PROBLEMS

Problem 1 (Book Problem 3.1)

Use the Comparison Lemma to construct a curve upper bounding the solution of the differential equation $\dot{x} = -x + \tan^{-1}(x)$.

Consider the differential equation

$$\dot{u} = -u + \frac{\pi}{2} \implies u(t) = u_0 e^{-t} + \frac{\pi}{2}\left(1 - e^{-t}\right)$$

Since

$$\dot{u} = -u + \frac{\pi}{2} \geq -u + \tan^{-1}(u)$$

Setting $u_0 = x_0$ and applying Comparison Lemma yields

$$x(t) \leq u(t) = x_0 e^{-t} + \frac{\pi}{2}\left(1 - e^{-t}\right)$$

Problem 2 (Book Problem 3.3)

Let $\dot{x} = f(x)$, with $f : D \to \mathbb{R}^n$ locally Lipschitz continuous on D, with D compact, $0 \in D$, and suppose that $x(t) \in D$ for all $t \geq 0$ and for all $x(0) = x_0 \in D$. Additionally, suppose that $f(0) = 0$. Under these conditions, establish that:

$$\left|\frac{d}{dt}\|x(t)\|^2\right| \leq 2L\|x(t)\|^2 \quad t \geq 0$$

Use this to prove that:

$$\|x(t)\| \leq e^{Lt}\|x(0)\| \quad t \geq 0$$

First, consider

$$\begin{aligned}
\frac{d}{dt}\|x(t)\|_2^2 &= \frac{d}{dt}x^\top x \\
&= 2x^\top \dot{x} \\
&= 2x^\top f(x)
\end{aligned}$$

Now, considering the magnitude of the rate of change:

$$\begin{aligned}
\left|\frac{d}{dt}\|x(t)\|_2^2\right| &= |2x^\top f(x)| \\
&\leq 2\|f(x)\|_2\|x\|_2 \quad \text{(Cauchy Schwarz)} \\
&= 2\|f(x) - f(0)\|_2\|x\|_2 \quad f(0) = 0 \\
&\leq 2L\|x\|_2^2 \quad \text{Local Lipschitz continuity}
\end{aligned}$$

Consider the comparison system

$$\dot{y} = 2Ly \implies y(t) = e^{2Lt}y_0$$

Applying Comparison Lemma, we conclude

$$\begin{aligned}
\|x(t)\|_2^2 &\leq e^{2Lt}\|x_0\|_2^2 \\
\|x(t)\|_2 &\leq e^{Lt}\|x_0\|
\end{aligned}$$

since the square root preserves ordering.

Problem 3 (Book Problem 3.10)

This problem will establish a variant on the Gronwall-Bellman Inequality (Theorem 3.1) where some of the assumptions on the functions in the theorem are relaxed; specifically, we will relax the assumption that $\lambda(t)$ be a positive and non-decreasing function and that $y(t)$ be nonnegative.
Let $I = [a, b] \subset \mathbb{R}$ be an interval, $\lambda : I \to \mathbb{R}$ be continuous, and $\varphi : I \to \mathbb{R}$ be continuous and nonnegative. If $y : I \to \mathbb{R}$ satisfies:

$$y(t) \leq \lambda(t) + \int_a^t \varphi(s)y(s)ds \quad \forall t \in I$$

Prove the following:

(a) That $y(t)$ satisfies the bound:

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\varphi(s)e^{\int_s^t \varphi(\tau)d\tau}ds$$

Define $z(t) = \lambda(t) + \int_a^t \varphi(s)y(s)ds$, noting that

$$\begin{aligned}
&z(a) = \lambda(a) \\
&y(t) \leq z(t) \\
&\dot{z}(t) = \dot{\lambda}(t) + \varphi(t)y(t) \leq \dot{\lambda}(t) + \varphi(t)z(t)
\end{aligned}$$

From the last relation, we can solve the differential equation in z

$$\begin{aligned}
\dot{z}(t) - \varphi(t)z(t) &\leq \dot{\lambda}(t) \\
\dot{z}(t)e^{-\int_a^t \varphi(\tau)d\tau} - \varphi(t)z(t)e^{-\int_a^t \varphi(\tau)d\tau} &\leq \dot{\lambda}(t)e^{-\int_a^t \varphi(\tau)d\tau} \\
\frac{d}{dt}\left[z(t)e^{-\int_a^t \varphi(\tau)d\tau}\right] &\leq \dot{\lambda}(t)e^{-\int_a^t \varphi(\tau)d\tau}
\end{aligned}$$

Both sides of this equation can be integrated, the right hand side requiring integration by parts

$$\int_a^t \dot{\lambda}(s)e^{-\int_a^s \varphi(\tau)d\tau}ds = \lambda(t)e^{-\int_a^t \varphi(\tau)d\tau} - \lambda(a) + \int_a^t \lambda(s)\varphi(s)e^{-\int_a^s \varphi(\tau)d\tau}ds$$

Integrating the left hand side trivially,

$$\int_a^t \frac{d}{dt}\left[z(t)e^{-\int_a^t \varphi(\tau)d\tau}\right]dt = z(t)e^{-\int_a^t \varphi(\tau)d\tau} - z(a)$$

and recalling that $z(a) = \lambda(a)$ we can simplify

$$\begin{aligned}
z(t)e^{-\int_a^t \varphi(\tau)d\tau} - z(a) &\leq \lambda(t)e^{-\int_a^t \varphi(\tau)d\tau} - \lambda(a) + \int_a^t \lambda(s)\varphi(s)e^{-\int_a^s \varphi(\tau)d\tau}ds \\
z(t)e^{-\int_a^t \varphi(\tau)d\tau} &\leq \lambda(t)e^{-\int_a^t \varphi(\tau)d\tau} + \int_a^t \lambda(s)\varphi(s)e^{-\int_a^s \varphi(\tau)d\tau}ds \\
z(t) &\leq \lambda(t) + e^{\int_a^t \varphi(\tau)d\tau} \int_a^t \lambda(s)\varphi(s)e^{-\int_a^s \varphi(\tau)d\tau}ds \\
y(t) \leq z(t) &\leq \lambda(t) + \int_a^t \lambda(s)\varphi(s)e^{\int_s^t \varphi(\tau)d\tau}ds
\end{aligned}$$

And the first relation is shown.
(b) In the special case that $\lambda(t) \equiv \lambda$ is constant, then

$$y(t) \leq \lambda e^{\int_a^t \varphi(\tau)d\tau}$$

Now, this setup matches the setup for the Gronwall-Bellman Lemma, except that the statement of this lemma in the course notes requires $\lambda \geq 0$. However, an identical result can be obtained without this restriction; proceeding identically to part (a) we now have $z(t) = \lambda + \int_a^t \varphi(s)y(s)ds$, with the same properties of z (except the time derivative of λ is now zero)

$$\begin{aligned}
&z(a) = \lambda \\
&y(t) \leq z(t) \\
&\dot{z}(t) = \varphi(t)y(t) \leq \varphi(t)z(t)
\end{aligned}$$

Now, we can proceed as before

$$\begin{aligned}
\dot{z}(t) - \varphi(t)z(t) &\leq 0 \\
\dot{z}(t)e^{-\int_a^t \varphi(\tau)d\tau} - \varphi(t)z(t)e^{-\int_a^t \varphi(\tau)d\tau} &\leq 0 \\
\frac{d}{dt}\left[z(t)e^{-\int_a^t \varphi(\tau)d\tau}\right] &\leq 0 \\
\int_a^t \frac{d}{dt}\left[z(t)e^{-\int_a^t \varphi(\tau)d\tau}\right]dt &\leq 0 \\
z(t)e^{-\int_a^t \varphi(\tau)d\tau} - z(a) &\leq 0 \\
z(t)e^{-\int_a^t \varphi(\tau)d\tau} &\leq \lambda \\
z(t) &\leq \lambda e^{\int_a^t \varphi(\tau)d\tau}
\end{aligned}$$

And the desired result is shown.
(c) If in addition $\varphi(t) \equiv \varphi > 0$ is constant, then

$$y(t) \leq \lambda e^{\varphi(t-a)}$$

This follows trivially from (b) by evaluating the integral in the exponent

$$\int_a^t \varphi d\tau = \varphi(t-a)$$

Problem 4 continued on next page. . .

giving

$$y(t) \leq \lambda e^{\varphi(t-a)}$$

Problem 4 (Book Problem 4.1)

Prove Proposition 4.1, i.e. that flows satisfy the identity, composability, and reversability properties.

Identity (trivially):

$$\varphi_0(x_0) = x(0) = x_0$$

Composability: If $t \in I(x_0)$, then $y = \varphi_t(x_0) = x(t)$ is well defined. If $s \in I(y)$, then $\varphi_s(y) = x'(s)$ is well defined, where x' is the solution to the IVP with initial condition $x(t)$. Flowing s time units from $x(t)$ is the same as flowing $s + t$ units from x_0, so we have

$$\varphi_s(\varphi_t(x_0)) = \varphi_s(y) = x'(s) = x(t+s) = \varphi_{s+t}(x_0)$$

Reversability: If $t \in I(x_0)$; since I is a maximum interval of existence, it is then guaranteed for $-t \in I(\varphi_t(x_0))$. Following the same argument about composing flows above yields the desired result.

Problem 5 (Book Problem 4.8)

Consider the system in Example 1.3.

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) - \alpha x_2 \left(\frac{x_2^2}{2} - \cos(x_1) + c\right) \end{bmatrix}$$

for $c \in (0, 1)$, and $\alpha > 0$ What does the linearization say about the system dynamics? What behavior does the linearization fail to capture?

Note that $f(0) = 0$. Linearizing about equilibrium

$$\begin{aligned} Df(0) &= \begin{bmatrix} 0 & 1 \\ -\cos(x_1) - \alpha x_2 \sin(x_1) & -\alpha\left(\frac{x_2^2}{2} - \cos(x_1) + c\right) - \alpha x_2^2 \end{bmatrix}_{x=0} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(c-1) \end{bmatrix} \end{aligned}$$

Compute the eigenvalues of the linearization

$$\begin{aligned} det(Df(0) - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda - \alpha(c-1) \end{vmatrix} \\ &= \lambda(\lambda + \alpha(c-1)) + 1 \\ &= 0 \\ \implies \lambda &= \frac{-\alpha(c-1) \pm \sqrt{\alpha^2 c^2 - 4}}{2} \end{aligned}$$

The linearization implies that solutions to the system are unstable, as since $-\alpha(c-1) > 0$, at least one eigenvalue has positive real part, and the linearization is unstable. However, as discussed in Example 1.3, this system contains an asymptotically stable periodic orbit. The linearization contains no information that this system has any periodic orbit. The information from the linearization only captures the fact the the origin is locally unstable.