I. A. Maron - Problems in Calculus of One Variable - Problem 6.8.14

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Problem

Let the function $f(x)$ be positive on the interval $[a, b]$. Prove that the expression

$$\int_a^b f(x)dx \int_a^b \frac{dx}{f(x)} \tag{1}$$

reaches the least value only if $f(x)$ is constant on this interval.

Solution

Consider

$$\int_a^b \sqrt{f(x)}dx \int_a^b \frac{dx}{\sqrt{f(x)}}$$

and apply Cauchy-Bunyakovsky-Schwarz inequality to get

$$0 \leq \left(\int_a^b \sqrt{f(x)} \frac{dx}{\sqrt{f(x)}} \right)^2 \leq \int_a^b \left(\sqrt{f(x)} \right)^2 dx \int_a^b \frac{dx}{\left(\sqrt{f(x)} \right)^2}$$

$$0 \leq (b-a)^2 \leq \int_a^b f(x)dx \int_a^b \frac{dx}{f(x)}.$$

Hence, the least value of (1) is $(b-a)^2$. So,

$$(b-a)^2 = \underbrace{\int_a^b f(x)dx}_{c(b-a)} \underbrace{\int_a^b \frac{dx}{f(x)}}_{\frac{1}{c}(b-a)}$$

for some constant c. This configuration is only possible if f is constant on $[a, b]$.