PROBLEMS

Problem 1 (Book Problem 5.4)

Consider the dynamical system, $\dot{x} = f(x)$, introduced in Example 5.15.

$$\dot{x} = f(x) = \begin{bmatrix} -x_1 \\ -x_2 + x_1^k \\ x_3 + x_1^2 \end{bmatrix}$$

with $k \in \mathbb{N}_{\geq 1}$. For this system:

(a) Find an explicit expression for the stable and unstable manifolds, S and U, respectively, i.e., find functions $h_S : \mathbb{R}^3 \to \mathbb{R}^s$, $h_U : \mathbb{R}^3 \to \mathbb{R}^u$ defining these surfaces: $S = h_S^{-1}(0)$ and $U = h_U^{-1}(0)$.

(b) Establish that the surfaces S and U are, in fact, manifolds. **OPTIONAL:** Find local coordinates for these manifolds in a neighborhood of $0 \in \mathbb{R}^3$ via coordinate charts.

(c) Verify that the tangent spaces to the manifolds S and U at 0 is the stable and unstable subspaces, E^s and E^u, respectively.

Problem 2 (Will Original)

Give an example of a differentiable function h whose zero level set does not define a manifold. Specifically, find a smooth map $h : \mathbb{R}^n \to \mathbb{R}^m$, for which $E = \{x \in \mathbb{R}^n \mid h(x) = 0\} = h^{-1}(0)$, such that $\mathrm{rank}(h) = r$ at all but a finite number of points in E. Qualitatively, or intuitively, provide an explanation of what occurs at points where h loses or gains rank. For simplicity, you may fix n, m, and r (for instance $n = 2, m = 1, r = 1$). Qualitatively, or intuitively, how does this indicate that $h^{-1}(0)$ is not a manifold? (use an intuitive interpretation of a manifold, as a space which locally looks like $\mathbb{R}^n$)

Problem 3 (Book Problem 6.1)

Prove the properties of class $\mathcal{K}$ functions presented in Section 6.2 and, specifically, *invertibility* and *composability*.

Problem 4 (Book Problem 6.5)

Consider the second order differential equation:

$$\ddot{\theta} + p(\theta) = 0$$

for $\theta \in \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$ continuously differentiable with $p(0) = 0$ and $\frac{\partial p}{\partial \theta}(\theta) \neq 0$. This can be converted to an ODE as follows:

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -p(x_1) \end{bmatrix}$$

with $x_1 = \theta$ and $x_2 = \dot{\theta}$. Utilize the corresponding energy of the system given by:

$$E(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} p_1(s)ds$$

Problem 4 continued on next page...

to construct a Lyapunov function and give conditions on p for which the system is stable. What can you say about the asymptotic stability of the system?

Problem 5 (Book Problem 6.6)

Consider the differential equation:

$$\dot{x} = J(x)\nabla H(x)$$

where $J(x)$ is a skew-symmetric matrix for every $x \in \mathbb{R}^n$, and $\nabla H(x) = (DH(x))^\top$. This is a generalization of Hamilton's equations of motion since we can always take J to be:

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

to recover said equations. Using this interpretation we can regard H as the energy of this system. Show that solutions starting in a level-set of H remain in that level set for all future time, i.e, energy is preserved along solutions.

Consider now the differential equation

$$\dot{x} = J(x)\nabla H(x) + R(x)\nabla H(x)$$

where $R(x)$ is negative semi-definite. This can be interpreted as a generalization of Hamilton's equations with dissipation. Show that solutions starting in a sublevel set of H remain in that sublevel set forever, i.e, energy does not increase (and may decrease) along solutions.

PROJECT

This homework includes a set of tasks associated with the linearization of your system. Complete the following:

- Identify at least one equilibrium point for your system.
- Linearize your system about the equilibrium. Does Hartman Grobman apply? If so, what conclusions can be drawn.
- Linearize your system about a non-equilibrium point (to obtain a system of the form $\delta x \approx A\delta x + C$). Does Hartman Grobman apply?

If your system does not naturally have any equilibrium points, you may need to apply a steady-state controller to establish one.

OPTIONAL PROBLEMS

Optional problems will not be graded (and solutions will not be released). They may be of increased difficulty, or be problems which are omitted from the homework to keep it from being too long. These problems may make for good study material or qualifying exam preparation. Since solutions will not be released, feel free to ask questions about these questions in office hours (at then end, after required homework problem questions have been covered).

Problem 6 (Book Problem 6.3)

The goal of this problem is to establish that Proposition 6.1 gives an alternative definition of stability. In particular, that proposition proved that (6.1) $\Longleftarrow$ (6.5). Establish that this is necessary and sufficient by proving that (6.1) $\Longrightarrow$ (6.5). To establish this:

(a) Use (6.1) to construct a function $\gamma \in \mathcal{K}$ such that:

$$\forall \varepsilon > 0 \quad \|x(t_0) - x_0\| < \gamma(\varepsilon) \qquad \Longrightarrow \qquad \|x(t) - x^*\| < \varepsilon$$

(b) Use Part (a) to conclude that (6.1) $\Longrightarrow$ (6.5).

Problem 7 (Book Problem 6.4)

A group, G, is a set (finite or infinite) together with a group operation, denoted by $a \cdot b$, and defined for all $a, b \in G$ while satisfying:
<u>Closure</u>: For $a, b \in G$, $a \cdot b \in G$
<u>Associativity</u>: For $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
<u>Identity</u>: There exists an element $1 \in G$ suth that $1 \cdot a = a \cdot 1 = a$ for all $a \in G$
<u>Inverse</u>: For each $a \in G$ there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$
Show that the set of class $\mathcal{K}$ functions is a group. Similarly, show that the set of class $\mathcal{K}_\infty$ functions is also a group, i.e., a subgroup of the group K (since it is closed under the group operation and inverse operation).

Problem 8 (Book Problem 6.10)

(ADVANCED) Consider a manifold M and a vector field $X : M \to TM$, on which we wish to define stability in a manner that does not require the Euclidean norm (since the notion of distance on manifolds is not Euclidean in nature). In this content, define and show the following:

(a) Define the notion of an equilibrium point, x^*, for a vector field: $X : M \to TM$.

(b) Let U and V denote open, connected, and bounded neighborhoods of x^*. An equilibrium point $x^* \in M$ of a vector field $X : M \to TM$ on a manifold is stable if:

$$\forall U \subset M,\ \exists V \subset U \text{ with } x_0 \in V \quad \text{s.t.} \quad x(t_0) \in V \cap M \implies x(t) \in U \cap M \quad \forall t \geq t_0$$

Show that when $M = \mathbb{R}^n$, this is equivalent to the definition of stability given in Definition 6.1.

(c) Formulate asymptotic and exponential stability in a similar fashion, and without the use of norms.

(d) Repeat parts (b) and (c) but through the use of class K functions as in Proposition 6.1.