CRUX - Problem 4677

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Problem

Evaluate

$$I = \int_0^\infty \arctan\left(\frac{2x}{x^2+1}\right)\frac{x}{x^2+4}dx.$$

Solution

We show that

$$I = \frac{\pi \ln(2\sqrt{2}-1)}{2}.$$

From arctangent addition formula

$$\arctan\left(\frac{2x}{x^2+1}\right) = \arctan((\sqrt{2}+1)x) - \arctan((\sqrt{2}-1)x).$$

Additionally we know that for $x > 0$,

$$\arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - \arctan(x).$$

Hence,

$$\arctan\left(\frac{2x}{x^2+1}\right) = \arctan\left(\frac{\sqrt{2}+1}{x}\right) - \arctan\left(\frac{\sqrt{2}-1}{x}\right) = \int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{x}{a^2+x^2}da.$$

Consequently,

$$\begin{aligned} I &= \int_0^\infty \left(\int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{x^2}{(a^2+x^2)(x^2+4)}da\right)dx \\ &= \int_{\sqrt{2}-1}^{\sqrt{2}+1} \left(\int_0^\infty \frac{x^2}{(a^2+x^2)(x^2+4)}dx\right)da. \end{aligned}$$

Using partial fractions we get

$$
\begin{aligned}
I &= \int_{\sqrt{2}-1}^{\sqrt{2}+1} \left(\frac{a^2}{a^2-4} \int_0^\infty \frac{dx}{a^2+x^2} - \frac{4}{a^2-4} \int_0^\infty \frac{dx}{x^2+4} \right) da \\
&= \int_{\sqrt{2}-1}^{\sqrt{2}+1} \left(\frac{1}{a^2-4} \int_0^\infty \frac{dx}{1+\frac{x^2}{a^2}} - \frac{4}{a^2-4} \int_0^\infty \frac{dx}{x^2+4} \right) da \\
&= \int_{\sqrt{2}-1}^{\sqrt{2}+1} \left[\frac{a \arctan(\frac{x}{a}) - 2\arctan(\frac{x}{2})}{a^2-4} \right]_0^\infty da \\
&= \frac{\pi}{2} \int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{da}{a+2} = \frac{\pi}{2} \Big[\ln(a+2) \Big]_{\sqrt{2}-1}^{\sqrt{2}+1} = \frac{\pi \ln(2\sqrt{2}-1)}{2}.
\end{aligned}
$$