Weighted Graph Aggregation: Implications and Applications

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We study the problem of aggregating weighted directed graphs. We give its theoretical implications, finding its place among existing social choice and AI literature and well-known aggregation problems; we then discuss examples of its practical applications, and demonstrate the predictive power of weighted graph aggregation.

1 INTRODUCTION

In social choice theory, given a group of individuals' preferences over some alternatives, a *preference aggregation* rule or *social choice function* maps the individuals' ordinal preferences to one (winning) alternative or to a ordering of the alternatives [2].

Over the last half-century, studies from various disciplines re-think and generalize this problem to wider use cases, most notably to *cardinal preference aggregation* [32], *graph aggregation* [14], *judgment aggregation* [9], and *quantitative judgment aggregation* [38]. In this paper, we identify a missing link in the existing theory and propose the problem of *weighted graph aggregation*. When given a decision-making interpretation, it can be used to model aggregation of *incomplete* and *intransitive* preferences [26, 36], with edges representing pairwise comparisons and edge weights representing 'strength' of the preferences. Beyond the decision-making context, we can also use it to aggregate other forms of information represented by weighted direct graphs.

In section 2, we reexamine the existing classes of aggregation problem and their percise relationships, and identify where weighted graph aggregation fits in the existing theory. In section 3, we formally define weighted graph aggregation, draw (literally) a complete picture of the relationship among classes of aggregation problem and discuss its applications. In section 4, we demonstrate the predictive power of weighted graph aggregation.

2 RELATED WORK

2.1 Judgment Aggregation

How can a group of individuals make collective *judgments* on a set of *propositions* based on their individual judgments? [23] The initial observation that motivated this problem had its origins in the area of jurisprudence [21], but its present theory was due to List and Pettit [24]. They introduced a first formal model of judgment aggregation, combining an axiomatic approach to the study of aggregation rules, as common in social choice theory [2].

Let $\mathcal{L}$ be a set of all propositional formulas built from a finite set of propositional variables p, q, etc. using the usual connectives $\neg$, $\wedge$, etc. An *agenda* is a finite nonempty set $\Phi \subseteq \mathcal{L}$ that does not contain any double-negated formulas and is closed under negation. An example of agenda would be $\Phi = \{p, \neg p, q, \neg, q, p \rightarrow q, \neg(p \rightarrow q), p \wedge q, \neg(p \wedge q)\}$.

Let $N = \{1, \ldots, n\}$ be a finite set of *individuals* (or *judges*). Individual i's *judgment set* for agenda Φ is a subset $J_i \subseteq \Phi$. A judgment set is called *consistent* if all propositions in it can be simultaneously true (e.g. $\{p, q, p \rightarrow q\}$), and *complete* if it contains a member of each proposition-negation pair in Φ (e.g. $\{p, q, \neg(p \rightarrow q), \neg(p \wedge q)\}$). Let $\mathcal{J}(\Phi)$ denote the set of all complete and consistent subsets of Φ.

An *aggregation rule* $f : \mathcal{J}(\Phi)^n \rightarrow 2^\Phi$ (2^Φ denotes the powerset of Φ) maps each profile of individual judgment sets $(J_1, \ldots, J_n) \in \mathcal{J}(\Phi)^n$ to a single collective judgment set. Note that, by

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this definition, the resulting judgment set need not be complete and consistent, but the individual sets in the profile are always assumed to have these properties. An aggregation rule is *collectively rational* if the output is complete and consistent whenever the inputs are complete and consistent. [6]

However, many have argued that both output conditions are reasonable not only from a pragmatic perspective but also from a fundamental philosophical one (see [22, 27] for consistency, see [23] for completeness). Nevertheless, possible relaxation of input conditions (not all complete and consistent profile are admissible [12]) and that of output conditions (the collective judgment set can be inconsistent [11] or incomplete [16]) had been studied extensively to allow the possibility for aggregation rules with favorable properties.

2.2 Graph Aggregation

Graph aggregation is the process of computing a single collective graph that constitutes a good compromise between several input graphs, each provided by a different source. It is an emerging topic in the field of artificial intelligence, first proposed by Endriss and Grandi [13].

Given a set of individuals $N = \{1, \ldots, n\}$ and a set of vertices V. Individual i specifies a directed graph $G_i = (V, E_i)$, defined by a set of edges $E_i \subseteq V \times V$. This gives rise to a profile $E = (E_1, \ldots, E_n)$. Let $2^{V\times V}$ denote the powerset of $V\times V$, the set of all graphs. An aggregation rule $f : (2^{V\times V})^n \to 2^{V\times V}$ maps each profile of graphs into a single collective graph. An aggregation method F is *collectively rational* w.r.t. some graph properties P (e.g. transitivity) if $F(E)$ satisfies P whenever each and every $E_i \in E$ do. In other words, the properties are preserved during aggregation process. [14]

2.2.1 *Relationship to Judgment Aggregation.*

Trivially, it is possible to embed graph aggregation into judgment aggregation, by defining agenda Φ as the set of propositions of the form $\{(u, v)\}$, where $u, v \in V$, corresponding to a possible edge defined on V. A judgment set, therefore, correspond to a directed graph in the graph aggregation problem. [14]

It is worth noting that, judgment aggregation mostly concern with collective rationality (of the aggregation rules) w.r.t. consistency and completeness. Graph aggregation, on the other hand, may or may not concern about preserving transitivity ($\forall u, v, w$, if $(u, v), (v, w) \in E$, then $(u, w) \in E$) and completeness ($\forall u, v$, $(u, v) \in E$ or $(v, u) \in E$), the corresponding graph properties. Aggregation of undirected graph, for example, concerns with preserving *symmetry* ($\forall u, v$, if $(u, v) \in E$, then $(v, u) \in E$). [14] However, such collective rationality can be easily handled by judgment aggregation by defining the analogous set of propositions $\{(u, v), (v, u)\}$ to be logically inconsistent. In a word, every graph aggregation problem can be solved by a judgment aggregation rule (that solves judgment aggregation problem in which each proposition is 'existence of an edge').

2.3 Preference Aggregation

The fundamental idea of voting/social choice/preference aggregation dates back to the Middle Ages [6], but it was the seminal work of Arrow et al. [2] in the middle of the 20 century that introduced the first mathematical framework and axiomatic approach for analyzing voting rules.

Let $N = \{1, \ldots, n\}$ be a finite set of *individuals* (or *voters*), and let A be a finite set of *alternatives* (or *candidates*). The set of all weak orders $\succsim$ on A, that is, the set of all binary relations on A that are complete ($\forall a \neq b \in A$, $a \succsim b$ or $b \succsim a$) and transitive ($\forall a, b, c \in A$, if $a \succsim b$ and $b \succsim c$, then $a \succsim c$), is denoted as $\mathcal{R}(A)$. The *preference* of individual i is a weak order $\succsim_i \in \mathcal{R}(A)$.

An *aggregation rule* (or *social welfare function*) $f : \mathcal{R}(A)^n \to \mathcal{R}(A)$ maps each profile $P = (\succsim_1, \ldots, \succsim_n)$ of preferences to a single collective weak order. Note that, by this definition, individual rankings are not *strict* - voters may express indifference to two alternative by specifying $a \succsim b$ and $b \succsim a$.

Many classical voting rules are not designed to aggregate weak orders; instead, they can only take as input *linear orders*, which are weak orders that in addition are antisymmetric ($\forall a, b \in A$, if $a \succsim b$ and $b \succsim a$, then $a = b$). Under such rules, no voter may express indifference to two alternatives. [6]

2.3.1 Relationship to Judgment Aggregation. It turns out that preference aggregation problems can be formally represented within the model of judgment aggregation. The idea is that preference orderings can be represented as sets of accepted preference ranking propositions of the form 'a is preferable to b'. [10]

Formally, define agenda Φ as the set of all propositions of the form 'a is preferable to b' and their negations 'a is not preferable to b', where $a \neq b \in A$. We then define consistency of a set of propositions that correspond to the definition of weak order. For example, {a is preferable to b, b is preferable to c, c is preferable to a} shall be defined as inconsistent. Now each consistent and complete judgment set on Φ uniquely represents a weak order of the alternatives. For instance, the judgment set {a is preferable to b, b is preferable to a, a is preferable to c, c is not preferable to a, b is preferable to c, c is not preferable to b} uniquely represents the weak order of '$a \succsim b, b \succsim a, a \succsim c, b \succsim c$'.

To account for anitsymmetricity in linear orders, all we need to do is to define all sets of propositions of the form {a is preferable to b, b is preferable to a} to be inconsistent. A more detailed correspondence between preference and judgment aggregation concepts under the constructed embedding is summarized in Table 2 of [10]. In a word, every preference aggregation problem can be solved by a judgment aggregation rule that is collectively rationl w.r.t. completeness and consistency (defined accordingly).

It is also worth noting that complete judgment aggregation can be viewed as approval voting, a special case of preference aggregation. [5]

2.3.2 Relationship to Graph Aggregation. It is not hard to see that every weak order can be represented by a complete and transitive directed graph in which vertices represent alternatives and edges represent preference relations. Similarly, every linear order can be represented by a complete, transitive and antisymmetric ($\forall u, v$, if $(u, v) \in E$ and $(v, u) \in E$, then $u = v$) directed graph. In a word, every preference aggregation problem can be solved by a graph aggregation rule that is collectively rational w.r.t. transitivity and completeness. [14]

In fact, graph aggregation, when not concern about preserving transitivity and/or completeness, can also model aggregation of *intransitive* and/or *incomplete* preferences.

Intransitive (or nontransitive, or cyclic) preferences have been a topic of curiosity, study, and debate over decades among economists [1], decision theorists [15], philosophers [33] and psychologists [30]. Most decision theorists insist on transitivity or weak order as a foundational principle of normative theory. Attempts were made to explain intransitive preferences with mixture models [30] and noise during neural information processing [34], etc., but it was Kendall [19] who first raise the question of how to aggregate intransitive (but complete) preferences. Problems regarding intransitive preference aggregation or complete directed graph aggregation is referred to as *tournament aggregation* in operation research and decision theory literature. Monjardet [25] was the first to take an axiomatic approach to the tournament aggregation problem, and to generalize Arrow's impossibility theorem.

Incomplete preferences, on the other hand, may arise from the fact that some alternatives are *incomparable* or from missing information. [28]. Recent AI literature has concerned about computing winner under such conditions [20], and Pini et al. [29] generalize Arrow's theorem to the incomplete setting.

In any case, Endriss and Grandi [14] argues that graph aggregation problem is interesting even when it does not at all represent a preference relation, complete or transitive or not. For example, it can be applied to aggregation of social or economic networks or of argumentation or logic.

2.4 Quantitative Judgment Aggregation

Judgments evaluate propositions by labeling it true or false; quantitative judgments, on the other hand, evaluate *issues* on a numerical scale. Depending on the issue of interest, they are essential building blocks of, for example, financial forecasts in the stock market, sales estimates, and climate change predictions. [31]

It is immediate that the aggregation of complete quantitative judgments on some alternatives is very similar to the process of *cardinal preference aggregation* (or *cardinal voting*), in which individuals give *ratings* (in some scale) to alternatives. [17] The only difference is that traditionally the output of a cardinal preference aggregation rule (or *social welfare functional*) is a weak order, even though the weak order is usually inferred from the collective ratings computed by the rule. Nevertheless, for pedagogical purposes, we will refer to the aggregation process that outputs the collective ratings as *cardinal preference aggregation (raw output).*

2.4.1 Societal Tradeoff Problem. Conitzer et al. [7] proposed *Societal Tradeoff*, an application of quantitative judgment aggregation in which the judgments are (1) on pairwise comparisons, similar to how we formulated graph aggregation with judgment aggregation, and (2) consistent, which will be defined in the next paragraph.

Formally, let A be a finite set of alternatives and $N = \{1, \ldots, n\}$ a finite set of individuals. Let $t_i^{ab} \in [1, \infty)$ denote i's *tradeoff value* between a (unordered) pair of alternative a and b. For example, it may represent that i believes 'a is t_i^{ab} times more preferable to b' or 'a is t_i^{ab} units more preferable to b'. Therefore, either t_i^{ab} or t_i^{ba} exist in t_i, the vector of all i's tradeoff values. An aggregation rule f maps a profile $P = (t_1, \ldots, t_n)$ of tradeoff vectors to a single collective tradeoff vector.

The definition of consistency then follows. If the tradeoff values represent 'by how many times an individual prefers an alternative over another', ideally the input tradeoff vectors and the collective tradeoff vector are *consistent by multiplication* (if $t_i^{ab} \in t_i$ and $t_i^{bc} \in t_i$, then $t_i^{ac} \in t_i$ and $t_i^{ac} = t_i^{ab} t_i^{ac}$); if the tradeoff values represent 'by how many units an individual prefers an alternative over another', ideally the input tradeoff vectors and the collective tradeoff vector are *consistent by addition* (if $t_i^{ab} \in t_i$ and $t_i^{bc} \in t_i$, then $t_i^{ac} \in t_i$ and $t_i^{ac} = t_i^{ab} + t_i^{ac}$). Completeness of the inputs and output, on the other hand, is not require.

In fact, Conitzer et al. [8] suggest that inconsistent input tradeoff vectors should also be considered admissible. *Distance-based rules* - familiar from preference aggregation, judgment aggregation, and belief merging - output consistent collective vectors (or, in the language of judgment aggregation, are collectively rational) by definition, but Endriss and Grandi [14] dismiss them due to the drawback of 'typically being computationally intractable'. Zhang et al. [37], on the other hand, go on to develop approximation algorithm for a distance-based rule to aggregate input tradeoff vectors, which may not be consistent, into a consistent vector.

2.4.2 Relationship to Cardinal Preference Aggregation. The societal tradeoff problem when the inputs are complete and consistent is closely related to cardinal preference aggregation (raw output), which, as stated earlier, is an intermediate step of cardinal preference aggregation. To derive the exact relationship, we first recall the definition of the problem of cardinal preference aggregation.

Given a set $N = \{1, \ldots, n\}$ of individuals and a set S of m alternatives. Individual i's utility (cardinal preference, welfare, rating) is a function $W_i : S \longrightarrow \mathbb{R}^m$, and $W = (W_1, \ldots, W_n)$ is a cardinal preference profile. Let $\mathcal{D}$ denote the set of all profiles and $\mathcal{R}$ the set of all weak orders on S. An *aggregation rule* (or *social welfare functional*) $f : \mathcal{D} \rightarrow \mathcal{R}$ maps every profile to a weak order. [6]

In a world where utilities are *ordinally measurable*, individual utility is measured on an ordinal scale and no interpersonal comparisons are possible. Only the weak order implied by one's utility function is observed; no interpersonal comparisons of utility levels are possible. Thus, if we say a rule f is *invariant* to the vector of transformations $\phi = (\phi_1, \ldots, \phi_n)$ iff for any W, we have $f(W) = f(\phi(W))$, then in the world described above, aggregation rules shall be invariant w.r.t. any vector ϕ of positive monotonic transformations. On the other end of the spectrum, in a world where utilities are *perfectly measurable*, aggregation rules shall be invariant w.r.t. only the vector of identity transforms. An example of this world would be people reporting utility functions in dollar term - all utilities are measured on the same absolute scale. As a consequence, the numerical values of utilities are significant and it is possible to make any kind of intrapersonal or interpersonal utility comparison. In the middle of this spectrum, the 'von Neumann-Morgenstern' world where utilities are *cardinally measurable*, aggregation rules shall be invariant w.r.t. any vector of positive monotonic affine transformations, i.e. $\phi_i = a_i + b_i W_i$. Here, intrapersonal comparisons of utility levels and utility differences are possible, but because the transforms are chosen independently across individuals, no interpersonal comparisons are possible. [3]

The 'world' of interest in this paper is *ratio-scale measurable* utilities [35] and *translation-scale* measurable utilities [4]. With ratio-scale measurability, each person's utility is measurable on a ratio scale and these scales can be chosen independently. Utility ratios are interpersonally comparable but utility differences are not, and the aggregation method shall be invariant w.r.t. any vector ϕ of transformations for which $\phi_i(W_i) = b_i W_i$. Hence, for each individual, only the (complete and consistent by multiplication) ratios of the ratings reported are measured and aggregated. With translation-scale measurability, for each individual, there is an absolute interpersonally comparable unit in which utility is measured. Interpersonal comparisons of utility differences are possible but interpersonal comparisons of utility levels are not, and the aggregation method shall be invariant w.r.t. any vector ϕ of transformations for which $\phi_i(W_i) = W_i + a_i$. Hence, for each individual, only the (complete and consistent by addition) differences of the rating reported are measured and aggregated. [3]

The only difference now, between the two versions of cardinal preference aggregation problems and the two versions of societal tradeoff problem, is that the output of cardinal preference aggregation rules are weak orders. This is where the problem of cardinal preference aggregation (raw output) we defined earlier fills the gap in the theory. We define that a cardinal preference aggregation (raw output) rules f is *invariant* to the vector of transformations $\phi = (\phi_1, \ldots, \phi_n)$ iff for any W, we have $\phi_0(f(W)) = f(\phi(W))$ where $\phi_i(W_i) = b_i W_i$ in the ratio-scale-measurable world or $\phi_i(W_i) = a_i + W_i$ in the translation-scale-measurable world, for all $i = 0, \ldots, n$. It follows immediately that these two versions of cardinal preference aggregation (raw output) problems are equivalent to the multiplicative and additive versions of the societal tradeoff problem with the extra requirement that the input and output tradeoff vectors are complete (and, by default, consistent), i.e. $|t_i| = \binom{m}{2}$.

3 WEIGHTED GRAPH AGGREGATION

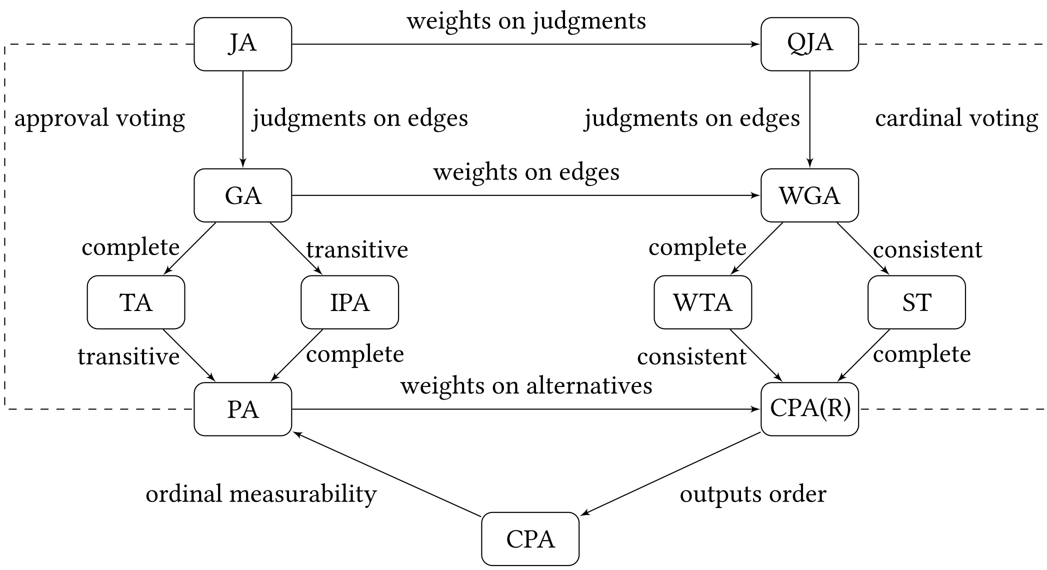
We are now ready to introduce the problem of weighted graph aggregation, in which the input and output are weighted direct graphs.

In this section, we present the problem formally, find its place among existing theories and discuss its potential application.

3.1 The Problem

Given a set of individuals $N = \{1, \ldots, n\}$ and a set of vertices (or, in a decision-making context, alternatives) V. Individual i specifies a weighted directed graph $G = (V, E)$, defined by weights on a set of edges $W_i \subseteq V \times V \times \mathbb{R}$. This gives rise to a profile $W = (W_1, \ldots, W_n)$. Let $2^{V \times V \times \mathbb{R}}$ denote the power set of $V \times V \times \mathbb{R}$, the set of all weighted directed graphs defined on V. An aggregation method is a function $F : (2^{V\times V\times\mathbb{R}})^n \to 2^{V\times V\times\mathbb{R}}$, mapping every profile of weighted directed graphs into a single collective weighted directed graph.

3.2 In A Bigger Picture



The above figure draws relationship between classes of aggregation problems, including judement aggregation (JA) [23], graph aggregation (GA) [14], tournament aggregation (TA) [25], incomplete preference aggregation (IPA) [28], preference aggregation (PA) [2], quantitative judgment aggregation (QJA) which is equivalent to cardinal preference aggregation (raw output) (CPA(R)), weighted graph aggregation (WGA), weighted tournament aggregation (WTA) [18], societal tradeoff problem (ST) [8], and cardinal preference aggregation (CPA) [32]. From the picture, it is easy to see that weight graph aggregation, in addition to its graph-theoretic interpretation, can also be viewed as:

- Aggregation of quantitative judgments on existence of edges
- Aggregation of incomplete weighted tournaments
- Aggregation of inconsistent tradeoff vectors (into a potentially inconsistent societal tradeoff vector)

3.3 Application

Much of the application scenarios for weighted graph aggregation sketched in this section are inspired by [14] and [38].

3.3.1 Cardinal Preferences. Our main example for a weighted graph aggregation problem is going to be preference aggregation as classically studied in social choice theory. In this context, vertices are interpreted as alternatives available and the graphs considered are weighted (indicating, for example, strength of preferences) weak orders on these alternatives. Here, the input and output graphs are, by classical economic assumptions, complete and consistent. Our aggregation rules then reduce to so-called social welfare functionals. Depending on whether the graphs are consistent by multiplication or by addition, the problem is equivalent to cardinal preference aggregation with ratio-scale measurability or with translation-scale measurability. If we then relax the consistency and completeness constraint, the idea of aggregation of incomplete, inconsistent weighted preferences follows.

The theory of tournament aggregation concerns with aggregation of complete directed graphs (which may or may not be transitive). Many research on the this problem have given the graphs a social-choice-theoretic interpretation [25], and if we consider weights on edges, this problem is essentially a subset of the use case described above. Our next application, however, concerns with aggregation of actual sports tournaments.

3.3.2 *Sports Tournaments.* Zhang et al. [38] suggest that the societal tradeoff problem (or, in the language of their paper, quantitative judgment aggregation) can be applied to evaluate contestants under the settings where (1) contests are reasonably frequent, (2) the contests provide a numerical score, (3) the outcomes vary between contests, and (4) not every contestant appears in every contest. The score corresponds to an individual contestant, for example, can be the finishing time if she is a marathon runner. The tradeoff vector would then be the time difference between every two runners. Our application, on the other hand, concerns with sports that directly involve pairwise contests from which the winner is inferred from some scores, common among many popular team sports, e.g. football, basketball.

More formally, if we see teams as vertices and pairwise outcomes (e.g. point differential in a match) as weighted edges, then the end result of a season can be seen as a weighted directed graph, which contains more information than the usual end-of-season standing, a ranking of number of wins.

We argue (and will demonstrate in the next section) that cycles of wins are very common in team sports. Therefore, it makes sense that a prediction of pairwise outcomes contains cycles too. The intuition is that a team may be better than another in some aspects that determine the outcome, but not all. For example, a basketball team with good perimeter offense can out-score a team with good interior offense more likely than not, but may struggle against a team with good perimeter defense. However, a good perimeter defense does not help much playing against a good interior offense, hence a cycle. This is an example of multi-attribute decision-making, one of the many scenarios that cyclic outcomes arise [15].

In addition to going beyond the societal tradeoff problem, we argue that incomplete outcomes, going beyond the weighted tournament aggregation problem, is also common in sport. Unlike in professional sports where there are a few teams and have long season, college sports usually consists of many teams and have short seasons. Two teams may not face each other at all in some seasons, but when a new season comes and we want to predict an outcome of the match, weighted graph aggregation comes in handy.

3.3.3 *Social Networks.* We may also think of each of the weighted graphs in a profile as a different social network relating members of the same population. For example, we may aggregated weighted networks of work relations, family relations and online purchasing behavior, into a single weighted meta-network that describes relationships at a global level, with weights representing the ‘strength’ of relationships or influence factors. Alternatively, we may wish to aggregate several graphs representing snapshots of the same social network at different points in time for predictive purpose.

4 EXPERIMENTAL RESULT

4.1 The Setup

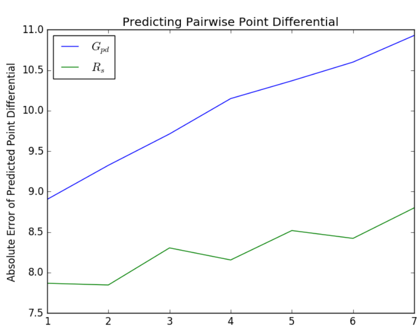
We demonstrate the predictive power of various aggregation framework, including weighted graph aggregation, similar to the methodology in [38]. We predict outcomes of NBA basketball games by aggregation results from previous seasons. We gather the NBA regular season game results from 2005 to 2018, 14 seasons in total. Since each team play against one another at least 1 time per season, this problem is reduced to weighted tournament aggregation. Our goal is to predict pairwise outcome in a season by aggregation outcomes in the previous ones. For each season we construct the following graphs and rankings:

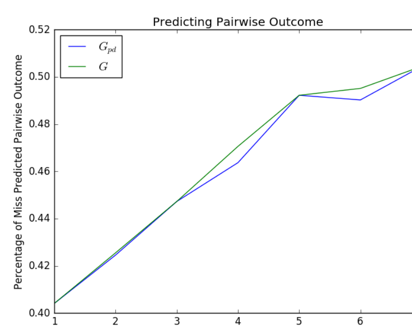
- Weighted directed graph G_{pd} where an edge exist from team A to B if A scores more points on average than B in their head-to-head matches this season. The edge weight is a positive number indicating their average per-game point differential.
- Directed graph G that is G_{pd} but without the weight.
- Weighted Ranking R_s where the ranking is induced by each team's average points scored per-game.
- Ranking R that is R_s but without the weight.

It is worth noting that team A can lose its pairwise competition against B in a season while scoring more pairwise points per game. Consider a season when A and B play 3 times against each other, and the score was 100-70,90-100 and 80-90. A loses 2 of the 3 games, but scores 3.3 more points per game. In this example, we only care about the latter.

Somewhat unsurprisingly, our first observation is that every one of the 30 teams in each of the 14 seasons is in some cycles, which justifies the assumption of intransitivity/inconsistency. With G_{pd}, G and R_s, we predict average pairwise point differentials in a new season by applying the majority (median) rule on 1 to 7 previous seasons. We compare the weightless graph induced by G_{pd} and G against the actual outcome (which team scores more point), and G_{pd} and R_s against the actual numerical outcome (what is the point differential).

4.2 Result and Discussion





The results are not nearly ideal, but it provides new understanding of and intuition to the problem.

- Aggregation is not a good gateway to prediction, at least in our very problem. We observe that, regardless of the aggregation framework used, the accuracy of prediction decreases as we aggregate more prior seasons. The best prediction in fact is to look at just 1 season before. This makes sense because of the dynamic of player trading, draft picks, retirements, etc.
- Weighted graphs G_{pg} carry more information than the weightless G, but not by much. From the second figure, we can see that aggregating pairwise point differential changes little to noting compared to choosing the majority edge direction, in terms of prediction of the outcome.
- In the first figure, when we aggregate and predict the weighted ranking, we look at the average points scored by each team in a season. Since such measurement are relatively 'interpersonally' comparable (since every team scores about 100 points a game with no

extreme outliers), it simply carries more information then pairwise point differential, which is too much to overcome even with the introduction of inconsistency.

5 CONCLUSION AND FUTURE WORK

In this work, we propose and illustrate the problem of weighted graph aggregation, and provide a complete picture of the relationship among classes of aggregation problems. We believe that this framework opens up many interesting theoretical and practical problems:

- We failed to justify a use case of weighted graph aggregation with predictive power. We believe the main reason is that when ratings on alternatives (or weights on vertices) are perfectly or even cardinally measurable, utilities become very interpersonally comparable and thus carry much more information than a relative pairwise measurement. It would be interesting to identify a use case where individual utilities reported are naturally ratio-scale or translation-scale measurable, and see if allowing intransitive/inconsistent preferences indeed aids prediction.
- In addition to predictive power, we can also use an axiomatic approach, similar to the study of other classes of aggregation problem. For example, close variant of Arrow's impossibility theorem have been proven in JA [24], GA [14], TA [25], IPA [29], CPA [32], and, of course, PA [2] by Arrow himself. The axiomatic boundaries in QJA and all other subclasses of weighted aggregation remain unexplored.

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