Introduction to Probability and Statistics

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Progress

- Last set of lectures
    - Basic concepts
    - Discrete random variables
    - Continuous random variables
    - Generation of random variables and Probabilistic Inequalities
    - Two random variables
    - Multivariate random variables
- Today
    - Sum of random variables
    - Central limit theorem
    - Types of Convergence

Sum Statistically Independent Random Variables

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- Let $W = X + Y$, where X and Y are two independent random variables
- The distribution function of W is given by

$$\begin{aligned} F_W(w) &= \int_{-\infty}^{\infty} f_Y(y) F_X(w-y) dy \\ &= \int_{-\infty}^{\infty} f_X(x) F_Y(w-x) dx \end{aligned}$$

- The density function of W is given by

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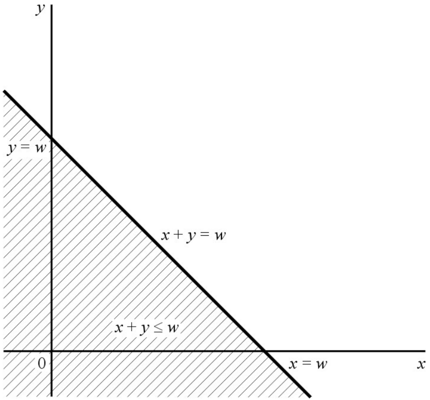

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- Consider two independent gaussian random variables X and Y with the following density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

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$$= \frac{e^{-\frac{w^2}{4}}}{\sqrt{4\pi}} \quad \Longrightarrow \text{Normal distribution with } \bar{W} = 0 \text{ and } \sigma_W^2 = 2$$

Example

- Consider two independent exponential random variables X and Y with the following joint density function

$$f_{X,Y}(x,y) = e^{-x-y}; \quad 0 \leq x, y \leq \infty$$

- Find the PDF of $W = X + Y$

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$$f_W(w) \int_{0}^{w} e^{-y-(w-y)} dy$$

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$$f_W(w) = u(w) we^{-w} \Longrightarrow \text{Gamma distribution with } \bar{W} = 2$$

Example

- Consider two independent uniform random variables X and Y with the following density functions

$$f_X(x) = \begin{cases} \frac{1}{a} & 0 \leq x \leq a \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{b} & 0 \leq y \leq b \\ 0 & \text{elsewhere} \end{cases}$$

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$$f_X((w-y)) = \begin{cases} \frac{1}{a} & 0 \leq w - y \leq a \\ 0 & \text{elsewhere} \end{cases} \Rightarrow f_X((w-y)) = \begin{cases} \frac{1}{a} & w - a \leq y \leq w \\ 0 & \text{elsewhere} \end{cases}$$

Example

- The constraints are

$$\begin{array}{ccccc} w-a & \leq & y & \leq & w \\ 0 & \leq & y & \leq & b \\ 0 & \leq & w & \leq & a+b \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} \max(0, w-a) & \leq & y & \leq & \min(w, b) \\ 0 & \leq & w & \leq & a+b \end{array}$$

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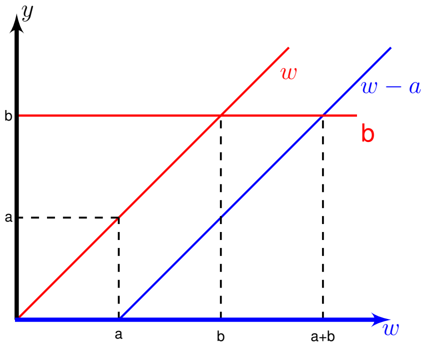

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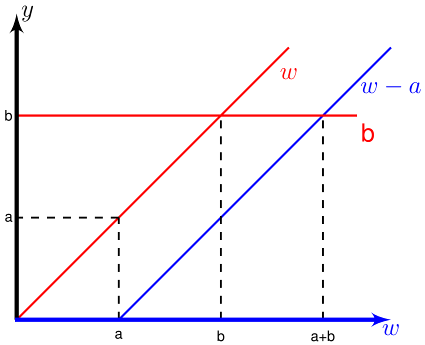

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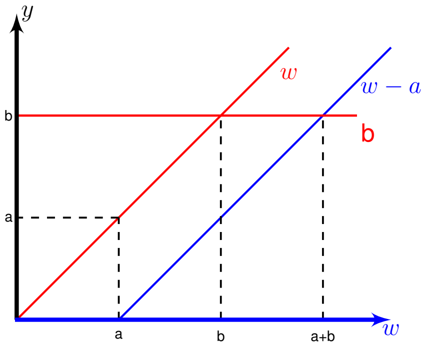

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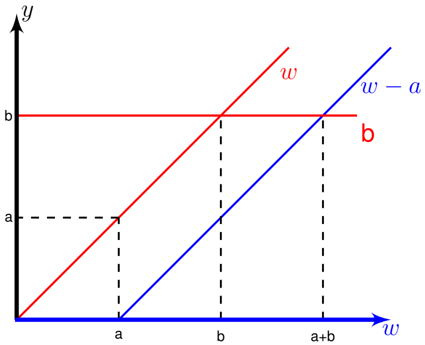

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Sum Statistically Independent Random Variables

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- Let $W = X + Y$, where X and Y are two independent random variables
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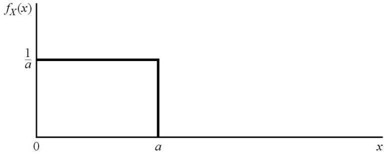
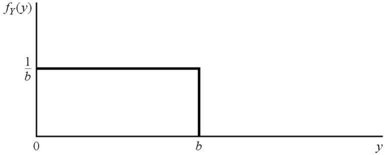


(a)



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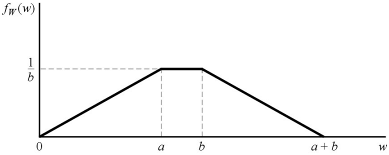

(c)

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$$f_W(w) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) * \cdots * f_{X_N}(x_N)$$

- Exploiting the Fourier transform identities

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Progress...

- Last section
  - Sum of two random variables
- Current section
  - Law of large number
  - Central limit theorem

Calculating Sample Averages

- Consider eleven samples are taken for the daily used data in Mega Bytes as $\mathbf{x} = \{0.1,\ 0.4,\ 0.9,\ 1.4,\ 2.0,\ 2.8,\ 3.7,\ 4.8,\ 6.4,\ 9.2,\ 12.0\}$
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- The estimation error for the mean and variance are

$$\frac{|\hat{\bar{X}} - \bar{X}|}{\bar{X}} = 0.675\% \quad \text{and} \quad \frac{|\widehat{\sigma_X^2} - \sigma_X^2|}{\sigma_X^2} = 7.8\%$$

Sample mean

- Consider a randomly selected sample of size n from a certain population
- The mean value of such sample is

$$M_n = \hat{\bar{X}}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

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$$\mathsf{Var}[M_n] = \mathsf{Var}\left(\frac{\mathbb{E}X_1 + X_2 + \cdots + X_n}{n}\right) = \frac{\mathsf{Var}(X)}{n}$$

Sample mean

- For very large sample size $n \to \infty$ we have

$$\lim_{n \to \infty} M_n = \frac{X_1 + X_2 + \cdots + X_n}{n} = \bar{X}$$

- This is because

$$\lim_{n \to \infty} \text{Var}[M_n] = \frac{\text{Var}(X)}{n} = 0$$

Weak Law of Large Number

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- Consider the mean estimator of a sample of size n

$$M_n = \hat{\hat{X}}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

  where X_i for all i are identically distributed with finite mean and are pairwise independent

- Then, for any $\epsilon > 0$, the following is true

$$\lim_{n \to \infty} \mathbb{P}(|\hat{\hat{X}}_n - \bar{X}| > \epsilon) = 0$$

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- Proof: using Chebyshev's inequality

$$\mathbb{P}(|M_n - \bar{X}| > \epsilon) \leq \frac{\mathsf{Var}(M_n)}{\epsilon^2} = \frac{\mathsf{Var}(X)}{n\epsilon^2}$$

Then

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- Caution: Keep in mind that you used Chebyshev's inequality

Central Limit Theorem

Central Limit Theorem: i.i.d case

- Let $X_i, i = 1, 2, 3, \ldots$ be independent and identically distributed (i.i.d) random variables with finite means $\bar{X}$ and finite variances σ_X
- Let $Y = X_1 + X_2 + \cdots + X_n$
- Define $W = \frac{Y-\bar{Y}}{\sigma_Y} = \frac{Y-n\bar{X}}{\sqrt{n}\sigma_x}$ such that

$$\bar{W} = 0 \qquad \text{and} \qquad \sigma_W^2 = 1$$

- As $N \to \infty$, for every c

$$F_W(c) \to \Phi(c)$$
$$\mathbb{P}(W < c) \to \mathbb{P}(Z < c)$$

where Z is a standardized gaussian random variable

Example

Assumptions:

- $X_1, X_2 \ldots$ are iid Bernoulli(p).
- $Z_n = \dfrac{X_1 + X_2 + \ldots + X_n - np}{\sqrt{np(1-p)}}$.

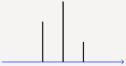
We choose $p = \frac{1}{3}$.

$$Z_1 = \frac{X_1 - p}{\sqrt{p(1-p)}}$$

PMF of Z_1

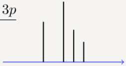
$$Z_2 = \frac{X_1 + X_2 - 2p}{\sqrt{2p(1-p)}}$$

PMF of Z_2



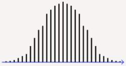
$$Z_3 = \frac{X_1 + X_2 + X_3 - 3p}{\sqrt{3p(1-p)}}$$

PMF of Z_3



$$Z_{30} = \frac{\sum_{i=1}^{30} X_i - 30p}{\sqrt{30p(1-p)}}$$

PMF of Z_{30}



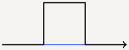
Example

Assumptions:

- $X_1, X_2 \ldots$ are iid Uniform(0,1).
- $Z_n = \dfrac{X_1 + X_2 + \ldots + X_n - \frac{n}{2}}{\sqrt{\frac{n}{12}}}$.

$$Z_1 = \frac{X_1 - \frac{1}{2}}{\sqrt{\frac{1}{12}}}$$

PDF of Z_1

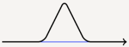


$$Z_2 = \frac{X_1 + X_2 - 1}{\sqrt{\frac{2}{12}}}$$

PDF of Z_2

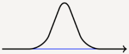
$$Z_3 = \frac{X_1 + X_2 + X_3 - \frac{2}{3}}{\sqrt{\frac{3}{12}}}$$

PDF of Z_3



$$Z_{30} = \frac{\sum_{i=1}^{30} X_i - \frac{30}{2}}{\sqrt{\frac{30}{12}}}$$

PDF of Z_{30}



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- Using the CLT

$$\mathbb{P}(|M_n - \bar{X}| > 0.01) = \mathbb{P}\left(\left|\frac{M_n - n\bar{X}}{n}\right| > 0.01\right)$$
$$= \mathbb{P}\left(\left|\frac{M_n - n\bar{X}}{\sqrt{n}\sigma}\right| > \frac{0.01\sqrt{n}}{\sigma}\right)$$
$$\approx \mathbb{P}(|Z| > \frac{0.01\sqrt{n}}{\sigma}) \leq \mathbb{P}(|Z| > \frac{0.01\sqrt{n}}{2}) = \frac{\mathbb{P}(Z > \frac{0.01\sqrt{n}}{0.5})}{2}$$

Example

- Mathematically, we need to have $\mathbb{P}(|M_n - \bar{X}| > 0.01) \leq 0.05$?
- Using the CLT

$$
\begin{aligned}
\mathbb{P}(|M_n - \bar{X}| > 0.01) &= \mathbb{P}\left(\left|\frac{M_n - n\bar{X}}{n}\right| > 0.01\right) \\
&= \mathbb{P}\left(\left|\frac{M_n - n\bar{X}}{\sqrt{n}\sigma}\right| > \frac{0.01\sqrt{n}}{\sigma}\right) \\
&\approx \mathbb{P}(|Z| > \frac{0.01\sqrt{n}}{\sigma}) \leq \mathbb{P}(|Z| > \frac{0.01\sqrt{n}}{2}) = \frac{\mathbb{P}(Z > \frac{0.01\sqrt{n}}{0.5})}{2}
\end{aligned}
$$

- Hence, we need to find n that satisfies

Example

- Mathematically, we need to have $\mathbb{P}(|M_n - \bar{X}| > 0.01) \leq 0.05$?
- Using the CLT

$$\mathbb{P}(|M_n - \bar{X}| > 0.01) = \mathbb{P}\left(\left|\frac{M_n - n\bar{X}}{n}\right| > 0.01\right)$$
$$= \mathbb{P}\left(\left|\frac{M_n - n\bar{X}}{\sqrt{n}\sigma}\right| > \frac{0.01\sqrt{n}}{\sigma}\right)$$
$$\approx \mathbb{P}(|Z| > \frac{0.01\sqrt{n}}{\sigma}) \leq \mathbb{P}(|Z| > \frac{0.01\sqrt{n}}{2}) = \frac{\mathbb{P}(Z > \frac{0.01\sqrt{n}}{0.5})}{2}$$

- Hence, we need to find n that satisfies

$$\Phi(0.02\sqrt{n}) = 0.975$$

Example

$$F_Z(0.02\sqrt{N}) = 0.975$$

TABLE B-1
Values of $F(x)$ for $0 \le x \le 3.89$ in steps of 0.01

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9773	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000

Example

$$F_Z(0.02\sqrt{N}) = 0.975$$

Using Table

$$0.02\sqrt{N} = 1.96$$

TABLE B-1
Values of $F(x)$ for $0 \leq x \leq 3.89$ in steps of 0.01

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
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1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
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2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
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2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000

Example

$$F_Z(0.02\sqrt{N}) = 0.975$$

Using Table

$$0.02\sqrt{N} = 1.96$$

$$N = 9604$$

TABLE B-1
Values of $F(x)$ for $0 \leq x \leq 3.89$ in steps of 0.01

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
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0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9773	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000

Central limit theorems (CLT)

- Consider a case where $Y = \sum_{i=1}^{n} a_i X_i$ for very large n
- Exact characterization of Y becomes overwhelming
- CLT can be used to find probabilities of Y

$$
\begin{aligned}
\mathbb{P}\{y_1 \leq Y \leq y_2\} &= \mathbb{P}\{\frac{y_1 - n\bar{X}}{\sigma\sqrt{n}} \leq \frac{Y - n\bar{X}}{\sigma\sqrt{n}} \leq \frac{y_2 - n\bar{X}}{\sigma\sqrt{n}}\} \\
&= \mathbb{P}\{\frac{y_1 - n\bar{X}}{\sigma\sqrt{n}} \leq Z \leq \frac{y_2 - n\bar{X}}{\sigma\sqrt{n}}\} \\
&= \Phi\left(\frac{y_2 - n\bar{X}}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{y_1 - n\bar{X}}{\sigma\sqrt{n}}\right)
\end{aligned}
$$

Continuity Correction for CLT

- The CLT applies to both discrete and continuous random variables
- To improve the CLT accuracy for discrete RVs, we apply the continuity correction
- Consider that the random variable Y only takes integer values $\{1, 2, 3, \cdots\}$, then

$$\mathbb{P}\{y_1 - \frac{1}{2} \leq Y \leq y_2 + \frac{1}{2}\}$$

Recall: Weak Law of Large Number

Recall: Weak Law of Large Number

- Consider the sample mean estimator

$$\hat{\hat{X}}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$$

  where X_i for all i are identically distributed with finite mean and are pairwise independent
- Then, for any $\epsilon > 0$, the following is true

$$\lim_{N\to\infty} \mathbb{P}(|\hat{\hat{X}}_N - \bar{X}| \leq \epsilon) = 1$$

Strong Law of Large Numbers

Strong Law of Large Numbers

- Consider the sample mean estimator

$$\hat{\bar{X}}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$$

where X_i for all i are identically distributed with finite mean and are pairwise independent

- Then the following is true

$$\lim_{N\to\infty} \mathbb{P}(\hat{\bar{X}}_N = \bar{X}) = 1$$

Progress

- Last section
  - Sum of two random variables
  - Law of large number
  - Central limit theorem
- Current section
  - Convergence of random variables

Convergence

Definition 7.2.

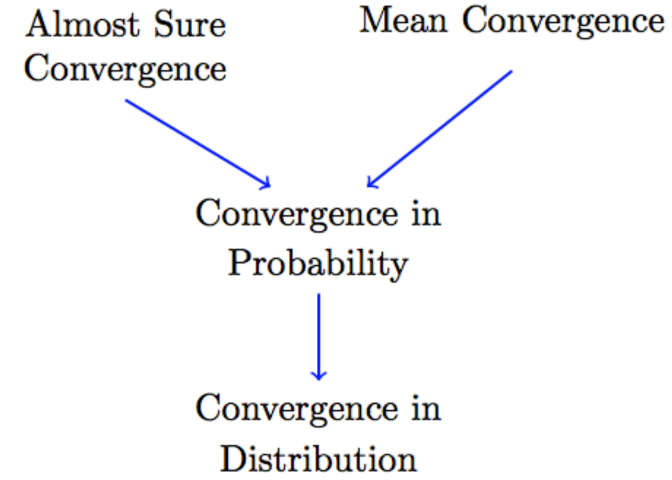
A sequence $a_1, a_2, a_3, \cdots$ converges to a limit L if

$$\lim_{n\to\infty} a_n = L.$$

That is, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon, \qquad \text{for all } n > N.$$

Convergence



Convergence

Before discussing convergence for a sequence of random variables, let us remember what convergence means for a sequence of real numbers. If we have a sequence of real numbers $a_1, a_2, a_3, \cdots$, we can ask whether the sequence converges. For example, the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots$$

is defined as

$$a_n = \frac{n}{n+1}, \qquad \text{for } n = 1, 2, 3, \cdots$$

This sequence converges to 1. We say that a sequence a_1, a_2, a_3, $\cdots$ converges to a limit L if a_n approaches L as n goes to infinity.

Convergence

Convergence in Distribution

A sequence of random variables $X_1, X_2, X_3, \cdots$ converges **in distribution** to a random variable X, shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

for all x at which $F_X(x)$ is continuous.

Convergence

Theorem 7.1 Consider the sequence X_1, X_2, X_3, $\cdots$ and the random variable X. Assume that X and X_n (for all n) are non-negative and integer-valued, i.e.,

$$
\begin{aligned}
&R_X \subset \{0, 1, 2, \cdots\}, \\
&R_{X_n} \subset \{0, 1, 2, \cdots\}, \qquad \text{for } n = 1, 2, 3, \cdots .
\end{aligned}
$$

Then $X_n \xrightarrow{d} X$ if and only if

$$
\lim_{n \to \infty} P_{X_n}(k) = P_X(k), \qquad \text{for } k = 0, 1, 2, \cdots .
$$

Convergence

Convergence in Probability

A sequence of random variables $X_1, X_2, X_3, \cdots$ converges **in probability** to a random variable X, shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n\to\infty} P\Big(|X_n - X| \geq \epsilon\Big) = 0, \qquad \text{for all } \epsilon > 0.$$

Convergence

Convergence in Mean

Let $r \geq 1$ be a fixed number. A sequence of random variables X_1, X_2, X_3, $\cdots$ converges **in the rth mean** or **in the L^r norm** to a random variable X, shown by $X_n \xrightarrow{L^r} X$, if

$$\lim_{n \to \infty} E\left(|X_n - X|^r\right) = 0.$$

If $r = 2$, it is called the **mean-square convergence**, and it is shown by $X_n \xrightarrow{m.s.} X$.

Convergence

Almost Sure Convergence

A sequence of random variables $X_1, X_2, X_3, \cdots$ converges **almost surely** to a random variable X, shown by $X_n \xrightarrow{a.s.} X$, if

$$P\left(\left\{s \in S : \lim_{n\to\infty} X_n(s) = X(s)\right\}\right) = 1.$$

Questions?