### Maximum Likelihood Estimation and KL Divergence

Let denote the empirical data distribution over a space of inputs

and outputs . For example, in an image recognition task, can be an image and can be whether the image contains a cat or not. Let be a probabilistic classifier parameterized by , e.g., a logistic regression classifier with coefficients Show that the following equivalence holds:

(1)

Where denotes the KL-Divergence:

(2)

To help you get started consider this: We rely on the known property that if is a strictly monotonically decreasing function, then the following two problems are equivalent:

Expanding the KL-Divergence term on the RHS of (1), we get:

substituting the term in (1)

Since does not depend on , we can remove it from the optimization,

Since is strictly monotonically decreasing, we can use the hint, and have:

Completing the proof.

### Conditional Independence and Parameterization

Consider a collection of n discrete random variables , where the number of outcomes for is .

(a) [1.50 points (Written)] Without any conditional independence assumptions, what is the total number of independent parameters needed to describe the joint distribution over ?

(b) [1.50 points (Written)] Under what independence assumptions is it possible to represent the joint distribution with total number of independent parameters?

(c) [2 points (Written)]

Let denote the topological sort for a Bayesian network for the random variables . Let be a positive integer in . Suppose, for every , the random variable is conditionally independent of all ancestors given the previous ancestors in the topological ordering.

Mathematically, we impose the independence assumptions

(7)

for . For , we impose no conditional independence of with respect to its ancestors. Derive the total number of independent parameters to specify the joint distribution over . You can express the answer using summation and product symbols.

1. Without conditional independence assumptions, we need to specify the probability of every possible combination of outcomes for all variables. However, since the probabilities must sum to 1, the last probability can be determined from the others, so, the total number of independent parameters required is:
2. When all variables are mutually independent, the joint distribution can be factored into the product of individual marginal distributions:

For each variable , we need parameters to specify its marginal distribution since the last probability for each variable will be determined by the sum to 1 constraint. So, the total number of parameters required under the mutually independent assumption is:

1. Given the conditional independence assumptions

(7)

for , the total number of independent parameters required to specific the joint distribution over is:

For , we only need to specify the conditional distribution of with respect to its previous ancestors. The parameters required is:

Hence the total number of independent parameters required to specify the joint distribution over is:

### Monte Carlo Integration

A latent variable generative model specifies a joint probability distribution between a set of observed variables x ∈ and a set of latent variables . From the definition of conditional probability, we can express the joint distribution as . Here, is referred to as the prior distribution over z and is the likelihood of the observed data given the latent variables. One natural objective for learning a latent variable model is to maximize the marginal likelihood of the observed data given by:

(12)

When is high dimensional, evaluation of the marginal likelihood is computationally intractable even if we can tractably evaluate the prior and the conditional likelihood for any given and We can however use Monte Carlo to estimate the above integral. To do so, we sample samples from the prior and our estimate is given as:

(13)

1. [1 point (Written)] An estimator is an unbiased estimator of if and only if θ. Show that is an unbiased estimator of .
2. [1 point (Written)] Is an unbiased estimator of ? Prove why or why not.

Hint: The proof is short, estimator of using the definition of an unbiased estimator and Jensen’s Inequality

1. From (13), we have

Taking the expectation:

By the linearity of expectation:

Since each is sampled from , for any :

= = p(x)

Therefore,

(14)

1. No. is not an unbiased estimator of . Jensen’ inequality for a concave function states that , since logarithm is a concave function:

From (14) above, we know , so:

is not an unbiased estimator of ; it is tends to underestimate .