Lecture 7

Lyapunov's Method

The goal of this lecture is to introduce one of the fundamental tools in nonlinear analysis and control: Lyapunov's method. The idea behind this approach is that, rather than analyzing the full-order dynamics of the system, we find a "simpler" representation of the system—as encoded by a *Lyapunov function*—and show that stability properties of the full-order dynamics can be inferred from this simpler representation. This is the intuition behind Lyapunov's approach to establishing the stability of nonlinear systems' equilibria.

7.1 Establishing Stability with Lyapunov functions

We now introduce the main result of this lecture, and one of the foundational results in nonlinear dynamics and control: Lyapunov's method for establishing stability. For simplicity of exposition, and without loss of generality, we will assume that $x^* = 0$ (which can always be accomplished with a simple coordinate change: $x \mapsto x - x^*$).

Let $E \subseteq \mathbb{R}^n$ be an open and connected set containing the equilibrium point, $x^* = 0 \in E$, and let $\mathtt{V} : E \to \mathbb{R}$ be a continuously differentiable function. For a nonlinear system $\dot{x} = f(x)$, the derivative of $\mathtt{V}$ along its solutions is given by:

$$\dot{\mathtt{V}}(x) = \sum_{i=1}^{n} \frac{\partial \mathtt{V}}{\partial x_i}\Big|_x \dot{x}_i = \sum_{i=1}^{n} \frac{\partial \mathtt{V}}{\partial x_i}\Big|_x f_i(x) = \frac{\partial \mathtt{V}}{\partial x}\Big|_x f(x),$$

where, again, x denotes the solution and we therefore applied the chain rule. The following result, Lyapunov's theorem, rigorously captures the intuition given in Section 6.3.

Theorem 7.1 (Lyapunov). *Let $\dot{x} = f(x)$ where $f : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on the open and connected set $E \subseteq \mathbb{R}^n$. Consider the equilibrium point $x^* = 0 \in E$ and the continuously differentiable function $\mathtt{V} : E \to \mathbb{R}$ satisfying $\mathtt{V}(0) = 0$. If the following conditions are satisfied:*

- $\mathtt{V}(x) > 0$ *for all* $x \in E \backslash \{0\}$*;*
- $\dot{\mathtt{V}}(x) \leq 0$ *for all* $x \in E$*,*

then the equilibrium $x^* = 0$ is stable. If, in addition, the following condition is satisfied:

- $\dot{\mathtt{V}}(x) < 0$ *for all* $x \in E \backslash \{0\}$*,*

then the equilibrium $x^* = 0$ is asymptotically stable.

We will prove this theorem later in the lecture using class $\mathcal{K}$ functions, but first we introduce some additional terminology and the intuition behind Lyapunov's theorem.

Definition 7.1. A function $\mathtt{V} : E \subseteq \mathbb{R}^n \to \mathbb{R}$, $0 \in E$, satisfying the conditions of Theorem 7.1 is termed a Lyapunov function.

Intuition behind Lyapunov's theorem. Let $\mathtt{V} : E \subseteq \mathbb{R}^n \to \mathbb{R}$ be a Lyapunov function and consider the Lyapunov sublevel sets (as introduced in Section 6.3 of Lecture 6) defined for (potentially small) $c > 0$:

$$\Omega_c = \{x \in E \subseteq \mathbb{R}^n : \mathtt{V}(x) \le c\},$$
$$\partial\Omega_c = \{x \in E \subseteq \mathbb{R}^n : \mathtt{V}(x) = c\}.$$

Lyapunov's theorem can be understood in terms of the forward invariance properties of these sets. This is, in essence, the "classical" proof of stability via Lyapunov functions.

Forward Invariance: If $\dot{\mathtt{V}} \le 0$, then trajectories never leave Ω_c (but may be tangent to $\partial\Omega_c$), i.e., Ω_c is forward invariant. Since Ω_c is bounded (for small c), this implies that all solutions stay near the origin, i.e., it implies stability.

To make this more explicit (see Figure 7.1 for a graphical illustration), for a given $\varepsilon > 0$ we can consider a ball $B_\rho(0) \subset E$, with $\rho < \varepsilon$, and a $\beta > 0$ such that $\Omega_\beta \subset B_\rho(0)$ (this is possible by the continuity of $\mathtt{V}$ coupled with the fact that $\mathtt{V}(0) = 0$ and $\mathtt{V}(x) > 0$ for all $x \neq 0$). Moreover, given $x(0) \in \Omega_\beta$ the fact that $\dot{\mathtt{V}}(x) \le 0$ implies that $x(t) \in \Omega_\beta$ for all $t \ge 0$. This follows from the simple inequality:

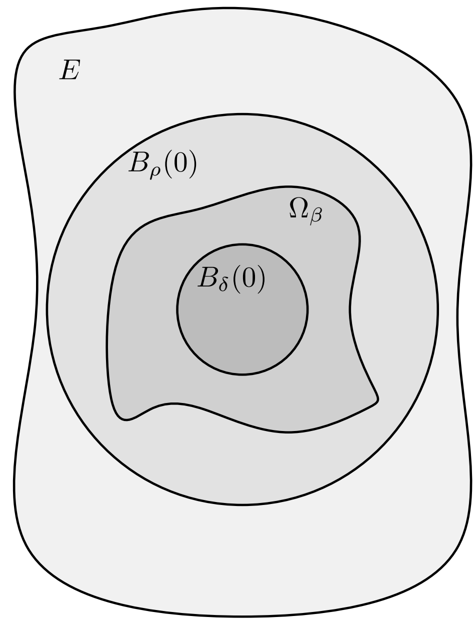

Figure 7.1. The sets considered in the proof of Lyapunov's theorem.

$$\mathtt{V}(x(t)) = \mathtt{V}(x(0)) + \int_0^t \underbrace{\dot{\mathtt{V}}(x(\tau))}_{\le 0} d\tau \le \mathtt{V}(x(0)) \le \beta. \tag{7.1}$$

In fact, we have proved the following important proposition:

Proposition 7.1. *Consider the ODE $\dot{x} = f(x)$ where $f : E \to \mathbb{R}^n$ is a locally Lipschitz function defined on an open and connected set $E \subset \mathbb{R}^n$ containing the equilibrium point $x^* = 0$. Given a Lyapunov function, $\mathtt{V}$, the Lyapunov sublevel set Ω_c is forward invariant.*

The importance of the forward invariance of Lyapunov sublevel sets can be intuitively understood both in the context of stability and asymptotic stability.

Stability: To establish results related to stability, we need to restrict our attention to open balls around the origin. In particular, given $\varepsilon > 0$ we can choose $\delta > 0$ such that $B_\delta(0) \subset \Omega_\beta$ wherein it follows from Proposition 7.1 that:

$$\begin{array}{lllllll} x(0) \in B_\delta(0) & \implies & x(0) \in \Omega_\beta & \implies & x(t) \in \Omega_\beta, \quad \forall t \ge 0 & \implies & x(t) \in B_\rho(0), \quad \forall t \ge 0, \\ \text{therefore}: & & \|x(0)\| < \delta & \implies & \|x(t)\| < \rho < \varepsilon, \quad \forall t \ge 0, & & \end{array}$$

and thus stability is established.

Asymptotic Stability: If $\dot{\mathtt{V}} < 0$, then the sets in the sequence:

$$\Omega_{c_1} \supset \Omega_{c_2} \supset \ldots \supset \Omega_{c_n} \supset \ldots, \qquad c_1 > c_2 > \ldots > c_n > \ldots,$$

are all forward invariant, and the solution moves from the set Ω_{c_i} at time t_i to the set $\Omega_{c_{i+1}}$ at some time $t_{i+1} > t_i$. Since $\mathtt{V}$ is only zero for $x = 0$, if $\lim_{n\to\infty} c_n = 0$ then:

$$\lim_{n\to\infty} \Omega_{c_n} = \{0\},$$

which implies that $\lim_{t\to\infty} \|x(t)\| = 0$, i.e, it implies asymptotic stability.

Positive and negative definite functions. Lyapunov functions are defined in terms of certain positivity and negativity (non-positivity) conditions. It is therefore useful to develop some language to precisely describe these properties.

Definition 7.2. A function $\mathtt{V} : E \subseteq \mathbb{R}^n \to \mathbb{R}$, $0 \in E$, is said to be:

Positive semi-definite if $\mathtt{V}(x) \geq 0$ for all $x \in E$ and $\mathtt{V}(0) = 0$.

Positive definite if $\mathtt{V}(x) \geq 0$ for all $x \in E$ and $\mathtt{V}(x) = 0$ if and only if $x = 0$.

Negative semi-definite if $\mathtt{V}(x) \leq 0$ for all $x \in E$ and $\mathtt{V}(0) = 0$.

Negative definite if $\mathtt{V}(x) \leq 0$ for all $x \in E$ and $\mathtt{V}(x) = 0$ if and only if $x = 0$.

Using the language introduced in Definition 7.2 we can restate Lyapunov's theorem as follows, with the same assumptions stated as in Theorem 7.1,

$$\begin{aligned} \mathtt{V} \textit{ positive definite and } \dot{\mathtt{V}} \textit{ negative semi-definite} \quad &\Longrightarrow \quad x^* = 0 \textit{ is stable;} \\ \mathtt{V} \textit{ positive definite and } \dot{\mathtt{V}} \textit{ negative definite} \quad &\Longrightarrow \quad x^* = 0 \textit{ is asymptotically stable.} \end{aligned}$$

Example 7.1. Let $\mathtt{V}(x) = x^T P x$ with $P^T = P \in \mathbb{R}^{n\times n}$. The following are equivalent:

- $\mathtt{V}$ is positive definite;
- All of the eigenvalues of P are real and positive;
- $P = B^T B$, with B invertible.

In this case, we call P a (symmetric) positive definite matrix and denote it by $P = P^T > 0$.

When $\mathtt{V}$ is only positive semi-definite, the eigenvalues of P are real and non-negative. Although it is still the case that $P = B^T B$, B is no longer invertible since at least one of its eigenvalues is zero. Note also that these characterizations extend to the negative (semi-)definite case since $\mathtt{V}$ is negative (semi-)definite if and only if $-\mathtt{V}$ is positive (semi-)definite.

Example 7.2. We return to the system introduced in Section 6.3 modeling a pendulum with damping. The Lyapunov function $\mathtt{V}$ given in (6.8) applied to $\dot{x} = f(x)$ given in (6.7) satisfied the following properties (derived in Example 6.9) as determined by γ:

- $\gamma = 0 \implies \dot{\mathtt{V}} = 0$ and stability follows by Lyapunov's theorem;
- $\gamma > 0 \implies \dot{\mathtt{V}} = -\gamma x_2^2$ from which we conclude that $\dot{\mathtt{V}} \leq 0$, i.e., negative semi-definiteness of $\dot{\mathtt{V}}$. We can, however, only conclude stability from Lyapunov's theorem.

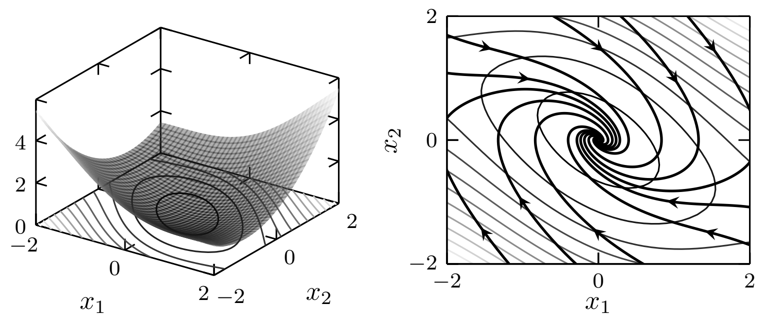

Figure 7.2. (Left) Level sets of the Lyapunov function given in Example 7.2 used to establish asymptotic stability for the pendulum, along with the behavior of the system relative to level sets of the Lyapunov function.

When a given Lyapunov function only yields a specific stability result, *it does not* imply that it is the strongest stability result that can be proved or that a system displays. To prove stronger stability properties it may be necessary to use a different Lyapunov function. To illustrate this point we draw inspiration in the constructions in Lecture 1 and Lecture 6, and specifically (6.8), and consider the different Lyapunov function candidate for the pendulum:

$$\mathtt{V}(x) \triangleq \frac{1}{2}x_2^2 - \cos(x_1) + 1 + \frac{1}{2}\gamma x_1 x_2, \tag{7.2}$$

which is positive definite on the domain $E = (-\pi, \pi) \times \mathbb{R}$ for $0 < \gamma < 2$. Computing the derivative of this function along solutions yields:

$$\dot{\mathtt{V}}(x) = -\frac{1}{2}\gamma\left(x_2^2 + \gamma x_1 x_2 + x_1 \sin(x_1)\right), \tag{7.3}$$

which is negative definite (this is easily checked locally). Therefore, $\mathtt{V}$ is a Lyapunov function establishing asymptotic stability (see Figure 7.2).

7.2 Lyapunov's Theorem: Modern Statement

In this section we prove Lyapunov's theorem using the modern approach—leveraging the power of the Comparison Lemma and the interplay between class $\mathcal{K}$ and $\mathcal{KL}$ functions. In essence, a Lyapunov function results in a 1-dimensional dynamical system:

$$\dot{\mathtt{V}} \leq -\alpha(\mathtt{V}),$$

with $\alpha \in \mathcal{K}$ and, by Lemmas 3.1 and 6.1, this implies that $\mathtt{V}$ evolves according to a class $\mathcal{KL}$ function, i.e., $\mathtt{V}(t) \leq \beta(\mathtt{V}(0), t)$. This, in turn, will imply that solutions of the original dynamical system will be bounded by functions of this form, hence establishing stability or asymptotic stability. This intuition is captured by the following theorem which is a different, yet equivalent, formulation of Lyapunov's Theorem 7.1; see Proposition 6.1 for the corresponding "modern" definition of stability.

Theorem 7.2 (Modern statement of Lyapunov's Theorem 7.1)**.** *Let $\dot{x} = f(x)$ where $f : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on the open and connected set $E \subseteq \mathbb{R}^n$. Consider the equilibrium point $x^* = 0 \in E$ and the continuously differentiable function $\mathtt{V} : E \to \mathbb{R}$. If the following conditions are satisfied:*

$$\begin{aligned} \alpha_1(\|x\|) \leq \quad \mathtt{V}(x) &\leq \alpha_2(\|x\|) && (7.4)\\ \dot{\mathtt{V}}(x) &\leq -\alpha_3(\|x\|), && (7.5)\end{aligned}$$

for $x \in B_\rho(0) \subset E$, $\rho > 0$, and α_i class $\mathcal{K}$ functions for $i = 1, 2, 3$ on $[0, \rho)$, then $x^* = 0$ is asymptotically stable. Moreover,

$$\|x(t)\| \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(t_0)\|), t - t_0)), \qquad \forall\ t \geq t_0, \tag{7.6}$$

where $\beta \in \mathcal{KL}$ is the solution to the IVP:

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \qquad y(t_0) = \mathtt{V}(x(t_0)). \tag{7.7}$$

Relating Classical and Modern Lyapunov's Method. Before proceeding to the proof of Theorem 7.2, it is important to discuss the equivalence with Theorem 7.1. In Theorem 7.1, $\mathtt{V}$ is required to be positive definite whereas in Theorem 7.2, $\mathtt{V}$ is assumed to satisfy the inequalities (7.4). These are, in fact, equivalent statements although the inequalities (7.4) will considerably simplify the proof Lyapunov's theorem while having the added benefit of also leading to the time-varying bound (7.6) on how far the solution is from the equilibrium $x^* = 0$. More formally, we have the following relationships:

Theorem 7.2 $\implies$ Theorem 7.1: Note that $\alpha_1(\|x\|) \leq \mathtt{V}(x)$ immediately implies that $\mathtt{V}(x) \geq 0$ since $\alpha_1 \in \mathcal{K}$ is positive definite. Moreover, for $x = 0$ we have $\alpha_1(\|0\|) = 0 = \alpha_2(\|0\|)$ and thus $\mathtt{V}(x) = 0$ for $x = 0$. Hence, $\mathtt{V}$ is positive semi-definite. To show that $\mathtt{V}$ is, in fact, positive definite note that $\mathtt{V}(x) = 0$ implies $\alpha_1(\|x\|) \leq 0$ and this can only occur for $x = 0$. Similar arguments show that $\dot{\mathtt{V}}$ is negative definite (when coupled with the fact that $f(0) = 0$).

Theorem 7.1 $\implies$ Theorem 7.2: If $\mathtt{V}$ is positive definite on $B_\rho(0)$, then consider the function:

$$\hat{\alpha}_1(r) \triangleq \min_{r \leq \|y\| \leq \rho} \mathtt{V}(y).$$

This function is clearly itself positive definite on $[0, \rho)$, increasing (not necessarily strictly so) and satisfies: $\hat{\alpha}(\|x\|) \leq \mathtt{V}(x)$. Moreover, every increasing positive definite function on $[0, \rho)$ is bounded below by a class $\mathcal{K}$ function, $\alpha_1 \in \mathcal{K}$, on $[0, \rho)$ (see Additional Reading): $\alpha_1(r) \leq \hat{\alpha}_1(r)$. Therefore: $\alpha_1(\|x\|) \leq \mathtt{V}(x)$. Conversely, consider:

$$\hat{\alpha}_2(r) \triangleq \max_{\|y\| \leq r} \mathtt{V}(y),$$

wherein $\mathtt{V}(x) \leq \hat{\alpha}_2(\|x\|)$. Thus function, like $\hat{\alpha}_1$, is increasing and positive definite on $[0, \rho)$ and every such positive definite function is also upper bounded by a class $\mathcal{K}$ function, $\hat{\alpha}_2(r) \leq \alpha_2(r)$, on $[0, \rho)$ (again, see Additional Reading). Therefore: $\mathtt{V}(x) \leq \alpha_2(\|x\|)$. To summarize, the above observations have established the following:

Lemma 7.1. *Let $\mathtt{V} : E \subset \mathbb{R}^n \to \mathbb{R}$ be continuous, then $\mathtt{V}$ is positive definite on $B_\rho(0) \subseteq E$ if and only if there exists class $\mathcal{K}$ functions α_1 and α_2 defined on $[0, \rho)$ satisfying:*

$$\alpha_1(\|x\|) \leq \mathtt{V}(x) \leq \alpha_2(\|x\|). \tag{7.8}$$

This implies that condition $\dot{\mathtt{V}} < 0$ in Theorem 7.1 can be replaced with (7.5), and the desired equivalence between Theorem 7.1 and Theorem 7.2 has been established.

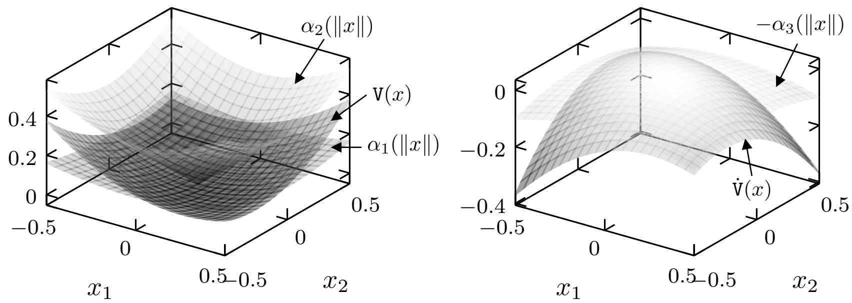

Figure 7.3. Illustration of Lemma 7.1 using the bounding class $\mathcal{K}$ functions found in Example 7.3. Specifically, the bounds for $\mathtt{V}$ and $\dot{\mathtt{V}}$ given in (7.9).

Example 7.3. Returning to Example 7.2, note that we can utilize the following bounds (inspired by the Taylor series of each of the respective functions):

$$\begin{array}{rcccl}\left(\frac{1}{2}-\frac{1}{24}\right)x_1^2 & \leq & 1-\cos(x_1) & \leq & \frac{1}{2}x_1^2 \\ \left(1-\frac{1}{6}\right)x_1^2 & \leq & x_1\sin(x_1) & \leq & x_1^2,\end{array}$$

valid for $|x_1| < 1$, to obtain the bounds for $\mathtt{V}$ in (7.2) and $\dot{\mathtt{V}}$ in (7.3):

$$\begin{array}{rcccl}\frac{1}{2}\left(x_2^2+\left(1-\frac{1}{12}\right)x_1^2+\gamma x_1x_2\right) & \leq & \mathtt{V}(x) & \leq & \frac{1}{2}\left(x_2^2+x_1^2+\gamma x_1x_2\right) \\ -\frac{1}{2}\gamma\left(x_2^2+\gamma x_1x_2+x_1^2\right) & \leq & \dot{\mathtt{V}}(x) & \leq & -\frac{1}{2}\gamma\left(x_2^2+\gamma x_1x_2+\left(1-\frac{1}{6}\right)x_1^2\right),\end{array}$$

valid for $\|x\| < 1$. These inequalities can be recast in matrix form:

$$\begin{array}{rcccl}\frac{1}{2}x^T\begin{bmatrix}\left(1-\frac{1}{12}\right) & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 1\end{bmatrix}x & \leq & \mathtt{V}(x) & \leq & \frac{1}{2}x^T\begin{bmatrix}1 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 1\end{bmatrix}x \\ & & \dot{\mathtt{V}}(x) & \leq & -\frac{1}{2}\gamma x^T\begin{bmatrix}\left(1-\frac{1}{6}\right) & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 1\end{bmatrix}x.\end{array}$$

Using the fact that for symmetric positive definite matrices, $P = P^T > 0$, there are the bounds: $\lambda_{\min}(P)\|x\| \leq x^TPx \leq \lambda_{\max}\|x\|^2$ (see Example 8.1), for $\|x\| < 1$ and $0 < \gamma < 2$ we obtained the desired bounds on $\mathtt{V}$ and $\dot{\mathtt{V}}$ by class $\mathcal{K}$ functions with domain $[0, 1)$:

$$\begin{array}{rcccl}\overbrace{\frac{1}{48}\left(23-\sqrt{144\gamma^2+1}\right)\|x\|^2}^{\alpha_1(\|x\|)} & \leq & \mathtt{V}(x) & \leq & \overbrace{\frac{2+\gamma}{4}}^{\alpha_2(\|x\|)}\|x\| \\ & & \dot{\mathtt{V}}(x) & \leq & -\underbrace{\gamma\frac{1}{24}\left(11-\sqrt{36\gamma^2+1}\right)\|x\|^2}_{\alpha_3(\|x\|)}.\end{array} \tag{7.9}$$

Therefore, Theorem 7.2 allows us to conclude that the pendulum (with damping) is asymptotically stable. As will be seen in Lecture 8, because the class $\mathcal{K}$ functions are of the form $\alpha_i(r) = k_i\|x\|^2$, we can actually conclude exponential stability.

Proof of Theorem 7.2. The main observation is that, by combining (7.4) and (7.5), we have:

$$\mathtt{V}(x) \leq \alpha_2(\|x\|) \quad\Longrightarrow\quad \alpha_2^{-1}(\mathtt{V}(x)) \leq \|x\| \quad\Longrightarrow\quad -\alpha_3(\alpha_2^{-1}(\mathtt{V}(x))) \geq -\alpha_3(\|x\|)$$

and therefore:

$$\dot{\mathtt{V}} \leq -\alpha_3 \circ \alpha_2^{-1}(\mathtt{V}), \tag{7.10}$$

where $\alpha_3 \circ \alpha_2^{-1} \in \mathcal{K}$. In order to prepare for using the Comparison Lemma we consider the IVP:

$$\begin{aligned} \dot{y} &= -\alpha_3(\alpha_2^{-1}(y)), \qquad y(t_0) = \mathtt{V}(x(t_0)), & (7.11) \\ \implies \qquad y(t) &= \beta(y(t_0), t - t_0) = \beta(\mathtt{V}(x(t_0)), t - t_0), & (7.12) \end{aligned}$$

with $\beta \in \mathcal{KL}$ according to[1] Lemma 6.1. Therefore, by the Comparison Lemma, if follows that:

$$\mathtt{V}(x(t)) \leq \beta(\mathtt{V}(x(t_0)), t - t_0), \qquad \forall t \geq t_0. \tag{7.13}$$

Combining (7.13) with (7.4) leads to:

$$\begin{aligned} \|x(t)\| &\leq \alpha_1^{-1} \circ \mathtt{V}(x(t)) \\ &\leq \alpha_1^{-1} \circ \beta(\mathtt{V}(x(t_0)), t - t_0), \qquad \forall t \geq t_0 \\ &\leq \alpha_1^{-1} \circ \beta(\alpha_2(\|x(t_0)\|), t - t_0), \qquad \forall t \geq t_0, \end{aligned} \tag{7.14}$$

from which we conclude asymptotic stability (from Proposition 6.1) since $\alpha_1^{-1} \circ \beta(\alpha_2(r), s) \in \mathcal{KL}$. Moreover, this is inequality (7.6) appearing in Theorem 7.2. □

Remark 7.1. Note that the methods used to prove Theorem 7.2 could be used to establish stability in the case when (7.5) is replaced by:

$$\dot{\mathtt{V}}(x) \leq 0. \tag{7.15}$$

Indeed, in this case the Comparison Lemma with $t_0 = 0$ yields:

$$\mathtt{V}(x(t)) \leq \mathtt{V}(x(0)),$$

and stability follows from (7.4):

$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|)), \tag{7.16}$$

since $\alpha_1^{-1} \circ \alpha_2 \in \mathcal{K}$. Thus we have stability according to Proposition 6.1.

Remark 7.2. The proof of Theorem 7.2 was based on the "modern" definition of stability, i.e., Proposition 6.1. However, we could have also used the "classic" definition of stability, i.e., Definition 6.1. This would make the proof more complex, which points to the advantage of using class $\mathcal{K}$ functions. This follows from first having to establish stability of the solutions and only afterwards proving asymptotic stability—the classical definition requires both stability and the limiting behavior to be established, while this is combined in the modern definition of stability. For completeness, we illustrate these steps.

Given $\varepsilon > 0$, analogously to the intuition behind Theorem 7.1, choose $\rho < \varepsilon$ such that $B_\rho(0) \subset E$ and $\delta > 0$ satisfying $\delta \leq \alpha_2^{-1}(\alpha_1(\rho))$. Therefore, using the constructions in Remark 7.1, we have that:

$$\|x(0)\| < \delta \quad \implies \quad \|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|)) < \alpha_1^{-1}(\alpha_2(\delta)) \leq \rho < \varepsilon,$$

establishing classic stability according to Definition 6.1. Additionally, by (7.14):

$$\lim_{t\to\infty} \|x(t)\| = \lim_{t\to\infty} \alpha_1^{-1}(\beta(\alpha_2(\|x(0)\|), t)) = 0,$$

establishing asymptotic stability according to the classic definition.

[1] For the purposes of this proof, we need not extend $\alpha_3 \circ \alpha_2^{-1}$ to the interval $[-\varepsilon, a)$. That is, as discussed in Remark 6.3, we can consider the open interval $(0, a)$ with initial conditions $y(t_0) = \mathtt{V}(x(t_0)) \in (0, a)$. This follows from the fact that $y(t_0) = \mathtt{V}(x(t_0)) = 0$ trivially implies that $V(x(t)) \equiv 0$ since $\mathtt{V}$ is a positive definite function and $x^* = 0$ is an equilibrium point.

7.3 Global and converse results

So far, we only considered local stability results. Global stability is defined as in Definition 6.1 except one now takes $E = \mathbb{R}^n$ (in the case of global asymptotic stability, the limiting behavior must hold for all initial conditions, independent of δ). Because $E = \mathbb{R}^n$ we have to ensure boundedness of the sublevel sets of all the relevant class $\mathcal{K}$ functions (this was not needed before since sublevel sets of class $\mathcal{K}$ functions are always locally bounded). We illustrate this conceptually through a simple example.

Example 7.4. Let us restrict our attention to $x \in \mathbb{R}$ and consider the function:

$$\mathtt{V}(x) = e^{-x}x^2.$$

This function is positive-definite and a Lyapunov function for the equilibrium $x^* = 0$ of:

$$\dot{x} = -D\mathtt{V}(x) = e^{-x}(x-2)x, \tag{7.17}$$

since:

$$\dot{\mathtt{V}} = D\mathtt{V}(x)\dot{x} = -\left(D\mathtt{V}(x)\right)^2 \leq 0.$$

Although we conclude from Theorem 7.2 and Remark 7.1 that $x^* = 0$ is a stable equilibrium, we cannot conclude it is globally stable even though $\mathtt{V}$ is positive-definite on $E = \mathbb{R}$ and $\dot{\mathtt{V}}$ is negative semi-definite on $E = \mathbb{R}$. This can be appreciated in Figure 7.4 by considering any initial condition $x(0) > 2$. Since $-D\mathtt{V}(x) > 0$ for any $x > 2$, solutions starting to the right of 2 will not be bounded and will keep growing.

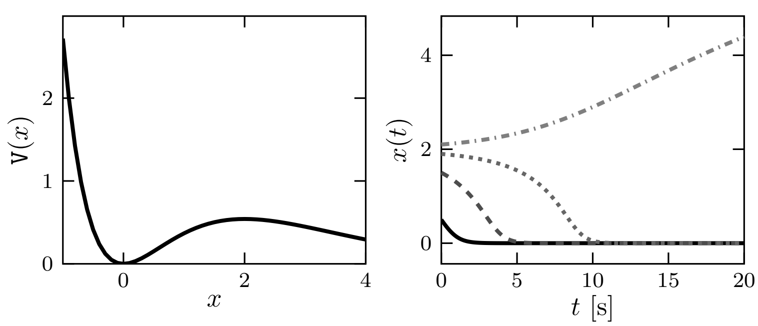

Figure 7.4. Left: Lyapunov function $\mathtt{V}$. Right: different solutions of (7.17). We can see that solutions starting below 2 converge and above 2 diverge.

Given the previous discussion, we must strengthen the class $\mathcal{K}$ functions in the statement of Theorem 7.2 to class $\mathcal{K}_\infty$ in order to obtain its global version whose proof can be obtained *mutatis mutandis.*

Theorem 7.3 (Global version of Lyapunov's Theorem 7.2)**.** *Let $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz continuous function. Consider the equilibrium point $x^* = 0 \in \mathbb{R}^n$ and the continuously differentiable function $\mathtt{V} : \mathbb{R}^n \to \mathbb{R}$. If the inequalities* (7.4) *and* (7.5) *are satisfied with class $\mathcal{K}_\infty$ functions, i.e., $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$, then the solution β to* (7.7) *is a class $\mathcal{KL}_\infty$ function and for all $x(t_0) \in \mathbb{R}^n$:*

$$\|x(t)\| \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(t_0)\|), t - t_0)), \qquad \forall\ t \geq t_0, \tag{7.18}$$

thus the system is globally asymptotically stable. If (7.5) *is relaxed to $\dot{\mathtt{V}} \leq 0$ then the system is globally stable.*

Given the emphasis placed, in this lecture, on proving stability using Lyapunov functions, it would be a disappointment if there exist stable or asymptotically stable equilibria whose stability properties cannot be proved via Lyapunov functions. This raises the question of a converse to Theorem 7.2, i.e., a result guaranteeing the existence of Lyapunov functions. The classical way of showing existence of Lyapunov functions consists of constructing one such function by (for $t_0 = 0$):

$$\mathtt{V}(x_0) = \int_0^{+\infty} \gamma\left(\|x(\tau)\|\right) d\tau, \tag{7.19}$$

where $x(t)$ is the solution of the IVP $\dot{x} = f(x)$, $x(0) = x_0$ having $x^* = 0$ as a globally asymptotically stable equilibrium point with f being locally Lipschitz continuous. The function γ is suitably chosen to satisfy (among other) the following properties:

- γ is a class $\mathcal{K}$ function (hence positive definite);
- the integral (7.19) converges.

We can now see that $\mathtt{V}$ is positive definite since it is the integral of a positive definite function. Moreover:

$$\dot{\mathtt{V}}(x(t)) = \frac{d}{dt}\int_t^{+\infty} \gamma\left(\|x(\tau)\|\right) d\tau = \gamma\left(\|x(\tau)\|\right)\Big|_t^{+\infty} = 0 - \gamma\left(\|x(t)\|\right),$$

thereby showing that $\dot{\mathtt{V}}$ is negative definite. Hence, we conclude that asymptotic stability of an equilibrium can always be proved by constructing a Lyapunov function. A similar results holds for stable equilibria. It should be noted, however, that building Lyapunov functions is an art since there is no systematic procedure for constructing them, e.g., the Lyapunov function in (7.19) cannot be calculated, in general, as in nonlinear systems we don't have explicit access to $x(t)$. We will see in Lecture 8 that one can (perhaps very locally) construction Lyapunov functions from the linearized system. Additionally, in Lecture 9, converse Lyapunov results will be explored in detail based on a different approach from the one outlined above.

Additional Reading

Lyapunov's theorem dates back to the 1892 where it was first introduced by Aleksandr Lyapunov [131] (this paper was reprinted in [132]). A more formal treatment of the proof of Theorem 7.1, as outlined in the intuition behind Lyapunov's theorem, can be found in [110]. There are many other good treatments of Lyapunov's theorem, e.g., in [182, 38]. A proof of the inequality 7.8, i.e., of the fact that positive definite functions are bounded above and below by class $\mathcal{K}$ functions can be found in [88] (see also [110], cf. Lemma 4.3). Note that the fact that a positive definite function on $[0, \rho)$ can be upper and lower bounded by class $\mathcal{K}$ functions is further discussed in [108] (Lemma 1) and proven in [**?**] (see Lemma 2.5). The converse results presented in Section 7.3 have their origins in [138]. The interested reader is referred to [109] for a survey of existing converse results.

Problems for Lecture 7

[P7.1] Prove that the conditions given in Example 7.1 are equivalent.

[P7.2] Establish that if $\mathtt{V}(x) = x^T P x$ is positive semi-definite, then $P = B^T B$ but that B is no longer invertible (as was the case when $\mathtt{V}$ is positive definite).

[P7.3] In the proof of Theorem of Theorem 7.2, the class $\mathcal{K}$ functions α_i, $i = 1, 2, 3$ were considered with domain $[0, \rho)$. These domains, and the corresponding domains associated with the composite functions were not explicitly discussed in the proof for simplicity of exposition. Specify the domain and range of all class $\mathcal{K}$ and class $\mathcal{KL}$ functions used in the proof. Justify why all of the corresponding arguments take values in these respective domains for an appropriately chosen ρ.

[P7.4] Consider a function $\psi : [0, \rho) \to [0, \infty)$ that is continuous, positive definite, and increasing (not necessarily strictly increasing). Show that there exists class $\mathcal{K}$ functions $\alpha_i : [0, \rho) \to [0, \infty)$, $i = 1, 2$, such that:

$$\alpha_1(r) \leq \psi(r) \leq \alpha_2(r),$$

for all $r \in [0, \rho)$.

[P7.5] [**Advanced Problem**] Let $\alpha : [0, \rho) \to [0, \infty)$ be a class $\mathcal{K}$ function. Then there exists class $\mathcal{K}$ functions $\alpha_1, \alpha_2 : [0, \rho) \to [0, \infty)$ that are smooth (continuously differentiable) satisfying:

$$\alpha(r) - \varepsilon \leq \alpha_1(r) \leq \alpha(r) \leq \alpha_2(r) \leq \alpha(r) + \varepsilon,$$

for any $\varepsilon > 0$.

[P7.6] ([33]) Consider a nonlinear system $\dot{x} = f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^n$, and assume the existence of a function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying $\alpha_1(\|x\|) \leq W(x) \leq \alpha_3(\|x\|)$ and $\dot{W} \leq \lambda W$ with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\lambda \in \mathbb{R}_{\geq 0}$. By drawing inspiration from the proof of Lyapunov's theorem, show that the solutions of $\dot{x} = f(x)$ exist for all $t \geq 0$. In fact, the converse of this results holds: if solutions exist for all time then a function W with the above properties is guaranteed to exist.

[P7.7] Continuing Problem P2.5 and Problem P4.9, recall that we considered two dynamical systems:

$$\underbrace{\dot{x} = f(x)}_{\text{Nominal System}} \qquad \leftrightarrow \qquad \underbrace{\dot{y} = \overbrace{\frac{f(y)}{1 + \|f(y)\|}}^{\triangleq g(y)}}_{\text{Transformed System}}, \tag{7.20}$$

for $x, y \in \mathbb{R}^n$ and $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable. Show that stability is preserved under this transformation. That is, $V : E \subseteq \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for the Nominal system ($\dot{V} \leq 0$) if and only if it is a Lyapunov function for the Transformed system. Is this also true for asymptotic stability?

[P7.8] The equations of motion for mechanical systems can be written in the following Hamiltonian form:

$$\dot{x} = J(x)\nabla H(x),$$

where $J(x)$ is a skew-symmetric matrix, i.e., $J(x) = -J^T(x)$, $H : \mathbb{R}^n \to \mathbb{R}$ is the Hamiltonian, and ∇H is $(DH(x))^T$. Show that H is constant along trajectories, i.e., $H(x(t)) = c$ for some $c \in \mathbb{R}$ that may depend on the solution $x(t)$, and all t for which the solution exists. Which assumptions would you place on the Hamiltonian so that it can be used as a Lyapunov function to establish stability?

[P7.9] Consider the dynamical system with $x \in \mathbb{R}^2$ given by:

$$\dot{x} = f(x) = \begin{bmatrix} -x_1^3 + x_2 \\ -ax_1 - bx_2 \end{bmatrix},$$

for $a, b \in \mathbb{R}_{>0}$. Show that the origin is globally asymptotically stable.

[P7.10] Consider the discrete-time dynamical system (see Lecture 2):

$$x_{k+1} = F(x_k), \qquad k \in \mathbb{Z}_{\geq 0}, \tag{7.21}$$

for $F : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable. Furthermore, assume that $x^* = 0$ is an equilibrium point (fixed point), i.e., $F(0) = 0$.

For a function $\mathtt{V} : \mathbb{R}^n \to \mathbb{R}$, the rate of change along solutions of the discrete-time dynamical system is given by:

$$\nabla \mathtt{V}(x) \triangleq \mathtt{V}(F(x)) - \mathtt{V}(x).$$

In the context of discrete-time dynamical systems:

(a) Restate the definitions of stability and asymptotic stability.

(b) Show that the origin of (7.21) is *stable* if $\mathtt{V}$ is a positive definite function and $\nabla \mathtt{V}(x)$ is negative semidefinite.

(c) Show that the origin of (7.21) is *asymptotically stable* if $\mathtt{V}$ is a positive definite function and $\nabla \mathtt{V}(x)$ is negative definite.