

A Type Theory for Comprehension Categories with Applications to Subtyping

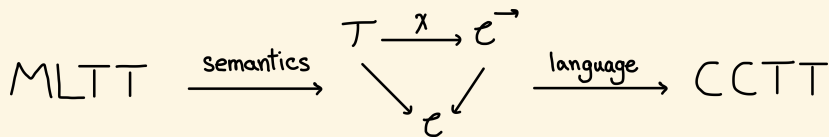
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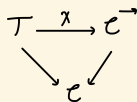
Proofs and Algorithms seminar, École Polytechnique

24 March, 2025

Overview



The “semantics” arrow does not use all the features of



Two options:

1. Restrict the comprehension categories: usually done
2. Make the type theory more expressive: CCTT

1. Design rules of a type theory which reflect the structure of comprehension categories
2. Prove soundness by giving an interpretation of the type theory in any comprehension category
3. Extend Coraglia and Emmenegger's work [CE24] by giving rules that capture coercive subtyping
4. Develop rules for Π -, Σ - and Id -types and give soundness results wrt each suitable semantic structure
5. Extend the rules with subtyping for type formers
6. Define suitable semantic structure for subtyping for each type former and show soundness wrt to these

Preprint is available on arXiv: <https://arxiv.org/abs/2503.10868>

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Outline

1. Review: MLTT and Comprehension Categories
2. Our Work: Core Syntax CCTT
3. CCTT Captures Subtyping
4. Extending CCTT with Subtyping for Type Formers

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An intuitionistic theory of types

Per Martin-Löf

Department of Mathematics, University of Stockholm

From “Syntax and Semantics of Dependent Types”¹:

$\vdash \Gamma \text{ ctxt}$	Γ is a valid context
$\Gamma \vdash \sigma \text{ type}$	σ is a type in context Γ
$\Gamma \vdash M : \sigma$	M is a term of type σ in context Γ
$\vdash \Gamma = \Delta \text{ ctxt}$	Γ and Δ are definitionally equal contexts
$\Gamma \vdash \sigma = \tau \text{ type}$	σ and τ are definitionally equal types in context Γ
$\Gamma \vdash M = N : \sigma$	M and N are def. equal terms of type σ in context Γ .

¹Martin Hofmann. *Syntax and Semantics of Dependent Types*. Publications of the Newton Institute. Cambridge University Press, 1997.

Comprehension Categories

Comprehension Category [Jac93, Definition 4.1]

A *comprehension category* consists of a category \mathcal{C} , a (cloven) fibration $p : \mathcal{T} \rightarrow \mathcal{C}$, and a functor $\chi : \mathcal{T} \rightarrow \mathcal{C}^{\rightarrow}$ preserving cartesian arrows, such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\chi} & \mathcal{C}^{\rightarrow} \\ & \searrow p & \swarrow \text{cod} \\ & \mathcal{C} & \end{array}$$

A comprehension category is *full* if χ is full and faithful.

A comprehension category is *split* if p is a split fibration.

Full split comprehension categories are models for MLTT.

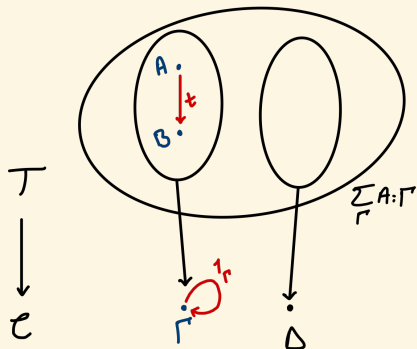
Comprehension Categories

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\chi} & \mathcal{C}^{\rightarrow} \\ & \searrow p \quad \swarrow \text{cod} & \\ & \mathcal{C} & \end{array}$$

1. \mathcal{C} : category of contexts and context morphisms
2. Fibre \mathcal{T}_{Γ} : category of types in context Γ
3. Substitution is captured by the reindexing functors
4. Extended context $\Gamma.A$ is given by $\text{dom} \circ \chi : A \mapsto \Gamma.A$
5. $\Gamma \vdash t : A$ is interpreted as sections of $\chi(A) : \Gamma.A \rightarrow \Gamma$ in \mathcal{C}

Vertical Morphisms

What about morphisms in a fibre \mathcal{T}_Γ ?



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Goal: Design rules that reflect all structure of (not-necessarily full) comprehension categories.

CCTT: Judgements

1. $\Gamma \text{ ctx}$
2. $\Gamma \vdash s : \Delta$
3. $\Gamma \vdash s \equiv s' : \Delta$
4. $\Gamma \vdash A \text{ type}$
5. $\Gamma | A \vdash t : B$
6. $\Gamma | A \vdash t \equiv t' : B$

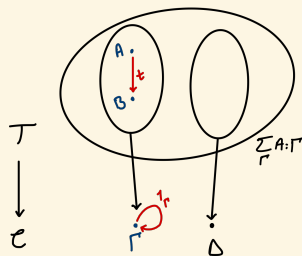
CCTT: Judgements

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4. $\Gamma \vdash A \text{ type}$
5. $\Gamma \mid A \vdash t : B$
6. $\Gamma \mid A \vdash t \equiv t' : B$

} $\Gamma \vdash t : A \ \& \ \Gamma \vdash t \equiv t' : A$ in MLTT

CCTT: Judgements

1. $\Gamma \text{ ctx}$
2. $\Gamma \vdash s : \Delta$
3. $\Gamma \vdash s \equiv s' : \Delta$
4. $\Gamma \vdash A \text{ type}$
5. $\Gamma|A \vdash t : B$
6. $\Gamma|A \vdash t \equiv t' : B$



Judgement 5: a morphism $\llbracket t \rrbracket : \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$ in the fibre $\mathcal{T}_{\llbracket \Gamma \rrbracket}$.

See the paper for the structural rules.

In the next section, we discuss some rules through the lens of subtyping.

Theorem

Every comprehension category models the rules of CCTT.

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We Put Our Subtyping Glasses on



Coraglia and Emmenegger [CE24] observe that the vertical morphisms can be thought of as **witnesses for coercive subtyping**.

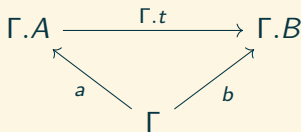
$$\Gamma \mid A \vdash t : B \quad \rightsquigarrow \quad \Gamma \vdash A \leq_t B$$

Subtyping: Subsumption

Proposition (Subsumption)

From the rules of CCTT, we can derive the following rule.

$$\frac{\Gamma \vdash A, B \text{ type} \quad \Gamma \vdash A \leq_t B \quad \Gamma \vdash a : A}{\Gamma \vdash \Gamma.t \circ a : B}$$



$\Gamma.t$ is like a coercion function for $A \leq_t B$.

Subtyping: Weakening and Substitution

We have the following rule in CCTT, which corresponds to **substitution for subtyping**.

$$\frac{\Delta \vdash A, B \text{ type} \quad \Delta \vdash A \leq_t B \quad \Gamma \vdash s : \Delta}{\Gamma \vdash A[s] \leq_{t[s]} B[s]}$$

Proposition (Weakening for Subtyping)

From the rules of CCTT, we can derive the following rule.

$$\frac{\Gamma \vdash A, A', B \text{ type} \quad \Gamma \vdash A \leq_t A'}{\Gamma.B \vdash A[\pi_B] \leq_{t[\pi_B]} A'[\pi_{B'}]}$$

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Subtyping for Type formers

1. Extend CCTT with a type former (e.g. Σ -types) and show soundness: naturally, no rules involving judgements of the form $\Gamma \vdash A \leq_t B$ get added.
2. Extend CCTT with subtyping for the type former and show soundness: we see how through an example!

Example: Σ -types

Extend CCTT with Σ -types, e.g.:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma.A \vdash B \text{ type}}{\Gamma \vdash \Sigma_A B \text{ type}} \text{ sigma-form}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma.A \vdash B \text{ type}}{\Gamma.A.B \vdash \text{pair}_{\Sigma_A B} : \Gamma.\Sigma_A B} \text{ sigma-intro}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma.A \vdash B \text{ type}}{\Gamma.\Sigma_A B \vdash \text{proj}_{\Sigma_A B} : \Gamma.A.B} \text{ sigma-elim}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma.A \vdash B \text{ type}}{\begin{array}{l} \Gamma.A.B \vdash \text{proj}_{\Sigma_A B} \circ \text{pair}_{\Sigma_A B} \equiv 1_{\Gamma.A.B} : \Gamma.A.B \\ \Gamma.\Sigma_A B \vdash \text{pair}_{\Sigma_A B} \circ \text{proj}_{\Sigma_A B} \equiv 1_{\Gamma.\Sigma_A B} : \Gamma.\Sigma_A B \end{array}} \text{ sigma-beta-eta}$$

$$\frac{\Delta \vdash A \text{ type} \quad \Delta.A \vdash B \text{ type} \quad \Gamma \vdash s : \Delta}{\Gamma \mid \Sigma_{A[s]} B[s.A] \tilde{\vdash} i_{\Sigma_A B, s} : (\Sigma_A B)[s]} \text{ subst-sigma}$$

Example: Subtyping for Σ -types

1. We want to have the following rule:

$$\frac{\begin{array}{l} \Gamma \vdash A, A' \text{ type} \quad \Gamma.A \vdash B \text{ type} \quad \Gamma.A' \vdash B' \text{ type} \\ \Gamma \vdash A \leq_f A' \quad \Gamma.A \vdash B \leq_g B'[\Gamma.f] \end{array}}{\Gamma \vdash \Sigma_A B \leq_{\Sigma(f,g)} \Sigma_{A'} B'}$$

Σ acts covariantly on both arguments.

Example: Subtyping for Σ -types

1. We want to have the following rule:

$$\frac{\begin{array}{c} \Gamma \vdash A, A' \text{ type} \quad \Gamma.A \vdash B \text{ type} \quad \Gamma.A' \vdash B' \text{ type} \\ \Gamma \vdash A \leq_f A' \quad \Gamma.A \vdash B \leq_g B'[\Gamma.f] \end{array}}{\Gamma \vdash \Sigma_A B \leq_{\Sigma(f,g)} \Sigma_{A'} B'}$$

Σ acts covariantly on both arguments.

2. The coercion function for $\Sigma_A B \leq_{\Sigma(f,g)} \Sigma_{A'} B'$ should act as follows:

$$\Gamma.\Sigma_A B \xrightarrow{\text{proj}_{\Sigma_A B}} \Gamma.A.B \xrightarrow{\chi_0 g} \Gamma.A.B'[\chi_0 f] \xrightarrow{\chi_0 f.B'} \Gamma.A'.B' \xrightarrow{\text{pair}_{\Sigma_{A'} B'}} \Gamma.\Sigma_{A'} B'$$

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3. Rules for functoriality for $\Sigma(-, -)$

Comprehension Categories with Subtyping for Σ -types

Definition

A comprehension category $(\mathcal{C}, \mathcal{T}, p, \chi)$ **has subtyping for Σ -types** if it has dependent sums and is equipped with a function giving for each $f : A \rightarrow A'$ in \mathcal{T}_Γ and $g : B \rightarrow B'[\chi_0 f]$ in $\mathcal{T}_{\Gamma.A}$, a morphism

$$\Sigma_f g : \Sigma_A B \rightarrow \Sigma_{A'} B'$$

in \mathcal{T}_Γ such that:

1. $\chi_0(\Sigma_f g)$ is the following composite

$$\Gamma.\Sigma_A B \xrightarrow{\text{proj}_{\Sigma_A B}} \Gamma.A.B \xrightarrow{\chi_0 g} \Gamma.A'.B'[\chi_0 f] \xrightarrow{\chi_0 f.B'} \Gamma.A'.B' \xrightarrow{\text{pair}_{\Sigma_{A'} B'}} \Gamma.\Sigma_{A'} B'$$

2. $\Sigma_{(-)}(-)$ preserves identities and composition

Theorem

Any comprehension category with subtyping for Σ -types models CCTT extended with subtyping for Σ -types.

Summary

1. We presented CCTT
2. CCTT captures coercive subtyping
3. We extended CCTT with Π (resp. Σ , Id) and subtyping for Π (resp. Σ , Id)
4. At each step we showed soundness wrt to the suitable semantic structure

Thank you for your attention!

References I

- [CE24] Greta Coraglia and Jacopo Emmenegger. “Categorical Models of Subtyping”. In: *29th International Conference on Types for Proofs and Programs (TYPES 2023)*. Ed. by Delia Kesner, Eduardo Hermo Reyes, and Benno van den Berg. Vol. 303. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024, 3:1–3:19. ISBN: 978-3-95977-332-4. DOI: [10.4230/LIPIcs.TYPES.2023.3](https://doi.org/10.4230/LIPIcs.TYPES.2023.3). URL: <https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.TYPES.2023.3>.
- [Hof97] Martin Hofmann. *Syntax and Semantics of Dependent Types*. Publications of the Newton Institute. Cambridge University Press, 1997.
- [Jac93] Bart Jacobs. “Comprehension Categories and the Semantics of Type Dependency”. In: *Theor. Comput. Sci.* 107.2 (1993), pp. 169–207. DOI: [10.1016/0304-3975\(93\)90169-T](https://doi.org/10.1016/0304-3975(93)90169-T). URL: [https://doi.org/10.1016/0304-3975\(93\)90169-T](https://doi.org/10.1016/0304-3975(93)90169-T).

Structural Rules: \mathcal{C}

Structural rules regarding the category of contexts

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash 1_\Gamma : \Gamma} \text{ ctx-mor-id} \quad \frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta}{\Gamma \vdash s' \circ s : \Theta} \text{ ctx-mor-comp}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\begin{array}{l} \Gamma \vdash s \circ 1_\Gamma \equiv s : \Delta \\ \Gamma \vdash 1_\Delta \circ s \equiv s : \Delta \end{array}} \text{ ctx-id-unit}$$

$$\frac{\Gamma, \Delta, \Theta, \Phi \text{ ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta \quad \Theta \vdash s'' : \Phi}{\Gamma \vdash s'' \circ (s' \circ s) \equiv (s'' \circ s') \circ s : \Phi} \text{ ctx-comp-assoc}$$

Structural Rules: \mathcal{T}

Structural rules regarding the category of types

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma \mid A \vdash 1_A : A} \text{ ty-mor-id}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B, C \text{ type} \quad \Gamma \mid A \vdash t : B \quad \Gamma \mid B \vdash t' : C}{\Gamma \mid A \vdash t' \circ t : C} \text{ ty-mor-comp}$$

$$\frac{\Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t : B}{\begin{array}{l} \Gamma \mid A \vdash t \circ 1_A \equiv t : B \\ \Gamma \mid A \vdash 1_B \circ t \equiv t : B \end{array}} \text{ ty-id-unit}$$

$$\frac{\Gamma \mid A \vdash t : B \quad \Gamma \mid B \vdash t' : C \quad \Gamma \mid C \vdash t'' : D}{\Gamma \mid A \vdash t'' \circ (t' \circ t) \equiv (t'' \circ t') \circ t : D} \text{ ty-comp-assoc}$$

Structural Rules: Context Extension

Structural rules regarding context extension

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma.A \text{ ctx}} \text{ ext-ty}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t : B}{\Gamma.A \vdash \Gamma.t : \Gamma.B} \text{ ext-tm}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma.A \vdash \Gamma.1_A \equiv 1_{\Gamma.A} : \Gamma.A} \text{ ext-id}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B, C \text{ type} \quad \Gamma \mid A \vdash t : B \quad \Gamma \mid B \vdash t' : C}{\Gamma.A \vdash \Gamma.(t' \circ t) \equiv \Gamma.t' \circ \Gamma.t : \Gamma.B} \text{ ext-comp}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma.A \vdash \pi_A : \Gamma} \text{ ext-proj}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t : B}{\Gamma.A \vdash \pi_B \circ \Gamma.t \equiv \pi_A : \Gamma} \text{ ext-coh}$$

Structural Rules: Substitution (Part 1)

Structural rules regarding substitution

$$\frac{\Gamma \vdash s : \Delta \quad \Delta \vdash A \text{ type}}{\Gamma \vdash A[s] \text{ type}} \text{ sub-ty} \quad \frac{\Delta \vdash A, B \text{ type} \quad \Gamma \vdash s : \Delta \quad \Delta \mid A \vdash t : B}{\Gamma \mid A[s] \vdash t[s] : B[s]} \text{ sub-tm}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A \text{ type}}{\Gamma \mid A[s] \vdash 1_{A[s]} \equiv 1_{A[s]} : A[s]} \text{ sub-pres-id}$$

$$\frac{\Gamma \vdash s : \Delta \quad \Delta \vdash A, B, C \text{ type} \quad \Delta \mid A \vdash t : B \quad \Delta \mid B \vdash t' : C}{\Gamma \mid A[s] \vdash (t' \circ t)[s] \equiv t'[s] \circ t[s] : C[s]} \text{ sub-pres-comp}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma \mid A[1_\Gamma] \vdash \tilde{i}_A^{\text{id}} : A} \text{ sub-id} \quad \frac{\Gamma \vdash u : \Delta \quad \Delta \vdash u' : \Theta \quad \Theta \vdash A \text{ type}}{\Gamma \mid A[u' \circ u] \vdash \tilde{i}_{A, u', u}^{\text{comp}} : A[u'][u]} \text{ sub-comp}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t : B}{\Gamma \mid A[1_\Gamma] \vdash t[1_\Gamma] \equiv i_B^{\text{id}^{-1}} \circ t \circ i_A^{\text{id}} : B[1_\Gamma]} \text{ sub-tm-id}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Theta \vdash A, B \text{ type} \quad \Gamma \vdash u : \Delta \quad \Delta \vdash u' : \Theta \quad \Theta \mid A \vdash t : B}{\Gamma \mid A[u' \circ u] \vdash t[u' \circ u] \equiv i_{B, u', u}^{\text{comp}^{-1}} \circ t[u'][u] \circ i_{A, u', u}^{\text{comp}} : B[u' \circ u]} \text{ sub-tm-comp}$$

Structural Rules: Substitution (Part 2)

Structural rules regarding substitution

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta \quad \Gamma \vdash t : \Gamma.A[s] \quad \Gamma \vdash \pi_{A[s]} \circ t \equiv 1_\Gamma : \Gamma}{\Gamma \vdash (s, t) : \Delta.A} \text{ sub-ext}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta.A}{\Gamma \vdash p_2(s) : \Gamma.A[\pi_A \circ s]} \text{ sub-proj}$$
$$\Gamma \vdash \pi_{A[\pi_A \circ s]} \circ p_2(s) \equiv 1_\Gamma : \Gamma$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta.A}{\Gamma \vdash (\pi_A \circ s, p_2(s)) \equiv s : \Delta.A} \text{ sub-eta}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta \quad \Gamma \vdash t : \Gamma.A[s] \quad \Gamma \vdash \pi_{A[s]} \circ t \equiv 1_\Gamma : \Gamma}{\begin{array}{l} \Gamma \vdash \pi_A \circ (s, t) \equiv s : \Delta \\ \Gamma \vdash p_2(s, t) \equiv t : \Gamma.A[s] \end{array}} \text{ sub-beta}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A, B \text{ type} \quad \Delta \mid A \vdash t : B \quad \Gamma \vdash s : \Delta}{\Gamma.A[s] \vdash \Gamma.t[s] \equiv p_2(\Delta.t \circ (s \circ \pi_{A[s]}, \Gamma.i_{A, s, \pi_{A[s]}}^{\text{comp}} [\pi_{A[s]}] \circ p_2(1_{\Gamma.A[s]}))) : \Gamma.B[s]} \text{ tm-sub-coh}$$

Structural Rules: \equiv (Part 1)

Structural rules regarding \equiv being a congruence relation

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma \vdash s \equiv s : \Delta} \text{ ctx-eq-refl}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s_1, s_2 : \Delta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta}{\Gamma \vdash s_2 \equiv s_1 : \Delta} \text{ ctx-eq-sym}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s_1, s_2, s_3 : \Delta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta \quad \Gamma \vdash s_2 \equiv s_3 : \Delta}{\Gamma \vdash s_1 \equiv s_3 : \Delta} \text{ ctx-eq-trans}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash s_1, s_2 : \Delta \quad \Delta \vdash t : \Theta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta}{\Gamma \vdash t \circ s_1 \equiv t \circ s_2 : \Theta} \text{ ctx-comp-cong-1}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash t : \Delta \quad \Delta \vdash s_1, s_2 : \Theta \quad \Delta \vdash s_1 \equiv s_2 : \Theta}{\Gamma \vdash s_1 \circ t \equiv s_2 \circ t : \Theta} \text{ ctx-comp-cong-2}$$

Structural Rules: \equiv (Part 2)

Structural rules regarding \equiv being a congruence relation

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t : B}{\Gamma \mid A \vdash t \equiv t : B} \text{ ty-eq-refl}$$

$$\frac{\Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t_1, t_2 : B \quad \Gamma \mid A \vdash t_1 \equiv t_2 : B}{\Gamma \mid A \vdash t_2 \equiv t_1 : B} \text{ ty-eq-sym}$$

$$\frac{\Gamma \mid A \vdash t_1, t_2, t_3 : B \quad \Gamma \mid A \vdash t_1 \equiv t_2 : B \quad \Gamma \mid A \vdash t_2 \equiv t_3 : B}{\Gamma \mid A \vdash t_1 \equiv t_3 : B} \text{ ty-eq-trans}$$

$$\frac{\Gamma \mid A \vdash u_1, u_2 : B \quad \Gamma \mid B \vdash v : C \quad \Gamma \mid A \vdash u_1 \equiv u_2 : B}{\Gamma \mid A \vdash v \circ u_1 \equiv v \circ u_2 : C} \text{ ty-comp-cong-1}$$

$$\frac{\Gamma \mid A \vdash v : B \quad \Gamma \mid B \vdash u_1, u_2 : C \quad \Gamma \mid B \vdash u_1 \equiv u_2 : C}{\Gamma \mid A \vdash u_1 \circ v \equiv u_2 \circ v : C} \text{ ty-comp-cong-2}$$

Structural Rules: \equiv (Part 3)

Structural rules regarding \equiv being a congruence relation

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t_1, t_2 : B \quad \Gamma \mid A \vdash t_1 \equiv t_2 : B}{\Gamma.A \vdash \Gamma.t_1 \equiv \Gamma.t_2 : \Gamma.B} \text{ ext-cong}$$

$$\frac{\Gamma \vdash s : \Delta \quad \Delta \mid A \vdash t_1, t_2 : B \quad \Delta \mid A \vdash t_1 \equiv t_2 : B}{\Gamma \mid A[s] \vdash t_1[s] \equiv t_2[s] : B[s]} \text{ sub-cong-tm}$$

$$\frac{\Gamma \vdash s_1, s_2 : \Delta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta \quad \Gamma \vdash t_1, t_2 : A[s] \quad \Gamma \vdash t_1 \equiv t_2 : A[s]}{\Gamma \vdash (s_1, t_1) \equiv (s_2, t_2) : \Delta.A} \text{ sub-ext-cong}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s_1, s_2 : \Delta.A \quad \Gamma \vdash s_1 \equiv s_2 : \Delta.A}{\begin{array}{l} \Gamma \vdash p_1(s_1) \equiv p_1(s_2) : \Delta \\ \Gamma \vdash p_2(s_1) \equiv p_2(s_2) : A[s] \end{array}} \text{ sub-proj-cong}$$

Structural Rules: \equiv (Part 4)

Structural rules regarding there being no judgement for equality of types

$$\frac{\Delta \vdash A \text{ type} \quad \Gamma \vdash u \equiv u' : \Delta}{\Gamma \mid A[u] \vdash i_{A,u,u'}^{\text{sub}} : A[u']} \text{ sub-cong}$$
$$\Gamma \mid A[u'] \vdash i_{A,u,u'}^{\text{sub}^{-1}} \equiv i_{A,u',u}^{\text{sub}} : A[u]$$

$$\frac{\Delta \vdash A \text{ type} \quad \Gamma \vdash u : \Delta}{\Gamma \mid A[u] \vdash i_{A,u,u}^{\text{sub}} \equiv 1_{A[u]} : A[u]} \text{ sub-cong-id}$$

$$\frac{\Theta \vdash A \text{ type} \quad \Gamma \vdash t, t' : \Delta \quad \Delta \vdash u : \Theta \quad \Gamma \vdash t \equiv t' : \Delta}{\Gamma \mid A[u][t] \vdash i_{A,u,t'}^{\text{comp}} \circ i_{A,t,t'}^{\text{sub}} \equiv i_{A,u \circ t, u \circ t'}^{\text{sub}} \circ i_{A,u,t}^{\text{comp}} : A[u \circ t']} \text{ sub-cong-comp-1}$$

$$\frac{\Theta \vdash A \text{ type} \quad \Gamma \vdash t : \Delta \quad \Delta \vdash u, u' : \Theta \quad \Gamma \vdash u \equiv u' : \Delta}{\Gamma \mid A[u][t] \vdash i_{A,u',t}^{\text{comp}} \circ i_{A,u,u'}^{\text{sub}}[t] \equiv i_{A,u \circ t, u' \circ t}^{\text{sub}} \circ i_{A,u,t}^{\text{comp}} : A[u' \circ t]} \text{ sub-cong-comp-2}$$