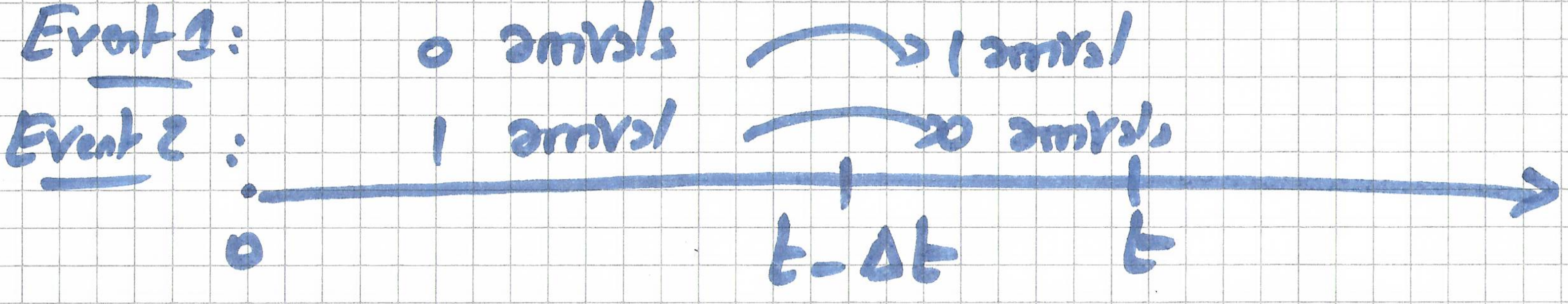
$$P[N(0)=0] = P_0(0) = 1$$

$$1 = K e^0 \Rightarrow \boxed{K=1}$$

$$\Rightarrow \boxed{P_0(t) = P[N(t)=0] = e^{-\lambda t}}$$

. Let us focus on

$$P[N(t)=1] \triangleq P_1(t)$$



These two events are mutually exclusive

So we can write

$$P_1(t) = P[N(t-\Delta t)=0 \text{ and } N(t)-N(t-\Delta t)=1]$$

$$+ P[N(t-\Delta t)=1 \text{ and } N(t)-N(t-\Delta t)=0]$$

$\downarrow$ Axiom 2
independence

$$= P[N(t-\Delta t)=0] \cdot P[N(t)-N(t-\Delta t)=1]$$

$$+ P[N(t-\Delta t)=1] \cdot P[N(t)-N(t-\Delta t)=0]$$

$$\Big\downarrow \text{Axiom 2 stationarity}$$

$$= P[N(t-\Delta t)=0] \cdot P[N(t+\Delta t)-N(t)=1]$$

$$+ P[N(t-\Delta t)=1] \cdot P[N(t+\Delta t)-N(t)=0]$$

$$\Big\downarrow \text{Axiom 3}$$

$$\Rightarrow P_1(t) = P_0(t-\Delta t) \cdot \lambda \Delta t + P_1(t-\Delta t) \cdot (1-\lambda \Delta t)$$

$$\frac{P_1(t) - P_1(t-\Delta t)}{\Delta t} = -\lambda\, P_1(t-\Delta t) + \lambda\, P_0(t-\Delta t)$$

$$\Big\downarrow \; \Delta t \to 0$$

$$\boxed{\frac{d P_1(t)}{dt} + \lambda\, P_1(t) = \lambda\, P_0(t)}$$

More generally, and using the same steps we can show by induction that

$P_k(t) \triangleq P[N(t) = k]$ follows a similar differential equation:

$$\frac{d\,P_k(t)}{dt} + \lambda\, P_k(t) = \lambda\, P_{k-1}(t)$$

for $k = 1, 2, 3, \ldots$

and with $P_0(t) = e^{-\lambda t}$

→ we can solve this set of differential equations recursively and get:

$$P_k(t) = P[N(t) = k] = \frac{(\lambda t)^k}{k!}\, e^{-\lambda t}$$

for $k = 0, 1, 2, \ldots$

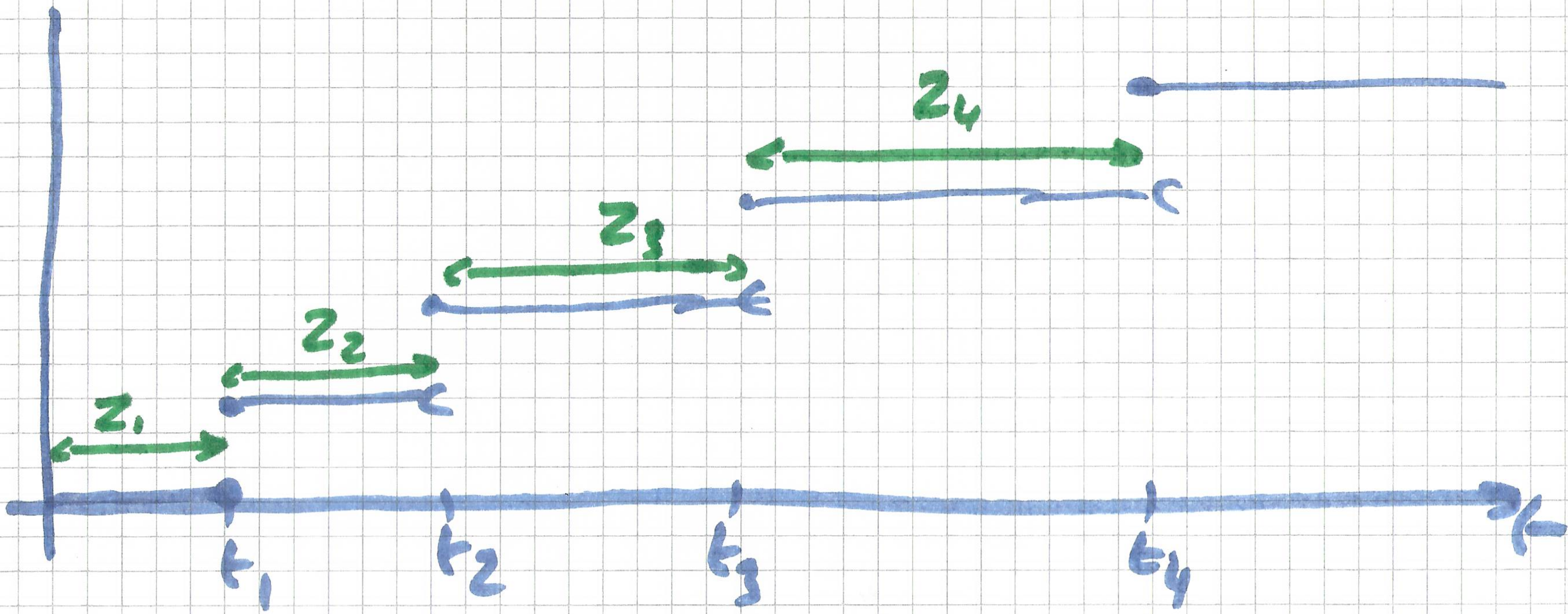
Note that

- $E[N(t)] = \lambda t$ function of time $\Rightarrow$ $N(t)$ is NOT WSS

- $\lambda = \dfrac{E[N(t)]}{t}$: Average number of arrivals per second = Rate of the Poisson R.P.

$\boxed{\text{10. 6. C Interarrival Times}}$

Consider a typical realization of $N(t)$



→ the times $t_1, t_2, t_3, \ldots$ are called the <u>arrival times</u>.

→ the time intervals

$$Z_1 = t_1 - 0 \qquad Z_3 = t_3 - t_2$$
$$Z_2 = t_2 - t_1 \qquad Z_4 = t_4 - t_3 \ldots$$

∴ $Z_n = t_n - t_{n-1}$ are called the <u>interarrival times</u>.

<u>Theorem</u>

The interarrival times $Z_1, Z_2, \dots, Z_k$ are i.i.d RVs with each having an exponential distribution with parameter λ.

<u>i.e</u>

1) $$f_{Z_k}(z_k) = \begin{cases} \lambda e^{-\lambda z_k} & z_k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2, 3, \ldots$

with $$E[Z_k] = \frac{1}{\lambda}$$

$$Var[Z_k] = \frac{1}{\lambda^2}$$

2) $$f_{Z_1 Z_2 \ldots Z_K}(z_1, z_2, \ldots, z_K) = \prod_{k=1}^{K} f_{Z_k}(z_k) = \lambda^K \cancel{\prod_{k=1}} e^{-\lambda \sum_{k=1}^{K} z_k}$$

$$\lambda \geq 0 \quad z_1 \geq 0 \quad \ldots \quad z_K \geq 0$$

10.6.d Arrival Times

→ The k^{th} arrival time T_k is defined as the time for $t=0$ until the k^{th} arrival time (i.e waiting time until the k^{th} arrival)

$$T_k = \sum_{i=1}^{k} Z_i$$

- $Z_i \sim Exp(\lambda)$
- Z_i's are independent

Thus

T_k is the sum of k i.i.d
$Exp(\lambda)$ RVs

$$\Downarrow$$

T_k follows a Gamma distribution
given by

$$f_{T_k}(t) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \quad t \geq 0$$

<u>Proof:</u> use MGF or C.F approach
for sum of i.i.d RVs.

Remarks:

1. Mean

$$E[T_k] = E[Z_1 + Z_2 + \cdots + Z_k]$$

$$= \underbrace{E(Z_1)}_{= 1/\lambda} + \underbrace{E(Z_2)}_{1/\lambda} + \cdots + \underbrace{E(Z_k)}_{1/\lambda}$$

$$\Downarrow$$

$$E[T_k] = \frac{k}{\lambda}$$

2. Variance

$$Var(T_k) = Var(Z_1 + Z_2 + \cdots + Z_k)$$

The Z_i's are independent $\Rightarrow$ uncorrelated

$$Var[T_k] = \sum_{i=1}^{k} \underbrace{Var[Z_i]}_{= 1/\lambda^2}$$

$$\Downarrow$$

$$\boxed{Var[T_k] = \frac{k}{\lambda^2}}$$

3. Arrival Times are <u>NOT</u> independent.

In particular

$$T_1 \leq T_2 \leq T_3 \leq \ldots$$