Lecture 5

Stable Manifolds

This lecture presents more advanced concepts, and at a faster pace, than previous lectures. Pointers to several good sources for the material discussed in this lecture are provided in the Additional Reading section. The concept of a manifold will be used in Lectures 14 and 15 when Poincaré maps will be defined on manifolds termed Poincaré sections.

This lecture further explores the relationship between linear and nonlinear systems via linearization methods. Whereas the previous lecture established an equivalence between a nonlinear system and its linearization around an equilibrium point, in this lecture we extend this equivalence to the state space and how it relates to the dynamics. The state space of linear systems is known to globally decompose into its stable and unstable subspaces (when the equilibrium point is hyperbolic), denoted by E^s and E^u respectively. This is also possible, albeit locally, for nonlinear systems where the role of E^s and E^u is played by manifolds whose tangent space coincides with the stable and unstable spaces of its linearization. Since these results require the language of smooth manifolds, this lecture devotes considerable attention to a gentle introduction to these concepts.

5.1 Stable and Unstable Subspaces

Linear Systems as Motivation. We begin by again considering a linear system of the form:

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n}.$$

Eigenvalues were considered in Section 4.3 of the previous lecture in the context of linearization. Here, we additionally consider the notion of a generalized eigenvector associated with an eigenvalue.

Definition 5.1. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{R}^{n\times n}$, i.e., λ is a solution of $\det(\lambda I - A) = 0$. The (algebraic) multiplicity[1] $\mu(\lambda)$ of λ is the multiplicity λ as a root of $\det(\lambda I - A) = 0$ wherein $\sum_{\lambda\in\sigma(A)} \mu(\lambda) = n$ with $\sigma(A)$ being the spectrum of A, i.e., the set of its eigenvalues. A vector $w \in \mathbb{C}$ is a generalized eigenvector associated with λ if:

$$(A - \lambda I)^r w = 0, \qquad \forall r \in \{1, \ldots, \mu(\lambda)\}. \tag{5.1}$$

There are $\mu(\lambda)$ eignevectors associated with an eigenvalue λ of multiplicity $\mu(\lambda)$.

When $r = 1$ the notion of generalized eigenvector becomes the usual notion of eigenvector.

[1] The algebraic multiplicity should not be confused with the geometric multiplicity which is the number of eigenvectors associated with an eigenvalue: $\dim(\ker(A - \lambda I))$.

***Remark* 5.1.** Note that generalized Eigenvectors (with multiplicity $\mu(\lambda)$) are obtained by solving:

$$(A - \lambda I)w_r = w_{r-1}, \qquad w_0 = 0, \quad r = 1, \dots, \mu(\lambda).$$

For example, in the case when $\mu(\lambda) = 2$,

$$\begin{aligned} (A - \lambda I)w_1 &= 0 \\ (A - \lambda I)w_2 &= w_1 \\ \Longrightarrow \qquad (A - \lambda I)^2 w_2 &= (A - \lambda I)w_1 = 0. \end{aligned}$$

Example 5.1. Consider the matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solving the equation $\det(A - \lambda I) = (1 - \lambda)^2 = 0$ we determine it has the unique solution $\lambda = 1$, i.e., $\mu(\lambda) = 2$. The corresponding generalized eigenvectors are:

$$\begin{aligned} (A - \lambda I)w_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} w_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ (A - \lambda I)w_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} w_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

We now use generalized eigenvalues and eigenvectors to introduce the notion of stable, unstable, and center (corresponding to marginally stable) subspaces for linear systems.

Definition 5.2. Let $w_j = u_j \pm \mathbf{i}v_j \in \mathbb{C}^n$ be a generalized eigenvector of $A \in \mathbb{R}^{n \times n}$ corresponding to the eigenvalue $\lambda_j = a_j \pm \mathbf{i}b_j \in \mathbb{C}$, with k real eigenvalues and $n - k$ complex eigenvalues:

$$\begin{aligned} \text{Real Eigenvalues: } & b_j = 0 \implies v_j = 0 \qquad \text{for} \qquad j = 1, \dots, k, \\ \text{Complex Eigenvalues: } & b_j \neq 0 \implies v_j \neq 0 \qquad \text{for} \qquad j = k+1, \dots, m. \end{aligned}$$

Thus there are a total of m generalized Eigenvectors with $n = 2m - k$. The <u>stable</u>, <u>unstable</u>, and <u>center subspaces</u> are given by:

$$\begin{aligned} E^s &\triangleq \text{span}\{x \in \mathbb{R}^n \mid x = u_j \text{ or } x = v_j \text{ for } a_j < 0\}, && \text{(Stable Subspace)} \\ E^u &\triangleq \text{span}\{x \in \mathbb{R}^n \mid x = u_j \text{ or } x = v_j \text{ for } a_j > 0\}, && \text{(Unstable Subspace)} \\ E^c &\triangleq \text{span}\{x \in \mathbb{R}^n \mid x = u_j \text{ or } x = v_j \text{ for } a_j = 0\}. && \text{(Center Subspace)} \end{aligned}$$

The importance of the these spaces is that, for linear systems, they form a decomposition of the state space. The proof of this result follows from linear algebra, and is thus not in the scope of this lecture.

Theorem 5.1. *With the notation of Definition 5.2, $\mathbb{R}^n$ can be decomposed[2] into the stable, unstable, and center subspaces of the linear system $\dot{x} = Ax$, i.e.:*

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c.$$

Moreover, each of the subspaces E^l, $l \in \{s, u, c\}$, is invariant with respect to the flow e^{At}, that is:

$$e^{At}(E^l) \subseteq E^l, \qquad \forall t \in \mathbb{R}, \;\; l \in \{s, u, c\}.$$

[2] Given vector spaces $V_1, V_2 \subseteq \mathbb{R}^n$, their direct sum $V_1 \oplus V_2$ can be identified with $\mathbb{R}^n$ if for any $x \in \mathbb{R}^n$ there exists a unique $x_1 \in V_1$ and a unique $x_2 \in V_2$ such that $x = x_1 + x_2$.

The goal of this lecture is to present a generalization of the preceding result to nonlinear systems. This will require us to introduce manifolds and tangent spaces. Before doing so we present two short examples.

Example 5.2. Consider the linear system $\dot{x} = Ax$ with the same A matrix as in Example 5.1. Since the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 1$ we have $\mathbb{R}^2 = E^u = \mathrm{span}\{w_1, w_2\}$ where $w_1 = (1, 0)$ and $w_2 = (0, 1)$.

Example 5.3. Consider the linear system $\dot{x} = Ax$ with:

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2 \pm \mathbf{i}$ and the corresponding eigenvectors are:

$$w_1 = u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad w_2 = u_2 \pm \mathbf{i} v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \pm \mathbf{i} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, $k = 1$, $m = 2$, and it follows that:

$$\begin{aligned} E^u &= \mathrm{span}\{u_1\}, \\ E^s &= \mathrm{span}\{u_2, v_2\}, \\ \mathbb{R}^3 = E^u \oplus E^s &= \mathrm{span}\{u_1, u_2, v_2\}. \end{aligned}$$

A graphical illustration of the resulting behavior of the system can be seen in Figure 5.1.

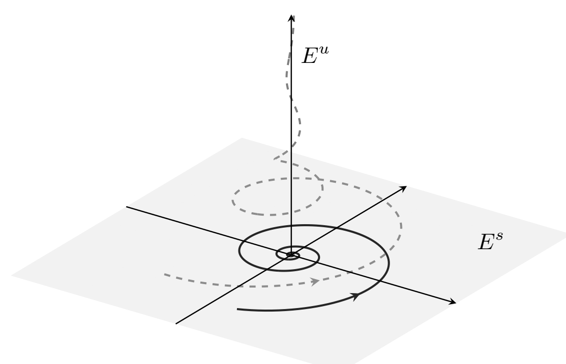


Figure 5.1. Graphical representation of the phase portrait associated with Example 5.3, indicating the stable and unstable subspaces.

5.2 Manifolds

To introduce stable and unstable manifolds for nonlinear systems, we must first introduce the notion of *manifold*. This section presents these mathematical preliminaries to set the stage for the stable manifold theorem.

Rank of a Function. For simplicity we will restrict our attention to n-dimensional *embedded submanifolds* of $\mathbb{R}^k$, with $n \leq k$, defined by level sets of smooth functions.

Example 5.4. In order to motivate the definitions that follow we revisit the unit circle $\mathbb{S}^1$ (introduced in Example 4.2 and 4.5) defined as:

$$\mathbb{S}^1 \triangleq \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\right\}.$$

We want to mathematically describe the fact that $\mathbb{S}^1$, locally, looks like $\mathbb{R}$. A convenient way of doing this is to define the function $h : \mathbb{R}^2 \to \mathbb{R}$ by:

$$h(x_1, x_2) = x_1^2 + x_2^2 - 1, \tag{5.2}$$

and note that $h^{-1}(0) = \{x \in \mathbb{R}^2 \mid h(x) = 0\}$ equals $\mathbb{S}^1$.

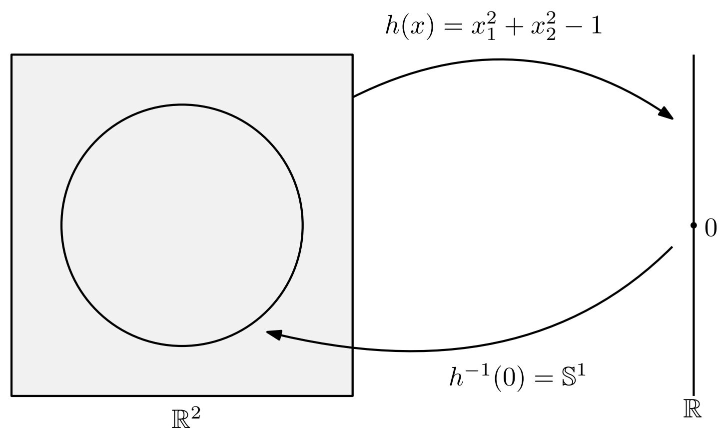


Figure 5.2. Illustration of defining the unit circle as the zero level set of smooth function.

If h was a linear map $H : \mathbb{R}^k \to \mathbb{R}^m$, then we can decompose $\mathbb{R}^k$ as $\ker H \oplus V$ where $\ker H = H^{-1}(0) = \left\{x \in \mathbb{R}^k \mid Hx = 0\right\}$ and V is any subspace of $\mathbb{R}^k$ such that $\ker H \oplus V = \mathbb{R}^k$. Importantly, $\mathrm{rank}(H) = \dim(V)$. Hence, H allows to decompose $\mathbb{R}^k$ into the kernel and subspace isomorphic to the image of H of dimension equal to the rank of H, as represented by the *Rank-nullity theorem*:

$$\mathrm{rank}(H) + \dim(\ker H) = k. \tag{5.3}$$

This decomposition based on H can be generalized to nonlinear maps h in virtue of the constant rank theorem. In order to state this result we first introduce the notion of rank for nonlinear maps.

Definition 5.3. Let $h : E \subseteq \mathbb{R}^k \to \mathbb{R}^m$ be a <u>smooth</u> (continuously differentiable) map. The <u>rank</u> of h at $x \in E$ is the rank of the matrix $Dh(x) \in \mathbb{R}^{m \times k}$. If the rank of $Dh(x)$ is constant for all $x \in E$, i.e., $\mathrm{rank}(Dh(x)) = r$ for all $x \in E$, then h has <u>constant rank</u> $\mathrm{rank}(h) = r$.

The function h is a <u>submersion</u> if $\mathrm{rank}(h) = m$ and an <u>immersion</u> if $\mathrm{rank}(h) = k$. This is often denoted by $h : E \twoheadrightarrow \mathbb{R}^m$ and $h : E \hookrightarrow \mathbb{R}^m$, respectively.

Example 5.5. Returning to the map (5.2) defining the unit circle we compute:

$$Dh(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}, \tag{5.4}$$

and note note that h has rank 1 for every $x \in \mathbb{R}^2$ such that $x \neq 0$, i.e., for $\mathbb{R}^2 \backslash \{0\}$, and rank zero at $x = 0$. In particular, the rank of h is equal to 1 (hence h is a submersion) for every $x \in \mathbb{S}^1$.

Example 5.6. Consider now the map $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x) = x_1 x_2$. The set $g^{-1}(0)$ is the union of the subspaces of $\mathbb{R}^2$ defined by $x_1 = 0$ and by $x_2 = 0$. Geometrically, we see that $g^{-1}(0)$ is, locally, like $\mathbb{R}$ except at the origin where the subspaces defined by $x_1 = 0$ and by $x_2 = 0$ meet. Algebraically, we can compute the Jacobian of g:

$$Dg(x) = \begin{bmatrix} x_2 & x_1 \end{bmatrix},$$

and observe that its rank is also 1 for every $x \in \mathbb{R}^2$ such that $x \neq 0$ and zero at $x = 0$. However, it is no longer the case that the rank of g is the same for every $x \in g^{-1}(0)$. In particular, the rank drops from 1 to 0 at the origin suggesting the origin is different from the other points in $g^{-1}(0)$. This has important ramifications on determining if $g^{-1}(0)$ is a manifold.

Example 5.7. Let $x : I \to E$ be the solution of an ODE $\dot{x} = f(x)$. Since $Dx(t) = \dot{x}(t) = f(x(t))$, there are only two possible values for the rank of the map $x : I \to E$: 0 when $f(x(t)) = 0$ and 1 when $f(x(t)) \neq 0$. Hence, we conclude that a solution of an ODE is an immersion provided the initial condition is not an equilibrium. We will see, after defining manifolds, that solutions are therefore manifolds of dimension 1.

Smooth Manifolds. With the notion of rank in hand, and the intuition provided by the previous examples, we now introduce the notion of smooth manifold.

Definition 5.4. A set $M \subseteq \mathbb{R}^k$ is a smooth manifold if there exists a smooth function $h : \mathbb{R}^k \to \mathbb{R}^m$ satisfying the following:

- $M = h^{-1}(0) = \left\{ x \in \mathbb{R}^k \mid h(x) = 0 \right\}$,
- $\mathrm{rank}(h) = r$ on M, i.e., h has constant rank on M: $\mathrm{rank}(Dh(x)) = r$ for every $x \in M$.

The dimension of M is given by $\dim(M) = k - r$. Conversely, if $\dim(M) = n$ then $\mathrm{rank}(h) = k - n$.

***Remark* 5.2.** For notational simplicity, and without loss of generality, for a manifold M of dimension n, $\dim(M) = n$, we will often take $m = k - n$ and wherein $\mathrm{rank}(h) = r = k - n$. The connections between the rank of h and the dimension of M can be understood intuitively through the rank-nullity theorem, i.e., the decomposition (5.3), by virtue of the fact that: $\mathrm{rank}(h) + \dim(M) = k$.

Example 5.8. Recall the functions h and g defined in Example 5.4 and Example 5.6, respectively. As was seen, the function h has constant rank 1 on $h^{-1}(0) = \mathbb{S}^1$ and so $\mathbb{S}^1$ is a smooth manifold of dimension 1. Conversely, g does not have constant rank on $g^{-1}(0)$ and thus we cannot conclude it is a smooth manifold.

Example 5.9. Smooth manifolds are modeled after $\mathbb{R}^n$ and is thus natural to expect $\mathbb{R}^n$ to be a manifold. This is simple to establish since we can consider the constant function $h : \mathbb{R}^n \to \mathbb{R}^0 = \{0\}$ sending every $x \in \mathbb{R}^n$ to $h(x) = 0 \in \mathbb{R}^0$. Since the rank of h is 0 at every $x \in \mathbb{R}^n$ we conclude that $\mathbb{R}^n = h^{-1}(0)$ is a manifold of dimension n.

Example 5.10. When studying the stability of linear systems (see Lecture 8) one considers quadratic Lyapunov functions $\mathtt{V} : \mathbb{R}^k \to \mathbb{R}$ with $\mathtt{V}(x) = x^T P x$ where P is a symmetric and positive definite matrix. To show the level set $\mathtt{V}^{-1}(c)$ is a manifold for every $c \in \mathbb{R}_{>0}$ it suffices to consider the function $W(x) = \mathtt{V}(x) - c$ and note that $DW(x) = D\mathtt{V}(x) : \mathbb{R}^k \to \mathbb{R}^1$ is the linear function mapping $y \in \mathbb{R}^k$ to $DW(x)y = y^T P x + x^T P y = 2x^T P y$. Since P is an invertible matrix we see the rank of $DW(x)$ is 1 for every $x \neq 0$. Hence, $\mathtt{V}^{-1}(c)$ will be a manifold of dimension $k - 1$ for every $c \in \mathbb{R}_{>0}$ provided that $x \in \mathtt{V}^{-1}(c)$ implies $x \neq 0$. However, this directly follows from the positive definiteness of P.

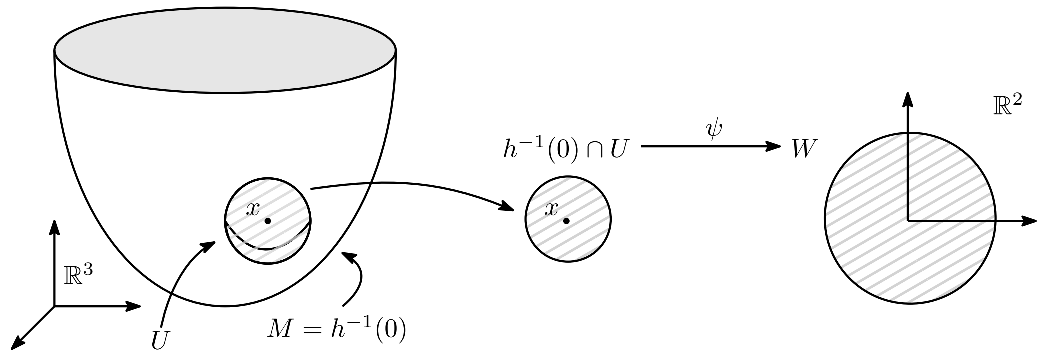


Figure 5.3. Graphical illustration of Proposition 5.1.

Example 5.11. Let $M_1 = h_1^{-1}(0)$ and $M_2 = h_2^{-1}(0)$ be manifolds where $h_1 : \mathbb{R}^{k_1} \to \mathbb{R}^{k_1 - n_1}$ and $h_2 : \mathbb{R}^{k_2} \to \mathbb{R}^{k_2 - n_2}$ are smooth maps. In this example we show that the Cartesian product $M_1 \times M_2$ is also a manifold. For this purpose we consider the map $h = (h_1, h_2) : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}^{k_1 - n_1} \times \mathbb{R}^{k_2 - n_2}$. It is simple to verify that h is a smooth map and that $M_1 \times M_2 = h^{-1}(0) = (h_1^{-1}(0), h_2^{-1}(0))$. Moreover, the rank of h is $k_1 - n_1 + k_2 - n_2$ since the matrix Dh is a block diagonal matrix with Dh_1 as its first block and Dh_2 as its second block. Thus, $\dim(M) = \dim(M_1) + \dim(M_2)$.

Local Coordinates via the Constant Rank Theorem Smooth manifolds of dimension n look, locally, like $\mathbb{R}^n$. To formally establish this, we will utilize the Constant Rank Theorem to show that functions h with constant rank $r = k - n$ locally look like maps whose level sets are obtained by setting the first r elements of the function to zero. Consequently, the remaining $k - r = n$ coordinates parameterize the manifold $h^{-1}(0)$. The end result of this correspondence are local coordinates on coordinate charts. This is illustrated in Figure 5.3 and made formal in the following.

Proposition 5.1. *Given a smooth manifold $M = h^{-1}(0) \subset \mathbb{R}^k$ with $\dim(M) = n$, for every $x \in M$ there exists an open set $U \subset \mathbb{R}^k$ containing x, an open set $W \subset \mathbb{R}^n$, and a diffeomorphism:*

$$\psi : h^{-1}(0) \cap U \to W \subset \mathbb{R}^n. \tag{5.5}$$

The local coordinates are points $q \in W$ wherein $\psi^{-1}(q) \in h^{-1}(0) = M$ and (U, ψ) is a coordinate chart.

We can paraphrase the previous result by saying that the manifold $h^{-1}(0)$ locally looks like the patch W of $\mathbb{R}^n$ where the correspondence between the local patch $h^{-1}(0) \cap U$ of $M = h^{-1}(0)$ and W is given by ψ. For this reason, (U, ψ) is termed a coordinate chart. It provides coordinates on $\mathbb{R}^n$ for the patch $h^{-1}(0) \cap U$.

Example 5.12. We illustrate the preceding result using again (5.2) in Example 5.4. Consider the point $(0, 1) \in \mathbb{S}^1$ and the set:

$$U_1 = (-3/4, 3/4) \times (1/2, 3/2). \tag{5.6}$$

Clearly, x belongs to U_1. We now construct a diffeomorphism $\psi_1 : \mathbb{S}^1 \cap U_1 \to W_1 = (-3/4, 3/4)$ by projecting points in $\mathbb{S}^1 \cap U_1$ onto the horizontal axis in $\mathbb{R}^2$, i.e.:

$$\psi_1(x_1, x_2) = x_1, \qquad \psi_1^{-1}(z) = \left(z, \sqrt{1 - z^2}\right). \tag{5.7}$$

Since $z \in (-3/4, 3/4)$, the square root is only being used for positive arguments and is thus a smooth function. We conclude that when working locally, on the set U_1, we can regard $\mathbb{S}^1$ as a copy of $\mathbb{R}$

embedded in $\mathbb{R}^2$ and parameterized by the coordinate $z = x_1$. The map φ_1 effectively "flattens" the segment of the unit circle that intersects U_1 to obtain the local coordinates $z \in W_1 \subset \mathbb{R}$ as illustrated in Figure 5.4.

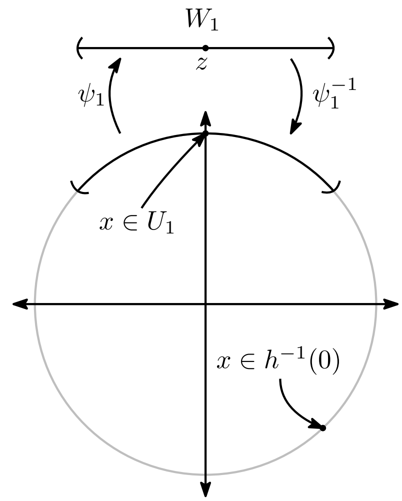


Figure 5.4. Illustration of the local coordinates defined on a segment of the unit circle.

We can extend this construction to the sets illustrated in Figure 5.5:

$$U_2 = (-3/4, 3/4) \times (-3/2, -1/2),\ U_3 = (1/2, 3/2) \times (-3/4, 3/4),\ U_4 = (-3/2, -1/2) \times (-3/4, 3/4),$$

so as to provide a chart for every point in $\mathbb{S}^1$ since $\mathbb{S}^1 \subset U_1 \cup U_2 \cup U_3 \cup U_4$; that is, this provides a "cover" of the circle. The reader can verify that diffeomorphisms $\psi_2 : \mathbb{S}^1 \cap U_2 \to W_2$, $\psi_3 : \mathbb{S}^1 \cap U_3 \to W_3$, and $\psi_4 : \mathbb{S}^1 \cap U_4 \to W_4$ can also be constructed based on the projections:

$$\psi_2(x_1, x_2) = x_1, \qquad \psi_3(x_1, x_2) = x_2, \qquad \psi_4(x_1, x_2) = x_2. \tag{5.8}$$

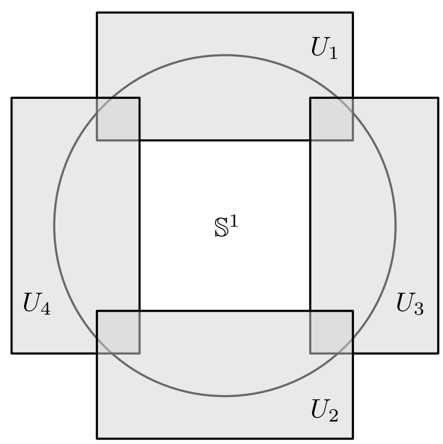


Figure 5.5. Illustration the covering of the unit circle by the cover consisting of the sets U_1, U_2, U_3, and U_4.

***Remark* 5.3.** Conceptually, one can also understand the relationship between maps ψ defining the local coordinates and the map h defining the manifold via immersions and submersions (termed an exact sequence):

$$W \subset \mathbb{R}^n \xhookrightarrow{\psi^{-1}} \mathbb{R}^k \xrightarrow{h} \mathbb{R}^{k-n}$$

wherein:

$$\mathrm{image}(\psi^{-1}) = \ker(h|_U) = h^{-1}(0) \cap U.$$

This is made apparent by the constant rank theorem. The proof is beyond scope of this lecture, but references providing the missing details can be found in Additional Reading

Theorem 5.2 (**Constant Rank Theorem**). *Let $h : E \subseteq \mathbb{R}^k \to \mathbb{R}^m$ be a smooth map with E an open set, and assume that* $\operatorname{rank}(h) = r$. *Then, for each $x \in E$ there exist an open set $U \subseteq \mathbb{R}^k$ containing x and an open set $V \subseteq \mathbb{R}^m$ containing $h(x)$, together with diffeomorphisms $\alpha : U \to \alpha(U) \subseteq \mathbb{R}^k$ and $\beta : V \to \beta(V) \subseteq \mathbb{R}^m$ such that for all $z \in \alpha(U) \subseteq \mathbb{R}^k$:*

$$\hat{h}(z) \triangleq \beta \circ h \circ \alpha^{-1}(z_1, \ldots, z_r, z_{r+1}, \ldots, z_k) = (z_1, \ldots, z_r, \underbrace{0, \ldots, 0}_{m-r}). \tag{5.9}$$

In the case of a manifold $M = h^{-1}(0)$ of dimension n with $h : \mathbb{R}^k \to \mathbb{R}^{k-n}$, it follows that $r = m = k - n$ and therefore $\hat{h}(z_1, \ldots, z_r, z_{r+1}, \ldots, z_k) = (z_1, \ldots, z_r)$.

***Remark* 5.4.** The Constant Rank Theorem can be illustrated via a diagram that commutes (the composition of all the functions labeling arrows from the same set and leading to the same set are the same function):

$$\begin{tikzcd} U \subseteq \mathbb{R}^k \arrow[rr, "h"] \arrow[d, shift right, "\alpha"'] & & V \subseteq \mathbb{R}^m \arrow[d, "\beta"] \\ \alpha(U) \arrow[u, shift right, "\alpha^{-1}"'] \arrow[rr, "\hat{h} = \beta \circ h \circ \alpha^{-1}"] & & \beta(V) \end{tikzcd} \tag{5.10}$$

***Remark* 5.5.** Note that since α and β are diffeomorphisms, locally $h^{-1}(0) \cong \hat{h}^{-1}(\beta(0)) \cong \hat{h}^{-1}(0)$ and $\hat{h}^{-1}(0)$ can be identified as a subset of $\mathbb{R}^{k-r}$. That is, given a point $z \in \hat{h}^{-1}(0)$ it follows that $z = (0, \ldots, 0, z_{r+1}, \ldots, z_k)$ and therefore $(z_{r+1}, \ldots, z_k) \in \mathbb{R}^{k-r}$ uniquely identifies points in $\hat{h}^{-1}(0)$ and hence $h^{-1}(0)$. (This will be formalized in the proof of Proposition 5.1.) We can therefore view $h^{-1}(0)$ locally as a subset of Euclidean space and, in the case when $r = k - n$, a subset of $\mathbb{R}^n$.

Proof of Proposition 5.1. To establish Proposition 5.1, we utilize the Constant Rank Theorem to first construct local coordinates for $\hat{h}$ which are then used to construct local coordinates for h.

For simplicity of exposition, and without loss of generality[3], assume that $\beta(0) = 0$. Without loss of generality (see Remark 5.2), assuming $r = m = k - n$ we have that:

$$\hat{h}(z) = \hat{h}(z_1, \ldots, z_{k-n}, z_{k-n+1}, \ldots, z_k) = (z_1, \ldots, z_{k-n}).$$

Therefore:

$$\hat{h}^{-1}(0) \cap \alpha(U) = \{z \in \alpha(U) \subset \mathbb{R}^k \ : \ z_1 = \cdots = z_{k-n} = 0\} \cong W \subset \mathbb{R}^n.$$

The symbol $\cong$ denotes the existence of a diffeomorphism between $\hat{h}^{-1}(0) \cap \alpha(U)$ and $W \subseteq \mathbb{R}^n$. Such a diffeomorphism is given by the projection $\pi : \alpha(U) \to \mathbb{R}^n$ with $\pi(z) = (z_{k-n+1}, \ldots, z_k)$. This is clearly a smooth map with inverse $\iota : \mathbb{R}^n \to \mathbb{R}^k$ given by $\iota(q) = (0, q)$, where $0 \in \mathbb{R}^{k-n}$. The reader can verify that $\pi \circ \iota(q) = q$. Hence, taking $W = \pi(\alpha(U))$ we obtain subset of $\mathbb{R}^n$ together with a diffeomorphism:

$$\begin{tikzcd} \hat{h}^{-1}(0) \cap \alpha(U) \arrow[r, shift left, "\pi"] & W = \pi(\alpha(U)) \subset \mathbb{R}^n. \arrow[l, shift left, "\iota"] \end{tikzcd}$$

[3] In the case when $\beta(0) \neq 0$, one considers $\hat{h}^{-1}(\beta(0))$ and the canonical embedding becomes $\iota(q) = (\beta(0), q)$. The maps that utilize ι must be modified accordingly but the proof remains the same.

It then follows that for any $q \in W$ we have $\hat{h}(\iota(q)) = \beta \circ h \circ \alpha^{-1}(\iota(q)) = 0$.

To find the diffeomorphism in (5.5), we repurpose these constructions for $\hat{h}$ and apply them to h. First note that for $q \in W = \pi(\alpha(U)) \subset \mathbb{R}^n$ we have:

$$x = \alpha^{-1} \circ \iota(q) \qquad \xRightarrow{\beta \circ h \circ \alpha^{-1}(\iota(q))=0} \qquad h(x) = \beta^{-1}(0) = 0 \qquad \Longrightarrow \qquad x \in h^{-1}(0) \cap U.$$

As a result we obtain the desired diffeomorphism:

$$h^{-1}(0) \cap U \underset{\psi^{-1} = \alpha^{-1} \circ \iota}{\overset{\psi \triangleq \pi \circ \alpha}{\rightleftarrows}} W = \pi(\alpha(U)) \subset \mathbb{R}^n.$$

Therefore, this map associates $h^{-1}(0)$ with a subset[4] of $\mathbb{R}^n$ with local coordinates $q \in \mathbb{R}^n$. $\square$

Example 5.13. We return to the unit circle, last considered in Example 5.12, with the purpose of constructing the map α appearing in the definition of $\hat{h}$ given in (5.9). In this example we can take β to be the identity map on $\mathbb{R}$ so that $\hat{h} = h \circ \alpha^{-1}$. This suggests that α^{-1} is given by:

$$\alpha^{-1}(z) = \left(z_1, \sqrt{1 - z_1^2}\right),$$

so that:

$$\hat{h}(z) = h \circ \alpha^{-1}(z) = z_1^2 + \left(\left(1 - z_1^2\right)^{\frac{1}{2}}\right)^2 - 1 = 0.$$

Hence, α is the diffeomorphism:

$$\alpha(x) = (x_1, x_1^2 + x_2^2 - 1).$$

Note how the first component of α is simply the projection ψ_1 in Example 5.12.

5.3 Stable Manifold Theorem

We now have the necessary framework to introduce stable and unstable manifolds, and therefore the stable manifold theorem.

Stable and Unstable Manifolds. Consider a nonlinear system $\dot{x} = f(x)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ a locally Lipschitz continuous map and $x^* = 0$ an equilibrium point. Further assume that $I(x) = \mathbb{R}$ for all $x \in \mathbb{R}^n$, i.e., assume that solutions are defined for all time. The stable and unstable manifolds are given by:

$$S \triangleq \left\{x \in \mathbb{R}^n \mid \lim_{t\to\infty} \varphi_t(x) = 0\right\}, \tag{5.11}$$

$$U \triangleq \left\{x \in \mathbb{R}^n \mid \lim_{t\to-\infty} \varphi_t(x) = 0\right\}, \tag{5.12}$$

where φ_t is the flow of $\dot{x} = f(x)$. Note that, as a result, S is forward invariant with respect to φ_t and U is backward invariant. These can also be defined locally when $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$ for E an open and connected subset $\mathbb{R}^n$:

$$\begin{aligned} S(E) &\triangleq \left\{x \in E \mid \varphi_t(x) \in E \quad \forall t \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \varphi_t(x) = 0\right\}, \\ U(E) &\triangleq \left\{x \in E \mid \varphi_{-t}(x) \in E \quad \forall t \geq 0 \quad \text{and} \quad \lim_{t\to-\infty} \varphi_t(x) = 0\right\}. \end{aligned}$$

[4]This also motivates the notion of dimensionality used, i.e., $\dim(h^{-1}(0)) = n$, since it can be associated locally with n-dimensional Euclidean space.

Stable Manifold Theorem. The stable manifold theorem characterizes the stable and unstable components of a nonlinear system in terms of the linearization.

Theorem 5.3. *Consider the nonlinear differential equation $\dot{x} = f(x)$ with $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable map and E an open and connected set. Let $x^* = 0$ be an hyperbolic equilibrium point, i.e., $f(0) = 0$ and all eigenvalues of $Df(0)$ have non-zero real part:*

$$\begin{array}{lll} \Re(\lambda_i) < 0 & \text{if} & i = 1, \ldots, k \\ \Re(\lambda_i) > 0 & \text{if} & i = k+1, \ldots, n. \end{array}$$

Then $S(E)$ is a k-dimensional manifold, $U(E)$ is an $n-k$-dimensional manifold.

***Remark* 5.6.** While beyond the scope of this lecture, if there exists eigenvalues such that $\Re(\lambda_i) = 0$, then there is also a *center manifold* C with dimension equal to the number of these eigenvalues.

Theorem 5.3 can be used to characterize the stable and unstable behavior of $\dot{x} = f(x)$.

Corollary 5.1. *Under the same assumptions of Theorem 5.3, let $\alpha, \beta \in \mathbb{R}_{>0}$ satisfy:*

$$\Re(\lambda_i) < -\alpha < 0 < \beta < \Re(\lambda_m),$$

for $i = 1, \ldots, k$ and $m = k+1, \ldots, n$. Given $\varepsilon > 0$, the exists $\delta > 0$ such that:

$$\begin{array}{lll} x_0 \in B_\delta(0) \cap S(E) & \Longrightarrow & \|\varphi_t(x_0)\| \leq \varepsilon e^{-\alpha t} \\ x_0 \in B_\delta(0) \cap U(E) & \Longrightarrow & \|\varphi_t(x_0)\| \leq \varepsilon e^{\beta t}. \end{array}$$

The stable manifold theorem, although asserting the existence of manifolds, considered dynamics defined on $\mathbb{R}^n$. There are many physical systems, however, whose state space is a manifold. In those cases one can either work locally on $\mathbb{R}^n$ using a coordinate chart or directly model the dynamics on the state space manifold in a global manner. The latter requires the concepts of tangent spaces and vector fields that we now introduce.

Tangent Spaces. Let M be a manifold and let $\gamma : I = (-a, a) \subset \mathbb{R} \to M$ be a smooth curve, for some $a \in \mathbb{R}_{>0}$, passing through the point $x \in M$ at $t = 0$, i.e., $\gamma(0) = x$. If we differentiate γ at $0 \in I$ we obtain a vector describing the velocity of γ at the point $x \in M$. Since γ describes a path in M, the velocity vector $\dot{\gamma}(0)$ must be tangent to M at x, otherwise the curve defined by γ would leave M. The tangent space of M at the point x is defined by the collection of all such tangent vectors. There is an abundance of tangent vectors, in fact, we will shortly see they form a vector space.

To formalize the notion of tangent space we note that, since $\gamma(t) \in M$ for all $t \in I$, we have $h \circ \gamma(t) = 0$ for all $t \in I$ and where h is the function defining M, i.e., $M = h^{-1}(0)$. Hence:

$$h \circ \gamma(t) \equiv 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\Big|_{t=0} (h \circ \gamma) = Dh(\gamma(0))\dot{\gamma}(0) \overbrace{=}^{\gamma(0)=x} Dh(x)\dot{\gamma}(0) = 0.$$

Noting that $Dh(x)$ is a linear map, we conclude the tangent vector $\dot{\gamma}(0)$ belongs to $\ker Dh(x)$.

Definition 5.5. Let $M = h^{-1}(0)$ be a smooth manifold. The <u>tangent space</u> of M at the <u>base point</u> x, denoted by T_xM, is given by:

$$T_xM \triangleq \ker Dh(x).$$

The <u>tangent bundle</u>, denoted by TM, is given by:

$$TM \triangleq \bigsqcup_{x \in M} T_xM = \bigcup_{x \in M} \{(x, v) \mid x \in M,\ v \in T_xM\},$$

with elements $(x, v) \in TM$ where $x \in M$ and $v \in T_xM$.

Several observations are in order:

- The tangent space at T_xM is a vector space since it was defined as the kernel of a linear map.
- The dimension of the vector space T_xM is the difference between the dimension of the domain and of the codomain of h, i.e., $k-(k-n)=n$ when $m=k-n$. This is the same as the dimension of M.
- The tangent map of h is the function $Th(x,v)\triangleq(h(x),Dh(x)v)$, and we note that TM can be written as $(Th)^{-1}(0)$. Since Th is a smooth map of rank $2(k-n)$ we conclude that TM is a manifold of dimension $2n$.

Example 5.14. We return, once more, to the circle $\mathbb{S}^1$ seen as an embedded submanifold of $\mathbb{R}^2$ defined by $h(x)=x_1^2+x_2^2-1$. If we consider the point $(0,1)\in\mathbb{S}^1$ and all the curves $\gamma: I\to\mathbb{S}^1$ satisfying $\gamma(0)=(0,1)$, we can observe that its derivatives $D\gamma(0)$ coincide with the set:

$$T_{(0,1)}\mathbb{S}^1=\left\{x\in\mathbb{R}^2 \mid x=(v,0), v\in\mathbb{R}\right\},$$

which can be identified with the vector space $\mathbb{R}$ "attached" to the base point $(0,1)$. This follows from the fact that $Dh(0,1)=[0\ \ 1]$ per (5.4).

Similarly, the set of vectors tangent to $\mathbb{S}^1$ at the point $(1,0)$ is given by:

$$T_{(1,0)}\mathbb{S}^1=\left\{x\in\mathbb{R}^2 \mid x=(0,v), v\in\mathbb{R}\right\},$$

which can again be identified with $\mathbb{R}$ although it is attached to a different base point. Hence, when forming the tangent bundle of $\mathbb{S}^1$ one must consider the collection of all base points and corresponding vector spaces.

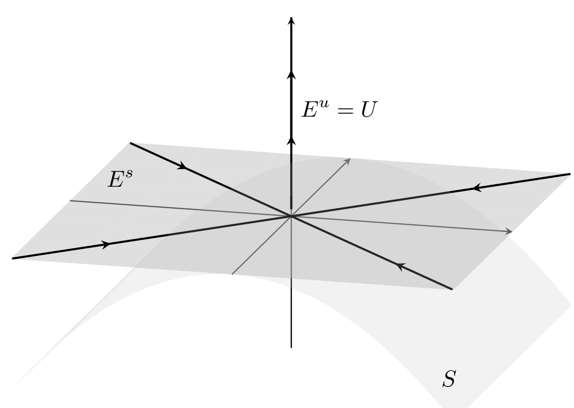
The following result illustrates the direct connection between tangent spaces and the linearization of a system.

Theorem 5.4 (**Stable Manifold Theorem Continued**). *Under the assumptions of Theorem 5.3, with $S(E)$ and $U(E)$ the stable and unstable manifold:*

$$T_0S(E)=E^s,\qquad T_0U(E)=E^u,$$

where E^s and E^u is the stable and unstable subspaces for $\dot{x}=Df(0)$, respectively.

Figure 5.6. Illustration of the stable and unstable manifolds, and the corresponding stable and unstable subspaces, associated with Example 5.15.



Example 5.15. Consider the nonlinear system (building upon Example 4.1):

$$\dot{x} = f(x) = \begin{bmatrix} -x_1 \\ -x_2 + x_1^k \\ x_3 + x_1^2 \end{bmatrix},$$

where $x^* = 0$ is an equilibrium point, and $k \in \mathbb{N}_{\geq 2}$. The Jacobian of f and its eigenvalues are:

$$Df(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \lambda_1 = \lambda_2 = -1, \quad \lambda_3 = 1.$$

The stable subspace E^s of the linearization at $x^* = 0$ is the $x_1 - x_2$ plane and the unstable subspace E^u is the x_3 axis. What does this say about the original nonlinear system? According to Theorem 5.3 and Theorem 5.4, these subspaces are tangent to the stable and unstable manifolds. Note that the stable and unstable subspaces do not change with $k \in \mathbb{N}_{\geq 2}$; this indicates both that the local information is uniformly captured while more global information is lost.

Vector Fields. If manifolds model the state space of nonlinear systems, we need a way of describing dynamics on manifolds. The relevant geometric object is called a <u>vector field</u>. It specifies how to assign a velocity vector to each point of the manifold and is thus a field of velocities or vectors, hence the name vector field. To introduce vector fields we need to emphasize that each point $x \in M$ has its own tangent space T_xM and this is the reason for the subscript x in T_xM. This can be observed in the equation $Th(x, v) = (h(x), Dh(x)v) = (0, 0)$ defining TM. The equation defining $v \in T_xM$ is parameterized by x. As x changes so do the solutions of $Dh(x)v = 0$. Geometrically, we describe this dependence on $x \in M$ by using the canonical projection:

$$\pi : TM \to M, \qquad \pi(x, v) = x,$$

that maps to each element (x, v) of TM to its base point $x \in M$.

Definition 5.6. A <u>vector field</u> on a manifold M is a smooth map:

$$X : M \to TM, \qquad \text{s.t.} \qquad \pi \circ X = \mathrm{id}_M,$$

where $\mathrm{id}_M : M \to M$ is the identity map on M.

It is pedagogical to write the equality $\pi \circ X = \mathrm{id}_M$ as the commutative diagram:

$$\begin{tikzcd} M \arrow[r, "X"] \arrow[d, "\mathrm{id}_M"'] & TM \arrow[dl, "\pi"] \\ M & \end{tikzcd} .$$

In this diagram we have the functions $\pi \circ X$ and id_M labeling the sequence of arrows leaving M and arriving at M. Hence, for this diagram to commute these functions must be the same, i.e., $\pi \circ X(x) = \mathrm{id}_M(x)$ for every $x \in M$.

To reconcile the notion of vector field with the notion of differential equation, we note that the differential equation $\dot{x} = f(x)$ associates the vector $f(x) \in \mathbb{R}^n$ to the point $x \in \mathbb{R}^n$. Hence, we can regard $\dot{x} = f(x)$ as the vector field $X(x) = (x, f(x)) \in T_xM \subset TM$. The membership $(x, f(x)) \in T_xM$ can be written as $Th(x, f(x)) = (h(x), Dh(x)f(x)) = (0, 0)$ showing that x must belong to M ($h(x) = 0$) and $f(x)$ must belong to T_xM ($Dh(x)f(x) = 0$).

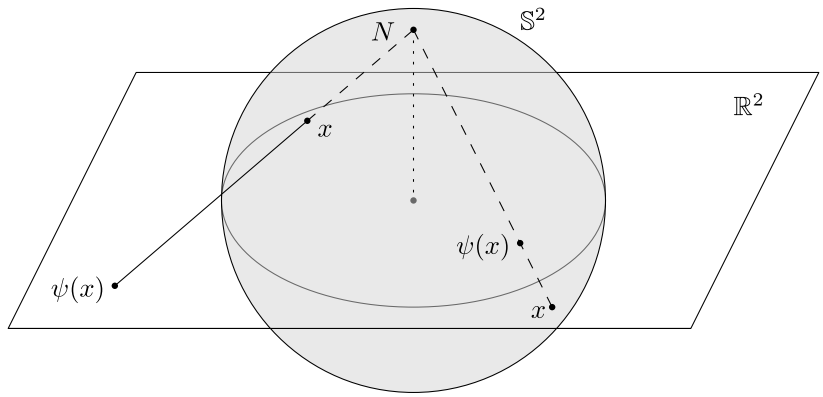


Figure 5.7. The sphere, $\mathbb{S}^2$, together with an illustration of the stereographic projection from the north pole.

Example 5.16. We now consider an example of a vector field defined on the 2-dimensional sphere:

$$\mathbb{S}^2 \triangleq \left\{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

One can verify that this is, in fact, a manifold by noting that $\mathbb{S}^2 = h^{-1}(0)$ where $h : \mathbb{R}^3 \to \mathbb{R}$ is defined as $h(x) = x_1^2 + x_2^2 + x_3^2 - 1$.

The stereographic projection is frequently used as a coordinate chart for $\mathbb{S}^2$ since it covers the entire sphere minus a point—in this case, the "north pole" given by $N = (0, 0, 1)$. For any point $x \in \mathbb{S}^2 \backslash \{N\}$ we construct a line passing through x and N and consider the point, z, where this line intersects the plane: $\{x \in \mathbb{R}^3 \mid x_3 = 0\} \cong \mathbb{R}^2$. This defines the map $x \mapsto z = \psi(x)$ given by:

$$\psi(x) = \frac{1}{1 - x_3} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5.13}$$

which, in turn, defines a local coordinate chart $\psi : \mathbb{S}^2 \backslash \{N\} \to \mathbb{R}^2$ with inverse:

$$\psi^{-1}(z) = \frac{1}{1 + z_1^2 + z_2^2} \begin{bmatrix} 2z_1 \\ 2z_2 \\ z_1^2 + z_2^2 - 1 \end{bmatrix}.$$

We can use this coordinate chart to map a vector field in $\mathbb{R}^2$ to one on the sphere. To illustrate how "simple" dynamics on $\mathbb{R}^2$ can result in "complex" dynamics on $\mathbb{S}^2$, consider the ODE on $\mathbb{R}^2$ given by:

$$\dot{z} = f(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{5.14}$$

This defines a vector field: $Z : \mathbb{R}^2 \to T\mathbb{R}^2$ with $Z(z) = (z, f(z))$. Since ψ^{-1} maps $\mathbb{R}^2$ to $\mathbb{S}^2 \backslash \{N\}$, we can use $T\psi^{-1} : T\mathbb{R}^2 \to T(\mathbb{S}^2 \backslash \{N\})$ to map the vector field Z to the vector field X on $\mathbb{S}^2$:

$$X(x) = T\psi^{-1} \circ Z \circ \psi(x) = (x, g(x)), \qquad g(x) \triangleq (1 - x_1^2 - x_3, -x_1 x_2, x_1 - x_1 x_3). \tag{5.15}$$

To see where the expression for X comes from, let us interpret Z as the ODE $\dot{z} = f(z)$ given above. Then, regarding ψ^{-1} as a change of coordinates (see Example 4.7) we obtain:

$$x(t) = \psi^{-1}(z(t)) \implies \dot{x}(t) = D\psi^{-1}(z(t))\dot{z}(t) = D\psi^{-1}(z(t)) f(z(t)) = \underbrace{D\psi^{-1}(\psi(x(t))) f(\psi(x(t)))}_{g(x(t)},$$

which is just the expression in (5.15). The final step consists in extending the definition of X to the whole sphere. This can be done by continuity since for any curve $\gamma : \mathbb{R}_{>0} \to \mathbb{S}^2$ such that $\lim_{t\to\infty} \gamma(t) = (0, 0, 1) = N$ we have:

$$\lim_{t\to\infty} X \circ \gamma(t) = (N, 0).$$

Hence, we define X at N to be 0. The readers can see a depiction of X in Figure 6.2. We have, therefore, shown that the constant vector field on $\mathbb{R}^2$ maps to a vector field on $\mathbb{S}^2$ with a single equilibrium point that is globally stable (on the sphere). This will be discussed further in the next lecture, and later in the context of the "hairy ball theorem" presented in Lecture 17.

Additional Reading

Perko, [166], does an excellent job presenting the stable and unstable manifold theorem. The reader is referred to this reference for proofs of the theorems presented on stable subspaces and stable manifolds, along with additional examples. For more on smooth manifolds, see the excellent book by Lee [128] or the equally excellent book by Abraham, Marsden, and Ratiu [1]. Specifically, The Constant Rank Theorem can be found in [128] (see Theorem 8.8). The advanced problems which treat solutions to vector fields through morphisms and commutative diagrams are inspired by categorical approaches to dynamical systems [8, 86]; see also Problem P5.10

Problems for Lecture 5

[P5.1] Following Example 5.12, show that the functions:

$$\psi_i : h^{-1}(0) \cap U_i \to W_i \subset \mathbb{R}, \qquad i = 2, 3, 4,$$

defined in (5.8) are, in fact, diffeomorphisms for the appropriate choice of W_i and therefore define local coordinate charts. In particular, utilizing the methods from the proof of Proposition 5.1, construct diffeomorphisms α_i defined on U_i, for $i = 2, 3, 4$, wherein $W_i = \pi_i(\alpha_i(U_i))$ and $\psi_i = \pi_i \circ \alpha_i$ where π_i is the appropriate projection.

[P5.2] The torus, $\mathbb{T} \subset \mathbb{R}^3$, is a surface that is defined by the function:

$$\mathbb{T} \triangleq \left\{ x \in \mathbb{R}^3 \mid h(x) = \left(\sqrt{x_1^2 + x_2^2} - R \right) + x_3^2 - r^2 = 0 \right\},$$

where R is the major radius, r is the minor radius, and $R > r$.

(a) Show that the torus is a manifold.

(b) Pick an open set $U \subset \mathbb{R}^3$ and find the diffeorphism $\psi : \mathbb{T} \cap U \to W \subset \mathbb{R}^2$ together with local coordinates on this chart.

[P5.3] A robotic system with n revolt joints has as its configuration space n copies of the unit circle, $\mathbb{S}^1$, with each copy of the circle corresponding to each joint. The local coordinates are the angle $\theta_i \in \mathbb{S}^1$ of each joint. This results in the configuration space:

$$Q = \underbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n \text{ times}}.$$

Prove that Q is a manifold. Note that the torus is given by $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, thus this problem provides a simple proof that the torus is a manifold.

[P5.4] Consider the dynamical system, $\dot{x} = f(x)$, introduced in Example 5.15. For this system:

(a) Find an explicit expression for the stable and unstable manifolds, S and U, respectively, i.e., find functions $h_S, h_U : \mathbb{R}^3 \to \mathbb{R}$ defining these surfaces: $S = h_S^{-1}(0)$ and $U = h_U^{-1}(0)$.

(b) Establish that the surfaces S and U are, in fact, manifolds. Find local coordinates for these manifolds in a neighborhood of $0 \in \mathbb{R}^3$ via coordiante charts.

(c) Verify that the tangent spaces to the manifolds S and U at 0 are the stable and unstable subspaces, E^s and E^u, respectivly.

[P5.5] Let X be a set and let $\sim \subseteq X \times X$ be a binary relation. We write $a \sim b$ iff $(a, b) \in \sim$. The binary relation $\sim$ is an equivalence relation if it satisfies the following properties for every $a, b, c \in X$:

$$\begin{aligned} \text{Reflectivity}: &\quad a \sim a \\ \text{Symmetry}: &\quad a \sim b \quad \Longrightarrow \quad b \sim a \\ \text{Transitivity}: &\quad a \sim b,\ b \sim c \quad \Longrightarrow \quad a \sim c. \end{aligned}$$

The equivalence class of an element $a \in X$ under $\sim$, denoted by $[a]$, is given by:

$$[a] \triangleq \{b \in X \mid a \sim b\},$$

and the quotient of X by $\sim$, denoted by $X/\sim$, is the set of all the equivalnce classes, i.e:

$$X/\sim \ \triangleq \{[a] | a \in X\}.$$

Thus if $a \sim b$ we have $[a] = [b]$. Using the notion of an equivalence relation:

(a) Consider the sets $A, B \subset \mathcal{V}$, with $\mathcal{V}$ a normed vector space, and let $f : A \to B$ be a surjective function. Define the binary relation $\sim \subseteq \mathcal{V} \times \mathcal{V}$ by $a \sim b$ iff $f(a) = f(b)$. Show that $\sim$ is an equivalence relation.

(b) For the equivalence relation given in Part (a), show that the induced function defined on the equivalence class:

$$\tilde{f} : A/\sim \ \to \ B, \qquad \tilde{f}([a]) := f(a)$$

is a bijection. If f is continuous, show that $\tilde{f}$ is a homeomorphism.

[P5.6] Construct an equivalence relation, $\sim$, on $\mathbb{R}^{n+1} \backslash \{0\}$ such that $\mathbb{S}^n = \left(\mathbb{R}^{n+1} \backslash \{0\}\right) / \sim$.

[P5.7] [**Advanced Problem**] Equivalence relations can be used to give an alternative formulation of the tangent space to a manifold M. Consider all the continuously differentiable curves:

$$\gamma : I = (-a, a) \subset \mathbb{R} \to M, \qquad \gamma(0) = x \in M, \qquad a \in \mathbb{R}_{>0}.$$

Let $\Gamma_x = \{\gamma \in C^1(I, M) \mid \gamma(0) = x\}$ be the set of all curves of this form mapping 0 to x. On the set Γ_x, define the binary relation $\sim$ by:

$$\gamma_1 \sim \gamma_2 \qquad \text{if} \qquad \dot{\gamma}_1(0) = \dot{\gamma}_2(0). \tag{5.16}$$

where here $\gamma_1(0) = \gamma_2(0) = x$. Therefore, two paths are equivalent if they have the same derivative (or tangent vector) at $t = 0$. Show the following:

(a) The relation $\sim \subset \Gamma_x \times \Gamma_x$ defined by (5.16) is an equivalence relation.

(b) For the T_xM to M at x, there is a homeomorphism:

$$T_xM \cong \Gamma_x / \sim .$$

[P5.8] [**Advanced Problem**] Maps between manifolds induce corresponding maps between tangent spaces: the pushforward or tangent maps. Given a smooth map between manifolds, $F : M \to N$, the pushforward is defined as a map that makes the following diagram commute:

$$\begin{tikzcd} TM \arrow[r, "TF"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "F"] & N \end{tikzcd} \tag{5.17}$$

From this implicit definition, show that for all $x \in M$, the pushforward $T_xF : T_xM \to T_{F(x)}N$ is explicitly given by the linear map: $T_xF = DF(x)$.

[P5.9] [**Advanced Problem**] Solutions to vector fields on manifolds can be framed in a way that does not require coordinates through the use of the push forward. Consider an interval $I = (-a, a)$, we can define the "unit clock" vector field via::

$$\mathbf{1} : I \to TI \cong I \times \mathbb{R}, \qquad \mathbf{1}(\tau) \triangleq (\tau, 1),$$

Thus, the unit clock is simply the vector constant vector field: $\dot{\tau} = 1$ which implies that $\tau(t) = t$ when $\tau(0) = 0$. Now consider a path $c : I \subset \mathbb{R} \to M$, with M a manifold. This path is a solution to a vector field $X : M \to TM$ if the following diagram commutes:

$$\begin{tikzcd} I \times \mathbb{R} \arrow[r, "Tc"] & TM \\ I \arrow[u, "\mathbf{1}"] \arrow[r, "c"] & M \arrow[u, "X"'] \end{tikzcd} \tag{5.18}$$

(a) Let $\dot{x} = f(x)$ be a differential equation which evolves on M, i.e., $f(x) \in T_xM$ for all $x \in M$. Show that this notion of solution for vector fields is equivalent to the standard notion of a solution to a differential equation. Specifically, that the following diagram commutes:

$$\begin{tikzcd} I \times \mathbb{R} \arrow[r, "Tc"] & TM \\ I \arrow[u, "\mathbf{1}"] \arrow[r, "c"] & M \arrow[u, "(\cdot, f(\cdot))"'] \end{tikzcd} \tag{5.19}$$

if and only if $\dot{c}(t) = f(c(t))$.

(b) Consider two dynamical systems, $\dot{x} = f(x)$ and $\dot{y} = g(y)$ for $x \in U \subset \mathbb{R}^n$ and $y \in V \subset \mathbb{R}^n$. Recall from Definition 4.4 that these two dynamical systems are orbitally equivalent if there exists diffeomorphism $H : U \to V$ together with a diffeomorphism $\tau(\cdot\,, x) : I_f(x) \to I_g(H(x))$ satisfying $\dot{\tau}(t, x) > 0$ such that:

$$H(\varphi^f_t(x)) = \varphi^g_{\tau(t,x)}(H(x)).$$

Find a diagram such that if the diagram commutes it implies that f and g are orbitally equivalent.

(c) Use part (b) to define orbital equivalence for vector fields $X : M \to TM$ and $Y : N \to TN$. That is, a diagram such that the diagram commuting implies that X and Y are orbitally equivalent.

[P5.10] [**Advanced Problem**] A Category C consists of the following: collection of objects, $\mathrm{Ob}(\mathsf{C})$ and a collection of morphisms $\mathrm{Hom}(A, B)$ between any two objects $A, B \in \mathrm{Ob}(\mathsf{C})$ where $F \in \mathrm{Hom}(A, B)$ is denoted by $F : A \to B$. Finally, a composition operation for any $A, B, C \in \mathrm{Ob}(\mathsf{C})$:

$$\begin{aligned} \circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) &\to \mathrm{Hom}(A, C) \\ (F, G) &\mapsto G \circ F \end{aligned}$$

satisfying the properties:

- Identity: For all $C \in \mathsf{C}$, there exits a morphism $1_C : C \to C$ such that for all $F : A \to B$: $1_B \circ F = F \circ 1_A$.
- Associativity: If $F : A \to B$, $G : B \to C$, and $H : C \to D$ then: $H \circ (G \circ F) = (H \circ G) \circ F$.

With the notion of a category:

(a) Define the category of dynamical systems, Dyn, as having objects pairs (M, X) where M is a manifold and X is a vector field on this manifold. A morphism between objects in Dyn, $F : (M, X) \to (N, Y)$, is a smooth map $F : M \to N$ such that the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "TF"] & TN \\ M \arrow[u, "X"] \arrow[r, "F"] & N \arrow[u, "Y"'] \end{tikzcd} \tag{5.20}$$

Define the composition operation and show that this is a well-defined operation, i.e., satisfies the identity and associativity properties.

(b) Consider the collection of objects in Dyn of the form $(I, \mathbf{1})$ for intervals $I = (-a, a)$, for $a \in \mathbb{R}_{>0}$. Given a dynamical system $(M, X) \in \mathrm{Ob}(\mathsf{Dyn})$, show that solutions to the dynamical system are just morphisms in Dyn of the form:

$$c : (I, \mathbf{1}) \to (M, X).$$

(c) An isomorphism between objects $A, B \in \mathsf{C}$ in a category is a morphism $F : A \to B$ such that there exists a morphism F^{-1} with $F \circ F^{-1} = 1_B$ and $F^{-1} \circ F = 1_A$. Show that if two dynamical systems, (M, X) and (N, Y) are isomorphic, i.e., there exists an isomorphism $H : (M, X) \to (N, Y)$, then the dynamical systems are orbitally equivalent. That is, isomorphic objects in Dyn are orbitally equivalent.