Problem Set 9

The Poisson Process

Date: Wednesday, 3^{rd} of July, 2019.

Problem 1:

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate $\lambda = 0.5$.

(a) Find the probability of no arrivals in $(3, 5]$

(b) Find the probability that there is exactly one arrival in each of the following time intervals: $(0, 1]$, $(1, 2]$, $(2, 3]$, and $(3, 4]$.

Problem 2:

The number of orders arriving at a service facility can be modeled by a Poisson process with intensity $\lambda = 10$ orders per hour.

(a) Find the probability that there are no orders between $10:30$ and 11.

(b) Find the probability that there are 3 orders between $10:30$ and 11 and 7 orders between $11:30$ and 12

Problem 3:

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ. Find the probability that there are two arrivals in $(0, 2]$ or three arrivals in $(4, 7]$.

Problem 4:

Let $N(t)$ be a Poisson process with rate $\lambda = 2$, and let $X_1, X_2, \ldots$ be the corresponding interarrival times.

(a) Find the probability that the first arrival occurs after $t = 0.5$, i.e. $P[X_1 > 0.5]$.

(b) Given that we have had no arrivals before $t = 1$, find $P[X_1 > 3]$.

(c) Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrival occurs after $t = 4$.

(d) I start watching the process at $t = 10$. Let T be the time of the first arrival that I see. In other words, T is the first arrival after $t = 10$. Find $\mathbb{E}[T]$ and $\mathrm{Var}[T]$.

(e) I start watching the process at time $t = 10$. Let T be the time of the first arrival I see. Find the conditional expectation and the conditional variance of T given that I am informed that the last arrival occurred at time $t = 9$.

Problem 5:

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson Process with rate λ. Find its covariance function

$$C_N(t_1, t_2) = \mathrm{Cov}\big(N(t_1), N(t_2)\big), \quad \text{for } t_1, t_2 \in [0, \infty)$$

Problem 6:

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Let $N(t)$ be the merged process $N(t) = N_1(t) + N_2(t)$.

(a) Find the probability that $N(1) = 2$ and $N(2) = 5$.

(b) Given that $N(1) = 2$, find the probability that $N_1(1) = 1$.

Problem 7:
Use a computer simulation to generate multiple realizations of a Poisson process with $\lambda = 1$. Use the simulation to estimate $P[T_2 \leq 1]$. Compare your result to the true value.

Problem 8:
Cars, trucks, and buses arrive at a toll booth as independent Poisson processes with rates $\lambda_c = 1.2 =$ cars/minute, $\lambda_t = 0.9$ trucks/minute, and $\lambda_b = 0.7$ buses/minute. In a 10-minute interval, what is the PMF of N, the number of vehicles (cars, trucks, or buses) that arrive?

Problem 9:
A corporate Web server records hits (requests for HTML documents) as a Poisson process at a rate of 10 hits per second. Each page is either an internal request (with probability 0.7) or an external request (with probability 0.3) from the Internet. Over a 10-minute interval, what is the joint PMF of I, the number of internal requests, and X, the number of external requests?

Problem 10:
Find the probability of 6 arrivals of a Poisson random process in the time interval $[7, 12]$ if $\lambda = 1$. Next determine the average number of arrivals for the same time interval.

Problem 11:
For a Poisson random process with an arrival rate of 2 arrivals per second, find the probability of exactly 2 arrivals in 5 successive time intervals of length 1 second each.

Problem 12:
If $N(t)$ is a Poisson counting random process, determine $\mathbb{E}[N(t_2) - N(t_1)]$ and $\text{Var}[N(t_2) - N(t_1)]$.

Problem 13:
A compound Poisson random process $X(t)$ is composed of random variables U_i that can take on the values ± 1 with $P[U_i = 1] = p$. What is the expected value of $X(t)$?