PROBLEMS

Problem 1 (Book Problem 5.4)

Consider the dynamical system, $\dot{x} = f(x)$, introduced in Example 5.15.

$$\dot{x} = f(x) = \begin{bmatrix} -x_1 \\ -x_2 + x_1^k \\ x_3 + x_1^2 \end{bmatrix}$$

with $k \in \mathbb{N}_{\geq 1}$. For this system:

(a) Find an explicit expression for the stable and unstable manifolds, S and U, respectively, i.e., find functions $h_S : \mathbb{R}^3 \to \mathbb{R}^s$, $h_U : \mathbb{R}^3 \to \mathbb{R}^u$ defining these surfaces: $S = h_S^{-1}(0)$ and $U = h_U^{-1}(0)$.

Compute the flow for the system:

$$\varphi_t(x_0) = \begin{bmatrix} x_{0,1}e^{-t} \\ x_{0,2}e^{-t} + \frac{x_{0,1}^k}{k+1}\left(e^{-t} - e^{-kt}\right) \\ x_{0,3}e^{t} + \frac{x_{0,1}}{3}\left(e^{t} - e^{-2t}\right) \end{bmatrix}$$

Note for this system this can be done by first solving the autonomous x_1 dynamics, and then solving for x_2 and x_3 dynamics using the time solutions of x_1 dynamics as forcing inputs.
Note that x_3 is unbounded unless

$$h_S(x_0) = x_{0,3} + \frac{x_{0,1}^2}{3} = 0$$

With initial conditions on this surface, the dynamics are stable; the stable dynamics manifold is therefore

$$\mathcal{S} = h_S^{-1}(x) = \left\{ x \in \mathbb{R}^3 \mid x_3 + \frac{x_1^2}{3} = 0 \right\}$$

Similarly, the function

$$h_U(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

zeros out the stable portions of the dynamics, such that trajectories limit backwards to zero, giving the unstable dynamics manifold:

$$\mathcal{U} = h_U^{-1}(x) = \left\{ x \in \mathbb{R}^3 \mid x_1 = 0, x_2 = 0 \right\}$$

(b) Establish that the surfaces S and U are, in fact, manifolds. Find local coordinates for these manifolds in a neighborhood of $0 \in \mathbb{R}^3$ via coordinate charts.

First, consider Jacobians of h_S and h_U:

$$Dh_U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Dh_S = \begin{bmatrix} \frac{2x_1}{3} & 0 & 1 \end{bmatrix}$$

Each of these Jacobians have constant rank, and therefore $\mathcal{S}, \mathcal{U}$ are smooth manifolds. The local coordinate charts about the origin are

$$\psi_S(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \psi_S^{-1}(z) = \begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix}$$
$$\psi_U(x) = x_1 \quad \psi_U^{-1}(z) = \begin{bmatrix} z \\ 0 \\ -\frac{z^2}{3} \end{bmatrix}$$

(c) Verify that the tangent spaces to the manifolds S and U at 0 is the stable and unstable subspaces, E^s and E^u, respectively.

The tangent space of the unstable manifold is the null space of its Jacobian evaluated at the origin,

$$T_0U = \text{null}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = \{x \in \mathbb{R}^3 \mid x = (0, 0, w), w \in \mathbb{R}\}$$

And the stable manifold is similarly, at the origin

$$T_0S = \text{null}\left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\right) = \{x \in \mathbb{R}^3 \mid x = (v, w, 0), v, w \in \mathbb{R}\}$$

Next, examine the stable and unstable subspaces of the linearization. The linearization of the system about the origin is

$$Df(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Indicating that the stable and unstable subspaces are

$$E^S = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$
$$E^U = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Note $T_0S = E^S$ and $T_0U = E^U$ as expected.

Problem 2 (Will Original)

Give an example of a differentiable function h whose zero level set does not define a manifold. Specifically, find a smooth map $h : \mathbb{R}^n \to \mathbb{R}^m$, for which $E = \{x \in \mathbb{R}^n \mid h(x) = 0\} = h^{-1}(0)$, such that $\text{rank}(h) = r$ at all but a finite number of points in E. Qualitatively, or intuitively, provide an explanation of what occurs at points where h loses or gains rank. For simplicity, you may fix n, m, and r (for instance $n = 2, m = 1, r = 1$). Qualitatively, or intuitively, how does this indicate that $h^{-1}(0)$ is not a manifold? (use an intuitive interpretation of a manifold, as a space which locally looks like $\mathbb{R}^n$)

$$h(x_1, x_2) = x_1 x_2$$

At the origin, the zero level set of this function is not locally homeomorphic to $\mathbb{R}^n$, and the structure is not a manifold.

Problem 3 continued on next page...

Problem 3 (Book Problem 6.1)

Prove the properties of class $\mathcal{K}$ functions presented in Section 6.2 and, specifically, *invertibility* and *composability*.

- Invertibility: Consider $\alpha \in \mathcal{K} : [0, a) \to \mathbb{R}$ for $a \geq 0$. Note $\alpha(0) = 0$, and $x > y \implies \alpha(x) > \alpha(y)$ (α is injective). Since α is continuous, it is surjective onto $[0, \alpha(a))$. As it is injective and surjective it is bijective, and α^{-1} is guaranteed to exist. Furthermore, $\alpha^{-1}(0) = 0$, and for $x > y \iff \alpha(x) > \alpha(y)$ implies that $\alpha^{-1}(x) > \alpha^{-1}(y) \iff x > y$, so α^{-1} is strictly increasing. Therefore, $\alpha^{-1} \in \mathcal{K}$
- Composability: Let $\alpha_1, \alpha_2 \in \mathcal{K}$ with domains $[0, \alpha)$. We have $\alpha_2(\alpha_1(0)) = \alpha_2(0) = 0$. For $x, y \in [0, \min\{a, \alpha_1^{-1}(a)\}$ we have

$$x < y \iff \alpha_1(x) < \alpha_1(y) \iff \alpha_2(\alpha_1(x)) < \alpha_2(\alpha_1(y))$$

  Therefore $\alpha_2 \circ \alpha_1 \in \mathcal{K} : [0, \min\{a, \alpha^{-1}(a)\}) \to \mathbb{R}_+$.

Problem 4 (Book Problem 6.5)

Consider the second order differential equation:

$$\ddot{\theta} + p(\theta) = 0$$

for $\theta \in \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$ continuously differentiable with $p(0) = 0$ and $\frac{\partial p}{\partial \theta}(\theta) \neq 0$. This can be converted to an ODE as follows:

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -p(x_1) \end{bmatrix}$$

with $x_1 = \theta$ and $x_2 = \dot{\theta}$. Utilize the corresponding energy of the system given by:

$$E(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} p_1(s)ds$$

to construct a Lyapunov function and give conditions on p for which the system is stable. What can you say about the asymptotic stability of the system?

In order for the energy function to be a Lyapunov function, it must be positive definite. A sufficient condition for ensuring this is $\frac{\partial p}{\partial \theta} > 0$. Under this condition, examine the time derivative along solutions of $V(x) = E(x)$:

$$\begin{aligned} V(\dot{x}) &= DV(x)(\dot{x}) \\ &= \begin{bmatrix} p(x_1) & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -p(x_1) \end{bmatrix} \\ &= 0 \end{aligned}$$

This indicates that the system is stable (in the sense of Lyapunov), but is not asymptotically stable - the system will evolve on Lyapunov level sets.

Problem 5 continued on next page...

Problem 5 (Book Problem 6.6)

Consider the differential equation:

$$\dot{x} = J(x)\nabla H(x)$$

where $J(x)$ is a skew-symmetric matrix for every $x \in \mathbb{R}^n$, and $\nabla H(x) = (DH(x))^\top$. This is a generalization of Hamilton's equations of motion since we can always take J to be:

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

to recover said equations. Using this interpretation we can regard H as the energy of this system. Show that solutions starting in a level-set of H remain in that level set for all future time, i.e, energy is preserved along solutions.
Consider now the differential equation

$$\dot{x} = J(x)\nabla H(x) + R(x)\nabla H(x)$$

where $R(x)$ is negative semi-definite. This can be interpreted as a generalization of Hamilton's equations with dissipation. Show that solutions starting in a sublevel set of H remain in that sublevel set forever, i.e, energy does not increase (and may decrease) along solutions.

Compute the time derivative of H along solution trajectories:

$$\begin{aligned}\frac{d}{dt}H(x(t)) &= DH(x)\dot{x} \\ &= DH(x)J(x)DH(x)^\top \\ &= 0\end{aligned}$$

Where the last equality holds by skew symmetry of $J(x)$. Therefore, we have $H(x(t)) \equiv H(x_0)$. If H is 'energy', as suggested by the prompt, energy is conserved along trajectories.
For the second part, again begin by computing the time derivative of H along trajectories

$$\begin{aligned}\frac{d}{dt}H(x(t)) &= DH(x)\dot{x} \\ &= DH(x)\left(J(x)DH(x)^\top + R(x)DH(x)^\top\right) \\ &= DH(x)R(x)DH(x)^\top \\ &\leq 0\end{aligned}$$

where the last inequality follows from the fact that $R(x) \preccurlyeq 0$ for all x. By Comparison Lemma, we have $H(x(t)) \leq H(x_0)$ for all time, and the desired statement is shown.