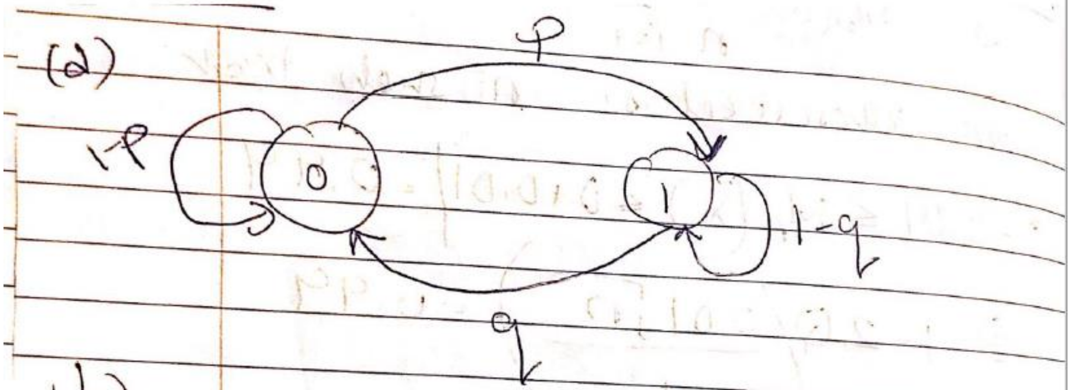
Problem Set 10 Solution

Problem 1:

(a)



(b)

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

(c)

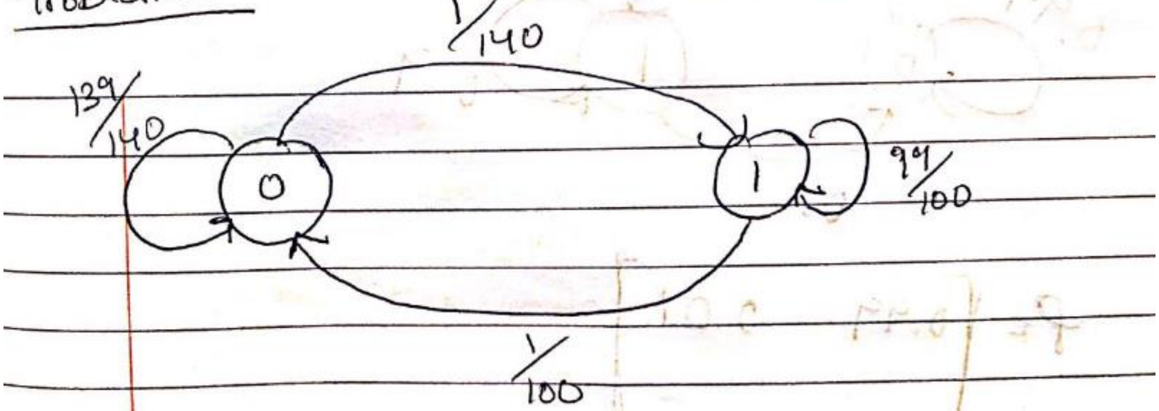
$$P^{(2)} = P^2 = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

$P^{(2)}$: two-step transition matrix

$$= \begin{bmatrix} (1-p)^2 + pq & p(1-p) + p(1-q) \\ q(1-p) + q(1-q) & pq + (1-q)^2 \end{bmatrix}$$

Problem 2:

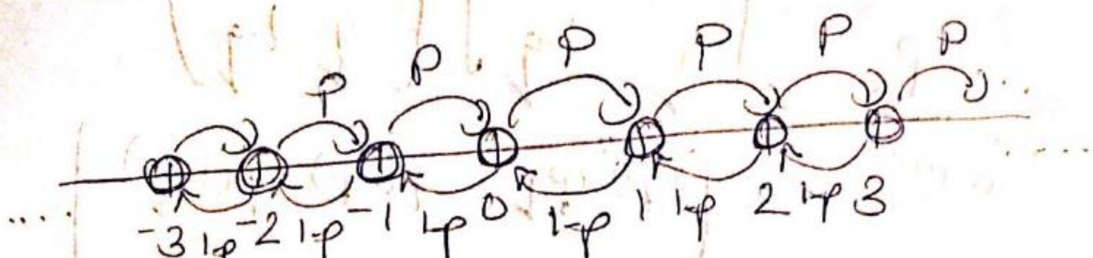
Problem 2



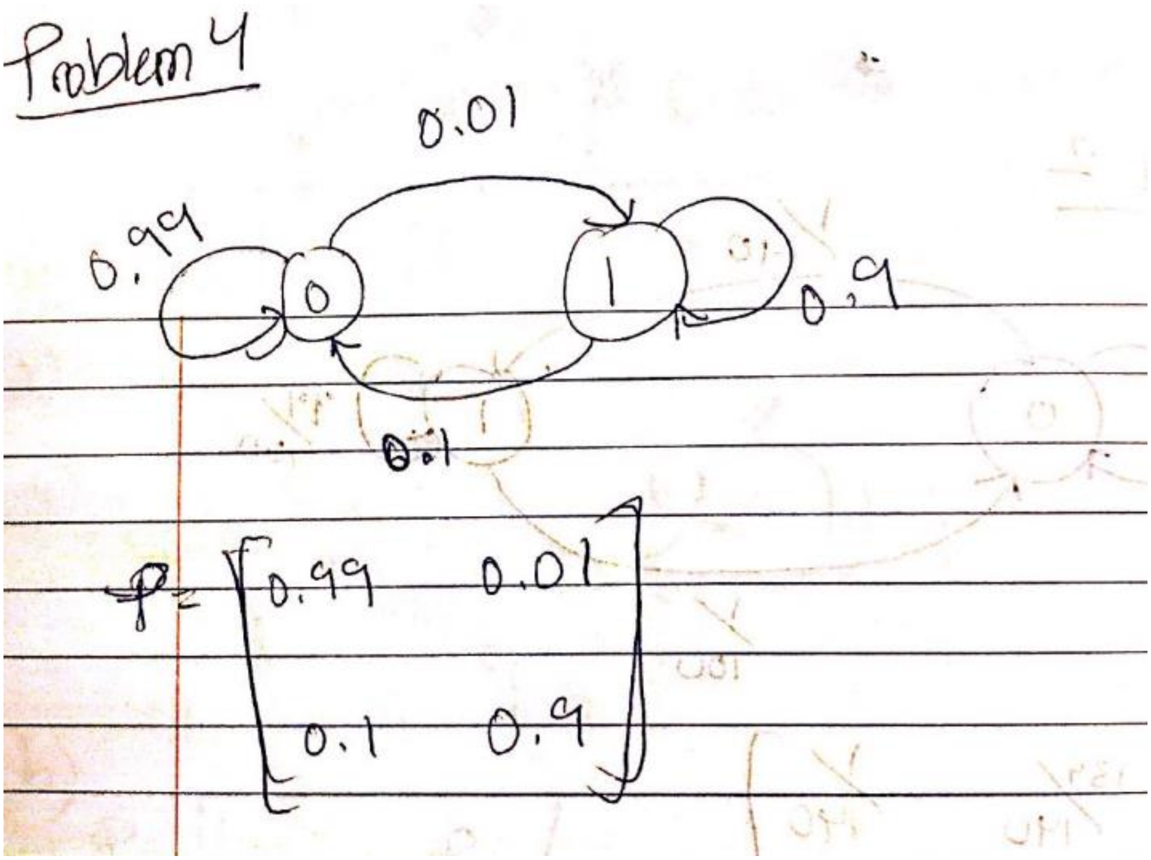
$$P=\begin{bmatrix} \frac{139}{140} & \frac{1}{140} \\ \frac{1}{100} & \frac{99}{100} \end{bmatrix}$$

Problem 3:

Problem 3



Problem 4:



Problem 5:

(a) The state transition matrix is given by

$$P = \begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

(b) First, we obtain

$$\begin{aligned} P(X_1 = 3) &= 1 - P(X_1 = 1) - P(X_1 = 2) \\ &= 1 - \frac{1}{2} - \frac{1}{4} \\ &= \frac{1}{4}. \end{aligned}$$

We can now write

$$\begin{aligned} P(X_1 = 3, X_2 = 2, X_3 = 1) &= P(X_1 = 3) \cdot p_{32} \cdot p_{21} \\ &= \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \\ &= \frac{1}{32}. \end{aligned}$$

(c) We can write

$$\begin{aligned} P(X_1 = 3, X_3 = 1) &= \sum_{k=1}^{3} P(X_1 = 3, X_2 = k, X_3 = 1) \\ &= \sum_{k=1}^{3} P(X_1 = 3) \cdot p_{3k} \cdot p_{k1} \\ &= P(X_1 = 3) \big[p_{31} \cdot p_{11} + p_{32} \cdot p_{21} + p_{33} \cdot p_{31} \big] \\ &= \frac{1}{4} \big[\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} \big] \\ &= \frac{3}{32}. \end{aligned}$$

Problem 6:

Communication is an *equivalence* relation. That means that

- every state communicates with itself, $i \leftrightarrow i$;
- if $i \leftrightarrow j$, then $j \leftrightarrow i$;
- if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

Therefore, the states of a Markov chain can be partitioned into communicating *classes* such that only members of the same class communicate with each other. That is, two states i and j belong to the same class if and only if $i \leftrightarrow j$.

Based on the above we can determine the communicating classes as follows:

There are four communicating classes in this Markov chain. Looking at Figure 11.10, we notice that states 1 and 2 communicate with each other, but they do not communicate with any other nodes in the graph. Similarly, nodes 3 and 4 communicate with each other, but they do not communicate with any other nodes in the graph. State 5 does not communicate with any other states, so it by itself is a class. Finally, states 6, 7, and 8 construct another class. Thus, here are the classes:

$$\begin{aligned}
\text{Class 1} &= \{\text{state 1}, \text{state 2}\},\\
\text{Class 2} &= \{\text{state 3}, \text{state 4}\},\\
\text{Class 3} &= \{\text{state 5}\},\\
\text{Class 4} &= \{\text{state 6}, \text{state 7}, \text{state 8}\}.
\end{aligned}$$

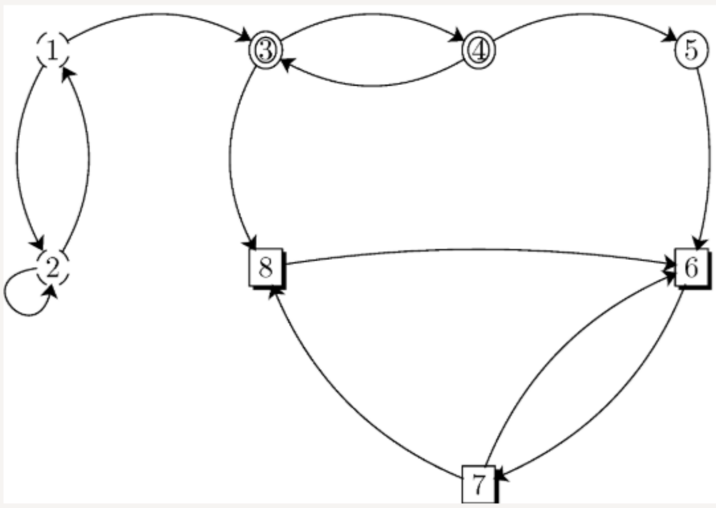


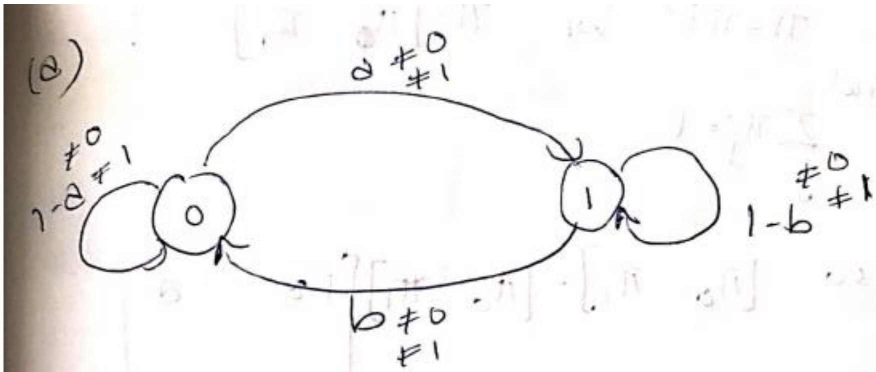
Figure 11.10 - Equivalence classes.

Problem 7:

a. $\text{Class } 1 = \{\text{state } 1, \text{state } 2\}$ is aperiodic since it has a self-transition, $p_{22} > 0$.
b. $\text{Class } 2 = \{\text{state } 3, \text{state } 4\}$ is periodic with period 2.
c. $\text{Class } 4 = \{\text{state } 6, \text{state } 7, \text{state } 8\}$ is aperiodic. For example, note that we can go from state 6 to state 6 in two steps $(6-7-6)$ and in three steps $(6-7-8-6)$. Since $\gcd(2,3) = 1$, we conclude state 6 and its class are aperiodic.

Problem 8:

(a)



for $0 < a+b < 2$

there is one class = {state 0, state 1}

and it is recurrent.

It is not periodic

The markov chain is irreducible and aperiodic.

b. By assumption $0 < a+b < 2$, which implies $-1 < 1-a-b < 1$. Thus,

$$\lim_{n\to\infty} (1-a-b)^n = 0.$$

Therefore,

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

We have

$$\begin{aligned}
\lim_{n\to\infty} \pi^{(n)} &= \lim_{n\to\infty} \left[\pi^{(0)} P^n\right] \\
&= \pi^{(0)} \lim_{n\to\infty} P^n \\
&= [\alpha \quad 1-\alpha] \cdot \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} \\
&= \left[\frac{b}{a+b} \quad \frac{a}{a+b} \right].
\end{aligned}$$

In the above example, the vector

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

is called the *limiting distribution* of the Markov chain. Note that the limiting distribution does not depend on the initial probabilities α and $1-\alpha$. In other words, the initial state (X_0) does not matter as n becomes large. Thus, for $i = 1, 2$, we can write

$$\lim_{n\to\infty} P(X_n = 0|X_0 = i) = \frac{b}{a+b},$$
$$\lim_{n\to\infty} P(X_n = 1|X_0 = i) = \frac{a}{a+b}.$$

Remember that we show $P(X_n = j|X_0 = i)$ by $P_{ij}^{(n)}$, which is the entry in the ith row and jth column of P^n.

(c) We solve

$\pi = \pi P$ for $\pi = [\pi_0 \quad \pi_1]$

and

$\sum_j \pi_j = 1$

so $[\pi_0 \quad \pi_1] = [\pi_0 \quad \pi_1] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$

and $\pi_0 = 1 - \frac{a}{a+b} = \frac{a+b-a}{a+b}$

$= \frac{b}{a+b}$

so

$\pi = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

We don't always have a limiting distribution that is also independent of the initial state. To have a limiting distribution, we should no more than one recurrent class in the chain (we can also have transient classes) that is aperiodic.

(d) and (e)

The two-state Markov chain discussed above is a "nice" one in the sense that it has a well-defined limiting behavior that does not depend on the initial probability distribution (PMF of X_0). However, not all Markov chains are like that. For example, consider the same Markov chain; however, choose $a = b = 1$. In this case, the chain has a periodic behavior, i.e.,

$$X_{n+2} = X_n, \quad \text{for all } n.$$

In particular,

$$X_n = \begin{cases} X_0 & \text{if } n \text{ is even} \\ \\ X_1 & \text{if } n \text{ is odd} \end{cases}$$

In this case, the distribution of X_n does not converge to a single PMF. Also, the distribution of X_n depends on the initial distribution. As another example, if we choose $a = b = 0$, the chain will consist of two disconnected nodes. In this case,

$$X_n = X_0, \quad \text{for all } n.$$

Here again, the PMF of X_n depends on the initial distribution. Now, the question that arises here is: when does a Markov chain have a limiting distribution (that does not depend on the initial PMF)? We will next discuss this question. We will first consider finite Markov chains and then discuss infinite Markov chains.

Problem 9

a. The chain is irreducible since we can go from any state to any other states in a finite number of steps.
b. Since there is a self-transition, i.e., $p_{11} > 0$, we conclude that the chain is aperiodic.
c. To find the stationary distribution, we need to solve

$$\begin{aligned} \pi_1 &= \frac{1}{4}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3, \\ \pi_2 &= \frac{1}{2}\pi_1, \\ \pi_3 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 + \frac{1}{2}\pi_3, \\ \pi_1 &+ \pi_2 + \pi_3 = 1. \end{aligned}$$

   We find

$$\pi_1 = \frac{3}{8}, \; \pi_2 = \frac{3}{16}, \; \pi_3 = \frac{7}{16}.$$

d. Since the chain is irreducible and aperiodic, we conclude that the above stationary distribution is a limiting distribution.

Problem 10

(a) The chain is irreducible since we can go from any state to any other states in a finite number of steps.

(b) The chain is aperiodic since there is a self-transition, e.g., $p_{11} > 0$.

(c) To find the stationary distribution, we need to solve

$$\pi_1 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3,$$
$$\pi_2 = \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3,$$
$$\pi_3 = \frac{2}{3}\pi_2,$$
$$\pi_1 + \pi_2 + \pi_3 = 1.$$

We find

$$\pi_1 = \frac{2}{7}, \ \pi_2 = \frac{3}{7}, \ \pi_3 = \frac{2}{7}.$$

(d) The above stationary distribution is a limiting distribution for the chain because the chain is irreducible and aperiodic.

Problem 11

Consider a finite Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Suppose that all states are either absorbing or transient. Let $l \in S$ be an absorbing state. Define

$$a_i = P(\text{absorption in } l | X_0 = i), \quad \text{for all } i \in S.$$

By the above definition, we have $a_l = 1$, and $a_j = 0$ if j is any other absorbing state. To find the unknown values of a_i's, we can use the following equations

$$a_i = \sum_k a_k p_{ik}, \quad \text{for } i \in S.$$

Based on the above result:

using $a_i = \sum_k a_k p_{ik} = a_0 p_{i0} + a_1 p_{i1} + a_2 p_{i2} + a_3 p_{i3}$

we have

$a_0 = a_0$

$a_1 = \frac{1}{3} a_0 + \frac{2}{3} a_2$

$a_2 = \frac{1}{2} a_1 + \frac{1}{2} a_3$

$a_3 = a_3$

so $a_2 = \frac{1}{4}$

by definition $a_0 = 1$ and $a_3 = 0$

so solving the system we have

$$\left.\begin{aligned} a_1 &= \frac{1}{3} + \frac{2}{3} a_2 \\ a_2 &= \frac{1}{2} a_1 \end{aligned}\right\} \Rightarrow a_1 = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{2} a_1\right)$$

$a_1 = \frac{1}{3} + \frac{1}{3} a_1$

$\frac{2}{3} a_1 = \frac{1}{3} \Rightarrow a_1 = \frac{1}{2}$

Since $a_i + b_i = 1$

we have

$b_0 = 0, b_1 = \frac{1}{2}, b_2 = \frac{3}{4}, b_3 = 1$

or by solving

$$b_i = \sum_{k=0}^{3} b_k p_{ik} = b_0 p_{i0} + b_1 p_{i1} + b_2 p_{i2} + b_3 p_{i3}$$

we have

$b_0 = 0$

$b_1 = \frac{1}{3} b_0 + \frac{2}{3} b_2$

$b_2 = \frac{1}{2} b_1 + \frac{1}{2} b_3$

$b_3 = b_3$

and by definition $b_0 = 0, b_3 = 1$

so

$b_1 = \frac{2}{3} b_2$

$b_2 = \frac{1}{2} b_1 + \frac{1}{2}$

$\Rightarrow b_2 = \frac{1}{3} b_2 + \frac{1}{2}$

$\frac{2}{3} b_2 = \frac{1}{2}$

$b_2 = \frac{3}{4}, b_1 = \frac{1}{2}$