Problem Set 10

Markov Chains

Date: Thursday, 4^{th} of July, 2019.

Problem 1:

The two-state Markov chain can be used to model a wide variety of systems that alternate between ON and OFF states. After each unit of time in the OFF state, the system turns on with probability p. After each unit of time in the ON state, the system turns OFF with probability q. Let 0 and 1 to denote the OFF and ON states.

(a) Sketch the Markov chain for this system.
(b) Obtain the transition probability matrix P.
(c) Obtain the 2-step transition matrix

Problem 2:

A packet voice communications system transmits digitized speech only during certain periods. In every 10-ms interval (referred to as a timeslot), the system decides whether the speaker is talking or silent. When the speaker is talking, a speech packet is generated; otherwise no packet is generated. If the speaker is silent in a slot, then the speaker is talking in the next slot with probability $p = 1/140$. If the speaker is talking in a slot, the speaker is silent in the next slot with probability $q = 1/100$. If states 0 and 1 represent silent and talking, sketch the Markov chain for this voice packet system and obtain its transition probability matrix.

Problem 3:

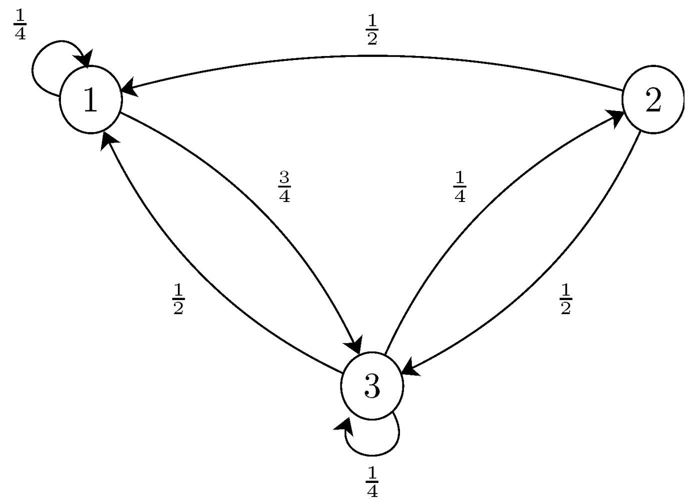
In a discrete random walk, a person's position is marked by an integer on the real line. Each unit of time, the person randomly moves one step, either to the right (with probability p) or to the left. Sketch the Markov chain.

Problem 4:

A wireless packet communications channel suffers from clustered errors. That is, whenever a packet has an error, the next packet will have an error with probability 0.9. Whenever a packet is error-free, the next packet is error-free with probability 0.99. Let $X_n = 1$ if the n^{th} packet has an error; otherwise, $X_n = 0$. Model the random process $\{X_n, n \geq 0\}$ using a Markov chain. Sketch the chain and find its transition probability matrix.

Problem 5:

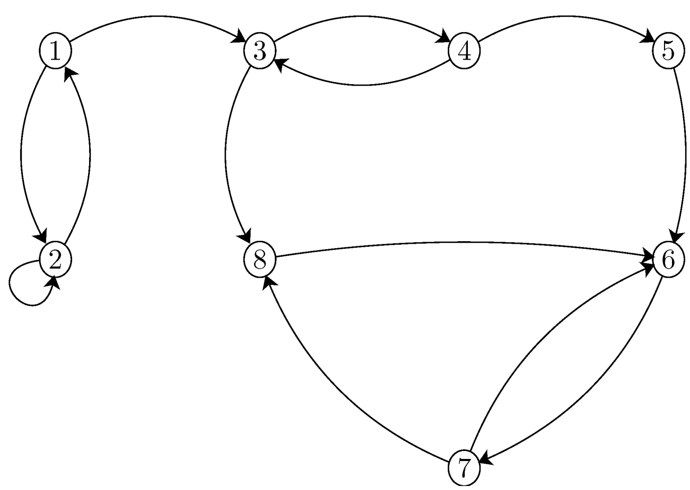
Consider the Markov chain with three states $S = 1, 2, 3$, that has the following state transition diagram Suppose $P[X_1 = 1] = 1/2$ and $P[X_1 = 2] = 1/4$.



(a) Find the state transition matrix for this chain.
(b) Find $P[X_1 = 3, X_2 = 2, X_3 = 1]$.
(c) Find $P(X_1 = 3, X_3 = 1)$.

Problem 6:

Consider the Markov chain shown in the figure below. It is assumed that when there is an arrow from state i to state j, then $p_{ij} > 0$. Find the communicating classes (also called equivalence classes) for this Markov chain.



Problem 7:

Consider the Markov chain in Problem 6.

(a) Is Class $1 = \{\text{state } 1,\ \text{state } 2\}$ aperiodic?
(b) Is Class $2 = \{\text{state } 3,\ \text{state } 4\}$ aperiodic?

(c) Is Class $4 = \{\text{state } 6, \text{ state } 7, \text{ state } 8\}$ aperiodic?

Problem 8:

Consider a Markov chain with two possible states, $S = \{0, 1\}$. In particular, suppose that the transition matrix is given by

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a+b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e., $\pi^{(0)} = [P(X_0 = 0)\ P(X_0 = 1)] = [\alpha\ 1-\alpha]$, where $\alpha \in [0, 1]$.

(a) Sketch the Markov chain. How many classes are in the chain, are they recurrent/transient, and are they periodic/aperiodic?

(b) Given that

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

and that the Markov chain has a limiting distribution given by

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

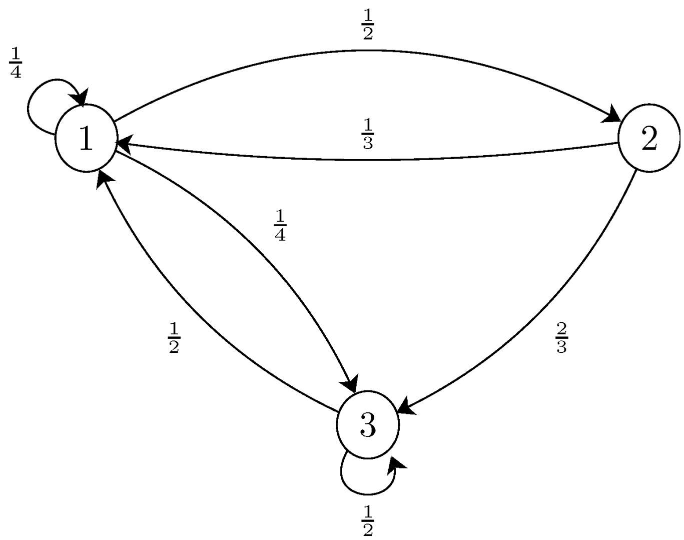
(c) Find the stationary distribution for this chain by solving the balance equations. Is this also the limiting distribution for this chain? Why or why not?

(d) Consider the case when $a = b = 1$. Sketch the Markov chain and comment on the existence of a limiting distribution for this Markov chain.

(e) Consider the case when $a = b = 0$. Sketch the Markov chain and comment on the existence of a limiting distribution for this Markov chain.

Problem 9:

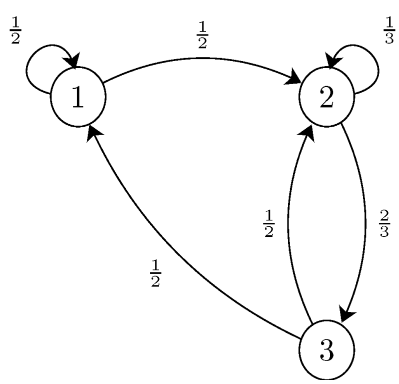
Consider the following Markov chain



(a) Is this chain irreducible?
(b) Is this chain aperiodic?
(c) Find the stationary distribution for this chain.
(d) Is the stationary distribution a limiting distribution for the chain?

Problem 10:

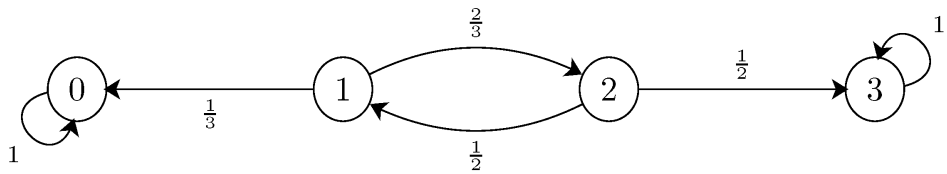
Consider the following Markov chain



(a) Is this chain irreducible?
(b) Is this chain aperiodic?
(c) Find the stationary distribution for this chain.
(d) Is the stationary distribution a limiting distribution for the chain?

Problem 11:

Consider the following Markov chain



(a) Find the state transition matrix of this chain.
(b) How many classes are there? For each class, mention if it is recurrent or transient.
(c) Define the following probabilities of absorption in state 0 conditioned on the initial state

$$\begin{aligned} a_0 &= P(\text{absorption in } 0|X_0 = 0), \\ a_1 &= P(\text{absorption in } 0|X_0 = 1), \\ a_2 &= P(\text{absorption in } 0|X_0 = 2), \\ a_3 &= P(\text{absorption in } 0|X_0 = 3). \end{aligned}$$

and compute them.

(d) Define the following probabilities of absorption in state 3 conditioned on the initial state

$$\begin{aligned} b_0 &= P(\text{absorption in } 3|X_0 = 0), \\ b_1 &= P(\text{absorption in } 3|X_0 = 1), \\ b_2 &= P(\text{absorption in } 3|X_0 = 2), \\ b_3 &= P(\text{absorption in } 3|X_0 = 3). \end{aligned}$$

and compute them.