Introduction to Probability and Statistics

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Markov Chains

- Let X_m for $m = \{0, 1, 2, 3, 4\}$ be a sequence of random variables.
- The random variable $X_n \in S = \{s_1, s_2, s_3, s_4, \cdots\}$ denotes the state of a system at time n
- Usually the system states are chosen as $S = \{0, 1, 2, 3, 4, \cdots, m\}$, where $X_n = i$ means that the system is at state i at time n.
- The system is denoted as a discrete tome Markov chain (DTMC) if the following memoryless properties are satisfied

Discrete-Time Markov Chains

Consider the random process $\{X_n, n = 0, 1, 2, \cdots\}$, where $R_{X_i} = S \subset \{0, 1, 2, \cdots\}$. We say that this process is a **Markov chain** if

$$P(X_{m+1} = j | X_m = i, X_{m-1} = i_{m-1}, \cdots, X_0 = i_0) = P(X_{m+1} = j | X_m = i),$$

for all $m, j, i, i_0, i_1, \cdots i_{m-1}$. If the number of states is finite, e.g., $S = \{0, 1, 2, \cdots, r\}$, we call it a **finite** Markov chain.

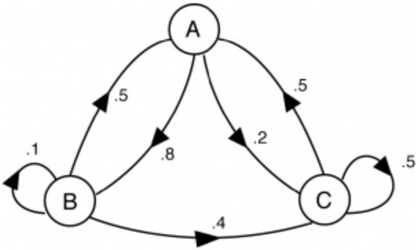
Markov Chains

- If $X_n = j$, we say that the process is in state j.
- The numbers $\mathbb{P}(X_{m+1} = j | X_m = i)$ are called transition probabilities.
- It is assumed that the transition probabilities do not depend on time.

$$p_{ij} = \mathbb{P}(X_{m+1} = j | X_m = i) \quad \text{for all} \quad m$$

- Irrespective of the time, the system goes from states i to j with probability p_{ij}
- The state probabilities satisfy the following important property

$$\sum_j p_{ij} = 1$$



Transition Matrix

- The transition probabilities are usually listed in a matrix, denoted as the state transition matrix
- Assuming $S = \{1, 2, \cdots, r\}$, the state transition matrix is given by

$$P = \begin{array}{c c} \text{states} & \begin{array}{ccccc} 1 & 2 & 3 & \cdots & r \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ r \end{array} & \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1r} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2r} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & p_{r3} & \cdots & p_{rr} \end{bmatrix} \end{array}$$

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- In general, the transition matrix is directly state as

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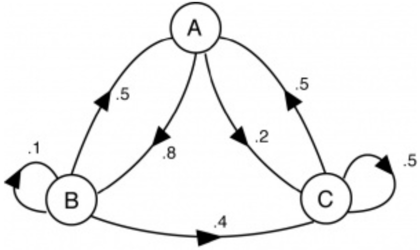
Example 1

- Find the transmission matrix of the following Markov chain



$$P = \begin{array}{c} \text{states} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0.8 & 0.2 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix} \end{array} \Longrightarrow \begin{bmatrix} 0 & 0.8 & 0.2 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

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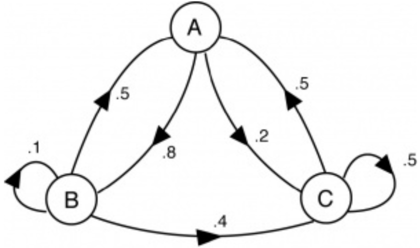

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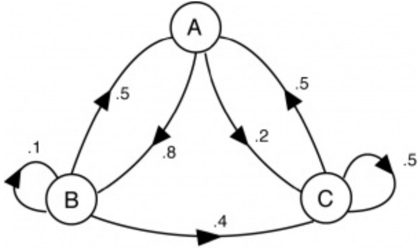

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states

- Check the sum of each row

$$\sum_{j=1}^{3} p_{ij} = 1$$

Example 2

- Check the validity and draw the corresponding state diagram of the following transmission matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

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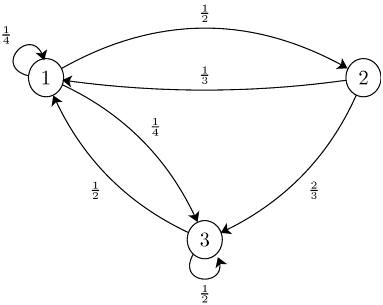

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  - $\mathbb{P}(X_4 = 3 | X_3 = 2)$
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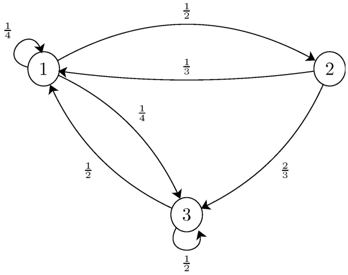

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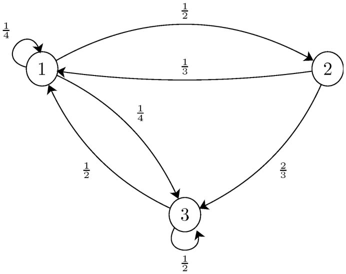

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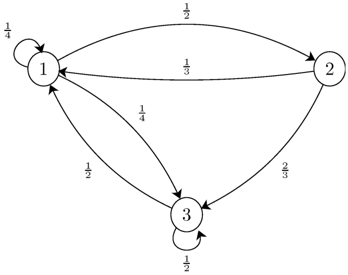
  - $\mathbb{P}(X_0 = 1, X_1 = 2, X_2 = 3)$



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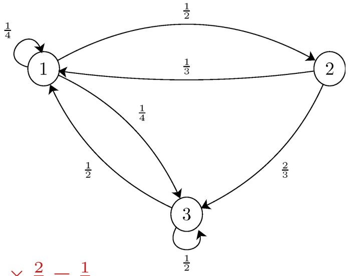
  - $\mathbb{P}(X_0 = 1, X_1 = 2)$ $\Rightarrow \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
  - $\mathbb{P}(X_0 = 1, X_1 = 2, X_2 = 3) \Rightarrow \frac{1}{3} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{6}$



Probability Distributions

- Consider a Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$, where $X_n \in S = \{1, 2, \cdots, r\}$.
- Define the state probability distribution for time step n via the row vector $\boldsymbol{\pi}^{(n)}$ as

$$\boldsymbol{\pi}^{(n)} = [P(X_n = 1) \quad P(X_n = 2) \quad \cdots \quad P(X_n = r)]$$

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- Using the law of total probability, we can write

$$\begin{aligned} \mathbb{P}(X_1 = j) &= \sum_{k=1}^{r} \mathbb{P}(X_1 = j | X_0 = k) \mathbb{P}(X_0 = k) \\ &= \sum_{k=1}^{r} p_{kj} \mathbb{P}(X_0 = k) \end{aligned}$$

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- Using vector notation, the state probability at time $n-1$ is given by

$$\boldsymbol{\pi}^{(1)} = \boldsymbol{\pi}^{(0)} P$$

Probability Distributions

- In general, we can write $\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}^{(n-1)} P$
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where $p_{ij}^{(n)} = \mathbb{P}(X_n = j | X_0 = i) = \mathbb{P}(X_{m+n} = j | X_m = i)$ is the transition probability from state i to state j in n steps

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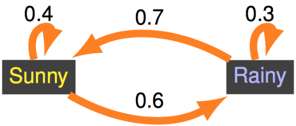
- The sum row condition still hold

$$\sum_{j=1}^{r} p_{ij}^{(n)} = 1$$

Example

- Consider that the following Markov chain with the initial state $\boldsymbol{\pi}^{(0)} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$



- Find
  - $\boldsymbol{\pi}^{(1)}$
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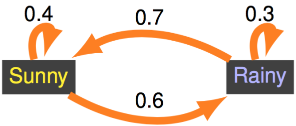

- Find
  - $\boldsymbol{\pi}^{(1)} = \boldsymbol{\pi}^{(0)} P = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \end{bmatrix}$
  - $\boldsymbol{\pi}^{(2)}$

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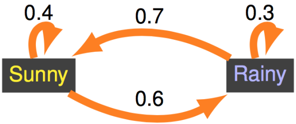

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  - $\boldsymbol{\pi}^{(2)} = \boldsymbol{\pi}^{(1)} P = \begin{bmatrix} 0.535 & 0.465 \end{bmatrix} \times \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.535 & 0.465 \end{bmatrix}$

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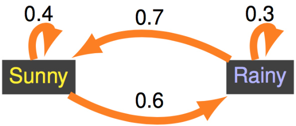

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  - $\boldsymbol{\pi}^{(2)} = \boldsymbol{\pi}^{(0)} P^2 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.42 \\ 0.49 & 0.51 \end{bmatrix} = \begin{bmatrix} 0.535 & 0.465 \end{bmatrix}$

n-Step Transition Matrix

- The n step transition matrix is given by

$$P^{(n)} = P^{n} = \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(2)} & p_{13}^{(n)} & \cdots & p_{1r}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & p_{23}^{(n)} & \cdots & p_{2r}^{(n)} \\ p_{31}^{(n)} & p_{32}^{(n)} & p_{33}^{(n)} & \cdots & p_{3r}^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{r1}^{(n)} & p_{r2}^{(n)} & p_{r3}^{(n)} & \cdots & p_{rr}^{(n)} \end{bmatrix}$$

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The Chapman-Kolmogorov equation can be written as

$$\begin{aligned} p_{ij}^{(m+n)} &= P(X_{m+n} = j | X_0 = i) \\ &= \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}. \end{aligned}$$

The n-step transition matrix is given by

$$P^{(n)} = P^n, \quad \text{for } n = 1, 2, 3, \cdots.$$

Example

- Consider a Markov chain with state space $S = \{0, 1\}$ and the following transmission matrix
$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$
- Draw the state transition diagram.
- Suppose that the system is in state 0 at time $n = 0$, find the probability that the system is in state 1 at time $n = 3$.

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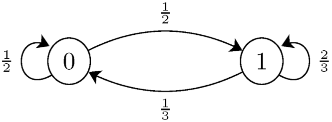

- $\boldsymbol{\pi}^{(0)} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

- $\boldsymbol{\pi}^{(3)}$

- $\mathbb{P}(X_3 = 1 | X_0 = 0) = \frac{43}{72}$

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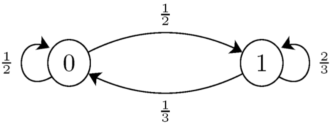

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- $\boldsymbol{\pi}^{(3)} = \boldsymbol{\pi}^{(0)} P^3 = \begin{bmatrix} 1 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \\ \frac{43}{108} & \frac{65}{108} \end{bmatrix} = \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}$
- $\mathbb{P}(X_3 = 1 | X_0 = 0) = \frac{43}{72}$

Classification of States

- Accessibility

We say that state j is **accessible** from state i, written as $i \to j$, if $p_{ij}^{(n)} > 0$ for some n. We assume every state is accessible from itself since $p_{ii}^{(0)} = 1$.

- Communication

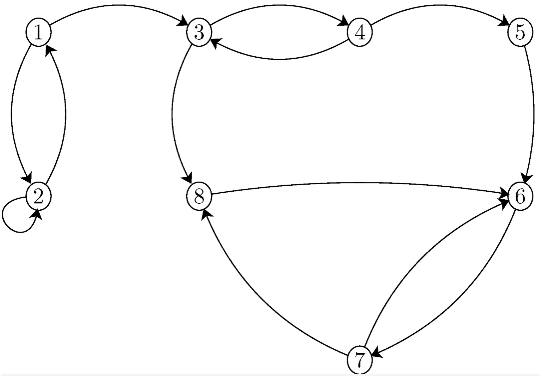
Two states i and j are said to **communicate**, written as $i \leftrightarrow j$, if they are **accessible** from each other. In other words,

$$i \leftrightarrow j \text{ means } i \to j \text{ and } j \to i.$$

- Each state communicates with itself $i \leftrightarrow i$
- If $i \leftrightarrow j$ then $j \leftrightarrow i$
- If $i \leftrightarrow k$ and $j \leftrightarrow k$, then $i \leftrightarrow j$

Classification of States

- Consider the Markov chain shown below
- It is assumed that when there is an arrow from state i to state j, then $p_{ij} > 0$
- Find the equivalence classes for this Markov chain



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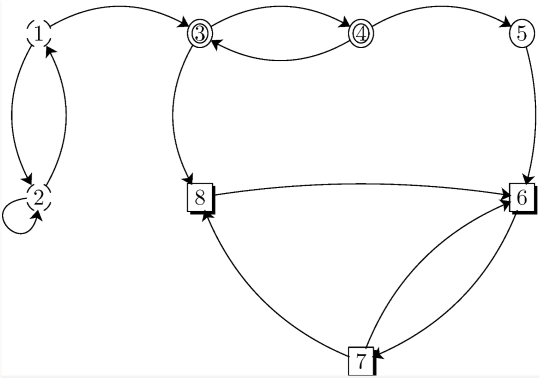


Classification of States

- Irreducible Markov chains

A Markov chain is said to be **irreducible** if all states communicate with each other.

- The states in Markov chains can either be recurrent or transient states

For any state i, we define

$$f_{ii} = P(X_n = i, \text{ for some } n \geq 1 | X_0 = i).$$

State i is **recurrent** if $f_{ii} = 1$, and it is **transient** if $f_{ii} < 1$.

Consider a discrete-time Markov chain. Let V be the total number of visits to state i.

a. If i is a recurrent state, then

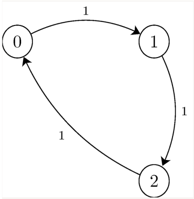
$$P(V = \infty | X_0 = i) = 1.$$

b. If i is a transient state, then

$$V | X_0 = i \sim Geometric(1 - f_{ii}).$$

Classification of States

- Periodicity



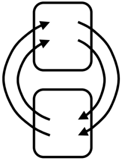
Consider a finite irreducible Markov chain X_n:

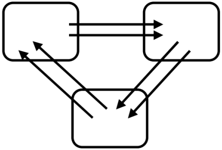
a. If there is a self-transition in the chain ($p_{ii} > 0$ for some i), then the chain is aperiodic.
b. Suppose that you can go from state i to state i in l steps, i.e., $p_{ii}^{(l)} > 0$. Also suppose that $p_{ii}^{(m)} > 0$. If $\gcd(l, m) = 1$, then state i is aperiodic.
c. The chain is aperiodic if and only if there exists a positive integer n such that all elements of the matrix P^n are strictly positive, i.e.,

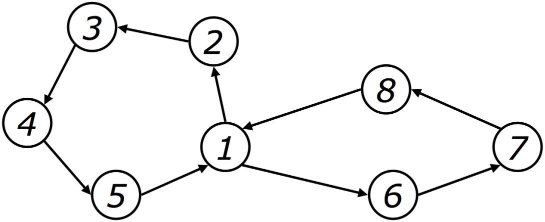
$$p_{ij}^{(n)} > 0, \quad \text{for all } i, j \in S.$$

Classification of States

- Periodicity

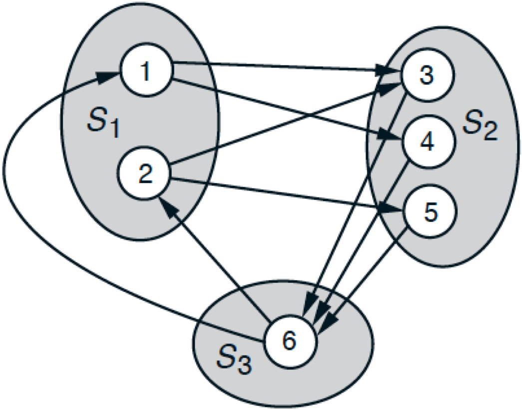






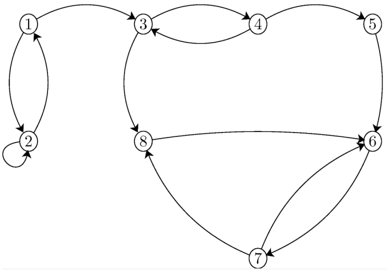
Classification of States

- Periodicity
- If $i \leftrightarrow j$, then $d(i) = d(j)$



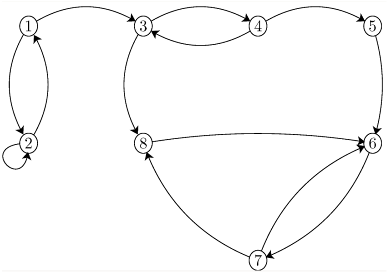
Example

- For the following Markov chain,
    - Is Class 1=state 1,state 2 aperiodic?
    - Is Class 2=state 3,state 4 aperiodic?
    - Is Class 4=state 6,state 7,state 8 aperiodic?



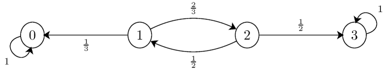
Example

- For the following Markov chain,
  - Is Class 1=state 1,state 2 aperiodic?
    aperiodic since it has a self-transition, $p_{22} > 0$
  - Is Class 2=state 3,state 4 aperiodic?
    periodic with period 2
  - Is Class 4=state 6,state 7,state 8 aperiodic?
    aperiodic. For example, note that we can go from state 6 to state 6 in two steps $(6 \to 7 \to 6)$ and in three steps $(6 \to 7 \to 8 \to 6)$. Since gcd(2,3)=1, we conclude state 6 and its class are aperiodic.



Absorbing Markov Chains

- For the following Markov chain,

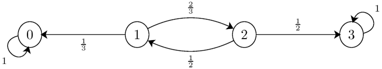


$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- There are three classes:
  1. state 0, which is a recurrent state
  2. states 1 and 2, both of which are transient
  3. state 3, which is a recurrent state
- States 0 and 3 are denoted as absorbing states

Absorbing Markov Chains

- For the following Markov chain,



- Let a_i be the probability of being absorbed in state 0 given that the system started at state i.

$$a_i = \mathbb{P}(\text{absorption in } 0 | X_0 = i)$$

- By definition, $a_0 = 1$ and $a_3 = 0$
- For the other two probabilities a_1 and a_2, we apply the law of total probability

$$a_i = \sum_k a_k p_{ik}$$

- Hence,

$$a_1 = \frac{1}{3}a_0 + \frac{2}{3}a_2 \quad \text{and} \quad a_2 = \frac{1}{2}a_1 + \frac{1}{2}a_3$$

- Hence, $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{4}$
- Similarly, let b_i be the the probability of being absorbed in state 3 given that the system started at state i.
- Applying the same steps $b_0 = 0$, $b_1 = \frac{1}{2}$, $b_2 = \frac{3}{4}$, $b_3 = 1$

Absorbing Markov Chains

- For the following Markov chain,

Absorption Probabilities

Consider a finite Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Suppose that all states are either absorbing or transient. Let $l \in S$ be an absorbing state. Define

$$a_i = P(\text{absorption in } l | X_0 = i), \qquad \text{for all } i \in S.$$

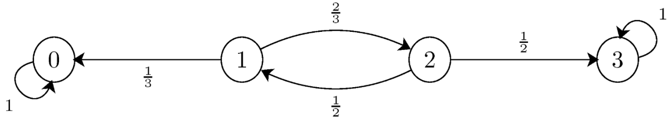
By the above definition, we have $a_l = 1$, and $a_j = 0$ if j is any other absorbing state. To find the unknown values of a_i's, we can use the following equations

$$a_i = \sum_k a_k p_{ik}, \qquad \text{for } i \in S.$$

- In general, a finite Markov chain might have several transient as well as several recurrent classes.
- In this case, replace each recurrent class with one absorbing state.
- Apply the same procedure, and then study the each recurrent class on its own

Mean Hitting Times

- Let's define t_i as the expected number of steps (time) needed until the chain hits state 0 or state 3 (i.e., get absorbed)



- By definition $t_0 = t_2 = 0$
- Using the law of total probability and recursion we have

$$
\begin{aligned}
t_1 &= 1 + \frac{1}{3}t_0 + \frac{2}{3}t_2 \\
&= 1 + \frac{2}{3}t_2 \\
t_2 &= 1 + \frac{1}{2}t_1 + \frac{1}{2}t_3 \\
&= 1 + \frac{1}{2}t_1
\end{aligned}
$$

- Hence, $t_1 = \frac{5}{2}$ and $t_2 = \frac{9}{4}$

Mean Hitting Times

- Mean hitting time

Mean Hitting Times

Consider a finite Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $A \subset S$ be a set of states. Let T be the first time the chain visits a state in A. For all $i \in S$, define

$$t_i = E[T | X_0 = i].$$

By the above definition, we have $t_j = 0$, for all $j \in A$. To find the unknown values of t_i's, we can use the following equations

$$t_i = 1 + \sum_k t_k p_{ik}, \qquad \text{for } i \in S - A.$$

Mean Return Times

- Assuming $X_0 = l$, let's define r_l as the expected number of steps needed until the chain returns to state l. Let

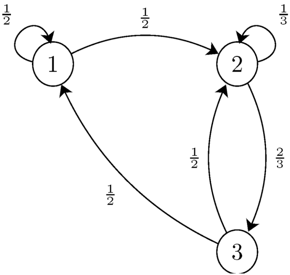
$$R_l = \min\{n \geq 1 : X_n = l\}, \quad \text{then} \quad r_l = \mathbb{E}[R_l | X_0 = l]$$

- Using the law of total probability, the mean hitting time, and recursion we can write

$$r_l = 1 + \sum_k p_{lk} t_k$$

Example

- Consider the following Markov chain. Let t_k be the expected number of steps until the chain hits state 1 for the first time given that $X_0 = k$
- . Let r1 be the mean return time to state 1
  - Find t_2 and t_3.
  - Find r_1.

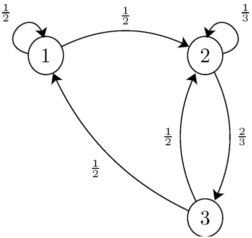


Example

- To find t_1 and t_2, we have

$$t_2 = 1 + \frac{1}{3}t_2 + \frac{2}{3}t_3$$

$$\begin{aligned} t_3 &= 1 + \frac{1}{2}t_1 + \frac{1}{2}t_2 \\ &= 1 + \frac{1}{2}t_2 \end{aligned}$$



- Hence $t_2 = 5$ and $t_3 = \frac{7}{2}$
- Then r_1 can be derive as

$$r_1 = 1 + \frac{1}{2}t_1 + \frac{1}{2}t_2 = \frac{7}{2}$$

Mean Return Times

- Mean return time

Mean Return Times

Consider a finite irreducible Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $l \in S$ be a state. Let r_l be the **mean return time** to state l. Then

$$r_l = 1 + \sum_k t_k p_{lk},$$

where t_k is the expected time until the chain hits state l given $X_0 = k$. Specifically,

$$\begin{aligned} t_l &= 0, \\ t_k &= 1 + \sum_j t_j p_{kj}, \qquad \text{for } k \neq l. \end{aligned}$$

Stationary and Limiting Distributions

- Recall the state probability distribution for time step n

$$\boldsymbol{\pi}^{(n)} = [P(X_n = 1) \quad P(X_n = 2) \quad \cdots \quad P(X_n = r)]$$

- We had $\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}^{(n-1)} P = \boldsymbol{\pi}^{(0)} P^n$
- In the limit when $n \to \infty$, the state probabilities may converges to a stationary distribution that are independent of the initial condition

Limiting Distributions

The probability distribution $\pi = [\pi_0, \pi_1, \pi_2, \cdots]$ is called the **limiting distribution** of the Markov chain X_n if

$$\pi_j = \lim_{n \to \infty} P(X_n = j | X_0 = i)$$

for all $i, j \in S$, and we have

$$\sum_{j \in S} \pi_j = 1.$$

Stationary and Limiting Distributions

- If exists, the stationary limiting distribution satisfies the following two equation

$$\boldsymbol{\pi} = \boldsymbol{\pi} P \qquad \text{and} \qquad \sum_i \pi_i = 1$$

- Note that $\boldsymbol{\pi} = \boldsymbol{\pi} P$ is equivalent to

$$\pi_j = \sum_{k \in S} \pi_k p_{kj}$$

- This implies that the probability of going to the state j in the next step (i.e., righthand side) is equal to the probability of being in state j now (lefthandsize)
- The condition $\sum_i \pi_i = 1$ is simply axiom number 2, which is the normalization condition.

Stationary and Limiting Distributions

- Stationary distributions exist for **irreducible** and **aperiodic** finite Markov chains

Consider a finite Markov chain $\{X_n, n = 0, 1, 2, \dots\}$ where $X_n \in S = \{0, 1, 2, \cdots, r\}$. Assume that the chain is irreducible and aperiodic. Then,

1. The set of equations

$$\pi = \pi P,$$
$$\sum_{j \in S} \pi_j = 1$$

   has a unique solution.
2. The unique solution to the above equations is the limiting distribution of the Markov chain, i.e.,

$$\pi_j = \lim_{n \to \infty} P(X_n = j | X_0 = i),$$

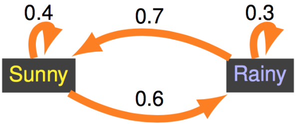
   for all $i, j \in S$.
3. We have

$$r_j = \frac{1}{\pi_j}, \quad \text{for all } j \in S,$$

   where r_j is the mean return time to state j.

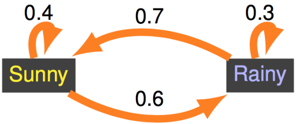
Example

- Find the stationary limiting distribution of the following Markov chain



Example

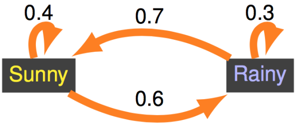
- Find the stationary limiting distribution of the following Markov chain



- $\boldsymbol{\pi} = \boldsymbol{\pi} P = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \Longrightarrow \pi_1 = \frac{7}{6}\pi_2$

Example

- Find the stationary limiting distribution of the following Markov chain



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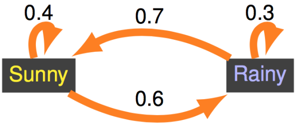
- Using the fact that $\pi_1 + \pi_2 = 1$

Example

- Find the stationary limiting distribution of the following Markov chain



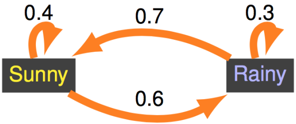
- $\boldsymbol{\pi} = \boldsymbol{\pi} P = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \Longrightarrow \pi_1 = \frac{7}{6}\pi_2$

- Using the fact that $\pi_1 + \pi_2 = 1$

- We have $\boldsymbol{\pi} = \begin{bmatrix} \frac{7}{13} & \frac{6}{13} \end{bmatrix}$

Example

- Find the stationary limiting distribution of the following Markov chain



- $\boldsymbol{\pi} = \boldsymbol{\pi} P = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \Longrightarrow \pi_1 = \frac{7}{6}\pi_2$

- Using the fact that $\pi_1 + \pi_2 = 1$

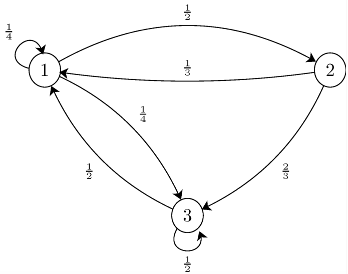
- We have $\boldsymbol{\pi} = \begin{bmatrix} \frac{7}{13} & \frac{6}{13} \end{bmatrix}$

- Validation

$$\begin{bmatrix} \frac{7}{13} & \frac{6}{13} \end{bmatrix} \times \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} = \begin{bmatrix} \frac{7}{13} & \frac{6}{13} \end{bmatrix}$$

Example2

- Consider the following Markov chain

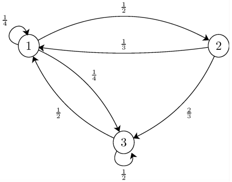


- Is this chain irreducible?
- Is this chain aperiodic?
- Find the stationary distribution for this chain.

Example2

- Applying $\boldsymbol{\pi} = \boldsymbol{\pi} P$, we have

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \times \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$



- Hence we have

$$\begin{aligned} \pi_1 &= \frac{1}{4}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 \\ \pi_2 &= \frac{1}{2}\pi_1 \\ \pi_3 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 + \frac{1}{2}\pi_3 \end{aligned}$$

- Also $\pi_1 + \pi_2 + \pi_3 = 1$
- Hence $\boldsymbol{\pi} = \begin{bmatrix} \frac{3}{8} & \frac{3}{16} & \frac{7}{16} \end{bmatrix}$

Markov Chains with Countably Infinite States

- Consider Markov chains with countably infinite state space $S \in \{0, 1, 2, 3, \cdots\}$
- In this case, we need to differentiate between two types of chains, which are **positive recurrent** and **null recurrent** Markov chains
- Only **positive recurrent** chains have stationary limiting distributions

Let i be a recurrent state. Assuming $X_0 = i$, let R_i be the number of transitions needed to return to state i, i.e.,

$$R_i = \min\{n \geq 1 : X_n = i\}.$$

If $r_i = E[R_i | X_0 = i] < \infty$, then i is said to be **positive recurrent**. If $E[R_i | X_0 = i] = \infty$, then i is said to be **null recurrent**.

Markov Chains with Countably Infinite States

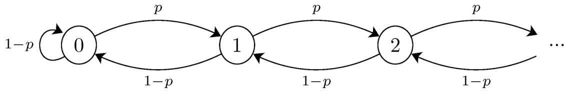
- We have the following theorem

Theorem 11.2
Consider an infinite Markov chain $\{X_n, n = 0, 1, 2, \dots\}$ where $X_n \in S = \{0, 1, 2, \cdots\}$. Assume that the chain is irreducible and aperiodic. Then, one of the following cases can occur:

1. All states are transient, and
$$\lim_{n \to \infty} P(X_n = j | X_0 = i) = 0, \text{ for all } i, j.$$
2. All states are null recurrent, and
$$\lim_{n \to \infty} P(X_n = j | X_0 = i) = 0, \text{ for all } i, j.$$
3. All states are positive recurrent. In this case, there exists a limiting distribution, $\pi = [\pi_0, \pi_1, \cdots]$, where
$$\pi_j = \lim_{n \to \infty} P(X_n = j | X_0 = i) > 0,$$
for all $i, j \in S$. The limiting distribution is the unique solution to the equations
$$\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}, \qquad \text{for } j = 0, 1, 2, \cdots,$$
$$\sum_{j=0}^{\infty} \pi_j = 1.$$

Birth Death Process

- Consider a system where we count the number of items within that system
- The items can only increased or decreased by one per time slot
- Birth death models are extensively applied in practical systems



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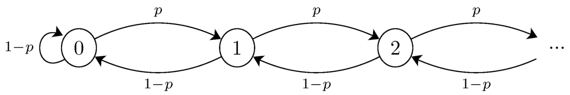

- Does the stationary steady state distribution exist?
- Consider that $0 < p < \frac{1}{2}$ find the steady state distribution

Questions?