Lecture 2

Solutions to Nonlinear Systems

Consider a nonlinear dynamical system of the form:

$$\dot{x} = f(x),$$

for $x \in \mathbb{R}^n$, $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and E an open and connected subset of $\mathbb{R}^n$.

Notation for differential equations. The notation used in this book for differential equations may cause, at first, some confusion since the symbol x is used with two different meanings: x can represent a point in $\mathbb{R}^n$ or a curve $x : I \to \mathbb{R}^n$ defined on some open subset I of $\mathbb{R}$. These two different meanings can be reconciled by interpreting $x \in \mathbb{R}^n$ as shorthand notation for $x(t) \in \mathbb{R}^n,\ \forall t \in I$. Similarly, $\dot{x} = f(x)$ is to be interpreted as shorthand notation for $\dot{x}(t) = f(x(t)),\ \forall t \in I$.

In the case when $f(x) = Ax$ with A a $n \times n$ real matrix, i.e., the system is *linear*, solutions are guaranteed to exist for all time. This is not true of nonlinear systems—solutions may not exist for all time and, even if they do exist, they are not guaranteed to be unique. The purpose of this lecture, therefore, is to establish conditions under which solutions to nonlinear systems exist and are unique. In particular, the solution to the Initial Value Problem (IVP) specified by the differential equation:

$$\dot{x} = f(x),$$

and the initial value:

$$x(t_0) = x_0,\ t_0 \in \mathbb{R},$$

is given by:

$$\begin{aligned}
\int_{t_0}^{t} \dot{x}(\tau)d\tau &= \int_{t_0}^{t} f(x(\tau))d\tau \\
&\Updownarrow \\
x(\tau)|_{t_0}^{t} &= \int_{t_0}^{t} f(x(\tau))d\tau \\
&\Updownarrow \\
x(t) &= x_0 + \int_{t_0}^{t} f(x(\tau))d\tau. \qquad (2.1)
\end{aligned}$$

Therefore, for most nonlinear systems it is not possible to explicitly solve for $x(t)$ since f is an arbitrarily nonlinear function. Moreover, while (2.1) being a solution may seem obvious, following from the fundamental theorem of calculus, showing that this is the unique solution to a nonlinear system is no easy task. It requires approximations to be constructed and proofs that these approximations converge to the solution must be given. This points out a very important lesson related to nonlinear systems: even showing apparently simple statements can be deceptively complex. Additionally, in Lecture 3, we will build on the results of this lecture to establish "robustness" of solutions to perturbations in the initial condition or in parameters on which the differential equation may depend.

Concretely, we begin by formally defining the problem that is of interest.

Definition 2.1. Consider the function $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ defined on the open and connected[1] subset E of $\mathbb{R}^n$. A continuously differentiable[2] function $x : I \to E$, defined on an open and connected subset I of $\mathbb{R}$, is a solution to $\dot{x} = f(x)$ if:

$$\dot{x}(t) = f(x(t)), \quad \forall t \in I.$$

Given $x_0 \in E$, an Initial Value Problem (IVP) is defined by the data:

$$\dot{x} = f(x), \qquad x(t_0) = x_0, \qquad t_0 \in \mathbb{R}, \tag{IVP}$$

and $x : I \to E$ is a solution to the IVP if it is a solution to $\dot{x} = f(x)$, $t_0 \in I$, and $x(t_0) = x_0$.

Solutions defined on open or closed sets? Solutions to IVPs, as introduced in Definition 2.1, are defined on open sets $I \subseteq \mathbb{R}$ so that the meaning of $\dot{x}$ is clear at every point $t \in I$. However, in multiple locations in this book it will be convenient to work with solutions defined on closed sets $J \subset \mathbb{R}$. We will do so with the understanding of the existence of an open set $I \subseteq \mathbb{R}$ containing J and over which the solution is defined.

The following examples illustrate that solutions may not exist and when they do, they may not be unique (as was seen in similar examples in Lecture 1).

Example 2.1. It can be easily verified through differentiation that the following two functions:

1. $x(t) = t^3$,
2. $x(t) \equiv 0$,

are solutions to the IVP:

$$\dot{x} = 3x^{2/3}, \qquad x(0) = 0.$$

The reason for the non-uniqueness, as will be seen, arises from the fact that although $3x^{2/3}$ is continuous, it is not Lipschitz continuous at $x = 0$ (see Definition 2.4).

Example 2.2. Nonexistence of solutions for all time can happen even for "simple" nonlinear systems. For example, consider the IVP given by:

$$\dot{x} = x^2, \qquad x(0) = 1,$$

and the corresponding solution:

$$x(t) = \frac{1}{(1-t)}.$$

The expression for $x(t)$ shows that solutions are not defined for all t, i.e., solutions only exist on the interval $t \in (-\infty, 1)$.

[1]The connectedness assumption can always be made without loss of generality. If E is the disjoint union of several connected components we can apply any result in this book requiring connectedness to each of the connected components.

[2]Other notions of solution exist that require less regularity from x. For example, by rewriting $\dot{x} = f(x)$ as $x(t) = x_0 + \int_{t_0}^{t} f(x(\tau))d\tau$ there is no need to differentiate x and the differentiability requirement can be weakened.

2.1 Solutions to Linear Systems

To motivate solutions for nonlinear systems, we begin at the most basic level: *linear systems*, i.e., dynamical systems for which $f(x) = Ax$ is a linear transformation represented by a matrix $A \in \mathbb{R}^{n\times n}$. For linear systems there is a clear understanding of when solutions exist, and a closed form for such solutions. The difference between characterizing solutions for linear and nonlinear systems highlights the difference between linear and nonlinear systems in general.

At the most basic level, solutions to general linear systems can be best understood by first considering a 1-dimensional linear system:

$$\dot{x} = \alpha x, \tag{2.2}$$

with $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. The solution is given by:

$$x(t) = ce^{\alpha t}, \tag{2.3}$$

with $c = x(0) \in \mathbb{R}$ being the initial condition or value. This motivates the following development.

Solutions to Linear Systems. Utilizing the framework of matrix exponentials presented in Appendix A, we can discuss solutions to general linear systems of the form $\dot{x} = Ax$ with *no* assumptions on the matrix A, i.e., solutions of linear systems can be completely characterized.

Theorem 2.1. *For any $a \in \mathbb{R}_{>0}$, the solution $x : I = (t_0 - a, t_0 + a) \to \mathbb{R}^n$ to the linear IVP:*

$$\dot{x} = Ax, \qquad x(t_0) = x_0, \qquad t_0 \in \mathbb{R}, \tag{2.4}$$

is unique and given by:

$$x(t) = e^{A(t-t_0)}x_0, \qquad \forall t \in I. \tag{2.5}$$

Proof. To establish the result, we must prove both existence *and* uniqueness of solutions.

Existence: To establish existence, we only need to show that $x(t) = e^{A(t-t_0)}x_0$ satisfies the ODE (2.4) since it clearly satisfies the initial condition: $x(t_0) = e^{A(t_0-t_0)}x_0 = x_0$. By Proposition A.4 in the appendix it follows that:

$$\dot{x}(t) = \frac{d}{dt}e^{A(t-t_0)}x_0 = Ae^{A(t-t_0)}x_0 = Ax(t).$$

Uniqueness: For simplicity of notation, take $t_0 = 0$, wherein it must be established that $x(t) = e^{At}x_0$ is in fact the only solution of (2.4). To see this, let $z(t)$ be any solution to $\dot{x} = Ax$ (not necessarily of the form given in (2.5)). Setting:

$$y(t) \triangleq e^{-At}z(t),$$

we obtain:

$$\begin{aligned}
\dot{y}(t) &= -Ae^{-At}z(t) + e^{-At}\dot{z}(t) \\
&\underbrace{=}_{\text{since } e^{-At}A = Ae^{-At}} -Ae^{-At}z(t) + Ae^{-At}z(t) \\
&= 0,
\end{aligned}$$

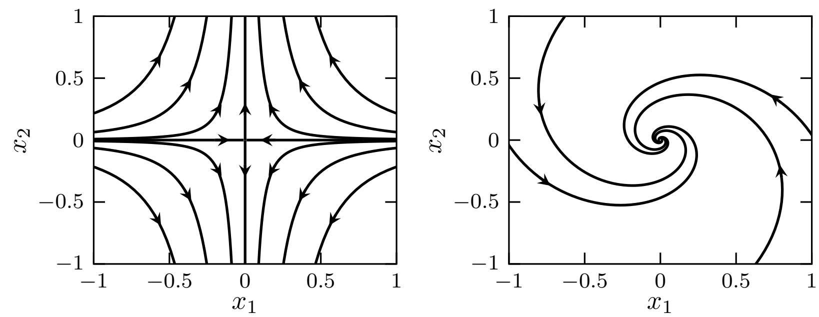

Figure 2.1. Phase portrait for the system considered in Example 2.3 (left) and Example 2.5 (right).

which implies that $y(t) = c$ for all $t \in I$, with $c \in \mathbb{R}^n$ a constant. But at $t = t_0 = 0$, $z(t_0) = x_0$, therefore:

$$y(0) = e^{-A0}x_0 = Ix_0 = x_0,$$

which implies that $c = x_0$. Hence, $y(t) \equiv x_0$, i.e., $y(t) = x_0$ for all $t \in I$ and:

$$x_0 = e^{-At}z(t) \qquad \underbrace{\Rightarrow}_{\text{by } \textit{Lem. A.3}} \qquad z(t) = e^{At}x_0,$$

thereby showing that solution z must in fact be of the form given in (2.5), i.e., the solution is unique. $\square$

Example 2.3. The solutions of the IVP $\dot{x} = Ax$, $x(t_0) = x(0) \in \mathbb{R}^2$, with the following A matrix:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},$$

are given by:

$$\begin{aligned} x_1(t) &= e^{-t}x_1(0) \\ x_2(t) &= e^{2t}x_2(0), \end{aligned}$$

since:

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}.$$

Example 2.4. Phase portraits are given by plotting the solution curves of a differential equation in state space, i.e., solutions for different initial conditions. Directionality in time is indicated by an arrow. The phase portrait for the system in Example 2.3 is given by plotting:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t}x_1(0) \\ e^{2t}x_2(0) \end{bmatrix},$$

for different values of $(x_1(0), x_2(0))$. For two-dimensional systems, it is easier to plot phase portraits by considering functional relationships between $x_1(t)$ and $x_2(t)$. For example, setting $x = x_1(t)$ and $y = x_2(t)$ it follows that y can be expressed as a function of x independent of time through the relationship:

$$y = \frac{x_1^2(0)x_2(0)}{x^2},$$

that is, $y = \frac{c}{x^2}$ for $c \in \mathbb{R}$ determined by different initial conditions. As a result, the phase portrait can be explicitly obtained via this relationship as shown in Figure 2.1.

Example 2.5. Consider the linear system:

$$\dot{x} = \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & -\frac{1}{2} \end{bmatrix} x \quad \text{with} \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then, by Example A.7 in the Appendix, the solution is given by:

$$\begin{aligned} x(t) &= e^{At}x_0 = e^{-\frac{1}{2}t} \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= e^{-\frac{1}{2}t} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}. \end{aligned} \tag{2.6}$$

The resulting phase portrait for the system can be seen in Figure 2.1, and consists of a *stable sink*.

2.2 Preliminaries

Before proceeding to the main existence and uniqueness result, some preliminary definitions are needed. To introduce continuity, we will consider *normed vector spaces* (see Appendix A).

Definition 2.2. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two normed vector spaces with norms $\| \, \|_1$ and $\| \, \|_2$, respectively. The function:

$$F : \mathcal{V}_1 \to \mathcal{V}_2,$$

is <u>continuous</u> at $x_0 \in \mathcal{V}_1$ if:

$$\forall \varepsilon > 0, \; \exists \delta > 0 \qquad \text{s.t.} \qquad \|x - x_0\|_1 < \delta \implies \|F(x) - F(x_0)\|_2 < \varepsilon.$$

Remark. The function $F : \mathcal{V}_1 \to \mathcal{V}_2$ is continuous on a subset $E \subseteq \mathcal{V}_1$ if it is continuous on each point in E, in which case we write $F \in C(E, \mathcal{V}_2)$.

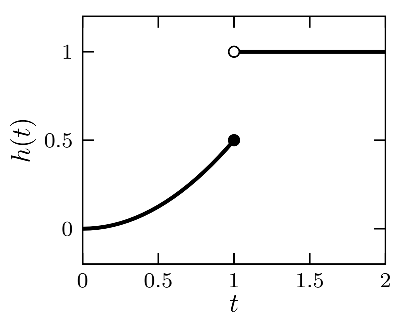

Figure 2.2. An example of a discontinuous function.

Example 2.6. The function $h : \mathbb{R} \to \mathbb{R}$ shown in Figure 2.2 is *not* continuous at $t_0 = 1$ because for $\varepsilon = \frac{1}{3}$ it is not possible to choose $\delta > 0$ such that $|t - t_0| < \delta$ implies that $|h(t) - h(t_0)| < \frac{1}{3}$. Indeed, when $\varepsilon = \frac{1}{3}$, we have that for all $\delta > 0$ there exists at least one $t \in B_\delta(t_0) = \{\tau \in \mathbb{R} \mid |\tau - t_0| < \delta\}$ (e.g., $t = t_0 + \frac{\delta}{2}$) that yields $|h(t) - h(t_0)| \geq \frac{1}{3}$.

Differentiability. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in \mathbb{R}^n$ if it is component-wise differentiable for each argument, i.e., if $\frac{\partial f_i}{\partial x_j}(x_0)$ is well defined for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. In this case, $Df(x_0)$ denotes the derivative of f at x_0 and it is uniquely given by:

$$Df(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix} \in \mathbb{R}^{m \times n}.$$

In the case when $f : \mathbb{R}^n \to \mathbb{R}$, we often use the notation $\left.\frac{\partial f}{\partial x}\right|_{x_0} = Df(x_0) \in \mathbb{R}^{1 \times n}$, or just $\frac{\partial f}{\partial x}$ when the point of evaluation is clear from context, to indicate the resulting row vector.

Definition 2.3. A function $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable on an open set E, denoted by $f \in C^1(E, \mathbb{R}^m)$, if $Df : E \subseteq \mathbb{R}^n \to \mathbb{R}^{m \times n}$ is continuous on E.

To establish existence and uniqueness of solutions to an IVP we need to assume a level of regularity for the right-hand side of the differential equation that lies between continuity and continuous differentiability.

Definition 2.4. The function $f : E \to \mathbb{R}^n$ defined on the open set $E \subseteq \mathbb{R}^n$ is said to be Lipschitz continuous on E if (for any norm $\|\cdot\|$ on $\mathbb{R}^n$) there exists a Lipschitz constant $L \in \mathbb{R}_{>0}$ such that for all $x, y \in E$ the following inequality holds:

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

The function f is said to be locally Lipschitz continuous if for each $x_0 \in E$ there exists a ball $B_\varepsilon(x_0)$ of radius $\varepsilon \in \mathbb{R}_{>0}$ around x_0 in E:

$$B_\varepsilon(x_0) \triangleq \{x \in \mathbb{R}^n : \|x - x_0\| < \varepsilon\} \subset E,$$

and a local Lipschitz constant $L_0 > 0$, that may depend on x_0 and ε, such that for every $x, y \in B_\varepsilon(x_0)$ the following inequality holds:

$$\|f(x) - f(y)\| \leq L_0\|x - y\|.$$

Lipschitz continuity implies continuity since every Lipschitz continuous function f is also continuous. To see why this is the case, it suffices to take $\delta = \varepsilon/L$ for any desired $\varepsilon \in \mathbb{R}_{>0}$. It then follows that for $\|x - x_0\| < \delta$ we have:

$$\|f(x) - f(x_0)\| \leq L\|x - x_0\| < L\delta = \varepsilon.$$

Example 2.7. The function $1/x$ defined on $(0, 1]$ is continuous but not Lipschitz continuous. To see why this is the case, note that Lipschitz continuity, with $y = 1$ (per the notation in Definition 2.4), requires:

$$\left|\frac{1}{x} - 1\right| = \left|\frac{1 - x}{x}\right| \leq L|x - 1|,$$

which implies:

$$L \geq \frac{1}{x},$$

for all $x \in (0, 1]$. It is clear that no such $L \in \mathbb{R}_{>0}$ exists. Although the function $1/x$ is continuous, its rate of growth increases without bound when x tends to 0 thereby preventing Lipschitz continuity.

Example 2.8 (**Linear Systems**)**.** Let $f(x) = Ax$ for $A \in \mathbb{R}^{m\times n}$, i.e., f is a linear function. Then f is Lipschitz continuous:

$$\|f(x) - f(y)\| = \|Ax - Ay\| = \|A(x - y)\| \leq \|A\|\|x - y\|,$$

where $L = \|A\|$ is the Lipschitz constant.

The next result shows that continuous differentiability implies Lipschitz continuity.

Theorem 2.2. *Consider the function $f : E \to \mathbb{R}^m$ defined on the open set $E \subseteq \mathbb{R}^n$. If f is continuously differentiable, i.e., $f \in C^1(E, \mathbb{R}^m)$, then f is locally Lipschitz continuous.*

This theorem points to an easy way to find a local Lipschitz constant. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and at some point $x_0 \in \mathbb{R}^n$, there exists $\varepsilon \in \mathbb{R}_{>0}$ and $L_0 \in \mathbb{R}_{>0}$ such that $x \in B_\varepsilon(x_0)$ implies $||Df(x)|| \leq L_0$. Then, for any $x, y \in B_\varepsilon(x_0)$ we have $\|f(x) - f(y)\| \leq L_0\|x - y\|$. This is made explicit in the following proof.

Proof. Let $x_0 \in E$ and consider a sufficiently small $\varepsilon \in \mathbb{R}_{>0}$ so that the topological closure:

$$\overline{B_\varepsilon(x_0)} \triangleq \{x \in E \mid \|x - x_0\| \leq \varepsilon\},$$

of the ball $B_\varepsilon(x_0)$ is contained in E. Such ε exists since E is an open subset of $\mathbb{R}^n$. By continuity of $\|Df\|$, this function achieves its maximum in the closed and bounded set $\overline{B_\varepsilon(x_0)}$ and we can define:

$$L_0 = \max_{x \in \overline{B_\varepsilon(x_0)}} \|Df(x)\|.$$

We now take any two points $y, z \in B_\varepsilon(x_0)$, and since $B_\varepsilon(x_0)$ is a convex set we have that any convex combination $\lambda y + (1 - \lambda)z$, $\lambda \in [0, 1]$, belongs to $B_\varepsilon(x_0)$. If we restrict f to the segment joining y to z we obtain the function $g(\lambda) = f(\lambda y + (1 - \lambda)z)$. We can now compute:

$$\begin{aligned}
\|f(y) - f(z)\| &= \|g(1) - g(0)\| \\
&= \left\| \int_0^1 \frac{d}{ds} g(s) ds \right\| \\
&\leq \int_0^1 \left\| \frac{d}{ds} g(s) \right\| ds \\
&\leq \int_0^1 \|Df(sy + (1 - s)z)(y - z)\| \, ds \\
&\leq \int_0^1 \|Df(sy + (1 - s)z)\| \, \|(y - z)\| \, ds \\
&\leq \int_0^1 L_0 \, \|y - z\| \, ds \\
&\leq L_0 \, \|y - z\| \, .
\end{aligned}$$

Here we used the chain rule to express the derivative of $g(s) = f(sy + (1 - s)x)$ with respect to s as the product of Df, evaluated at $sy + (1 - s)z$, with the derivative of $sy + (1 - s)z$ with respect to s, i.e., $y - z$. □

***Remark* 2.1.** To show that Lipschitz continuity does not imply continuous differentiability consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ for $x \leq 0$ and $f(x) = 0$ for $x > 0$. It is clearly Lipschitz continuous with Lipschitz constant $L = 1$. However, its derivative is not continuous at $x = 0$.

***Remark* 2.2.** The conclusions of Theorem 2.2 cannot be strengthened from local Lipschitz continuity to Lipschitz continuity. The function $f(x) = x^2$ is C^1 on the unbounded set $\mathbb{R}$. However, it is not Lipschitz continuous since this requires $|x^2| \leq L|x|$ when we take $y = 0$ in Definition 2.4. If there exists an $L \in \mathbb{R}_{>0}$ that satisfies this inequality for all $x \in \mathbb{R}$, then it also satisfies it for $x > 0$. However, $x > 0$ allows us to simplify $|x^2| \leq L|x|$ to $x \leq L$ which contradicts the assumption that L belongs to $\mathbb{R}_{>0}$, i.e., the assumption that L a finite constant.

Discrete-time Dynamical Systems. Dynamical systems can also evolve in discrete time, as opposed to the continuous time evolution considered thus far. We briefly give an overview of systems of this form in preparation for the concepts introduced in the next section. Importantly, note that this material is only required for Lectures 14 and 15 where discrete-time dynamical systems are obtained when continuous-time solutions intersect a manifold termed the Poincaré section, and Lecture 30 where event-driven controller implementations will be studied. Here we briefly motivate and define discrete-time dynamical systems.

The simplest example of a discrete-time dynamical system comes from the process of measuring the state of a continuous-time dynamical system. This is typically done using digital hardware governed by a clock. Therefore, state measurements are made at discrete instants of time that are integer multiples of a sampling period $T \in \mathbb{R}_{>0}$. Hence, if $x(t)$ is a solution of a dynamical system $\dot{x} = f(x)$, we can only measure the state at the time instants $0, T, 2T, \ldots$ which results in the sequence of measurements $x(kT)$ for $k \in \{0, 1, \ldots, \ell\} \subset \mathbb{N}$. This measurement process is time-driven, i.e., the decision to measure the state is solely dictated by time being equal to kT (an event driven approach is presented in Lecture 30). The result is a discrete-time dynamical system:

$$\overbrace{x((k+1)T)}^{x_{k+1}} = \underbrace{\overbrace{x(kT)}^{x_k} + \int_{kT}^{(k+1)T} f(x(\tau))d\tau}_{F(x_k)} \qquad \mapsto \qquad x_{k+1} = F(x_k). \tag{2.7}$$

This motivates the general notion of a discrete-time dynamical system:

Definition 2.5. A discrete-time dynamical system is a map:

$$F : E \subseteq \mathbb{R}^n \to \mathbb{R}^n, \tag{2.8}$$

where E is a connected subset of $\mathbb{R}^n$. Given an indexing set $I = \{k_0, k_0+1, \ldots, k_0+\ell\} \subset \mathbb{N}$, $k_0 \in \mathbb{N}$, a function $x : I \to E$ is a solution to the discrete-time dynamical system with initial condition $x_0 \in E$ if it satisfies the Initial Value Problem (IVP): for $x_k \triangleq x(k)$ for all $k \in I$:

$$x_{k+1} = F(x_k), \qquad x_{k_0} = x_0, \qquad k_0 \in I.$$

***Remark* 2.3.** Contrary to continuous-time dynamical systems, solutions are always unique and defined up to the point when the solution leaves the domain of F. In other words, finite escape times can still occur. Most results for discrete-time dynamical systems parallel those obtained for continuous time. However, it is important to be mindful of the fact that F does not need to be an invertible map. This is an important difference with respect to continuous time where, if it is possible to go from state x_0 to state x_f along a solution of $\dot{x} = f(x)$, then it is also possible to go from x_f to x_0 along a solution of $\dot{x} = -f(x)$. This difference manifests itself in many results that in continuous time are equivalences but in discrete time become implications only. Note this does not occur when a discrete-time dynamical system is obtained from the solution of a continuous-time dynamical system—as in (2.7)—through the measurement process that we previously described.

Example 2.9. Consider the linear dynamical system $\dot{x} = Ax$ with solution $x(t) = e^{At}x(0)$. The discrete-time dynamical system obtained by measuring the state x at the instants kT, $T \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$, is given by (see Appendix A for properties of the matrix exponential):

$$x_{k+1} = x((k+1)T) = e^{A(k+1)T}x(0) = e^{AT}e^{AkT}x(0) = e^{AT}x_k = F(x_k).$$

In this case F is the linear map $F(x_k) = e^{AT}x_k$. Finally, since F is obtained from a continuous-time dynamical system, we can observe that $F(x) = y$ is invertible with inverse $y \mapsto x = e^{-AT}y$.

2.3 Existence and Uniqueness

We now use the previously introduced notion of *Lipschitz continuity* to guarantee existence and uniqueness of solutions. The main idea is to successively approximate the solution to a nonlinear system and show that these approximations converge to the actual solution.

Theorem 2.3. *Consider a locally Lipshitz continuous function $f : E \to \mathbb{R}^n$ defined on the open and connected set $E \subseteq \mathbb{R}^n$. For any $x_0 \in E$ there exists $a \in \mathbb{R}_{>0}$ for which the IVP:*

$$\dot{x} = f(x), \qquad x(t_0) = x_0, \qquad t_0 \in \mathbb{R}, \tag{IVP}$$

has a unique solution, $x(t)$, on the interval $I = (t_0 - a, t_0 + a)$.

The key idea in proving this theorem is to use the method of *Picard iterations*, wherein an operator is defined on the infinite dimensional space of continuous functions from an interval to subsets of $\mathbb{R}^n$. This operator defines a discrete-time dynamical system and a fixed point of this operator corresponds to a solution of the (IVP). Therefore, if we can prove that it has a unique fixed point, existence and uniqueness of solutions will follow. This will be established using the contraction mapping principle (see Appendix A.3).

Solutions Through Approximation. To prove the existence of solutions, and provide a means to approximate them, we consider *Picard's method of approximation* iteratively defined by:

$$\begin{aligned} z_0(t) &= x_0, \\ z_{n+1}(t) &= x_0 + \int_{t_0}^{t} f(z_n(\tau))d\tau, \qquad n \in \mathbb{N}, \end{aligned} \tag{2.9}$$

which is well-defined since we successively integrate continuous functions of time on a bounded interval.

The *Picard operator*, defined on continuous functions from I to E, $C(I,E)$, is given by:

$$\begin{aligned} P : C(I,E) &\to C(I,E), \\ \varphi(t) &\mapsto P(\varphi)(t) = x_0 + \int_{t_0}^{t} f(\varphi(s))ds, \end{aligned} \tag{2.10}$$

where $I = (t_0 - a, t_0 + a)$ for $t_0 \in \mathbb{R}$ and $a \in \mathbb{R}_{>0}$. The method of Picard iteration is simply given by the *discrete-time dynamical system* on $C(I,E)$ for $n \in \mathbb{N}$:

$$z_{n+1} = P(z_n), \qquad z_0 = x_0.$$

A *fixed point* x^* of the Picard operator, if one exists, is a solution to the initial value problem (IVP):

$$x^* = P(x^*) \qquad \Longrightarrow \qquad x^*(t) = P(x^*)(t) = x_0 + \int_{t_0}^{t} f(x^*(s))ds.$$

Therefore, establishing existence and uniqueness of solutions to nonlinear systems amounts to showing that the Picard operator has a *unique* fixed point. Moreover, we will show that the sequence $z_{n+1} = P(z_n)$ is a Cauchy sequence (see Appendix A.3) and thus converges to the solution of the (IVP): $\lim_{n\to\infty} z_n = x^*$. Thus, Picard's method of approximation (2.9) converges to the actual solution and therefore can be used to calculate approximate solutions to the (IVP).

Example 2.10. Consider the IVP:

$$\dot{x} = \alpha x, \qquad x(0) = x_0,$$

for $x \in \mathbb{R}$, i.e., a 1-dimensional linear system. We saw in Section 2.1 that solutions to this system are given by $x(t) = e^{\alpha t}x(0)$. In this case, the method of Picard's iteration yields:

$$\begin{aligned}
z_0(t) &= x(0) \\
z_1(t) &= x(0) + \int_0^t \alpha x(0) ds = (1 + \alpha t)x(0) \\
z_2(t) &= x(0) + \int_0^t \alpha(1 + \alpha s)x(0) ds = \left(1 + \alpha t + \frac{1}{2}\alpha^2 t^2\right) x(0) \\
z_3(t) &= x(0) + \int_0^t \alpha \left(1 + \alpha s + \frac{1}{2}\alpha^2 s^2\right) x(0) ds = \left(1 + \alpha t + \frac{1}{2}\alpha^2 t^2 + \frac{1}{3!}\alpha^3 t^3\right) x(0) \\
&\vdots \\
z_n(t) &= \left(1 + \alpha t + \cdots + \frac{1}{n!}\alpha^n t^n\right) x(0) = \sum_{k=0}^{n} \frac{\alpha^k t^k}{k!} x(0).
\end{aligned}$$

Therefore:

$$\lim_{n\to\infty} z_n(t) = e^{\alpha t}x(0) = x(t).$$

Existence and Uniqueness for Nonlinear Systems. We now have the framework to prove Theorem 2.3. This theorem will essentially follow from the Contraction Mapping Principle (see Theorem A.1 in Appendix A) applied to the Picard operator (2.10), i.e., we only need to show that $P : C(I,E) \to C(I,E)$ is a contraction mapping. In this case, the fixed point x^* of P exists and is unique by Theorem A.1, and this fixed point is therefore the solution to the (IVP).

Proof of Theorem 2.3. Since f is locally Lipschitz continuous, there exists an ε-ball, $B_\varepsilon(x_0) \subset E$, and a (local) Lipschitz constant $L > 0$ such that for all $x, y \in B_\varepsilon(x_0)$:

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

Letting $b < \varepsilon$ we can define the closed and bounded (compact) set:

$$N_0 = \{x \in \mathbb{R}^n : \|x - x_0\| \leq b\} \subset B_\varepsilon(x_0).$$

The restriction of $\|f\|$ to N_0, being continuous, achieves its maximum M on this set:

$$M \triangleq \max_{x\in N_0} \|f(x)\|.$$

We now use N_0 and M to perform the proof in two steps using the closed interval $I = [t_0 - a, t_0 + a]$. The result will then hold, *a fortiori*, for the open interval $(t_0 - a, t_0 + a)$.

Step 1: P is well-defined: We first need to show that P is well-defined on the bounded closed set $N_0 \subset B_\varepsilon(x_0)$ in order to utilize the fact that it is locally Lipschitz continuous on this set. That is, we need to show that $P : C(I, N_0) \subset C(I, E) \to C(I, N_0)$ or, in other words, there exists $a \in \mathbb{R}_{>0}$ such that $P(x)$ belongs to $C(I, N_0)$:

$$\|P(x) - x_0\|_\infty \triangleq \max_{t \in I = [t_0 - a, t_0 + a]} \|P(x)(t) - x_0\| \leq b.$$

We claim this is achieved by taking $a < \frac{b}{M}$ as implied by the next four inequalities:

$$\begin{aligned}
\|P(x)(t) - x_0\| &\leq \left\| \int_{t_0}^{t} f(x(s)) ds \right\| \\
&\leq \int_{t_0}^{t} \|f(x(s))\| \, ds \\
&\leq Ma \\
&< b.
\end{aligned}$$

where the 3rd inequality follows from $t \in I = [t_0 - a, t_0 + a]$ and the fourth follows from $a < \frac{b}{M}$.

Step 2: P is a contraction mapping: To apply the Contraction Mapping Principle, and specifically Theorem A.1, we need to show that for every $x, y \in C(I, N_0)$ we have:

$$\|P(x) - P(y)\|_\infty \leq c\|x - y\|_\infty, \qquad 0 \leq c < 1.$$

Let $t \in I$ be such that:

$$\|P(x)(t) - P(y)(t)\| = \|P(x) - P(y)\|_\infty.$$

It then follows that:

$$\begin{aligned}
\|P(x) - P(y)\|_\infty &= \left\| \int_{t_0}^{t} \left(f(x(s)) - f(y(s))\right) ds \right\| \\
&\leq \int_{t_0}^{t} \|f(x(s)) - f(y(s))\| \, ds \\
&\leq \int_{t_0}^{t} L \, \|x(s) - y(s)\| \, ds \\
&\leq La \, \|x - y\|_\infty \, ,
\end{aligned}$$

therefore, choosing $a < \min\{b/M, 1/L\}$ ensures the Picard operator is a contraction mapping. The result then follows from Theorem A.1 asserting existence and uniqueness of a fixed point for P, i.e., (IVP) has a unique solution. □

Global Existence and Uniqueness. The presented results on existence and uniqueness were based on requiring as few assumptions on f as possible, i.e., local Lipschitz continuity. If we require a global version of this condition we obtain existence of solutions for all time.

Corollary 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function. Then, for any $a \in \mathbb{R}_{>0}$ the* (IVP) *has a unique solution for all $x_0 \in \mathbb{R}$ on the interval $(t_0 - a, t_0 + a)$.*

Proof. Since f is Lipschitz continuous, the proof of Theorem 2.3 can be greatly simplified, i.e., Step 1 is no longer necessary. Therefore, solutions exist and are unique for *for any* $x_0 \in \mathbb{R}^n$ and for all $t \in I = (t_0 - a, t_0 + a)$ for a given x_0 where $a < \frac{1}{L}$. By taking $a' < a$ we can assume the domain of a solution to be the closed interval $I = [t_0 - a', t_0 + a']$. But the interval, I, is independent of the initial condition x_0. Therefore, let $x_1(t)$ be the unique solution on $[t_0 - a', t_0 + a']$ and $x_2(t)$ be the unique solution on $[t_0 - a', t_0 + a']$ with $x_2(t_0) = x_1(t_0 + a)$. By uniqueness, we must have $x_2(t) = x_1(t + a')$ for $t \in [t_0 - a', t_0]$. Uniqueness implies these solutions must be the same, i.e., there exists an $x(t)$ defined on the interval $[t_0 - a', t_0 + 2a']$. Applying this argument iteratively (or by induction) yields the desired result. □

Returning to Linear Systems. In Section 2.1 it was shown that in the case when f is linear, i.e., $f(x) = Ax$, the (IVP) has a unique solution for all time. In fact, the results for nonlinear systems yield the same result with a much simpler proof.

Corollary 2.2. *For a linear system, $\dot{x} = Ax$, the IVP has a unique solution for all time $t \in \mathbb{R}$ given by $x(t) = e^{A(t-t_0)}x_0$.*

Proof. The function $f(x) = Ax$ is Lipschitz continuous with Lipschitz constant $L = \|A\|$ (see Example 2.8). Therefore, solutions exist and are unique for all $t \in \mathbb{R}$. Moreover, these solutions are given by $x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} P(x_{n-1})$, i.e., they are given by the limit of the sequence obtained from the Picard operator. It is easy to show that $e^{A(t-t_0)}x_0 = P(x^*)(t)$. □

Additional Reading

There is a large body of work on linear systems. This lecture was inspired by the overview of linear systems given in Perko [166], along with the specific methodology of proving existence and uniqueness. A more detailed introduction to the mathematical preliminaries discussed here can be found in Khalil [110]. The proof of existence and uniqueness using the Picard operator is inspired by Perko [166] (see also the original lectures by Picard [167]). It is important to note that there are many results related to the existence of solutions for nonlinear systems, some of which have weaker conditions on f than the methodology presented here. An example is given by Peano's Uniqueness Theorem (Theorem 1.3.1 in [2]; see also [92] for additional results on existence and uniqueness).

Problems for Lecture 2

[P2.1] For each of the functions f given below, find whether f is continuous, locally Lipschitz continuous, Lipschitz continuous, or continuously differentiable.

(a) $f(x) = \max\{0, x\}$; (Lipschitz continuous, not differentiable)

(b) $f(x) = e^x - x^p, p \in \mathbb{R}$; (locally Lipschitz continuous, not Lipschitz continuous)

(c) $f(x) = \tanh(x)$; (C^1 and globally Lipschitz continuous)

(d) $f(x) = \log(x)$ with domain $\mathbb{R}_{>0}$. (continuous but not locally Lipschitz continuous)

[P2.2] Let $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous on $\mathbb{R}$. Show that $f_1 + f_2$, $f_1 f_2$ and $f_2 \circ f_1$ are locally Lipschitz continuous. The symbol $f_1 \circ f_2$ denotes the composition of f_1 and f_2, i.e., $f_1 \circ f_2(x) = f_1(f_2(x))$.

[P2.3] [**Advanced Problem**] Let $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$, with E a compact set (closed and bounded). Show that if f is continuous then it achieves a maximum value on E: $M = \max_{x\in E} \|f(x)\|$.

[P2.4] Let A be an $n \times n$ matrix. Show that Picard's method of successive approximations converges to the solution $x(t) = e^{At}x_0$ of the initial value problem $\dot{x} = Ax$, $x(0) = x_0$. That is, Picard's method gives an alternative proof for existence and uniqueness of solutions to linear systems.

[P2.5] Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable for all $y \in \mathbb{R}^n$ and define $g(y)$ by:

$$g(y) = \frac{f(y)}{1 + \|f(y)\|}.$$

Show that $\dot{y} = g(y)$, with $y(0) = y_0$ has a *unique* solution defined for all $t \geq 0$. Problem P4.9 discusses how the solutions of $\dot{y} = g(y)$ and $\dot{x} = f(x)$ are related.

[P2.6] Which assumption of Theorem 2.3 is violated in Example 2.1?

[P2.7] Consider the nonlinear system (1.13) in Lecture 1 with $u = 0$. Show that solutions exist and are unique.

[P2.8] Consider the nonlinear system (1.44)-(1.46) in Lecture 1 with $u = 0$. Show that solutions exist and are unique.

[P2.9] There are nonlinear systems whose solutions exist for all $t \geq 0$. Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 - x_2 x_3 \\ \dot{x}_2 &= -x_1^2 - x_3^2 \\ \dot{x}_3 &= x_1 x_2 + x_2 x_3, \end{aligned}$$

and the function $H = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$. Show that H is constant along the solutions of the nonlinear system, i.e., $H(x(t)) = H(x(0))$. Use the fact that the manifold $H(x) = H(x(0))$ is compact to conclude existence of solutions for all $t \geq 0$.

[P2.10] Consider the system in Problem P1.7. Show that solutions exist and are unique by showing that the system is locally Lipschitz continuous. Is it Lipschitz continuous?