# Lecture 8

# Exponential Stability

This lecture begins by establishing conditions that ensure local exponential stability of nonlinear systems, wherein the proof can be viewed as a special case of Theorem 7.2. This result on exponential stability is then applied to linear systems to obtain necessary and sufficient conditions on stability. This will allow us to synthesize Lyapunov functions for the linearization of a nonlinear systems. The importance of this result is that it allows for a systematic way to generate Lyapunov functions for nonlinear systems that are valid locally and, importantly, it gives a constructive method for estimating the domain of attraction. This will be demonstrated by returning to our ongoing example of a pendulum, for which the constructions introduced in Lecture 1 will be revisited with the formal framework established in this lecture.

## 8.1 Exponential Stability

We begin by turning our attention to establishing exponential stability. Conceptually, we can understand the conditions for exponential stability through Theorem 7.2 in Lecture 7 coupled with stronger assumptions on the class $\mathcal{K}$ functions $\alpha_i$. That is, we need to ensure fast enough convergence, so the class $\mathcal{K}$ functions must be "sufficiently strictly increasing". In particular, we will require that these functions have the "canonical" form of a class $\mathcal{K}$ function, $\alpha_i(r) = k_i r^c$, with $c, k_i \in \mathbb{R}_{>0}$ (as motivated by Example 6.8).

**Establishing Exponential Stability.** We begin by recalling that an equilibrium point $x^* = 0$ of $\dot{x} = f(x)$ is exponentially stable if there exist constants $a, \lambda, M \in \mathbb{R}_{>0}$ such that::

$$\|x(t_0)\| < a \qquad \Longrightarrow \qquad \|x(t)\| \leq M e^{-\lambda(t-t_0)} \|x(t_0)\|, \qquad \forall t \geq t_0,$$

and we now present the main Lyapunov theorem for exponential stability.

**Theorem 8.1.** *Let $\dot{x} = f(x)$ where $f : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on the open and connected set $E \subseteq \mathbb{R}^n$. Consider the equilibrium point $x^* = 0 \in E$ and the continuously differentiable function $\mathtt{V} : E \to \mathbb{R}$. If the following conditions are satisfied:*

$$k_1 \|x\|^c \leq \mathtt{V}(x) \leq k_2 \|x\|^c \tag{8.1}$$

$$\dot{\mathtt{V}}(x) \leq -k_3 \|x\|^c, \tag{8.2}$$

*for $k_1, k_2, k_3, c \in \mathbb{R}_{>0}$, then $x^* = 0$ is exponentially stable. Moreover,*

$$\|x(t)\| \leq M e^{-\lambda(t-t_0)} \|x(t_0)\|, \quad \text{for} \quad M = \left(\frac{k_2}{k_1}\right)^{\frac{1}{c}}, \quad \lambda = \frac{k_3}{c k_2}, \qquad \forall t \geq t_0. \tag{8.3}$$

The proof of this theorem will follow from the same logic as the proof of Theorem 7.2 and can, in essence, be viewed as a special case of the constructions in that proof.

*Proof.* Utilizing $\alpha_i(r) = k_i r^c$ with inverse $\alpha_i^{-1}(r) = k_i^{-\frac{1}{c}} r^{\frac{1}{c}}$, Equation (7.10) becomes:

$$\dot{\mathtt{V}} \leq -\alpha_3(\alpha_2^{-1}(\mathtt{V})) = -\frac{k_3}{k_2}\mathtt{V},$$

which could also be calculated directly from (8.1) and (8.2). Application of the Comparison Lemma in this case, as in Equation (7.11), wherein:

$$\dot{y} = -\frac{k_3}{k_2} y,$$

yields (by Example 6.8) the following upper bound:

$$\mathtt{V}(x(t)) \leq \beta(\mathtt{V}(x(t_0)), t - t_0) = e^{-\frac{k_3}{k_2}(t-t_0)}\, \mathtt{V}(x(t_0)), \qquad \forall t \geq t_0.$$

Therefore, the application of the upper bound in Equation (7.6) yields:

$$\begin{aligned}
\|x(t)\| &\leq \alpha_1^{-1}(\beta(\alpha_2(\|x(t_0)\|), t - t_0)) \\
&= \alpha_1^{-1}\left(e^{-\frac{k_3}{k_2}(t-t_0)} k_2 \|x(t_0)\|^c\right) \\
&= k_1^{-\frac{1}{c}} e^{-\frac{k_3}{ck_2}(t-t_0)} k_2^{\frac{1}{c}} \|x(t_0)\| \\
&= \underbrace{\left(\frac{k_2}{k_1}\right)^{\frac{1}{c}}}_{M} e^{-\overbrace{\frac{k_3}{ck_2}}^{\lambda}(t-t_0)} \|x(t_0)\|.
\end{aligned}$$

□

***Remark* 8.1.** The global version of Theorem 8.1 follows by taking $E = \mathbb{R}^n$ since $\alpha_i \in \mathcal{K}_\infty$, with $\alpha_i(r) = k_i r^c$ as utilized in the proof of the theorem.

***Remark* 8.2.** The essential component on the proof of Theorem 8.1 is the inequality:

$$\dot{\mathtt{V}} \leq -\lambda \mathtt{V}(x),$$

for $\lambda > 0$, i.e., $\lambda = \frac{k_3}{k_2}$. The result is an exponentially stable linear system from which exponential stability of the noninear system follows from the comparison lemma. This simple observation: Lyapunov functions simply map to the "simplest" system that displays the desired properties (in this case exponential stability) is the core philosophy behind Lyapunov's method. It also indicates ways of extending and expanding Lyapunov theory. For example, if we instead consider the Lyapunov inequality:

$$\dot{\mathtt{V}} \leq -\lambda \mathtt{V}(x)^c,$$

the result is exponential stability for $c = 1$, rational stability for $c > 1$, and finite-time stability for $0 < c < 1$. Further generalizations of this idea will be seen in Lecture 12 in the context of barrier and safety functions.

**Example 8.1.** For $\mathtt{V}(x) = x^T P x$ with $P = P^T > 0$ a (symmetric and positive definite) matrix in $\mathbb{R}^{n\times n}$, we have that:

$$\lambda_{\min}(P)\|x\|^2 \leq \mathtt{V}(x) \leq \lambda_{\max}(P)\|x\|^2,$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the smallest and largest eigenvalues of $P$. Note that we can speak of smallest and largest since the eigenvalues of symmetric matrices are purely real and thus linearly ordered (according to Example 7.1). Hence, using the notation of Theorem 8.1, we have $c = 2$, $k_1 = \lambda_{\min}(P)$, and $k_2 = \lambda_{\max}(P)$.

## 8.2 Lyapunov's Method Applied to Linear Systems

Consider a linear system:

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}.$$

We will be interested in studying the Lyapunov function candidate:

$$\mathtt{V}(x) = x^T P x, \qquad \underbrace{P = P^T > 0}_{\text{Symmetric Positive Definite}}, \qquad P \in \mathbb{R}^{n\times n}. \tag{8.4}$$

Taking the derivative of $\mathtt{V}$ along solutions yields:

$$\begin{aligned}\dot{\mathtt{V}}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA) x.\end{aligned}$$

Therefore, if the matrix $A^T P + PA$ is negative definite, it implies that $\dot{\mathtt{V}}$ is negative definite. This motivates the following definition:

**Definition 8.1.** Given a positive definite matrix $Q = Q^T > 0$, a symmetric positive definite matrix $P = P^T > 0$ is a <u>Lyapunov matrix</u> if it satisfies the <u>Continuous-Time Lyapunov Equation</u>:

$$A^T P + PA = -Q. \tag{CTLE}$$

Therefore, given a $P$ satisfying the (CTLE), the time derivative of the Lyapunov function in Equation (8.4) is given by:

$$\dot{\mathtt{V}}(x) = -x^T Q x.$$

Utilizing the bounds established in Example 8.1 we have:

$$\lambda_{\min}(P)\|x\|^2 \leq \quad \mathtt{V}(x) \quad \leq \lambda_{\max}(P)\|x\|^2 \tag{8.5}$$

$$\dot{\mathtt{V}}(x) \quad \leq -\lambda_{\min}(Q)\|x\|^2. \tag{8.6}$$

We can thus conclude, from Theorem 8.1, that the origin is a globally exponentially stable equilibrium point of the linear system $\dot{x} = Ax$ (since $\mathtt{V}$ is radially unbounded) assuming there exists a Lyapunov matrix $P$. Moreover, the upper bound in Equation (8.3) becomes (for $t_0 = 0$):

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}t} \|x(0)\|. \tag{8.7}$$

We have established that existence of a solution of (CTLE) implies global exponential stability and, in fact, the converse is also true as evidenced in the following theorem:

**Theorem 8.2.** *For a linear system $\dot{x} = Ax$, $A \in \mathbb{R}^{n\times n}$ with an isolated equilibrium $x^* = 0$, i.e., $\ker A = \{0\}$, the following holds:*

$$x^* = 0 \textit{ is globally exponentially stable}$$

$$\Updownarrow$$

$$\textit{for every } Q = Q^T > 0, \textit{ there exists a unique } P = P^T > 0 \textit{ such that } A^T P + PA = -Q.$$

*Proof.* We have already established $\Uparrow$ with the exception of the fact that $P = P^T > 0$ solving the CTLE is unique. The uniqueness of $P$ can be seen by using the *Kronecker product*, denoted by $\otimes$, wherein (CTLE) becomes:

$$(I \otimes A + A \otimes I)\,\mathrm{Vec}(P) = -\mathrm{Vec}(Q), \tag{8.8}$$

with Vec being the mapping that vertically concatenates (stacks) the columns of its matrix argument into a vector. The Eigenvalues of $I \otimes A + A \otimes I \in \mathbb{R}^{n^2 \times n^2}$ are given by $\lambda_i(A) + \lambda_j(A)$ for $i, j = 1, \ldots, n$. Since $x^* = 0$ is an exponentially stable equilibrium, the eigenvalues of $A$ have negative real part and thus $\lambda_i(A) + \lambda_j(A)$ is non-zero for all $i, j = 1, \ldots, n$, from which we conclude that $I \otimes A + A \otimes I$ is nonsingular. Therefore, (8.8) can be solved as a linear system, thus both establishing the uniqueness of the solution to the CTLE and providing a method for the calculation of the Lyapunov matrix $P$.

It only remains to establish $\Downarrow$. For $Q = Q^T > 0$, let:

$$P \triangleq \int_0^\infty e^{A^T t} Q e^{At} dt,$$

which is well defined since $A$ is assumed to be stable, i.e., $e^{At}$ is bounded and converges to zero[1]. Note that $Q = B^T B$, with $B$ invertible, since $Q$ is positive definite. We will establish the following facts.

<u>Fact 1: $P = P^T > 0$:</u> Clearly $P$ is symmetric, so to show that it is positive definite we consider $\mathtt{V}(x) = x^T P x$. Assume by way of contradiction that $x^T P x = 0$ for some $x \neq 0$. Then:

$$\begin{aligned} x^T P x = \int_0^\infty x^T e^{A^T t} Q e^{At} x dt = 0 \qquad &\Leftrightarrow \qquad \int_0^\infty x^T e^{A^T t} B^T B e^{At} x dt = 0 \\ &\Leftrightarrow \qquad \int_0^\infty \|B e^{At} x\|^2 dt = 0 \\ &\Leftrightarrow \qquad \|B e^{At} x\| = 0. \end{aligned}$$

But $B$ is invertible, so we have a contradiction:

$$B e^{At} x = 0 \quad \Longrightarrow \quad e^{At} x = 0 \quad \Longrightarrow \quad x = 0, \qquad \boxed{\Rightarrow\Leftarrow}$$

since $\ker A = \{0\}$ and therefore $e^{At} x = 0$ would imply $x(t) \equiv 0$.

<u>Fact 2: $A^T P + PA = -Q$:</u> Directly substituting into the (CTLE) yields:

$$\begin{aligned} A^T P + PA &= \int_0^\infty A^T e^{A^T t} Q e^{At} dt + \int_0^\infty e^{A^T t} Q e^{At} A dt \\ &= \int_0^\infty \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) dt \\ &= \left[e^{A^T t} Q e^{At}\right]_0^\infty \\ &= -Q, \end{aligned}$$

because $\lim_{t \to \infty} e^{At} = 0$ by the assumption of exponential (hence asymptotic) stability.

[1]More formally, this can be established through the methods utilized in Fact 1 or by considering the Jordan decomposition and its use in expressing the solution (see Additional Reading).

Fact 3: $P$ is Unique: Suppose that $P$ is not unique, i.e., there exists another $\overline{P} = \overline{P}^T > 0$ solving the (CTLE). Therefore:

$$\begin{aligned} A^T(P - \overline{P}) + (P - \overline{P})A = 0 \quad &\Longrightarrow \quad e^{A^T t}A^T(P - \overline{P})e^{At} + e^{A^T t}(P - \overline{P})Ae^{At} = 0 \\ &\Longrightarrow \quad \frac{d}{dt}\left(e^{A^T t}(P - \overline{P})e^{At}\right) = 0 \\ &\Longrightarrow \quad e^{A^T t}(P - \overline{P})e^{At} = c. \end{aligned}$$

But evaluating at $t = 0$ implies that $c = P - \overline{P}$ and:

$$\lim_{t\to\infty} \underbrace{e^{A^T t}(P - \overline{P})e^{At}}_{=c} = 0 \qquad \Longrightarrow \qquad c = P - \overline{P} = 0 \qquad \Longrightarrow \qquad P = \overline{P},$$

so $P$ is unique. □

**Example 8.2** (Linearized Pendulum)**.** We now return to the linearized downward pendulum studied in Lecture 6 as the motivating example, i.e., in Section 6.3. In particular, the linearized dynamics are given by:

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) - \gamma x_2 \end{bmatrix} \qquad \Longrightarrow \qquad \dot{x} = Df(0)x = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix}}_{A_\gamma} x, \tag{8.9}$$

where we linearized around the equilibrium point $x^* = 0$ and, recall, $\gamma > 0$ is the damping in the system.

As in Lecture 1, and specifically (1.28), we begin by picking the matrix $Q = Q^T > 0$ to be:

$$Q = \gamma I = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}. \tag{8.10}$$

Solving the (CTLE):

$$A_\gamma^T P + PA_\gamma = -Q,$$

for $P$ yields:

$$P = \frac{1}{2}\begin{bmatrix} (\gamma^2 + 2) & \gamma \\ \gamma & 2 \end{bmatrix}. \tag{8.11}$$

We can calculate the eigenvalues of $P$ and $Q$:

$$\lambda_{\min}(Q) = \gamma, \quad \lambda_{\max,\min}(P) = \frac{1}{4}\left(\gamma^2 + 4 \pm \gamma\sqrt{\gamma^2 + 4}\right),$$

to conclude their positivity for all $\gamma > 0$ and, therefore, $x^* = 0$ is globally exponentially stable (for the linearization) by Theorem 8.2. The upper and lower bounds on $\mathtt{V}$ are shown in Figure 8.1.

## 8.3 Stability of the Linearization

We now return our attention to nonlinear systems given by $\dot{x} = f(x)$ with an equilibrium point $x^* = 0$. By considering the linearization of this system, $\dot{x} = Df(0)x$, we can synthesize a Lyapunov function of the form $\mathtt{V}(x) = x^T Px$ using the methods in Section 8.2. Importantly, this section will establish that such $\mathtt{V}$ is, locally, a Lyapunov function for the nonlinear system.

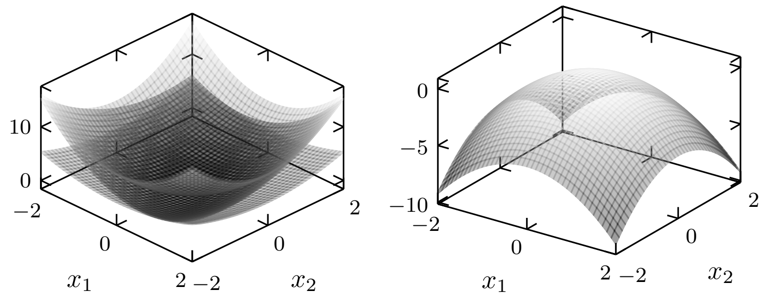


**Figure 8.1.** (Left) Upper and lower bounds on $\mathtt{V}$ provided by the maximum and minimum eigenvalues (with the constant $\gamma = \frac{2}{\sqrt{3}}$ used in Example 8.4) and (Right) The Lyapunov function $\dot{\mathtt{V}}$.

**Mean Value Theorem.** The main tool utilized to study the linearization of a nonlinear system with regard to Lyapunov's method is the *mean value theorem.* For $f : E \to \mathbb{R}^n$ with $E \subset \mathbb{R}^n$ and $x^* = 0 \in E$ an equilibrium point, consider a ball $B_\rho(0) = \{x \in E \ : \ \|x\| < \rho\} \subset E$. For any point $x \in B_\rho(0)$ we define the function $z_i(t) \triangleq f_i(tx)$ for $t \in [0,1]$, where $f_i$ is the $i^{\text{th}}$ component of $f$, i.e., $f_i : E \to \mathbb{R}$ and, therefore, $z_i : [0,1] \to \mathbb{R}$. It follows from the chain rule that:

$$\begin{aligned} f_i(x) - \underbrace{f_i(0)}_{=0} = z_i(1) - z_i(0) &= \int_0^1 \frac{d}{dt}\left(z_i(\tau)\right) d\tau \\ &= \int_0^1 \frac{\partial f_i}{\partial x}\Big|_{\tau x} x d\tau \\ &= \left(\int_0^1 Df(\tau x)_{(i,*)} d\tau\right) x, \end{aligned} \tag{8.12}$$

where $Df(\tau x)_{(i,*)}$ is the $i^{\text{th}}$ row of $Df(\tau x)$. Since this holds for all components of $f$ we have:

$$f(x) = \left(\int_0^1 Df(\tau x) d\tau\right) x.$$

Returning to the system $\dot{x} = f(x)$, we can write:

$$\dot{x} = f(x) = Df(0)x + \underbrace{\overbrace{\left(\left(\int_0^1 Df(\tau x) d\tau\right) - Df(0)\right)}^{G(x)} x}_{g(x) = G(x)x},$$

where $g(x)$ is the remainder term. This motivates the following Lemma:

**Lemma 8.1.** *Consider a continuously differentiable function $f : E \to \mathbb{R}^n$ defined on the open and connected set $E \subseteq \mathbb{R}^n$ with $f(0) = 0 \in E$. For any $\rho \in \mathbb{R}_{>0}$ such that $B_\rho(0) \subset E$ and for any $x \in B_\rho(0)$ there exists a decomposition:*

$$\dot{x} = f(x) = Df(0)x + g(x), \qquad \text{with} \qquad \lim_{x \to 0} \frac{\|g(x)\|}{\|x\|} = 0.$$

*Proof.* We have already established the decomposition, so we only need to show the limiting behavior of the remainder term $g(x)$. This follows from the fact that:

$$\lim_{x \to 0} \frac{\|g(x)\|}{\|x\|} \le \lim_{x \to 0} \left\| \left(\int_0^1 Df(\tau x) d\tau\right) - Df(0) \right\| = \left\| \lim_{x \to 0} \left(\int_0^1 Df(\tau x) d\tau\right) - Df(0) \right\| = 0.$$

□

**Example 8.3.** Returning to the pendulum in Example 8.2,

$$\int_0^1 Df(\tau x)d\tau = \begin{bmatrix} 0 & 1 \\ -\frac{\sin(x_1)}{x_1} & -\gamma \end{bmatrix}.$$

Hence,

$$f(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) - \gamma x_2 \end{bmatrix} \qquad \Longrightarrow \qquad g(x) = \begin{bmatrix} 0 \\ x_1 - \sin(x_1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 - \frac{\sin(x_1)}{x_1} & 0 \end{bmatrix}}_{G(x) = \int_0^1 Df(\tau x)d\tau - Df(0)} x.$$

**Lyapunov's Method Applied to the Linearization.** Through the decomposition provided by the mean value theorem, we can present the main result with respect to linearization: the linearized system can be used to obtain a Lyapunov function that is locally valid for the nonlinear system. The importance of this result is that it gives a systematic way of synthesizing Lyapunov functions for nonlinear systems, albeit only locally.

It is important to note that this result gives a constructive means for estimating the *domain of attraction* (of the origin in this case) which is a forward invariant set:

$$D_{\text{oa}} = \left\{ x \in E : x(0) = x \text{ and } \lim_{t \to \infty} x(t) = 0 \right\}.$$

In fact, we will construct a set of initial conditions whose corresponding solutions not only converge to the equilibrium, but converge exponentially.

**Theorem 8.3.** *Let $\dot{x} = f(x)$ where $f : E \to \mathbb{R}^n$ is a continuously differentiable function defined on the open and connected set $E \subseteq \mathbb{R}^n$. If the equilibrium $x^* = 0 \in E$ is exponentially stable for the linearization, $\dot{x} = Df(0)x$, then it is exponentially stable for $\dot{x} = f(x)$.*

*Moreover, given $Q = Q^T > 0$, the unique solution $P = P^T > 0$ of the Continuous-Time Lyapunov (CTLE) Equation:*

$$Df(0)^T P + PDf(0) = -Q, \tag{8.13}$$

*yields a Lyapunov function for the nonlinear system, $\mathtt{V}(x) = x^T P x$, that establishes exponential stability, with $\mathtt{V}$ satisfying:*

$$\begin{aligned} \lambda_{\min}(P)\|x\|^2 \leq \quad \mathtt{V}(x) \quad & \leq \lambda_{\max}(P)\|x\|^2 && (8.14)\\ \dot{\mathtt{V}}(x) \quad & \leq -\left(\lambda_{\min}(Q) - 2N\lambda_{\max}(P)\right)\|x\|^2 && (8.15)\\ \left\| \frac{\partial \mathtt{V}}{\partial x}\Big|_x \right\| \quad & \leq 2\lambda_{\max}(P)\|x\|, && (8.16) \end{aligned}$$

*for $x \in B_r(0) \subset E$ with $r > 0$ such that:*

$$N \triangleq \max_{\|x\| \leq r} \left\| \int_0^1 Df(\tau x)d\tau - Df(0) \right\| < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}. \tag{8.17}$$

*Therefore, for $\beta < \lambda_{\min}(P)r^2$ it follows that $\Omega_\beta = \{x \in E | \mathtt{V}(x) \leq \beta\} \subset B_r(0)$ and that $\Omega_\beta \subset D_{\text{oa}}$.*

*Proof.* Let $A = Df(0)$, wherein differentiating $\mathtt{V}$ along solutions yields:

$$\begin{aligned}
\dot{\mathtt{V}}(x) &= x^T P f(x) + f(x)^T P x \\
&= x^T P(Ax + g(x)) + (x^T A^T + g(x)^T) P x \\
&= x^T (PA + A^T P) x + 2x^T P g(x) \\
&= -x^T Q x + 2x^T P g(x).
\end{aligned}$$

It follows from the limiting behavior, $\lim_{x \to 0} \frac{\|g(x)\|}{\|x\|} = \lim_{x \to 0} \|G(x)\| = 0$ (which holds due to Lemma 8.1), that for any $N > 0$ there exists a $r > 0$ such that:

$$\begin{aligned}
\|x\| < r \quad \Longrightarrow \quad \max_{\|x\| \le r} \|G(x)\| \le N \quad &\Longrightarrow \quad \|g(x)\| \le \|G(x)\| \|x\| \le N\|x\| \quad \forall x \in B_r(0) \\
&\Longrightarrow \quad \|g(x)\| \le N\|x\| \quad \forall x \in B_r(0). \qquad (8.18)
\end{aligned}$$

Note that $\max_{\|x\| \le r} \|G(x)\|$ is well defined since $G$ is a continuous function and we are taking the maximum over a compact set. Using the *Cauchy-Schwartz* inequality[2]:

$$\begin{aligned}
x^T P g(x) \le |x^T P g(x)| &\le \|x\| \|P g(x)\| \qquad (8.19) \\
&\le \|x\| \|P\| \|g(x)\| \\
&\le N \lambda_{\max}(P) \|x\|^2, \qquad \forall \|x\| < r.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\dot{\mathtt{V}}(x) &\le -x^T Q x + 2N\lambda_{\max}(P)\|x\|^2 \\
&\le -\lambda_{\min}(Q)\|x\|^2 + 2N\lambda_{\max}(P)\|x\|^2 \\
&= -\underbrace{\left(\lambda_{\min}(Q) - 2N\lambda_{\max}(P)\right)}_{k_3} \|x\|^2, \qquad \forall \|x\| < r.
\end{aligned}$$

We now note that if $k_3 > 0$ we have established exponential stability. Since we can freely choose $N > 0$, we take:

$$N < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)},$$

so as to ensure $k_3 > 0$ wherein the resulting $r > 0$ is obtained from (8.18), which establishes (8.17) along with the bounds in (8.14) and (8.15). The bound (8.16) follows directly from the fact that $\mathtt{V}(x) = x^T P x$, wherein:

$$\left.\frac{\partial \mathtt{V}}{\partial x}\right|_x = 2x^T P \qquad \Longrightarrow \qquad \left\| \left.\frac{\partial \mathtt{V}}{\partial x}\right|_x \right\| \le 2\lambda_{\max}(P)\|x\|.$$

To show that the Lyapunov sublevel set satisfies $\Omega_\beta \subset D_{\text{oa}}$ for $\beta < \lambda_{\min}(P) r^2$, we note it follows from (8.14) that:

$$\mathtt{V}(x) \le \beta \qquad \Longrightarrow \qquad \lambda_{\min}(P)\|x\|^2 \le \mathtt{V}(x) \le \beta < \lambda_{\min}(P) r^2 \qquad \Longrightarrow \qquad \|x\| < r,$$

and therefore $\Omega_\beta \subset B_r(0)$. We conclude that $\Omega_\beta \subset D_{\text{oa}}$ since $\mathtt{V}$ is a valid Lyapunov function for the nonlinear system on $B_r(0)$ and, by Proposition 7.1, $\Omega_\beta$ is thus forward invariant. $\square$

[2]This states that for $x, y \in \mathbb{R}^n$, $|x^T y| = |\langle x, y \rangle| \le \|x\| \|y\|$, where $\langle \cdot, \cdot \rangle$ is the inner product.

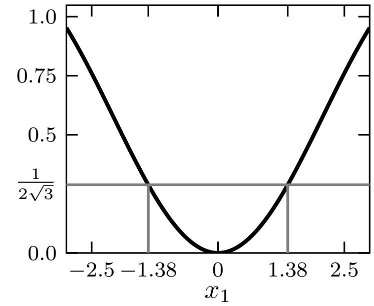


**Figure 8.2.** Illustration of the process used to calculate $r$ in the case of Example 8.4.

**Example 8.4.** Returning to the pendulum, and following Example 8.3, we first note that (utilizing the matrix norm in Example A.4 in Appendix A):

$$\left\| \int_0^1 Df(\tau x)d\tau - Df(0) \right\| = \left\| \begin{bmatrix} 0 & 0 \\ 1 - \frac{\sin(x_1)}{x_1} & 0 \end{bmatrix} \right\| = \left| 1 - \frac{\sin(x_1)}{x_1} \right|.$$

Therefore, to establish the neighborhood on which Theorem 8.3 is valid, i.e., the domain of attraction, we need to determine $r > 0$ such that:

$$\left| 1 - \frac{\sin(x_1)}{x_1} \right| < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \quad x \in B_r(0).$$

To calculate this concretely, pick $\gamma = \frac{2}{\sqrt{3}}$ wherein for the $Q$ and $P$ given in (8.10) and (8.11), respectively:

$$\lambda_{\min}(Q) = \frac{2}{\sqrt{3}}, \quad \lambda_{\min}(P) = \frac{2}{3}, \quad \lambda_{\max}(P) = 2,$$

and the bound becomes:

$$\left| 1 - \frac{\sin(x_1)}{x_1} \right| < \frac{1}{2\sqrt{3}} \qquad \Longrightarrow \qquad |x_1| < 1.38 \triangleq r,$$

where the bound on $x_1$ was determined numerically. This yields $N \approx 0.28$.

Therefore, for a ball $B_r(0)$ of radius $r$ the Lyapunov function obtained from the linear system $\mathtt{V}(x) = x^T P x$ with $P$ given in (8.11) and the nonlinear pendulum is locally exponentially stable on $B_r(0)$. Additionally, the bounds on $\mathtt{V}$ and $\dot{\mathtt{V}}$ are given as in (8.14) and (8.15). It is important to note that we have, as a result of these calculations, estimated the domain of attraction: $\Omega_\beta \subset B_{1.38}(0) \subset D_{\text{oa}}$, where $\beta = 0.92$. That is, $\mathtt{V}(x) = x^T P x \leq 0.92$ gives an (inner) approximation of the domain of attraction. This is illustrated in Figure 8.3.

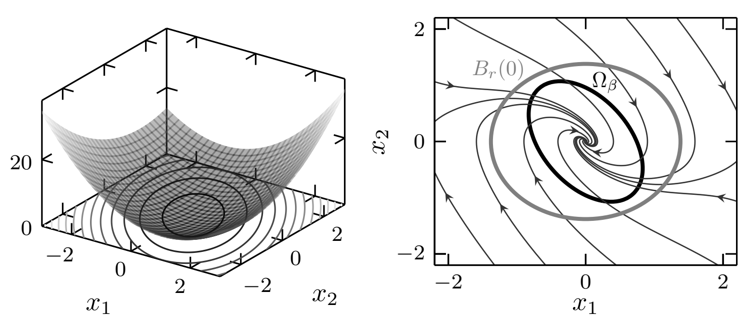


**Figure 8.3.** (Left) Level sets of the Lyapunov function given in Example 8.4 and the (Right) the domain of attraction and level sets of $\mathtt{V}$.

## Additional Reading

The results on proving the exponential stability of linear systems with Lyapunov functions can be found in [58]; in general, this book provides an excellent mathematical treatment of linear systems. The proof of Theorem 8.3 is standard and can be found in numerous texts, e.g., [110] and [182] (the later text also covers the linear case). The calculation of the domain of attraction via (8.17), while implicit in other proofs, is not typically explicitly stated. This implicit understanding between the linearization and a means to (conservatively) estimate the domain of attraction has proven useful, e.g., in the context of finite-time Lyapunov functions [163]. See [38] for an excellent reference on Lyapunov stability, especially in the context of exponential, rational, and finite-time stability.

## Problems for Lecture 8

[P8.1] For the following system with $x \in \mathbb{R}^2$,

$$\dot{x} = f(x) = \begin{bmatrix} -x_1 + x_2 \\ (x_1 + x_2)\sin(x_1) - 3x_2 \end{bmatrix}.$$

Show that the origin is exponentially stable, and estimate the domain of attraction.

[P8.2] Prove that if $A$ is a stable matrix, than $P = \int_0^\infty e^{A^T t} Q e^{At} dt$ is well-defined, i.e., bounded.

[P8.3] Consider the linear system:

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n},$$

where $A$ is stable. Hence, for $Q = Q^T > 0$, there exists a $P = P^T > 0$ solving the (CTLE). Using this $P$ we define the function $\mu$ by:

$$\mu(Q) = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}.$$

Show that $\mu(Q)$ has the following properties:

- $\mu(Q) = \mu(kQ)$ for all real constants $k > 0$.
- $\mu(I) \geq \mu(Q)$ for all $Q = Q^T > 0$ such that $\lambda_{\min}(Q) = 1$.
- $\mu(I) \geq \mu(Q)$ for all $Q = Q^T > 0$ [Hint: use the previous two facts].

As a result of the above properties, determine which $Q$ gives the fastest rate of convergence for the linear system.

[P8.4] Following from the previous problem, consider the perturbed linear system:

$$\dot{x} = Ax + g(x), \tag{8.20}$$

where $g(x)$ is a function satisfying $\|g(x)\| \leq \gamma \|x\|$ for some $\gamma > 0$. Show that if:

$$\gamma < \mu(Q),$$

the system (8.20) is exponentially stable. Determine which $Q$ for which the largest upper bound on $\gamma$ is obtained.

[P8.5] Consider the linear control system:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m,$$

Assume that $P = P^T > 0$ solves the *Continuous-Time Algebraic Riccati* (CARE) equation:

$$PA + A^T P + Q - PBB^T P = 0,$$

for some $Q = Q^T > 0$. Show that applying the feedback control law:

$$u(x) = -\frac{1}{2} B^T P x,$$

results in the origin being globally exponentially stable.

[P8.6] (Rational Stability [38, 89]) Let $\dot{x} = f(x)$ where $f : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on the open and connected set $E \subseteq \mathbb{R}^n$. The equilibrium point $x^* = 0 \in E$ is rationally stable if there exist $M, \delta, k, \eta \in \mathbb{R}_{>0}$, with $\eta < 1$, such that:

$$\|x(t_0)\| < \delta \qquad \Longrightarrow \qquad \|x(t)\| \leq M(1 + \|x(t_0)\|^k t)^{-\frac{1}{k}} \|x(t_0)\|^\eta, \quad \forall t \geq t_0. \tag{8.21}$$

The term rational refers to the particular form of the upper in the previous inequality. Consider a continuously differentiable function $\mathtt{V} : E \to \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned} k_1 \|x\|^{c_1} \leq \quad \mathtt{V}(x) \quad &\leq k_2 \|x\|^{c_2} && (8.22)\\ \dot{\mathtt{V}}(x) \quad &\leq -k_3 \|x\|^{c_3}, && (8.23) \end{aligned}$$

for $k_1, k_2, k_3, c_1, c_2, c_3 \in \mathbb{R}_{>0}$. Show that:

$$\frac{c_3}{c_2} > 1 \qquad \Longrightarrow \qquad x^* = 0 \text{ is rationally stable.}$$

[P8.7] (Finite-Time Stability [38, 89]) Let $\dot{x} = f(x)$ where $f : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on the open and connected set $E \subseteq \mathbb{R}^n$ with corresponding flow $\varphi_t(x)$ for $x \in E$ (see Lecture 4). The equilibrium point $x^* = 0 \in E$ is finite-time stable if there exists an open neighborhood of the origin $U \subseteq E$ and a *settling time function* $T : U \backslash \{0\} \to (0, \infty)$ such that $\varphi_t(x)$ is defined on $[0, T(x))$, i.e., $[0, T(x)) \subset I(x)$ with $I(x)$ the interval of existence, satisfying:

$$\forall x \in U \backslash \{0\} \qquad \Longrightarrow \qquad \varphi_t(x) \in U \backslash \{0\} \ \forall t \in [0, T(x)) \qquad \text{and} \qquad \lim_{t \to T(x)} \varphi_t(x) = 0. \tag{8.24}$$

The function $T(x)$ specifies the time it takes the solution to converge to the equilibrium $x^* = 0$ from $x$. Noting that $T(x) < +\infty$ we conclude that convergence occurs in finite time. Consider a continuously differentiable function $\mathtt{V} : E \to \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned} k_1 \|x\|^{c_1} \leq \quad \mathtt{V}(x) \quad &\leq k_2 \|x\|^{c_2} && (8.25)\\ \dot{\mathtt{V}}(x) \quad &\leq -k_3 \|x\|^{c_3}, && (8.26) \end{aligned}$$

for $k_1, k_2, k_3, c_1, c_2, c_3 \in \mathbb{R}_{>0}$. Show that if:

$$\frac{c_3}{c_2} < 1 \qquad \Longrightarrow \qquad x^* = 0 \text{ is finite-time stable.}$$

[P8.8] Consider again the discrete-time dynamical system given in Problem P7.10, i.e.:

$$x_{k+1} = F(x_k), \qquad k \in \mathbb{N}, \tag{8.27}$$

for $F : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable. Furthermore, assume that $x^* = 0$ is an equilibrium point (fixed point): $F(0) = 0$. We say that this system is exponentially stable if $\exists\ M > 0$ and $0 < \lambda < 1$ such that:

$$\|x_0 - x^*\| < \delta \qquad \Longrightarrow \qquad \|x_k - x^*\| \leq M\lambda^k \|x_0 - x^*\|.$$

Find conditions on a $\mathtt{V} : \mathbb{R}^n \to \mathbb{R}$, along with conditions on the rate of change along solutions to the discrete-time dynamical system:

$$\nabla\mathtt{V}(x) \triangleq \mathtt{V}(F(x)) - \mathtt{V}(x),$$

such that it guarantees exponential stability of (8.27).

[P8.9] Consider a linear discrete-time dynamical system:

$$x_{k+1} = Ax_k, \qquad k \in \mathbb{N},$$

for $A \in \mathbb{R}^{n \times n}$. For this linear system, the Discrete-Time Lyapunov Equation (DTLE) is given by, for a positive definite matrix $Q = Q^T > 0$:

$$APA^T - P = -Q,$$

with solution being the discrete-time Lyapunov matrix $P = P^T > 0$. Using the results of the previous problem, show that the Lyapunov function $\mathtt{V}(x) = x^T P x$ establishes the stability of the linear discrete-time system when $P$ is a solution to the discrete-time Lyapunov equation.

[P8.10] Let $F : \mathbb{R}^n \to \mathbb{R}^n$ with $x^* = 0$ a fixed point: $F(0) = 0$. Show that:

$$\begin{gathered} x_{k+1} = Df(0)x_k \text{ exponentially stable} \\ \Downarrow \\ x_{k+1} = F(x_k) \text{ locally exponentially stable} \end{gathered}$$

through the use of Lyapunov functions.