PROBLEM SOLUTIONS

Problem 1 (Book Problem 7.5)

Let $V : E \subset \mathbb{R}^n \to \mathbb{R}$ be a positive definite function on the open and connected set $E \subset \mathbb{R}^n$. Show that $r > 0$ such that $B_r(0) \subset E$, there exists class $\mathcal{K}$ functions $\alpha_1, \alpha_2 \in \mathcal{K}$ defined on $[0, r]$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

For all $x \in B_r(0)$. (Hint: consider the functions:

$$\psi_1(s) = \inf_{s \leq \|x\| < r} V(x) \quad \psi_2(s) = \sup_{\|x\| < s} V(x)$$

for $s \in [0, r)$ coupled with the previous problem (Book problem 7.4).)

First, consider the function ψ_1. Define the domain of the infimum,

$$\Omega_s = \{x \in E \mid s \leq \|x\| < r\}$$

Note that since $s_1 < s_2 \implies \Omega_{s_1} \supset \Omega_{s_2}$, the function $\psi_1(s)$ is increasing, as it takes infimum over smaller sets. Furthermore,

$$\psi_1(\|x\|) \leq V(x) \quad \forall x \in B_r(0)$$

since it is defined as an infimum over a set containing x. Next, consider ψ_2: similarly, defining

$$\Omega'_s = \{x \in E \mid \|x\| < s\}$$

we have $s_1 < s_2 \implies \Omega_{s_1} \subset \Omega_{s_2}$, the function $\psi_2(s)$ is increasing, as it takes the supremum over a larger set. Furthermore,

$$\psi_2(\|x\|) \geq V(x) \quad \forall x \in B_r(0)$$

since it is defined as the supremum over a set of which x is a limit point. Therefore, we have

$$\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|)$$

where the lower and upper bounding functions are increasing. Applying the result of problem 7.4, there exists class $\mathcal{K}$ functions bounding V.
I'll include a constructive proof of 7.4 as well:

$$\alpha_1(\|x\|) = \frac{\psi_1(\|x\|)}{1 + e^{-\|x\|}} \leq V(x) \leq \|x\| + \psi_2(\|x\|) = \alpha_2(\|x\|)$$

Note that $\alpha_1(0) = \alpha_2(0) = 0$, $\alpha_1(s) \leq \psi_1(s)$, $\alpha_2(s) \geq \psi_2(s)$, and since ψ_1, ψ_2 are increasing, α_1, α_2 are strictly increasing. Therefore, the inequalities hold and $\alpha_1, \alpha_2 \in \mathcal{K}$.

Problem 2 continued on next page. . .

Problem 2 (Book Problem 7.7)

Consider the two dynamical systems

$$\dot{x} = f(x) \quad \dot{y} = \frac{f(y)}{1 + \|f(y)\|}$$

for $x, y \in \mathbb{R}^n$ and $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable. Show that stability is preserved under this transformation. That is, $V : E \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for the nominal system if and only if it is a Lyapunov function for the transformed system. Is this also true for asymptotic stability?

First, consider a Lyapunov function V_x defined such that

$$\dot{V}(x) = \frac{dV}{dx} f(x) \leq 0$$

Consider applying the same Lyapunov function to the y system:

$$\begin{aligned}
\dot{V}_x(y) &= \left.\frac{dV_x}{dx}\right|_{x=y} \frac{f(y)}{1 + \|f(y)\|} \\
&\leq \left.\frac{dV_x}{dx}\right|_{x=y} f(y) \\
&\leq 0
\end{aligned}$$

Since $1 + \|f(y)\| \geq 1$ for all y. Therefore, the y system is stable. Now, consider a Lyapunov function for the y system, V_y defined such that

$$\dot{V}_y(y) = \frac{dV_y}{dy} \frac{f(y)}{1 + \|f(y)\|} \leq 0$$

Consider applying the same function to the x dynamics:

$$\begin{aligned}
\dot{V}_y(x) &= \left.\frac{dV_y}{dy}\right|_{y=x} f(x) \\
&= \left.\frac{dV_y}{dy}\right|_{y=x} \frac{f(x)}{1 + \|f(x)\|} (1 + \|f(x)\|)
\end{aligned}$$

The term $1 + \|f(x)\|$ does not change the sign of the last line, so this quantity is also non-positive, and the x system is also stable.
Both arguments above do not change if the original Lyapunov function has a negative definite (rather than negative semi-definite) time derivative. The argument extends to the asymptotically stable case.

Problem 3 (Book Problem 8.5)

Consider the linear control system

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$$

Assume that $P = P^\top \succ 0$ solves the Continuous Time Algebraic Ricatti Equation:

$$PA + A^\top P + Q - PBB^\top P = 0$$

for some $Q = Q^\top \succ 0$. show that applying the feedback control law:

$$u(x) = -\frac{1}{2} B^\top P x$$

Problem 3 continued on next page. . .

results in the origin being globally exponentially stable.

Consider the Lyapunov function $V(x) = x^\top P x$. Note that

$$\lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2$$

Next, consider the time derivative of V

$$\begin{aligned}
\dot{V}(x) &= x^\top P \left(A - \frac{1}{2}BB^\top P\right)x + x^\top \left(A - \frac{1}{2}BB^\top P\right)Px \\
&= x^\top \left(PA + A^\top P - PBB^\top P\right)x \\
&= -x^\top Q x \\
&\leq -\lambda_{min}(Q)\|x\|^2 \\
&\leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}V(x)
\end{aligned}$$

V is a Lyapunov function certifying global exponential stability of the system.

Problem 4 (Book Problem 8.6)

Rational Stability: Let $\dot{x} = f(x)$ where $f : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on the open and connected set $E \subseteq \mathbb{R}^n$. The equilibrium point $x^* = 0 \in E$ is rationally stable if there exists $M\delta, k, \eta \in \mathbb{R}_{>0}$ with $\eta < 1$ such that:

$$\|x(t_0)\| < \delta \quad \implies \quad \|x(t)\| \leq M\left(1 + \|x(t_0)\|^k t\right)^{-\frac{1}{k}} \|x(t_0)\|^\eta$$

Consider a continuously differentiable function $V : E \to \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned}
k_1\|x\|^{c_1} \leq V(x) &\leq k_2\|x\|^{c_2} \\
\dot{V}(x) &\leq -k_3\|x\|^{c_3}
\end{aligned}$$

for $k_1, k_2, k_3, c_1, c_2, c_3 \in \mathbb{R}_{>0}$. Show that

$$\frac{c_3}{c_2} > 1 \quad \implies \quad x^* = 0 \text{ is rationally stable.}$$

I didn't feel like parsing through this problem to find my algebraic mistake, but the solution approach is correct here. If anyone can provide an algebraically clean solution I would appreciate it. Manipulate the top right inequality:

$$\begin{aligned}
k_2\|x\|^{c_2} &\geq V(x) \\
\|x\| &\geq \left(\frac{V(x)}{k_2}\right)^{\frac{1}{c_2}} \\
-k_3\|x\|^{c_3} &\leq -k_3\left(\frac{V(x)}{k_2}\right)^{\frac{c_3}{c_2}}
\end{aligned}$$

This gives the inequality

$$\dot{V}(x) \leq -k_4 V(x)^{\frac{c_3}{c_2}}$$

with $k_4 = k_3 k_2^{-\frac{c_3}{c_2}}$. Next, aim to use comparison lemma with the system

$$\dot{y} = -k_4 y^{\frac{c_3}{c_2}}$$

This is a separable ODE, and can be solved by integration.

$$\begin{aligned}
\frac{dy}{dt} &= -k_4 y^{\frac{c_3}{c_2}} \\
-y^{-\frac{c_3}{c_2}}\frac{dy}{dt} &= k_4 \\
-\int_0^\tau y^{-\frac{c_3}{c_2}} dy &= \int_0^\tau k_4 dt \\
\left[\frac{1}{1-\frac{c_3}{c_2}} y^{1-\frac{c_3}{c_2}}\right]_0^\tau &= k_4\tau \\
y(\tau)^{1-\frac{c_3}{c_2}} - y(0)^{1-\frac{c_3}{c_2}} &= \left(1-\frac{c_3}{c_2}\right)k_4\tau \\
y(\tau) &= \left(y(0)^{1-\frac{c_3}{c_2}} + \left(1-\frac{c_3}{c_2}\right)k_4\tau\right)^{\frac{1}{1-\frac{c_3}{c_2}}}
\end{aligned}$$

Define $\gamma = \frac{c_3}{c_2} - 1 > 0$. Then this can be rewritten

$$y(t) = \left(y(0)^{-\gamma} + \gamma k_4 t\right)^{-\frac{1}{\gamma}}$$

Using comparison Lemma, we have that

$$\begin{aligned}
V(x) &\leq \left(V(x_0)^{-\gamma} + \gamma k_4 t\right)^{-\frac{1}{\gamma}} \\
k_1\|x\|^{c_1} &\leq \left((k_2\|x_0\|^{c_2})^{-\gamma} + \gamma k_4 t\right)^{-\frac{1}{\gamma}} \\
k_1\|x\|^{c_1} &\leq \left(1 + \gamma k_4 \left(k_2\|x_0\|^{c_2}\right)^{\gamma} t\right)^{-\frac{1}{\gamma}} k_2\|x_0\|^{c_2} \\
\|x\| &\leq \left(\frac{k_2}{k_1}\right)^{\frac{1}{c_1}} \left(1 + \gamma k_4 \left(k_2\|x_0\|^{c_2}\right)^{\gamma} t\right)^{-\frac{1}{c_1\gamma}} \|x_0\|^{\frac{c_2}{c_1}}
\end{aligned}$$

where the second inequality is obtained by applying the bounds on the Lyapunov function, and the next steps are obtained by manipulation. This inequality is of the form given in the question statement.

Problem 5 (Book Problem 9.5)

Let $\dot{x} = f(x)$ with $f : E = B_r(0) \to \mathbb{R}^n$ and the some conditions on f and E as in the statement of Theorem 9.3. The goal is to prove 9.15 and 9.16 using the V given in 9.18, i.e.

$$V(x) = \int_0^\infty \|\varphi_\tau(x)\|^2 d\tau$$

Specifically, show that there exists a $r_0 > 0$ with $r_0 < r$ such that V satisfies

$$\begin{aligned}
k_1\|x\|^2 \leq V(x) &\leq k_2\|x\|^2 \\
\dot{V}(x) &\leq -k_3\|x\|^2
\end{aligned}$$

for $k_1, k_2, k_3 > 0$.

Problem 5 continued on next page...

First, bound the integral

$$\begin{aligned}
V(x) &= \int_0^\infty \|\varphi_\tau(x)\|^2 d\tau \\
&\leq \int_0^\infty M^2 e^{-2\lambda\tau} \|x\|^2 d\tau \\
&= -\frac{M^2\|x\|^2}{2\lambda} \left[e^{-2\lambda\tau}\right]_0^\infty \\
&= \frac{M^2}{2\lambda}\|x\|^2
\end{aligned}$$

Consider a variant of HW 1 problem 2 (book problem 3.3)

$$\begin{aligned}
\frac{d}{dt}\|\varphi_t(x)\|^2 &= 2f(\varphi_t(x))^\top \varphi_t(x) \\
&\geq -2L\|\varphi_t(x)\|^2
\end{aligned}$$

Where L is the local Lipschitz constant for $B_r(0)$, guaranteed to exist since f is continuously differentiable on this domain. By comparison Lemma, we then have

$$\|\varphi_t(x)\|^2 \geq e^{-2Lt}\|x\|^2$$

Which we can use to give another bound on V:

$$\begin{aligned}
V(x) &= \int_0^\infty \|\varphi_t(x)\|^2 dt \\
&\geq \int_0^\infty e^{-2Lt}\|x\|^2 dt \\
&= -\frac{\|x\|^2}{2L}\left[e^{-2Lt}\right]_0^\infty \\
&= \frac{1}{2L}\|x\|^2
\end{aligned}$$

Establishing that

$$\frac{1}{2L}\|x\|^2 \leq V(x) \leq \frac{M^2}{2\lambda}\|x\|^2$$

Given that $\|\varphi_t(x)\|^2 \leq Me^{-2\lambda t}\|x\|^2$ we have that

Next, consider the time derivative of V

$$\begin{aligned}
\dot{V}(x) &= \lim_{\delta t\to 0} \frac{V(x(t+\delta t)) - V(x(t))}{\delta t} \\
&= \lim_{\delta t\to 0} \frac{\int_0^\infty \|\varphi_\tau(x(t+\delta t))\|^2 d\tau - \int_0^\infty \|\varphi_\tau(x(t))\|^2 d\tau}{\delta t} \\
&= \lim_{\delta t\to 0} \frac{\int_{\delta t}^\infty \|\varphi_\tau(x(t))\|^2 d\tau - \int_0^\infty \|\varphi_\tau(x(t))\|^2 d\tau}{\delta t} \\
&= \lim_{\delta t\to 0} \frac{-\int_0^{\delta t} \|\varphi_\tau(x(t))\|^2 d\tau}{\delta t} \\
&= \frac{d}{d\gamma}\int_0^\gamma \|\varphi_\tau(x)\|^2 d\tau\bigg|_{\gamma=0} \\
&= -\|\varphi_\tau(x)\|^2\bigg|_{\gamma=0} \\
&= -\|x\|^2
\end{aligned}$$

Defined in this way, the rate of change of the Lyapunov function is exactly $\dot{V}(x) = -\|x\|^2$. The result holds where the assumptions hold; $r_0 < r$ can be taken arbitrarily as long as L is a Lipschitz constant on $B_{r_0}(0)$.