PROBLEMS

Problem 1 (Book Problem 3.1)

Use the Comparison Lemma to construct a curve upper bounding the solution of the differential equation $\dot{x} = -x + \tan^{-1}(x)$.

Problem 2 (Book Problem 3.3)

Let $\dot{x} = f(x)$, with $f : D \to \mathbb{R}^n$ locally Lipschitz continuous on D, with D compact, $0 \in D$, and suppose that $x(t) \in D$ for all $t \geq 0$ and for all $x(0) = x_0 \in D$. Additionally, suppose that $f(0) = 0$. Under these conditions, establish that:

$$\left| \frac{d}{dt} \|x(t)\|^2 \right| \leq 2L\|x(t)\|^2 \quad t \geq 0$$

Use this to prove that:

$$\|x(t)\| \leq e^{Lt}\|x(0)\| \quad t \geq 0$$

Problem 3 (Book Problem 3.10)

This problem will establish a variant on the Gronwall-Bellman Inequality (Theorem 3.1) where some of the assumptions on the functions in the theorem are relaxed; specifically, we will relax the assumption that $\lambda(t)$ be a positive and non-decreasing function and that $y(t)$ be nonnegative.
Let $I = [a, b] \subset \mathbb{R}$ be an interval, $\lambda : I \to \mathbb{R}$ be continuous, and $\varphi : I \to \mathbb{R}$ be continuous and nonnegative. If $y : I \to \mathbb{R}$ satisfies:

$$y(t) \leq \lambda(t) + \int_a^t \varphi(s)y(s)ds \quad \forall t \in I$$

Prove the following:

(a) That $y(t)$ satisfies the bound:

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\varphi(s)e^{\int_s^t \varphi(\tau)d\tau}ds$$

(b) In the special case that $\lambda(t) \equiv \lambda$ is constant, then

$$y(t) \leq \lambda e^{\int_a^t \varphi(\tau)d\tau}$$

(c) If in addition $\phi(t) \equiv \varphi > 0$ is constant, then

$$y(t) \leq \lambda e^{\varphi(t-a)}$$

Problem 4 continued on next page...

Problem 4 (Book Problem 4.1)

Prove Proposition 4.1, i.e. that flows satisfy the identity, composability, and reversability properties.

Problem 5 (Book Problem 4.8)

Consider the system in Example 1.3.

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) - \alpha x_2 \left(\frac{x_2^2}{2} - \cos(x_1) + c\right) \end{bmatrix}$$

for $c \in (0, 1)$, and $\alpha > 0$ What does the linearization say about the system dynamics? What behavior does the linearization fail to capture?

PROJECT

The first task is to demonstrate that your system is locally or globally Lipschitz. Please report what the Lipschitz constant for your system is. The next task is to compute solutions to your system's differential equations using the method of Picard Iteration. This can be done in software of your choice. Please return a plot comparing the iterated solutions to solutions generated by a standard differential equation solver (Runge-Kutta 45, etc). Please note:

- A bounded domain can be specified for calculating a Lipschitz constant, it need not be over all of $\mathbb{R}^n$.
- Recall that continuously differentiable functions permit a way to compute Lipschitz constants outside of the standard definition.
- You can use optimization tools to evaluate your Lipschitz constant if necessary.
- For initializing the Picard Iteration, a constant function $\phi_0(t) \equiv x_0, \forall t$ can be used. It is is interesting to plot solutions spaced out in the iteration to see how they evolve.
- Recall that Picard Iteration can be used to generate solutions for extended intervals of time by concatenating solutions.

Additionally, please demonstrate the continuous dependence of solutions on initial conditions and parameters for your system. In particular, you should show that the time-varying bound given by Equation (3.15) in Theorem 3.2 is satisfied by the norm of the solution difference. A perturbation in the initial condition should be used, as well as a perturbation in at least one of the system parameters. Please explain how the bound μ is computed for your system. Return a plot demonstrating that the bound is satisfied, using software of your choice. Please note:

- You only need to perturb one parameter in your system, but you may perturb more if you choose to.
- As the bound grows exponentially, it is very conservative and will likely make visualization difficult. Feel free to plot a small time window at the beginning of the simulation demonstrating the bound holds.
- You can simulate your dynamics for this portion using a typical ODE solver (you need not use Picard Iterations).

OPTIONAL PROBLEMS

Optional problems will not be graded (and solutions will not be released). They may be of increased difficulty, or be problems which are omitted from the homework to keep it from being too long. These problems may make for good study material or qualifying exam preparation. Since solutions will not be released, feel free to ask questions about these questions in office hours (at then end, after required homework problem questions have been covered).

Problem 6 (Book Problem 3.4)

Complete the proof of the Comparison Lemma in the case when α is not assumed to be Lipschitz continuous. Specifically, utilize the constructions given in Remark 3.2 and prove that the implication in (3.4) holds.

Problem 7 (Book Problem 3.9)

Formalize the intuition given at the beginning of Section 3.3. In particular, for the differential equation $\dot{x} = f(x, \lambda)$ with $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^p$, lift this equation to

$$\dot{z} = g(x, \lambda) = \begin{bmatrix} f(x, \lambda) \\ 0 \end{bmatrix} \quad \text{for} \quad z = \begin{bmatrix} x \\ \lambda \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^p$$

For this transformed system, initial conditions, $z(t_0)$, consist of vectors $(x(t_0), \lambda)$ wherein the corresponding solution $z(t)$ of $\dot{z} = g(z)$ is given by $z(t) = (x(t), \lambda)$ where $x(t)$ is the solution of $\dot{x} = f(x, \lambda)$ for the chosen parameters λ. Show that Theorem 3.2 applied to $\dot{z} = g(z)$ guarantees continuity with respect to changes in initial values and parameters.

Problem 8 (Book Problem 4.3)

Prove Fact 4.1, i.e., that homeomorphisms preserves connectedness, openness and compactness.

Problem 9 (Book Problem 4.6)

As will be seen in Lecture 13, a solution $\varphi_t(x)$ to $\dot{x} = f(x)$, for $x \in \mathbb{R}^n$ is periodic if $\varphi_{t+T}(x) = \varphi_t(x)$ for some $T > 0$. The corresponding periodic orbit is given by:

$$O \triangleq \{\varphi_t(x) \in \mathbb{R}^n \mid t \in \mathbb{R}\} = \{\varphi_t(x) \in \mathbb{R}^n \mid t \in [0, T)]\}$$

Show that every periodic orbit is diffeomorphic to the unit circle.

Problem 10 (Book Problem 4.7)

Suppose that $\dot{x} = f(x)$ has a forward invariant set $S \subset \mathbb{R}^n$ and that:

$$\lim_{t \to \infty} \phi_t^f(x) = 0 \quad \forall x \in S$$

Show that for the dynamical system $\dot{y} = g(y)$, if f and g are topologically equivalent, then the dynamics dictated by g have a forward invariant set such that all flows with an initial condition in such set converge to a single point also in this set.

Problem 11 continued on next page...

Problem 11 (Book Problem 4.9)

Returning to Problem P2.4 in Lecture 2, show that for a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$, the dynamical system $\dot{x} = f(x)$ is orbitally equivalent to

$$\dot{y} = g(y) \triangleq \frac{f(y)}{1 + \|f(y)\|}$$