PROBLEMS

Problem 1 (Book Problem 2.1)

For each of the functions $f(x)$ given below, demonstrate whether f is continuous, locally Lipschitz continuous, Lipschitz continuous, or continuously differentiable on $\mathbb{R}$.

(a) $f(x) = x^2 + |x|$

- Continuously Differentiable? f is polynomial and thus cont. diff'able on $\mathbb{R}\backslash\{0\}$. However, at $x = 0$, we have

$$\lim_{x\to 0^-} \frac{df}{dx}(x) = -1$$
$$\lim_{x\to 0^+} \frac{df}{dx}(x) = 1$$

  And the derivative does not exist at $x = 0$.

- Locally Lipschitz Continuous: for every point $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ consider $x \in B_\varepsilon(x_0)$

$$\begin{aligned}
\|f(x_0) - f(x)\| &= |x_0^2 + |x_0| - x_0^2 - |x|| \\
&\le |x_0^2 - x^2| + ||x_0| - |x|| \quad \text{triangle inequality} \\
&\le |x_0^2 - x^2| + |x_0 - x| \quad \text{reverse triangle inequality} \\
&\le \left(\max_{\xi\in B_\varepsilon(x_0)} 2|\xi|\right)|x_0 - x| + |x_0 - x| \\
&\le \left(\max_{\xi\in B_\varepsilon(x_0)} 2|\xi| + 1\right)|x_0 - x| \\
&= (2(|x_0| + \varepsilon) + 1)|x_0 - x|
\end{aligned}$$

  such that $L_0 = (2(|x_0| + \varepsilon) + 1)$ is a local Lipschitz constant.

- Global Lipschitz Continuity: Since for all $x \in \mathbb{R}\backslash\{0\}$ we have

$$\left|\frac{df}{dx}(x)\right| \ge 2|x| - 1$$

  which grows unbounded with x, so there is no global Lipschitz constant for f, and f is not globally Lipschitz continuous.

- Continuity: Local Lipschitz continuity implies continuity.

(b) $f(x) = -x + a\sin(x)$

- Continuously Differentiable?

$$\frac{df}{dx} = -1 + a\cos(x)$$

  which is continuous, so f is continuously differentiable.

- Global Lipschitz Continuous: $|df/dx|$ is bounded by $1 + |a|$ for all $x \in \mathbb{R}$, so $L = 1 + |a|$ is a global Lipschitz constant.

Problem 1 continued on next page. . .

- Local Lipschitz Continuity: Global Lipschitz continuity implies local Lipschitz continuity.
- Continuity: Local Lipschitz continuity implies continuity.

(c) $f(x) = -x + 2|x|$

- Continuously Differentiable? As in (a) $|x|$ is not differentiable at $x = 0$.
- Global Lipschitz Continuous: for every point $x_0 \in \mathbb{R}$

$$\begin{aligned} \|f(x_0) - f(x)\| &= |-x_0 + 2|x_0| + x - 2|x|| \\ &\leq |-x_0 + x| + 2||x_0| - |x|| \quad \text{triangle inequality} \\ &\leq |-x_0 + x| + 2|x_0 - x| \quad \text{reverse triangle inequality} \\ &\leq 3|x_0 - x| \end{aligned}$$

  such that $L = 3$ is a global Lipschitz constant.

- Local Lipschitz Continuity: Global Lipschitz continuity implies Local Lipschitz continuity.
- Continuity: Local Lipschitz continuity implies continuity.

(d) $f(x) = \tan(x)$

- Continuity: $\lim_{x\to\pi^-} \tan(x) = \infty$ and $\lim_{x\to\pi^+} \tan(x) = -\infty$ so f is not continuous at π.
- Continuously Differentiable? Not continuous, so not continuously differentiable.
- Locally Lipschitz Continuous: Not continuous, so not locally Lipschitz.
- Global Lipschitz Continuity: Not continuous, so not globally Lipschitz.

Problem 2 (Book Problem 2.2)

Let $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous on $\mathbb{R}$. Show that $f_1 + f_2$, $f_1 f_2$, and $f_1 \circ f_2$ are locally Lipschitz continuous. $f_1 \circ f_2$ denotes function composition, i.e., $f_1 \circ f_2(x) = f_1(f_2(x))$ Assume that for all $x_0 \in \mathbb{R}$ and some $\varepsilon > 0$ there exists local Lipschitz constants $L_1(x_0, \varepsilon)$ and $L_2(x_0, \varepsilon)$

for f_1 and f_2 respectively.

- $f_1 + f_2$:

$$\begin{aligned} \|f_1(x_0) + f_2(x_0) - f_1(x_0) - f_2(x_0)\| &\leq \|f_1(x_0) - f_1(x)\| + \|f_2(x_0) - f_2(x)\| \quad \Delta \text{ inequality} \\ &\leq L_1(x_0, \varepsilon)|x_0 - x| + L_2(x_0, \varepsilon)|x_0 - x| \\ &= (L_1(x_0, \varepsilon) + L_2(x_0, \varepsilon))\,|x_0 - x| \end{aligned}$$

  So $f_1 + f_2$ is locally Lipschitz with constant $L_{f_1+f_2}(x_0 + \varepsilon) = L_1(x_0, \varepsilon) + L_2(x_0, \varepsilon)$

- $f_1 f_2$:

$$\begin{aligned} |f_1(x_0)f_2(x_0) - f_1(x)f_2(x)| &= |f_1(x_0)f_2(x_0) - f_1(x)f_2(x_0) + f_1(x)f_2(x_0) - f_1(x)f_2(x)| \\ &\leq |f_1(x_0)f_2(x_0) - f_1(x)f_2(x_0)| + |f_1(x)f_2(x_0) - f_1(x)f_2(x)| \\ &\leq |f_2(x_0)||f_1(x_0) - f_1(x)| + f_1(x)|f_2(x_0) - f_2(x)| \\ &\leq (|f_2(x_0)|L_1(x_0, \varepsilon) + |f_1(x)|L_2(x_0, \varepsilon))\,|x_0 - x| \end{aligned}$$

  Since f_1 and f_2 are locally Lipschitz continuous, they are continuous and thus bounded on closed sets $\overline{B}_\varepsilon(x_0)$. Thus, there exist bounds $M_1(x_0, \varepsilon), M_2(x_0, \varepsilon) > 0$ such that $|f_1(x)| \leq M_1(x_0, \varepsilon)$ and $|f_2(x)| \leq M_2(x_0, \varepsilon)$ for all $x \in B_\varepsilon(x_0)$. Therefore, $L(x_0, \varepsilon) = M_2(x_0, \varepsilon)L_1(x_0, \varepsilon) + M_1(x_0, \varepsilon)L_2(x_0, \varepsilon)$ is a local Lipschitz constant for $f_1 f_2$.

Problem 2 continued on next page. . .

- $f_2 \circ f_1$:

  The local Lipschitz continuity of f_1 ensures that

$$\begin{aligned} &\|f_1(x_0) - f_1(x)\| \leq L_1(x_0, \varepsilon)|x_0 - x| \\ \implies &f_1(x) \in f_1(x_0) + L_1(x_0, \varepsilon) \times (-\varepsilon, \varepsilon) \end{aligned}$$

  Define the open set $I(x_0, \varepsilon) = f_1(x_0) + L_1(x_0, \varepsilon) \times (-\varepsilon, \varepsilon)$, and use the Lipschitz continuity of f_2 to find

$$\begin{aligned} \|f_2 \circ f_1(x_0) - f_2 \circ f_1(x)\| &\leq L_2(f_1(x_0), L_1\varepsilon)\|f_1(x_0) - f_1(x)| \\ &\leq L_2(f_1(x_0), L_1\varepsilon)L_1(x_0, \varepsilon)|x_0 - x| \end{aligned}$$

  Thus $L_{f_2 \circ f_1}(x_0, \varepsilon) = L_2(f_1(x_0), L_1\varepsilon)L_1(x_0, \varepsilon)$ is a local Lipschitz constant for $f_2 \circ f_1$.

Problem 3 **(Book Problem 2.10)**

Consider the system in Problem P1.7, consider an iron ball levitating within a magnetic field created by a single electromagnet. A simplified model describing the vertical position of the ball is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{2m}\lambda^2 - g \\ -\frac{R}{c}(1 - x_1)\lambda + u \end{bmatrix}$$

where x_1 denotes the ball's height, x_2 the ball's velocity, m is the ball's mass, g is gravity's constant, λ is the magnetic flux linkage in the electromagnet, R is the resistance of the electromagnet's coil, c is a positive constant modeling the electromagnet's geometry and construction, and u is the voltage applied at the coil's terminals constituting the input to this system. This simplified model describes, e.g., a magnetically levitated train where the train mass has been lumped into the ball.
Show that solutions exist and are unique by showing that the system is locally Lipschitz continuous. Is it Lipschitz continuous? (Assume u is constant in time. What if u is a continuous function of time?)

The system is continuously differentiable

$$Df(x_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{m}\lambda_0 \\ \frac{R}{c}\lambda_0 & 0 & \frac{R}{c}x_{0,1} \end{bmatrix}$$

By Theorem 2.2 in the book, there exists a local Lipschitz constant for the system

$$L_0 = \max_{x \in B_\varepsilon(x_0)} \|Df(x)\|$$

Problem 4

Give a brief summary of your research interests, particularly (if applicable) what type of dynamical systems you work with, and what properties of those dynamical systems you care about.

(Will's brief answer:) I am interested in the control of highly agile dynamical systems, including systems with contact, such as walking and running robots. I like to operate at an intersection of theory and practice, and am interested in incorporating learning-based methods with model-based methods to obtain performant, dynamics informed controllers.

Problem 5 continued on next page...

Problem 5

Briefly discuss your motivation for taking this class.

(Will's brief answer:) Any system, when controlled on the edge of its physical capability, will require exploitation of nonlinear dynamics. While linear control techniques work well around equilibrium and as tracking controllers, designing high-performance, agile trajectories, or controlling systems in aggressive regimes of their capability space require analysis and use of highly nonlinear dynamics. I was excited to develop a fundamental understanding the tools required to perform this sort of analysis and control.