More Calculus Haha

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1 Double Exponential

1.1 Definition

$$y = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$$

1.2 Moment Generating Function

$$
\begin{aligned}
Mgf(t) &= \int_{-\infty}^{+\infty} e^{tx} \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\sigma} e^{tx-\frac{|x-\mu|}{\sigma}} dx \\
&= \int_{-\infty}^{\mu} \frac{1}{2\sigma} e^{tx-\frac{\mu-x}{\sigma}} dx \\
&+ \int_{\mu}^{\infty} \frac{1}{2\sigma} e^{tx-\frac{x-\mu}{\sigma}} dx \\
&= \int_{-\infty}^{\mu} \frac{1}{2\sigma} e^{tx-\frac{\mu-x}{\sigma}} dx \\
&+ \int_{\mu}^{\infty} \frac{1}{2\sigma} e^{tx-\frac{x-\mu}{\sigma}} dx \\
&= \int_{-\infty}^{\mu} \frac{1}{2\sigma} e^{\frac{\sigma tx-\mu+x}{\sigma}} dx \\
&+ \int_{\mu}^{\infty} \frac{1}{2\sigma} e^{\frac{\sigma tx-x+\mu}{\sigma}} dx \\
&= \frac{1}{2\sigma}\left(\int_{-\infty}^{\mu} e^{\frac{\sigma tx-\mu+x}{\sigma}} dx + \int_{\mu}^{\infty} e^{\frac{\sigma tx-x+\mu}{\sigma}} dx\right) \\
&= \frac{1}{2\sigma}\left(e^{-\frac{\mu}{\sigma}} \int_{-\infty}^{\mu} e^{\frac{(\sigma t+1)x}{\sigma}} dx + e^{\frac{\mu}{\sigma}} \int_{\mu}^{\infty} e^{\frac{\sigma tx-x}{\sigma}} dx\right)) \\
&= \frac{1}{2\sigma}\left(e^{-\frac{\mu}{\sigma}} \frac{\sigma}{\sigma t+1} e^{\frac{(\sigma t+1)x}{\sigma}} \Big|_{x=-\infty}^{\mu} + e^{\frac{\mu}{\sigma}} \frac{\sigma}{\sigma t-1} e^{\frac{(\sigma t-1)x}{\sigma}} \Big|_{x=\mu}^{\infty}\right) \\
&= \frac{1}{2\sigma}\left(e^{-\frac{\mu}{\sigma}} \frac{\sigma}{\sigma t+1} e^{\frac{(\sigma t+1)\mu}{\sigma}} - e^{\frac{\mu}{\sigma}} \frac{\sigma}{\sigma t-1} e^{\frac{(\sigma t-1)\mu}{\sigma}}\right)
\end{aligned}
$$

$$
= \frac{1}{2\sigma}\left(\frac{\sigma}{\sigma t+1} e^{\frac{(\sigma t)\mu}{\sigma}} - \frac{\sigma}{\sigma t-1} e^{\frac{(\sigma t)\mu}{\sigma}}\right) = \frac{1}{1-(\sigma t)^2} e^{t\mu}
$$

1.3 Means and Variance

Mean is μ, which is easy to show.

Variance ,easy to be proven by substitution and take derivative, is $2\sigma^2$

2 Fisher Distribution

2.1 Moments

nth-Moment is defined by the following integral:

$$\int_0^\infty x^n \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}(\frac{\nu_1}{\nu_2})^{\frac{\nu_1}{2}} \frac{x^{\frac{\nu_1-2}{2}}}{(1+(\frac{\nu_1}{\nu_2})x)^{\frac{\nu_1+\nu_2}{2}}}dx$$

$$\frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}(\frac{\nu_1}{\nu_2})^{\frac{\nu_1}{2}} \int_0^\infty x^{2n/2} \frac{x^{\frac{\nu_1-2}{2}}}{(1+(\frac{\nu_1}{\nu_2})x)^{\frac{\nu_1+\nu_2}{2}}}dx$$

Focusing on the integral,

$$\int_0^\infty \frac{x^{\frac{2n+\mu_1-2}{2}}}{(1+(\frac{\nu_1}{\nu_2})x)^{\frac{\nu_1+\nu_2}{2}}}dx$$

$$\int_0^\infty \frac{x^{\frac{2n+\nu_1-2}{2}}}{(1+(\frac{\nu_1}{\nu_2})x)^{\frac{\nu_1+\nu_2}{2}}}dx$$

Let $u = 1 + (\frac{\nu_1}{\nu_2})x$ then $(u-1)(\frac{\nu_2}{\nu_1}) = x$
and $du(\frac{\nu_2}{\nu_1}) = dx$, if $x = 0, u = 1$, if $x = \infty, u = \infty$

$$\int_1^\infty \frac{((u-1)(\frac{\nu_2}{\nu_1}))^{\frac{2n+\nu_1-2}{2}}}{u^{\frac{\nu_1+\nu_2}{2}}}(\frac{\nu_2}{\nu_1})du$$

$$(\frac{\nu_2}{\nu_1})^{\frac{2n+\nu_1}{2}} \int_1^\infty \frac{(u-1)^{\frac{2n+\nu_1-2}{2}}}{u^{\frac{\nu_1+\nu_2}{2}}}du$$

$$(\frac{\nu_2}{\nu_1})^{\frac{2n+\nu_1}{2}} \int_1^\infty \frac{(u-1)^{\frac{2n+\nu_1-2}{2}}}{u^{\frac{\nu_1+\nu_2}{2}}}du$$

Focusing on the integral, Let $v = u^{-1}$, $dv = -u^{-2}du = -v^2du$
$du = -v^{-2}dv$

$$\int_0^1 \frac{(v^{-1}-1)^{\frac{2n+\nu_1-2}{2}}}{v^{-1\frac{\nu_1+\nu_2}{2}}}v^{-2}dv$$

$$\int_0^1 (\frac{1-v}{v})^{\frac{2n+\nu_1-2}{2}} v^{-2} v^{\frac{\nu_1+\nu_2}{2}} dv$$

$$\int_0^1 (1-v)^{\frac{2n+\nu_1-2}{2}} v^{\frac{-2n-\nu_1+2}{2}} v^{-2} v^{\frac{\nu_1+\nu_2}{2}} dv$$

$$\int_0^1 (1-v)^{\frac{2n+\nu_1-2}{2}} v^{\frac{\nu_2-2n-2}{2}} dv$$

Which as it turns out to be,

$$\mathbf{B}(\frac{\nu_2-2n}{2}, \frac{2n+\nu_1}{2})$$

Plug in back,

$$\frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}(\frac{\nu_1}{\nu_2})^{\frac{\nu_1}{2}}(\frac{\nu_2}{\nu_1})^{\frac{2n+\nu_1}{2}}\mathbf{B}(\frac{\nu_2-2n}{2},\frac{2n+\nu_1}{2})$$

Which will be,

$$\frac{\Gamma(\frac{\nu_2-2n}{2})\Gamma(\frac{2n+\nu_1}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}(\frac{\nu_2}{\nu_1})^n$$