Problem Set 6

Multiple RVs and Multivariate Gaussians

Date: Sunday, 30^{th} of June, 2019.

Problem 1:

Let X, Y, Z be three jointly continuous random variables with joint PDF

$$f_{XYZ}(x, y, z) = \begin{cases} c(x + 2y + 3z) & 0 \leq x, y, z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant c.
(b) Find the marginal PDF of X.

Problem 2:

For a random vector $\mathbf{X}$, show that

$$\mathbf{C_X} = \mathbf{R_X} - \mathbf{E}[\mathbf{X}]\mathbf{E}[\mathbf{X^T}]. \quad (1)$$

Problem 3:

Let $\mathbf{X}$ be an n-dimensional random vector and the random vector $\mathbf{Y}$ be defined as

$$\mathbf{Y} = \mathbf{AX} + \mathbf{b}, \quad (2)$$

where $\mathbf{A}$ is a fixed m by n matrix and $\mathbf{b}$ is a fixed m-dimensional vector. Show that

$$\mathbf{C_Y} = \mathbf{AC_X A^T}. \quad (3)$$

Problem 4:

Let $\mathbf{X}$ and $\mathbf{Y}$ be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x, y) = \begin{cases} \frac{3}{2}x^2 + y & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

and let the random vector $\mathbf{U}$ be defined as

$$\mathbf{U} = \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Find the correlation and covariance matrices of $\mathbf{U}$.

Problem 5:

Let X, Y and Z be three jointly continuous random variables with joint PDF

$$f_{XYZ}(x,y,z) = \begin{cases} x+y, & 0 \le x,y,z \le 1 \\ 0, & \text{otherwise} \end{cases}. \tag{4}$$

(a) Find the joint PDF of X and Y.
(b) Find the marginal PDF of X.
(c) Find the conditional PDF of $f_{XY|Z}(x,y|\,z)$ using

$$f_{XY|Z}(x,y|\,z) = \frac{f_{XYZ}(x,y,z)}{f_Z(z)}. \tag{5}$$

(d) Are X and Y independent of Z?

Problem 6:

Random vector $\mathbf{X}$ has the PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 6e^{-\mathbf{a}^T\mathbf{x}}, & \mathbf{x} \ge 0 \\ 0, & \text{otherwise} \end{cases}. \tag{6}$$

where $\mathbf{a} = [1\ 2\ 3]^T$. What is the CDF of $\mathbf{X}$?

Problem 7:

The random variables Y_1, Y_2, Y_3, Y_4 have the joint PDF

$$f_{Y_1,Y_2,Y_3,Y_4}(y_1,y_2,y_3,y_4) = \begin{cases} 4, & 0 \le y_1 \le y_2 \le 1, 0 \le y_3 \le y_4 \le 1 \\ 0, & \text{otherwise} \end{cases}. \tag{7}$$

Find the marginal PDFs $f_{Y_1,Y_4}(y_1,y_4)$, $f_{Y_2,Y_3}(y_2,y_3)$, and $f_{Y_3}(y_3)$.

Problem 8:

The random variables Y_1, Y_2, Y_3, Y_4 have the joint PDF in Problem 7. Are Y_1, Y_2, Y_3, Y_4 independent random variables?

Problem 9:

Find the expected value $\mathbb{E}[\mathbf{X}]$, the correlation matrix $\mathbf{R}_\mathbf{X}$, and the covariance matrix $\mathbf{C}_\mathbf{X}$ of the 2-dimensional random vector $\mathbf{X}$ with PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 2, & 0 \le x_1 \le x_2 \le 1 \\ 0, & \text{otherwise} \end{cases}. \tag{8}$$

Problem 10:

Given random vector $\mathbf{X}$ defined in Problem 9, let $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$$

find the expected value $\mu_\mathbf{Y}$, the correlation $\mathbf{R}_\mathbf{Y}$, and the covariance $\mathbf{C}_\mathbf{Y}$.

Problem 11:

For $\mathbf{X}$ and $\mathbf{Y}$ defined as in Problem 9 and 10 respectively, compute

(a) The cross-correlation matrix $\mathbf{R}_{\mathbf{XY}}$ and the cross-covariance matrix $\mathbf{C}_{\mathbf{XY}}$.

(b) The cross-correlation coefficients ρ_{Y_1,Y_3} and ρ_{X_2,Y_1}.

Problem 12:

Lets X, Y, and Z be three jointly continuous random variables with joint PDF

$$f_{XYZ}(x,y,z) = \begin{cases} \frac{1}{3}(x+2y+3z) & 0 \leq x,y,z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint PDF of X and Y, $f_{XY}(x,y)$.

Problem 13:

Consider the outdoor temperature at a certain weather station. On May 5, the temperature measurements in units of degrees Fahrenheit taken at 6 AM, 12 noon, and 6 PM are all jointly Gaussian random variables, X_1, X_2, and X_3 with variance 16 degrees^2. The expected values are 50 degrees, 62 degrees, and 58 degrees respectively. The covariance matrix of the three measurements is

$$\mathbf{C_X} = \begin{bmatrix} 16.0 & 12.8 & 11.2 \\ 12.8 & 16.0 & 12.8 \\ 11.2 & 12.8 & 16.0 \end{bmatrix}$$

Write the joint PDF of X_1 and X_2.

Problem 14:

Continuing from Problem 13, use the formula $Y_i = \frac{5}{9}(X_i - 32)$ to convert the three temperature measurements to degrees Celsius.

(a) What is $\boldsymbol{\mu}_{\mathbf{Y}}$, the expected value of random vector $\mathbf{Y}$?
(b) What is $\mathbf{C}_Y$, the covariance of random vector $\mathbf{Y}$?
(c) Write the joint PDF of $\mathbf{Y}$ using vector notation.

Problem 15:

Let $[X\ Y]^T$ have a standard bivariate Gaussian PDF and consider the affine transformation

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} \sigma_W & 0 \\ 0 & \sigma_Z \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \mu_W \\ \mu_Z \end{bmatrix}$$

Use the Method of Transformations to derive the PDF of $[W\ Z]^T$.

Problem 16:

Let X, Y, and Z be three independent random variables with $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y, Z \sim$ Uniform$(0,2)$. We also know that

$$\mathbb{E}[X^2Y + XYZ] = 13$$
$$\mathbb{E}[XY^2 + ZX^2] = 14$$

Find μ and σ.

Problem 17:

Let $\mathbf{X} = [X_1 \ X_2]^T$ be a normal random vector with the following mean vector and covariance matrix

$$\mathbf{m} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}. \tag{9}$$

Let also

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \mathbf{AX} + \mathbf{b}. \tag{10}$$

(a) Find $P(0 \leq X_2 \leq 1)$
(b) Find the expected value vector of $\mathbf{Y}$, $\mathbf{m}_{\mathbf{Y}} = E\mathbf{Y}$.
(c) Find the covariance matrix of $\mathbf{Y}$, $\mathbf{C}_{\mathbf{Y}}$.
(d) Find $P(Y_3 \leq 4)$.

Problem 18:

Let X_n, $n = 1, \cdots, N$ be a set of N uncorrelated normal random variables with mean μ_n and variance σ_n^2. Let

$$Y = \sum_{n=1}^{N} a_n X_n,$$

where a_n are real constants.
(a) Show that Y is a Gaussian random variable by showing it has a Gaussian characteristic function.
(b) Find the mean and variance of Y.
(c) Find the probability that $Y \leq 10$.

Problem 19:

Assume the following joint PMF of the random variables X and Y:

	$y = 0$	$y = 1$
$x = 0$	$1/8$	$1/8$
$x = 1$	$1/4$	$1/2$

Write a computer program to generate realizations of X and Y. Use your program to estimate $p_{XY}(0, 0)$.