Lecture 3

Comparison Principles

Despite the existence and uniqueness of solutions of certain nonlinear systems, these results do not speak to how robust solutions are to a given initial condition and uncertain parameters. Yet, this is a fundamental question that must be addressed. In particular, for repeatability and robustness, "small" changes in initial conditions and parameters should result in "small" changes in the resulting solutions. In other words, for the system dependent on parameters λ:

$$\dot{x} = f(x, \lambda), \qquad x(0) = x_0, \quad \lambda \in \mathbb{R}^p, \tag{3.1}$$

it is important for solutions to have continuous dependence on x_0 and on the vector of parameters λ. To study systems of this form, we will compare them to "nearby" systems, i.e., to systems that can be obtained by adding a small "perturbation". When this perturbation is bounded we would like to conclude that its effect on solutions will remain bounded. The fundamental comparison principles discussed in this lecture will enable us to reach such a conclusion. This will also be central to several key proofs in nonlinear systems and control, including the proof of stability via Lyapunov functions.

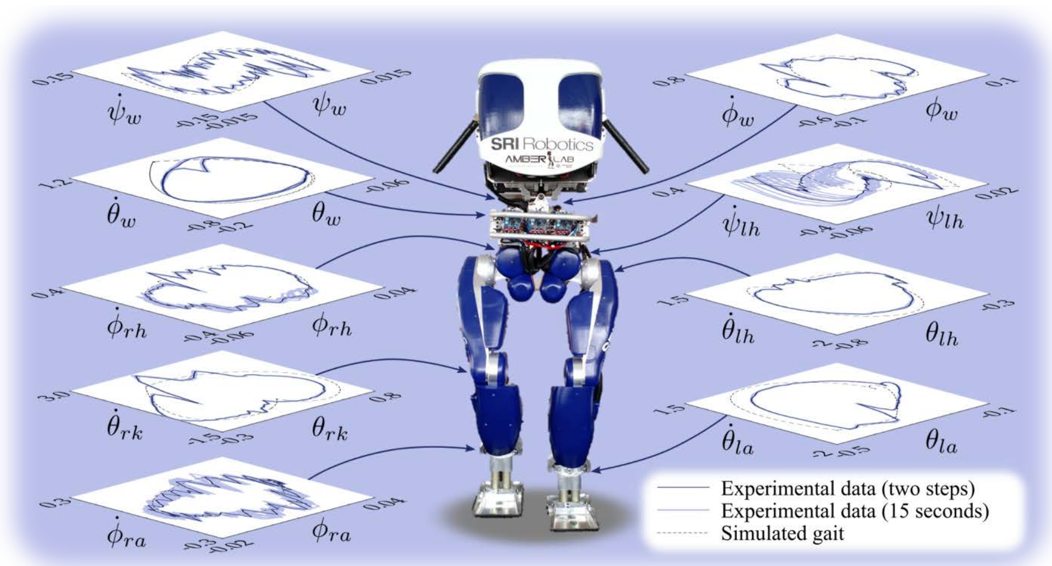


Figure 3.1. Illustration of periodic orbits on DURUS both in simulation and experimentally. This figure illustrates the importance of comparison principles; the goal is for different dynamical systems to display the same qualitative behavior—in this case the modeled system and the physical system.

As a motivating conceptional example, consider the humanoid robot DURUS shown in Figure 3.1. Gaits, represented by stable periodic orbits, are generated for this robot through a (hybrid) dynamical system model. Yet, this model only approximates the actual dynamics of the physical hardware. Therefore, again at a conceptual level, one can view the difference between the idealized mathematical model, and the real physical system via a perturbation on the nominal dynamics:

$$\underbrace{\dot{x} = f(x)}_{\text{model dynamics}}, \qquad \underbrace{\dot{x} = f(x) + \overbrace{g(x)}^{\text{perturbation}}}_{\text{physical dynamics}}.$$

If we can bound the difference between these systems (represented abstractly by some unknown function $g(x)$), then the hope is that these two systems will have the same qualitative behavior. This is what was observed on DURUS as seen in the periodic orbits in Figure 3.1 which compare the simulated system, based upon an idealized mathematical model, with the actual dynamics observed experimentally. This is the motivation for comparison lemmas that lay the groundwork for establishing continuity of solutions with respect to changes in initial condition, perturbations to the model, and perturbations of parameters.

3.1 Comparison Lemmas

Comparison lemmas play an essential role in nonlinear dynamics and control. They are framed as inequalities on one-dimensional continuous functions and, as a result, they may initially appear to be irrelevant to higher dimensional nonlinear systems. Yet, many of the results in nonlinear systems provide a "projection" of high dimensional quantities to a low (and often one-dimensional) representation. This will be seen in this lecture in the context of continuity of solutions, wherein a norm will map solutions to $\mathbb{R}$. Later, these comparison lemmas will play an essential role in establishing stability via Lyapunov functions; in this case, the Lyapunov function will provide the "projection" from a high dimensional space (the set of states) to a one-dimensional system and comparison lemmas will be utilized to bound the evolution of this low-dimensional representation.

Comparison Lemma. We begin by introducing the Comparison Lemma which lets us upper bound a function v by a solution of a differential equation provided that v does not grow faster than the velocity prescribed by the right-hand side of the differential equation. While this is a seemingly innocuous statement, it will be essential in establishing the stability of nonlinear systems via Lyapunov functions.

Lemma 3.1 **(Comparison Lemma).** *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous function, consider the IVP:*

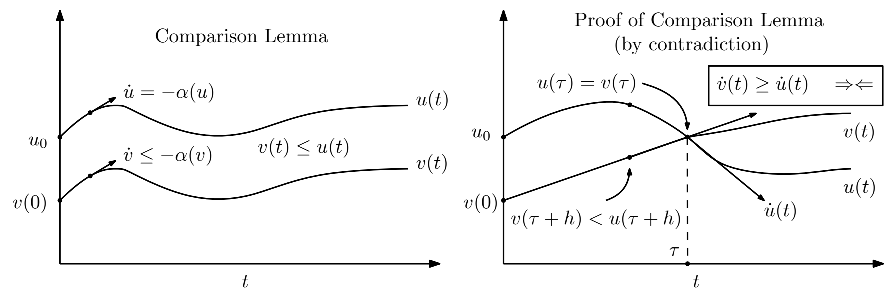
$$\dot{u} = -\alpha(u), \qquad u(0) = u_0,$$

and assume it has a unique solution, $u(t)$, on $[0, a]$, $a \in \mathbb{R}_{>0}$. If $v : [0, a] \to \mathbb{R}$ is a continuously differentiable function, then the following implication holds:

$$\dot{v}(t) \leq -\alpha(v(t)), \qquad v(0) \leq u_0 \qquad \Longrightarrow \qquad v(t) \leq u(t), \quad \forall t \in [0, a]. \tag{3.2}$$

***Remark* 3.1.** We will prove a weaker version of the Comparison Lemma where $\dot{v}(t) \leq -\alpha(v(t))$ and $v(0) \leq u_0$ in (3.2) are strengthened to $\dot{v}(t) < -\alpha(v(t))$ and $v(0) < u_0$ and the conclusion is also strengthened to $v(t) < u(t)$. At the end of this lecture we will complete the proof of the Comparison Lemma, as stated, under the additional hypothesis that α is Lipschitz continuous.

Figure 3.2. Illustration of the Comparison Lemma (left) and the proof by contradiction of this lemma. Namely, assuming that at some time, τ, $v(\tau) \geq u(\tau)$, but this implies that $\dot{v}(\tau) \geq \dot{u}(\tau)$ contradicting the assumption that $\dot{v}(t) < -\alpha(v(t))$.



Proof of Lemma 3.1 (Assuming $\dot{v}(t) < -\alpha(v(t))$ and $v(0) < u_0$). To prove that $v(t) < u(t)$, we proceed by contradiction and consider the set:

$$J = \{t \in [0, a] \mid v(t) \geq u(t)\}.$$

Note that the Comparison Lemma states that J is empty. Hence, for the sake of contradiction, we assume $\dot{v}(t) < -\alpha(v(t))$, $v(0) < u_0$, and that J is non-empty. Define $\tau \in \mathbb{R}$ by $\tau = \min J$, (τ is finite, by non-emptiness of J, and belongs to J since the inequality defining J is not strict). We note that when $t = \tau$ we have $v(\tau) = u(\tau)$ and thus $\tau > 0$ since $v(0) < u(0)$. Therefore, for $t \in [0, \tau)$ we have $v(t) < u(t)$. We now note that for sufficiently small negative $h \in \mathbb{R}_{<0}$:

$$\begin{array}{rrcl} & v(\tau + h) & < & u(\tau + h) \\ \Longrightarrow & v(\tau + h) - v(\tau) & < & u(\tau + h) - u(\tau) \\ \Longrightarrow & \frac{v(\tau+h) - v(\tau)}{h} & > & \frac{u(\tau+h) - u(\tau)}{h}, \\ \overset{\lim\ h \to 0^-}{\Longrightarrow} & \dot{v}(\tau) & \geq & \dot{u}(\tau) = -\alpha(u(\tau)) = -\alpha(v(\tau)) \end{array}$$

thereby contradicting the assumption $\dot{v}(t) < -\alpha(v(t))$. □

***Remark* 3.2.** The stated version of the Comparison Lemma is based on the above proof coupled with the use of a continuous function α_λ, dependent on and continuous with respect to a parameter $\lambda \in \mathbb{R}_{\geq 0}$, satisfying:

$$-\alpha(u) = -\alpha_0(u) \qquad \text{and} \qquad -\alpha(u) < -\alpha_\lambda(u), \quad \forall \lambda > 0. \tag{3.3}$$

The inequality $\dot{v}(t) \leq -\alpha(v(t))$ implies $\dot{v}(t) < -\alpha_\lambda(v(t))$. Finally, assume that α_λ is chosen so that $\dot{u}_\lambda = -\alpha_\lambda(u_\lambda)$ has a unique solution $u_\lambda(t)$ with initial condition $u_\lambda(0) = u_0$. Then a continuity argument, coupled with the above proof (establishing that $v(t) < u_\lambda(t)$), can be used to show the limiting behavior:

$$\dot{v}(t) \leq -\alpha(v(t)) = \lim_{\lambda \to 0} -\alpha_\lambda(v(t)) \qquad \Longrightarrow \qquad v(t) \leq u(t) = \lim_{\lambda \to 0} u_\lambda(t). \tag{3.4}$$

For further details, see Additional Reading.

Gronwall-Bellman Lemma. The Comparison Lemma is conceptually easy to understand, but it does not capture all comparisons that are needed for nonlinear systems. This motivates the Gronwall-Bellman inequality which allows a set of functions to be compared in integral form. Note that, unlike the Comparison Lemma, this result does not rely on differential equations so the assumptions on the functions can be relaxed.

Lemma 3.2. *Let $I = [a, b] \subset \mathbb{R}$ be an interval, let $y : I \to \mathbb{R}$ and $\varphi : I \to \mathbb{R}_{\geq 0}$ be continuous functions with $\varphi(t)$ nonnegative, and let $c \in \mathbb{R}_{\geq 0}$. If $y : I \to \mathbb{R}$ satisfies:*

$$y(t) \leq c + \int_a^t \varphi(s)y(s)ds, \quad \forall t \in [a, b], \tag{3.5}$$

then it follows that:

$$y(t) \leq ce^{\int_a^t \varphi(s)ds}, \quad \forall t \in [a, b]. \tag{3.6}$$

Proof. The right-hand side of (3.5) defines the function:

$$z(t) \triangleq c + \int_a^t \varphi(s)y(s)ds,$$

that satisfies the following properties:

1. $z(a) = c$;
2. $y(t) \leq z(t)$ (by (3.5));
3. $\dot{z}(t) = \varphi(t)y(t) \leq \varphi(t)z(t)$ (by the previous inequality and non-negativity of φ).

From the last property of z we obtain:

$$\dot{z}(t) - \varphi(t)z(t) \leq 0, \tag{3.7}$$

and multiplication by $e^{-\int_a^t \varphi(s)ds}$ provides:

$$\underbrace{\dot{z}(t)e^{-\int_a^t \varphi(s)ds} - \varphi(t)z(t)e^{-\int_a^t \varphi(s)ds}}_{=\frac{d}{dt}\left(z(t)e^{-\int_a^t \varphi(s)ds}\right)} \leq 0. \tag{3.8}$$

We can thus integrate:

$$\int_a^t \frac{d}{dt}\left(z(t)e^{-\int_a^t \varphi(s)ds}\right)dt \leq 0 \qquad \Longrightarrow \qquad z(t)e^{-\int_a^t \varphi(s)ds} - z(a) \leq 0. \tag{3.9}$$

The desired inequality now follows by multiplying (3.9) by $e^{\int_a^t \varphi(s)ds}$, using the equality $z(a) = c$, and the inequality $y(t) \leq z(t)$. $\square$

***Remark* 3.3.** The sequence of steps from (3.7) to (3.9) could also have been established by employing the Comparison Lemma 3.1. A direct proof was possible given that (3.7) leads to the linear time-varying ODE for which the closed form solution $e^{\int_a^t \varphi(s)ds}$ is available.

The previous Lemma (proved by Bellman and inspired by earlier work of Gronwall) will be used to prove the following more general inequality also due to Bellman.

Theorem 3.1 (Gronwall-Bellman Inequality). *Let $I = [a, b] \subset \mathbb{R}$ be an interval, let $y : I \to \mathbb{R}_{\geq 0}$ and $\varphi : I \to \mathbb{R}_{\geq 0}$ be continuous and nonnegative functions, and let $\lambda : I \to \mathbb{R}_{>0}$ be a continuous, positive, and non-decreasing function. If $y : I \to \mathbb{R}$ satisfies:*

$$y(t) \leq \lambda(t) + \int_a^t \varphi(s)y(s)ds, \quad \forall t \in [a, b], \tag{3.10}$$

then it follows that:

$$y(t) \leq \lambda(t)e^{\int_a^t \varphi(s)ds}, \quad \forall t \in [a, b]. \tag{3.11}$$

***Remark* 3.4.** Additionally, it follows from Theorem 3.1 that if $\varphi(t) \triangleq \varphi > 0$, i.e., φ is a constant, then:

$$y(t) \leq \lambda(t)e^{\varphi(t-a)}, \quad \forall t \in [a,b]. \tag{3.12}$$

Proof. Define the function z by $z(t) = y(t)/\lambda(t)$. This is always possible since λ is a positive function and hence never zero. In particular, z satisfies:

$$z(t) \leq 1 + \int_a^t \varphi(s)\frac{y(s)}{\lambda(t)}ds = 1 + \int_a^t \varphi(s)\frac{\lambda(s)z(s)}{\lambda(t)}ds \leq 1 + \int_a^t \varphi(s)z(s)ds,$$

where the last inequality follows from $\lambda(s)/\lambda(t) \leq 1$ in virtue of λ being a non-decreasing function and $\varphi z = \varphi y/\lambda$ being non-negative. We now conclude from Lemma 3.2 applied to z that:

$$z(t) \leq e^{\int_a^t \varphi(s)ds},$$

and multiplication by $\lambda(t)$ provides the desired inequality. □

3.2 Continuity of Solutions

Continuity for Initial Conditions. Solutions are continuous with respect to initial conditions if solutions starting near the initial condition $x_0 \in E$ are defined on the same interval, and stay close to each other on the whole interval. Informally speaking, *small changes in the initial condition should result in small changes in the final position.*

Definition 3.1. Let $x(t)$ be a solution to $\dot{x} = f(x)$ with $x(t_0) = x_0$, defined on $I = [t_0, t_1]$. Then $x(t)$ is <u>continuous with respect to its initial condition</u> if:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \qquad \text{s.t.} \qquad \|z_0 - x_0\| < \delta \quad \Longrightarrow \quad \|z(t) - x(t)\| < \varepsilon, \quad \forall t \in I,$$

where $z(t)$ is the unique solution to $\dot{z} = f(z)$ with $z(t_0) = z_0$.

Implicit in this definition, and all subsequente definitions related to continuity, is that these conditions must hold for all initial conditions sufficiently close to the point of interest. In particular, with the notation of Definition 3.1, the implication must hold for all initial conditions $z_0 \in E$ such that $\|z_0 - x_0\| < \delta$.

The following important theorem shows continuity of initial conditions over bounded time intervals.

Theorem 3.2. *Consider a Lipschitz continuous function $f : E \to \mathbb{R}^n$, with Lipschitz constant L, defined on an open and connected set $E \subseteq \mathbb{R}^n$. Let $x(t)$ and $z(t)$ be solutions to the initial value problems:*

$$\dot{x} = f(x), \qquad x(t_0) = x_0, \tag{3.13}$$

$$\dot{z} = f(z) + g(z), \qquad z(t_0) = z_0, \tag{3.14}$$

where g is a perturbation of f such that $x(t), z(t) \in E$ for all $t \in [t_0, t_1]$. Suppose that $\|g(z)\| \leq \mu$ for all $z \in E$ with $\mu \in \mathbb{R}_{\geq 0}$. Then, the following inequality holds:

$$\|x(t) - z(t)\| \leq \|x_0 - z_0\|e^{L(t-t_0)} + \mu(t - t_0)e^{L(t-t_0)}, \qquad \forall t \in [t_0, t_1]. \tag{3.15}$$

Theorem 3.2 states that the growth in the error $\|x(t)-z(t)\|$, in general, is governed by: (1) change of f with x as measured by the Lipschitz constant L; (2) difference in initial condition as measured by $\|x_0 - z_0\|$; and (3) size of the perturbation g as measured by μ. In addition, it includes as a corollary continuity with respect to initial conditions as given in Definition 3.1. To see this, we note that by taking $g(z) \equiv 0$, and thus $\mu = 0$, for any ε we can pick $\delta = \frac{\varepsilon}{e^{L(t_1-t_0)}}$.

Proof. The solutions to $x(t)$ and $z(t)$ are given by:

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} f(x(s))ds \\ z(t) &= z_0 + \int_{t_0}^{t} (f(z(s)) + g(z(s)))ds. \end{aligned}$$

Therefore, we can compare the difference of these solutions as follows:

$$\begin{aligned} \|x(t) - z(t)\| &= \left\| x_0 - z_0 + \int_{t_0}^{t} f(x(s))ds - \int_{t_0}^{t} f(z(s))ds - \int_{t_0}^{t} g(z(s))ds \right\| \\ &\leq \|x_0 - z_0\| + \int_{t_0}^{t} \|f(x(s)) - f(z(s))\|ds + \int_{t_0}^{t} \underbrace{\|g(z(s))\|}_{\leq \mu} ds \\ \implies \underbrace{\|x(t) - z(t)\|}_{y(t)} &\leq \underbrace{\|x_0 - z_0\| + \mu(t-t_0)}_{\lambda(t)} + \int_{t_0}^{t} \underbrace{L}_{\varphi} \underbrace{\|x(s) - z(s)\|}_{y(s)} ds, \end{aligned} \tag{3.16}$$

where the elements y, λ, φ are defined using the notation of Theorem 3.1 and, therefore, the Gronwall-Bellman Inequality directly yields the desired result. $\square$

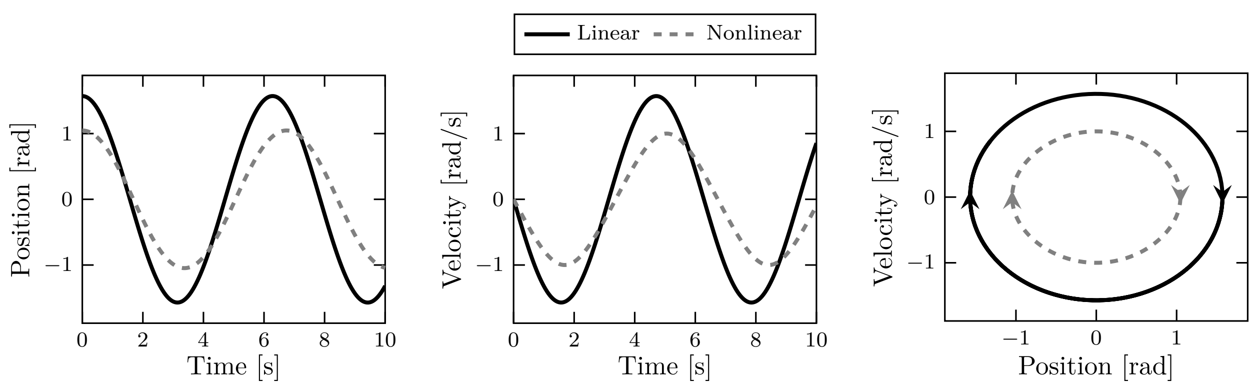


Figure 3.3. Simulation of the linearized and nonlinear pendulum from different initial conditions. Theorem 3.2 guarantees that the norm of the difference between these solutions is bounded.

Example 3.1. To illustrate the result of Theorem 3.2, consider the downward pendulum with dynamics introduced in Lecture 1 given in (1.5) and the corresponding linearization (with dynamics given in (1.27) with $\gamma = 0$). Using the notation of Theorem 3.2, we can write the linearized downward pendulum and the nonlinear downward pendulum, respectively, as follows:

$$\begin{aligned} \text{Linearized Pendulum:} \qquad \dot{x} &= f(x) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \\ \text{Nonlinear Pendulum:} \qquad \dot{z} &= f(z) + g(z) = \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -\sin(z_1) + z_1 \end{bmatrix}. \end{aligned}$$

Figure 3.4. The norm of the difference between solution $x(t)$ of the linear system and the solution $z(t)$ of the nonlinear system is bounded according to Theorem 3.2. As the figure indicates, this error bound is very conservative.

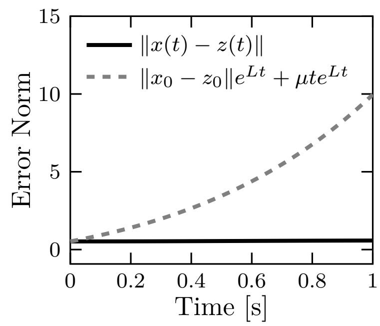


Figure 3.3 shows the simulation of the two systems, $x(t)$ and $z(t)$, from different initial conditions: x_0 and z_0. We can see that they both evolve on periodic orbits. Importantly, according to Theorem 3.2, Figure 3.4 shows how the norm of the difference of two solutions is bounded; in this case $L = 1$ and $\mu = \pi$ since $\|g(z)\| = |\sin(z_1) - z_1| \leq \pi$ for any $z_1 \in [-\pi, \pi)$.

3.3 Continuous Dependence on Parameters

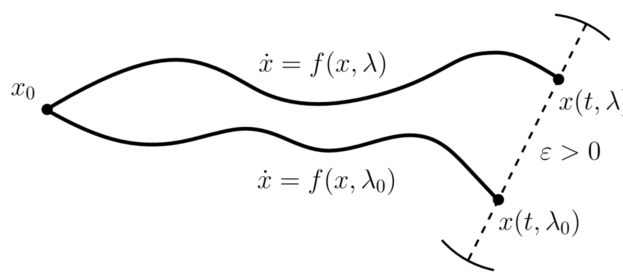
In this section we consider differential equations depending on parameters, i.e., differential equations of the form $\dot{x} = f(x, \lambda)$. In this setting, we wish to bound changes in the solutions caused by changes in parameters.

Definition 3.2. Let $x(t, \lambda_0)$ be a solution to $\dot{x} = f(x, \lambda_0)$ with $x(t_0, \lambda_0) = x_0$, defined on $[t_0, t_1]$ for $t_0, t_1 \in \mathbb{R}$, $t_1 > t_0$. The solution has a continuous dependence on the parameters $\lambda \in \mathbb{R}^p$ if:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \qquad \text{s.t.} \qquad \|\lambda - \lambda_0\| < \delta \implies \|x(t, \lambda) - x(t, \lambda_0)\| < \varepsilon, \qquad \forall t \in [t_0, t_1],$$

where $x(t, \lambda)$ is the solution to the $\dot{x} = f(x, \lambda)$ with $x(t_0, \lambda) = x_0$.

Figure 3.5. Illustration showing how the solution of the nonlinear dynamical system can change with the parameters λ, and the corresponding notion of continuity: $\|\lambda - \lambda_0\| < \delta$ implies $\|x(t, \lambda) - x(t, \lambda_0)\| < \varepsilon$.



The following theorem provides sufficient conditions on $\dot{x} = f(x, \lambda)$ for solutions to have a continuous dependence on both the initial conditions and the parameters.

Theorem 3.3. *Consider a continuous function $f : E \times B_c(\lambda_0) \subset \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$, locally Lipschitz continuous in its first argument, for some $\lambda_0 \in \mathbb{R}^p$, $c \in \mathbb{R}_{>0}$, and an open and connected set $E \subseteq \mathbb{R}^n$. Let $y(t, \lambda_0)$ be the solution of $\dot{y} = f(y, \lambda_0)$ with $y(t_0, \lambda_0) = y_0 \in E$ defined on $[t_0, t_1]$ for $t_0, t_1 \in \mathbb{R}$, $t_1 > t_0$. For all $\varepsilon > 0$, there exists a $\delta > 0$ such that:*

$$\|z_0 - y_0\| < \delta \text{ and } \|\lambda - \lambda_0\| < \delta \qquad \implies \qquad \|z(t, \lambda) - y(t, \lambda_0)\| < \varepsilon, \qquad \forall t \in [t_0, t_1], \quad (3.17)$$

where $z(t, \lambda)$ is the unique solution of $\dot{z} = f(z, \lambda)$ with $z(t_0, \lambda) = z_0 \in E$.

Theorem 3.3 in essence means that no matter how small $\varepsilon > 0$ is, we can find a $\delta > 0$ that keeps the solutions within ε of each other for all time in $[t_0, t_1]$ even under both perturbations of the initial conditions and the parameters (see Figure 3.5). Before proving this result, we note that under the slightly stronger assumption that f is locally Lipschitz continuous in both arguments, a simpler proof of Theorem 3.3 is possible.

***Remark* 3.5.** Consider the differential equation $\dot{x} = f(x, \lambda)$ with $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^p$, and assume that f is locally Lipschitz continuous in both arguments: x and λ. We can lift this differential equation to $\dot{z} = g(z)$ with $z = (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p$ and $g(z) = g(x, \lambda) = (f(x), 0)$. If we choose the initial value $z(t_0)$ to be $(x(t_0), \lambda)$, then the solution $z(t)$ of $\dot{z} = g(z)$ is given by $z(t) = (x(t), \lambda)$, i.e., the solution $z(t)$ provides the solution $x(t)$ of $\dot{x} = f(x, \lambda)$ for the chosen parameters λ. Changes in initial state $x(t_0)$ and parameters λ can now be described in the differential equation $\dot{z} = g(z)$ simply as changes in its initial condition $z(t_0) = (x(t_0), \lambda)$. Therefore, Theorem 3.2 applied to $\dot{z} = g(z)$ guarantees continuity with respect to changes in initial values and parameters.

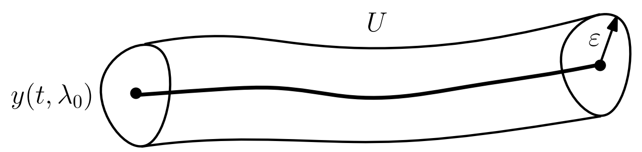


Figure 3.6. Illustration of a tube U of radius ε around $y(t, \lambda_0)$

Proof of Theorem 3.3. Continuity of $y(t, \lambda_0)$ in t and compactness of $[t_0, t_1]$ implies that $y(t, \lambda_0)$ is bounded on the interval $[t_0, t_1]$. Therefore, we can define a tube $U \subset E$ around $y(t, \lambda_0)$ of radius $\varepsilon > 0$ by:

$$U = \bigcup_{t \in [t_0, t_1]} \overline{B_\varepsilon(y(t, \lambda_0))} = \left\{ x \in E : \min_{t \in [t_0, t_1]} \|x - y(t, \lambda_0)\| \leq \varepsilon \right\}, \tag{3.18}$$

which is shown pictorially in Figure 3.6. Importantly, U is compact (since the union of compact sets is compact) and we will use this property throughout the proof.

Given $\varepsilon > 0$, let $a > 0$ be any positive real number satisfying:

$$a < \frac{\varepsilon}{(1 + t_1 - t_0)} e^{-L(t_1 - t_0)} < \varepsilon,$$

and let z_0 be any initial condition satisfying $\|z_0 - y_0\| < a$ (thus $z_0 \in U$). Since f is continuous in λ (although not necessarily locally Lipschitz continuous), we have[1]:

$$\exists \beta > 0 \qquad \text{s.t.} \qquad \|\lambda - \lambda_0\| < \beta \implies \|f(x, \lambda) - f(x, \lambda_0)\| < a, \ \forall x \in U. \tag{3.19}$$

The main idea is to utilize Theorem 3.2 through the following observation:

$$\begin{aligned} \dot{y} &= f(y, \lambda_0) \\ \dot{z} &= f(z, \lambda) + f(z, \lambda_0) - f(z, \lambda_0) = f(z, \lambda_0) + \underbrace{f(z, \lambda) - f(z, \lambda_0)}_{g(z)}, \end{aligned} \tag{3.20}$$

where $\|g(z)\| \leq a$ by (3.19).

[1] In the implication in (3.19), since we are utilizing the continuity of f with respect to λ, β is technically a function of x wherein the constant β given is obtained by taking $\beta = \min_{x \in U} \beta(x)$ which is well-defined since β is continuous and U is compact. The continuity of β is non-trivial to show, and is best done using a formulation of continuity based upon class $\mathcal{K}$ functions (see Lecture 6).

Now the solution, $z(t,\lambda)$, exists for all time t such that $z(t,\lambda) \in U$ (since U is compact, and thus there is a Lipschitz constant on this set). We will show that $z(t,\lambda) \in \mathrm{Int}(U)$ for all $t \in [t_0, t_1]$ by way of contradiction: suppose that $z(t,\lambda) \in \partial U$ at time τ for $t_0 < \tau \leq t_1$, i.e., at time τ the solution $z(\tau,\lambda)$ arrives at the boundary of the U. But on the time interval $[t_0, \tau]$, and by Theorem 3.2, it follows that:

$$\begin{aligned} \|z(t,\lambda) - y(t,\lambda_0)\| &\leq a e^{L(\tau - t_0)} + a(\tau - t_0) e^{L(\tau - t_0)} \\ &= a(1 + \tau - t_0) e^{L(\tau - t_0)} \\ &\leq a(1 + t_1 - t_0) e^{L(t_1 - t_0)} \\ &< \varepsilon, \end{aligned} \tag{3.21}$$

by the choice of a. This implies that $z(\tau,\lambda)$ belongs to the interior of U contradicting the claim that it belongs to the boundary. Therefore, it follows that $z(t,\lambda) \in \mathrm{Int}(U)$ for all time in which $y(t,\lambda_0)$ is defined, i.e., for all $t \in [t_0, t_1]$ and the corresponding bound $\|z(t,\lambda) - y(t,\lambda_0)\| < \varepsilon$ is satisfied for all $t \in [t_0, t_1]$, again by (3.21). Thus, picking $\delta = \min\{a, \beta, c\}$, with c given in the statement of the theorem, completes the proof. □

Completing the proof of the Comparison Lemma. Theorem 3.3 gives us the ability to complete the proof of Lemma 3.1 as stated—under the stronger assumption that α is locally Lipschitz continuous. The main idea, as in Remark 3.2, is to consider the following perturbed version of $\dot{u} = -\alpha(u)$ given by:

$$\dot{z} = \overbrace{-\alpha(z) + \lambda}^{-\alpha_\lambda(z)}, \quad z(0) = u_0 + \lambda, \quad \lambda > 0, \tag{3.22}$$

with solution $z(t,\lambda)$. As a result of Theorem 3.3 (with $\lambda_0 = 0$ and $c > 0$ arbitrary) it follows that:

$$\forall \varepsilon > 0, \ \exists \delta > 0 \qquad \text{s.t.} \qquad 0 < \lambda < \delta \implies |z(t,\lambda) - u(t)| < \varepsilon, \qquad \forall t \in [0, a] \tag{3.23}$$

This allows us to prove the Comparison Lemma:

Proof of Lemma 3.1 (with α locally Lipschitz). It follows by construction, i.e., from (3.22), that:

$$v(0) \leq u_0 < z(0) \quad \text{and} \quad \dot{v}(t) \leq -\alpha(v(t)) < -\alpha_\lambda(v(t)).$$

These inequalities and the proof of Lemma 3.1 with strict inequalities given in Section 3.1 provide:

$$v(t) < z(t,\lambda) \quad \forall t \in [0, a],$$

which holds for all $\lambda > 0$.

Now, by way of contradiction, suppose that there exists an $h \in (0, a]$ such that $v(h) > u(h)$. From (3.23), picking $\varepsilon = (v(h) - u(h))$ we conclude the existence of $\lambda > 0$ for which:

$$|z(t,\lambda) - u(t)| < \varepsilon, \quad \forall t \in [0, a]. \tag{3.24}$$

Therefore:

$$v(h) - z(h,\lambda) = \underbrace{(v(h) - u(h))}_{\varepsilon} + \underbrace{(u(h) - z(h,\lambda))}_{<\varepsilon} > 0 \qquad \Longrightarrow \qquad v(h) > z(h,\lambda),$$

which contradicts the fact that $v(t) < z(t,\lambda)$. □

Example 3.2. To illustrate Theorem 3.3, consider the downward pendulum with damping, i.e., the system governed by the equations given in (1.26), specifically:

$$\dot{x} = \begin{bmatrix} x_2 \\ -\sin(x_1) - \gamma x_2 \end{bmatrix}, \tag{3.25}$$

where $\gamma > 0$ is the damping coefficient of the system; we will utilize this parameter in the context of Theorem 3.3. First, one can select $\gamma = 0$ and simulate the dynamics from different initial conditions, i.e., $x_0 = (\frac{\pi}{6}, 0)$ and $x_0 = (\frac{\pi}{4}, 0)$. From Figure 3.7, we can see the pendulum evolves on two different orbits, and the norm of the difference of these two solutions is bounded. To illustrate continuity with respect to the parameter γ, let $\gamma = 1$ and the pendulum start at $x_0 = (\frac{\pi}{4}, 0)$. Due to the existence of damping, the energy of the system goes to zero asymptotically, and the solution goes to the origin. Comparing this behavior with the behavior in the case when $\gamma = 0$, the difference of these two solutions is also bounded as predicted by Theorem 3.3. This *does not* mean that they have the same qualitative behavior (see Lecture 4 for a formal notion of equivalences between dynamical systems, wherein qualitative behavior is preserved).

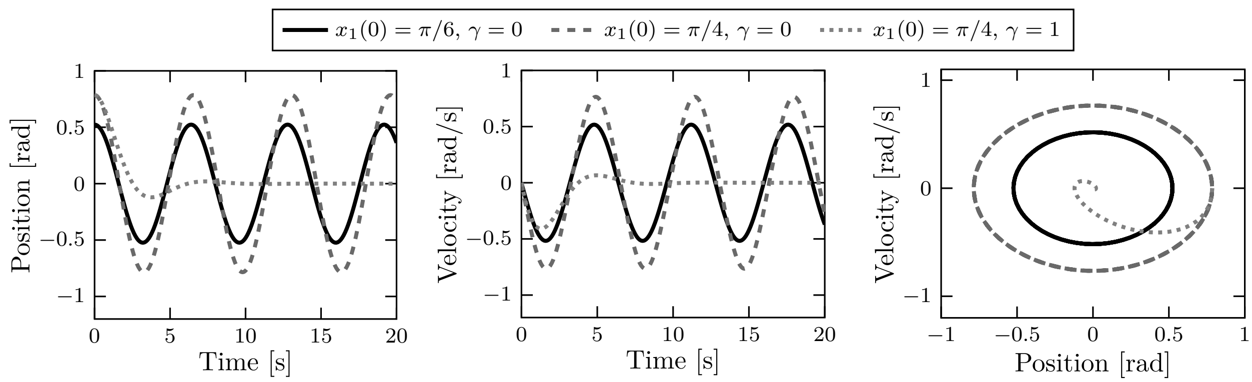


Figure 3.7. Solutions of the pendulum dynamics for different initial conditions and values of γ, illustrating the continuity of solutions with respect to varying initial conditions and parameters.

Additional Reading

The proofs of some of the results in this section follow from Khalil [110], specifically of Theorem 3.3. The Gronwall-Bellman Lemma is proved by following [31] and this reference also provides a proof for the Comparison Lemma as stated in Lemma 3.1. There are other variants of comparison lemmas, e.g., in [1] (exercise 4.1-9) and [124] (Theorem 1.4.1). Additional texts that establish a variety of inequalities for differential equations include [124] and [31].

Problems for Lecture 3

[P3.1] Use the Comparison Lemma to construct a curve upper-bounding the solution of the differential equation $\dot{x} = -x + \tan^{-1}(x)$.

[P3.2] For linear systems, $\dot{x} = Ax$, continuity of solutions with respect to initial conditions can be established directly by noticing that if $x(t)$ and $y(t)$ are solutions, then so is $z(t) = x(t) - y(t)$. Use the closed form expression of $z(t)$ to establish continuity of solutions with respect to initial conditions on bounded time intervals. Which additional assumption would you require to reach the same conclusion on unbounded time intervals?

[P3.3] Let $\dot{x} = f(x)$, with $f : D \to \mathbb{R}^n$ locally Lipschitz continuous on D, with D compact, $0 \in D$, and suppose that $x(t) \in D$ for all $t \geq 0$ and for all $x(0) = x_0 \in D$. Additionally, suppose that $f(0) = 0$. Under these conditions, establish that:

$$\left| \frac{d}{dt} \|x(t)\|_2^2 \right| \leq 2L\|x(t)\|_2^2, \qquad t \geq 0.$$

Use this to prove that:

$$\|x(t)\|_2 \leq e^{Lt}\|x(0)\|_2, \qquad t \geq 0.$$

[P3.4] [**Advanced Problem**] Complete the proof of the Comparison Lemma in the case when α is not assumed to be Lipschitz continuous. Specifically, utilize the constructions given in Remark 3.2 and prove that the implication in (3.4) holds.

[P3.5] Consider the electrical circuit in Figure 3.8. The resistor satisfies the linear constitutive relation $v = R\frac{dq}{dt}$ while the capacitor is described by the nonlinear constitutive relation $v = \frac{1}{C}q + \alpha q^3 + \beta q^5$ where v denotes voltage across the element's terminals and q denotes charge. What is the nonlinear differential equation describing the temporal evolution of the charge stored in the capacitor? Show (e.g., using the Comparison Lemma) that the capacitor completely discharges over time.

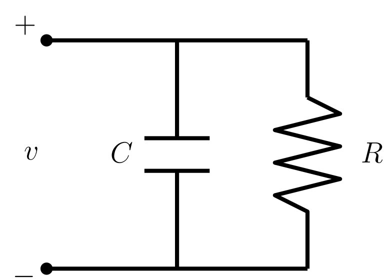


Figure 3.8. Electrical circuit for Problem P3.5.

[P3.6] This problem will establish variants of the Comparison Lemma. Specifically, let $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous in the first argument and continuous in the second and consider the IVP:

$$\dot{u} = -\alpha(u, t), \qquad u(0) = u_0,$$

where $u(t)$ is a corresponding unique solution on $[0, a]$, $a \in \mathbb{R}_{>0}$. Prove the following:

(a) If $v : [0, a] \to \mathbb{R}$ is a continuously differentiable function satisfying:

$$\dot{v}(t) \leq -\alpha(v(t), t), \qquad v(0) \leq u_0,$$

then $v(t) \leq u(t)$ for all $t \in [0, a]$.

(b) If $v : [0, a] \to \mathbb{R}$ is a continuously differentiable function satisfying:

$$\dot{v}(t) \geq -\alpha(v(t), t), \qquad v(0) \geq u_0,$$

then $v(t) \geq u(t)$ for all $t \in [0, a]$.

[P3.7] Use Problem P3.6 to prove Lemma 3.2, (as suggested by Remark 3.3).

[P3.8] Consider the levitated ball described in the problem section of Lecture 1, equations (1.44)-(1.46). If you regard the input u as a constant parameter, do solutions depend continuously on this parameter?

[P3.9] Formalize the intuition given in Remark 3.5. In particular, for the differential equation $\dot{x} = f(x, \lambda)$ with $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^p$, and under the assumption of Lipschitz continuity of f in both arguments, lift this equation to:

$$\dot{z} = g(x, \lambda) = \begin{bmatrix} f(x) \\ 0 \end{bmatrix} \qquad \text{for} \qquad z = \begin{bmatrix} x \\ \lambda \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^p.$$

For this transformed system, initial conditions, $z(t_0)$, consist of vectors $(x(t_0), \lambda)$ wherein the corresponding solution $z(t)$ of $\dot{z} = g(z)$ is given by $z(t) = (x(t), \lambda)$ where $x(t)$ is the solution of $\dot{x} = f(x, \lambda)$ for the chosen parameters λ. Show that Theorem 3.2 applied to $\dot{z} = g(z)$ guarantees continuity with respect to changes in initial values and parameters.

[P3.10] This problem will establish a variant on the Gronwall-Bellman Inequality (Theorem 3.1) where some of the assumptions on the functions in the theorem are relaxed; specifically, we will relax the assumption that $\lambda(t)$ be a positive and non-decreasing function and that $y(t)$ be nonnegative.

Let $I = [a, b] \subset \mathbb{R}$ be an interval, $\lambda : I \to \mathbb{R}$ be continuous, and $\varphi : I \to \mathbb{R}$ be continuous and nonnegative ($\varphi(t) \geq 0$ for all $t \in I$). If $y : I \to \mathbb{R}$ satisfies:

$$y(t) \leq \lambda(t) + \int_a^t \varphi(s) y(s) ds \qquad \forall t \in [a, b].$$

Prove the following:

(a) That $y(t)$ satisfies the bound:

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\varphi(s) e^{\int_s^t \varphi(\tau)d\tau} ds.$$

(b) In the special case that $\lambda(t) \equiv \lambda$ (a constant), then

$$y(t) \leq \lambda e^{\int_s^t \varphi(\tau)d\tau}.$$

(c) If in addition $\varphi(t) \equiv \varphi > 0$ (a constant), then

$$y(t) \leq \lambda e^{\varphi(t-a)}.$$

Hint: Consider the function:

$$v(s) = e^{-\int_a^s \varphi(\tau)d\tau} \int_a^s \varphi(\tau) y(\tau) d\tau.$$