Lecture 6

Notions of Stability

This lecture introduces stability for nonlinear systems about an equilibrium point—a fundamental concept in engineering. In particular, we note the following important properties related to the stability of nonlinear systems and how these differ from linear systems:

- For a linear system, $\dot{x} = Ax$ with $A \in \mathbb{R}^{n \times n}$, one can completely characterize stability properties in terms of the eigenvalues of A; these results are global, necessary, and sufficient.

- For a nonlinear system, $\dot{x} = f(x)$ with $f(0) = 0$, there is no algebraic or semi-algebraic criterion (such as negativity of the eigenvalue's real-part) fully characterizing stability properties.

- For general nonlinear systems, stability is not necessarily a global phenomenon and multiple equilibria may lead to quite complex behavior.

Therefore, for nonlinear systems a different paradigm (from checking eigenvalues) must be considered: *Lyapunov's method*. Before introducing this method in the next lecture, we will introduce an important class of functions needed to prove Lyapunov's theorem: class $\mathcal{K}$ functions. This class of functions allows us to recast the definition of stability in a form that streamlines the use of Lyapunov functions to establish different stability properties. They also provide a means for establishing forward set invariance, i.e., *safety*, of nonlinear systems. Finally, we will conclude the lecture with an example motivating the use of Lyapunov functions.

6.1 Types of Stability

Let $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, be a dynamical system where $f : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on the open and connected set $E \subseteq \mathbb{R}^n$. As introduced in Lecture 4, an equilibrium (the plural is equilibria) point for $\dot{x} = f(x)$ is a point $x^* \in E$ satisfying $f(x^*) = 0$. Equilibria are quite important as they define very special solutions, $x(t) \equiv x^*$, that start at x^* and remain at x^* forever, i.e., they define invariant sets (see Definition 4.2). Hence, it is natural to ask if this behavior is shared by trajectories starting close by, i.e., will solutions with initial conditions close to x^* remain close to x^*? Or, perhaps, even converge to x^*?

Types of Stability. In this lecture (and throughout the book) we consider three different types of stability. They are all local notions in the sense they only need to hold on an open ball. Global notions of stability will be discussed in Lecture 7.

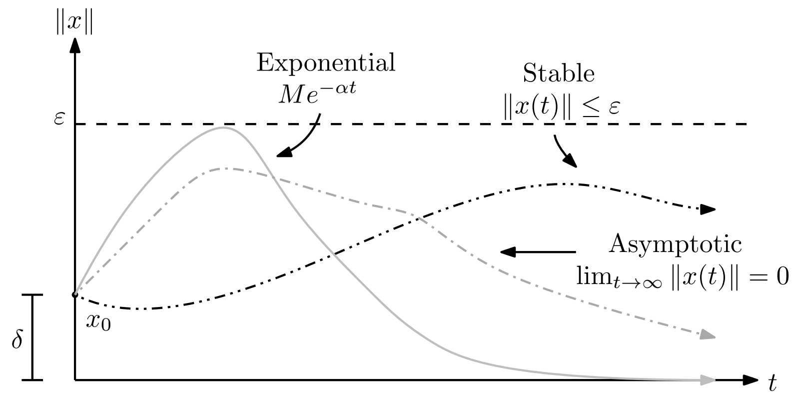


Figure 6.1. Illustration of the different types of stability given in Definition 6.1.

Definition 6.1. An equilibrium point x^* of $\dot{x} = f(x)$, with $f : E \to \mathbb{R}^n$ for $E \subseteq \mathbb{R}^n$ an open and connected set, is said to be:

Stable if:

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \qquad \text{s.t.} \qquad \|x(t_0) - x^*\| < \delta \quad \Longrightarrow \quad \|x(t) - x^*\| < \varepsilon, \quad \forall t \geq t_0; \tag{6.1}$$

Asymptotically Stable if it is *stable* and (by reducing δ if needed):

$$\|x(t_0) - x^*\| < \delta \quad \Longrightarrow \quad \lim_{t\to\infty} \|x(t) - x^*\| = 0; \tag{6.2}$$

Exponentially Stable if there exist $M, \lambda, \delta \in \mathbb{R}_{>0}$ such that:

$$\|x(t_0) - x^*\| < \delta \quad \Longrightarrow \quad \|x(t) - x^*\| \leq Me^{-\lambda(t-t_0)}\|x(t_0) - x^*\|, \quad \forall t \geq t_0. \tag{6.3}$$

***Remark* 6.1.** The notion of stability requires solutions starting δ-close to x^* at $t = t_0$ to be defined for all $t \geq t_0$ and remain ε-close to x^*. The reader should verify how much the notion of stability resembles the notion of continuity in Definition 2.2.

***Remark* 6.2.** It is also instructive to verify that exponential stability implies asymptotic stability. Since $e^{-\lambda(t-t_0)}$ is a decreasing function of t we have $e^{-\lambda(t-t_0)} \leq e^{-\lambda(t_0-t_0)} = 1$. Hence, inequality (6.3) provides the bound $\|x(t) - x^*\| \leq M\|x(t_0) - x^*\|$ from which we conclude stability by taking $\delta = \varepsilon/M$ in (6.1). Moreover, since $\lim_{t\to\infty} e^{-\lambda(t-t_0)} = 0$ we have from (6.3):

$$\lim_{t\to\infty} \|x(t) - x^*\| \leq 0,$$

which implies (6.2).

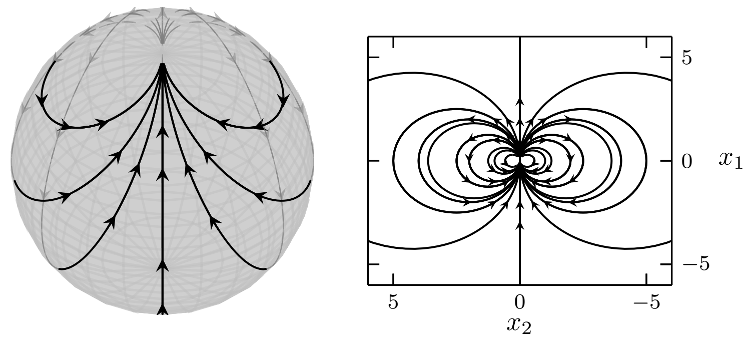
Example 6.1. A simple example may help solidify the different stability concepts. Consider the scalar nonlinear system:

$$\dot{x} = f(x) = -x^3, \qquad x \in \mathbb{R},$$

which has the single equilibrium point $x^* = 0$. We can compute its solution in closed-form, for $t_0 = 0$, as:

$$x(t) = \frac{x_0}{\sqrt{1 + 2tx_0^2}}.$$

Figure 6.2. Illustration of a system where all solutions converge asymptotically to the "north pole" but is not stable (left). The dynamics (specifically, the phase portrait) are shown locally on the right.



In order to determine if x^* is stable we note that $|x(t)| \leq |x_0|$ since the denominator is no smaller than 1 for $t \geq t_0 = 0$. Hence, for any ε we can always choose $\delta = \varepsilon$ to satisfy (6.1) and conclude stability. Moreover, since $\lim_{t\to\infty} x(t) = 0$ we conclude from (6.2) that asymptotic stability also holds. We now investigate exponential stability which requires:

$$|x(t)| \leq M e^{-\lambda t} |x_0|.$$

Substituting the closed-form expression of $x(t)$ in the previous inequality leads to:

$$\frac{1}{\sqrt{1 + 2tx_0^2}} \leq M e^{-\lambda t} \quad \Longrightarrow \quad \frac{e^{\lambda t}}{\sqrt{1 + 2tx_0^2}} \leq M,$$

which is a contradiction since:

$$\lim_{t\to\infty} \frac{e^{\lambda t}}{\sqrt{1 + 2tx_0^2}} = \infty.$$

Therefore, the equilibrium is not exponentially stable despite being asymptotically stable. This is in sharp contrast with linear systems where asymptotic and exponential stability are equivalent.

Example 6.2. Examination of the conditions for asymptotic stability reveals that it requires both stability and the proper limiting behavior: $\lim_{t\to\infty} \|x(t) - x^*\| = 0$. It is tempting to think that the limiting behavior suffices to conclude stability, but this is not the case. To illustrate this point, consider again the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, first introduced in Example 5.16 and defined by: $\mathbb{S}^2 = \left\{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\right\}$. This is a manifold of dimension 2 (see Definition 5.4), and a local coordinate chart covering the entire sphere minus the north pole can be obtained via stereographic projection. It follows from the hairy ball theorem (Example 17.5 in Lecture 17) that any vector field on the sphere must have at least one equilibrium point. The obvious examples are vector fields with two equilibrium points, but one can construct a vector field on this manifold with a single equilibrium point at the "north pole" as visualized in Figure 6.2 and described in detail in Example 5.16 with the vector field on the sphere given in (5.15). This provides an example of a system that is not stable (there are solutions inside any ball around the north pole that escape such balls) but for which all solutions converge asymptotically to the equilibrium point.

One can visualize this behavior locally via a system defined on $\mathbb{R}^2$; this can be thought of as the vector field on the sphere projected to a local coordinate chart (in this case the arctic coordinate chart and given by $\psi_a(x_1, x_2, x_3) = (x_1, x_2)$). Specifically, one obtains an ODE on $\mathbb{R}^2$ of the form (with associated phase portrait shown in Figure 6.2):

$$\dot{x} = f(x) = \begin{bmatrix} \frac{1}{2}(x_1^2 - x_2^2) \\ x_1 x_2 \end{bmatrix}. \tag{6.4}$$

This represents the dynamics associated with a point dipole. To reconstruct the vector field on the sphere from its projection depicted on the right of Figure 6.2, one would identify all the points at infinity with the south pole (this is formally done through an equivalence relation identifying all such points at infinity). It is interesting to note that in Example 5.16 we began with a constant vector field (5.14) on $\mathbb{R}^2$, which was pushed forward to create a non-trivial vector field (5.15) on $\mathbb{S}^2$, with a single equilibrium point, N, that is not stable despite that fact that all solutions converge to this point in the limit, and finally, a different choice of local coordinates resulted in the equation (6.4) describing a point dipole. This indicates the importance of coordinates, and geometry, in dynamics—this will be explored further in Lecture 17.

Stability for Linear Systems. The stability picture is greatly simplified for linear systems. We first observe that for such systems, i.e., for ODEs $\dot{x} = Ax$ with $A \in \mathbb{R}^{n\times n}$, an isolated equilibrium $x = 0$ satisfies:

$$x = 0 \text{ is Asymptotically Stable} \qquad \Leftrightarrow \qquad x = 0 \text{ is Exponentially Stable.}$$

Moreover, there is a complete characterization of stability for linear systems—a characterization that follows from the Stable Manifold Theorem presented in Lecture 5.

Fact 6.1. *Let the equilibrium $x = 0$ of $\dot{x} = Ax$, $A \in \mathbb{R}^{n\times n}$, be isolated, i.e., $Ax = 0$ iff $x = 0$. Then, $x = 0$ is (globally) exponentially stable if and only if for all eigenvalues $\lambda(A)$ of A, $\Re(\lambda(A)) < 0$.*

Although stability for linear systems can be characterized via eigenvalues, we will later provide characterizations utilizing Lyapunov functions to illustrate how they can be specialized to linear systems.

6.2 A detour into class $\mathcal{K}$ and $\mathcal{KL}$ functions.

The formalization of stability in Definition 6.1 is quite classical and has appeared in countless papers and books. However, we will be using a more recent, yet equivalent, formalization that greatly simplifies several key arguments related to Lyapunov characterizations of stability. This more recent formalization relies on the notion of class $\mathcal{K}$ and $\mathcal{KL}$ functions that we now introduce.

Definition 6.2. A continuous function $\alpha : [0, a) \to [0,\infty)$, where $a \in \mathbb{R}_{>0} \cup \{\infty\}$, is said to belong to class $\mathcal{K}$, denoted by $\alpha \in \mathcal{K}$, if it satisfies two properties:

Maps zero to zero: $\alpha(0) = 0$.

Strictly monotonically increasing: For all $r_1, r_2 \in [0, a)$, $r_1 < r_2 \implies \alpha(r_1) < \alpha(r_2)$.

A function $\alpha : [0,\infty) \to [0,\infty)$ is said to belong to class $\mathcal{K}_\infty$, denoted by $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and it satisfies[1]:

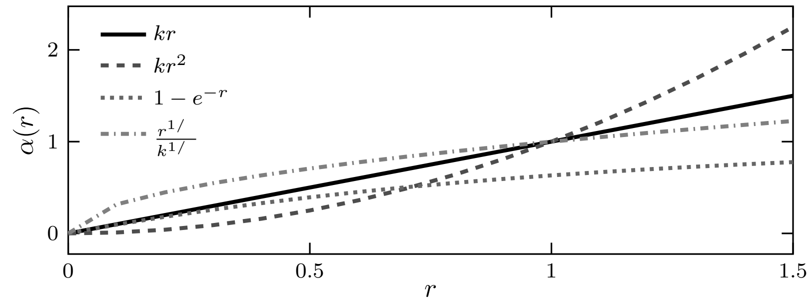
Radially unbounded: $\lim_{r\to\infty} \alpha(r) = \infty$.

Example 6.3. The simplest example of a class $\mathcal{K}_\infty$ function is the linear gain, so prevalent in the analysis of linear systems: $\alpha(r) = kr$ for $k \in \mathbb{R}_{>0}$. One can understand class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions as nonlinear generalizations of the linear gain. One common generalization is given by $\alpha(r) = kr^c \in \mathcal{K}_\infty$ for $k, c \in \mathbb{R}_{>0}$. See Figure 6.3 for an example of these and other class $\mathcal{K}$ functions.

Example 6.4. Another example of a class $\mathcal{K}$ function is $\alpha(r) = 1 - e^{-r}$. We note that $\alpha(0) = 0$ and monotonicity can be verified by computing its derivative $\frac{d\alpha}{dr} = e^{-r}$ which is positive for any $r \in \mathbb{R}_{\geq 0}$. However, α is not of class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = 1 \neq \infty$.

[1]Topologically, α is a proper function, i.e., the pre-image of a compact set is compact.

Figure 6.3. Examples of class $\mathcal{K}$ functions with $k = 1$ and $c = 2$.



Properties of Class $\mathcal{K}$ functions. Class $\mathcal{K}$ functions have certain closure properties that make them very convenient to work with. Consider the functions $\alpha_1, \alpha_2 \in \mathcal{K}$ with domain $[0, a)$ and the functions $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$. The following properties hold.

Invertibility: Class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions are invertible (since they are strictly increasing) and:

- $\alpha_1^{-1} \in \mathcal{K}$ with domain: $[0, \alpha_1(a))$,
- $\alpha_3^{-1} \in \mathcal{K}_\infty$.

Composability: Composition preserves $\mathcal{K}$ and $\mathcal{K}_\infty$ functions:

- $\alpha_2 \circ \alpha_1 \in \mathcal{K}$ with domain: $[0, \min\{a, \alpha_1^{-1}(a)\})$,
- $\alpha_4 \circ \alpha_3 \in \mathcal{K}_\infty$.

Example 6.5. The inverse of the class $\mathcal{K}_\infty$ function $\alpha(r) = kr^c$ for $k, c > 0$ is:

$$\alpha^{-1}(r) = \frac{1}{k^{\frac{1}{c}}} r^{\frac{1}{c}},$$

which is also a $\mathcal{K}_\infty$ function. Similarly, $\alpha_2 \circ \alpha_1(r) = k_2 {k_1}^{c_2} r^{c_1 c_2} \in \mathcal{K}_\infty$ for the functions $\alpha_i(r) = k_i r^{c_i}$, $i = 1, 2$.

In order to describe asymptotic stability we need another class of functions.

Definition 6.3. A function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$, denoted by $\beta \in \mathcal{KL}$, if the following hold:

- for every $s \in [0, \infty)$, $\beta(\cdot, s) : [0, a) \to [0, \infty)$ is a class $\mathcal{K}$ function;
- for every $r \in [0, a)$, $\beta(r, \cdot)$ is decreasing and $\lim_{s\to\infty} \beta(r, s) = 0$.

When $\beta(\cdot, s) \in \mathcal{K}_\infty$ we say that β belongs to class $\mathcal{KL}_\infty$, denoted by $\beta \in \mathcal{KL}_\infty$.

Example 6.6. The simplest example of a class $\mathcal{KL}$ function is $\beta(r, s) = Me^{-\lambda s} r$ with $M, \lambda \in \mathbb{R}_{>0}$. When we fix s, we obtain the linear gain $r \mapsto Me^{-\lambda s} r$ with which we are already familiar with. When we fix r, we obtain a decaying exponential, a decreasing function of s that converges to zero as s tends to infinity. See Figure 6.3 for a plot of this class $\mathcal{KL}$ function.

Example 6.7. Class $\mathcal{KL}$ functions can be seen as nonlinear generalizations of $Me^{-\lambda s} r$. One such generalization is obtained through the composition $\alpha \circ \beta$ with $\alpha \in \mathcal{K}$ and $\beta \in \mathcal{KL}$. When we fix s, $\alpha \circ \beta(r, s)$ is of class $\mathcal{K}$ since it is the composition of two such functions. When we fix r, we have that $\alpha \circ \beta(r, s)$ is decreasing with s since β is decreasing and α is monotonic. Moreover, $\lim_{s\to\infty} \alpha \circ \beta(r, s) = \alpha(\lim_{s\to\infty} \beta(r, s)) = \alpha(0) = 0$ where the first equality follows from continuity of α. Therefore, $\alpha \circ \beta \in \mathcal{KL}$.

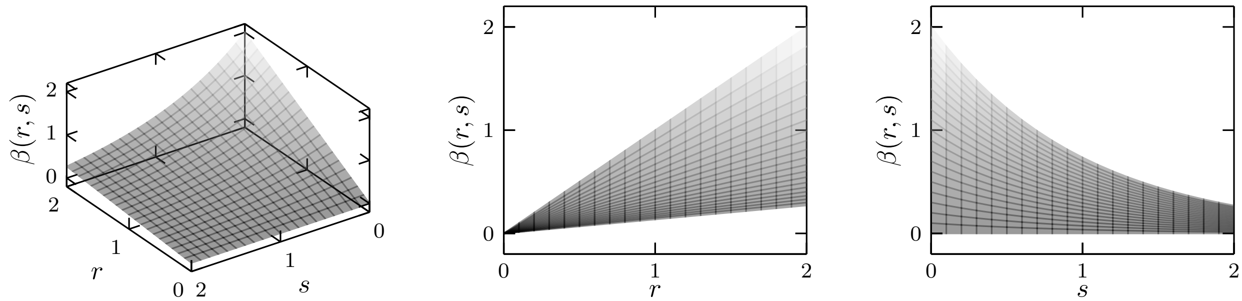


Figure 6.4. Surface plot of the class $\mathcal{KL}$ function $\beta(r,s) = Me^{-\lambda s}r$ with $M = 1$ and $\lambda = 1$.

Dynamics Governed by Class $\mathcal{K}$ Functions. The following result shows that class $\mathcal{K}$ and $\mathcal{KL}$ functions are two sides of the same coin[2].

Lemma 6.1. *Let $\alpha \in \mathcal{K}$ be defined on $[0,a)$, $a \in \mathbb{R}_{>0} \cup \{\infty\}$, and assume that for some $\varepsilon > 0$, α can be continuously extended to a function $\alpha : [-\varepsilon, a) \to \mathbb{R}$ with $\alpha(y) \le 0$ for all $y \in [-\varepsilon, 0]$. Consider the IVP:*

$$\dot{y} = -\alpha(y), \qquad y(t_0) = y_0 \in \mathbb{R},$$

If $y_0 \in [0,a)$, the IVP has a unique solution for all $t \ge t_0 \in \mathbb{R}_{\ge 0}$ given by:

$$y(t) = \beta(y_0, t - t_0),$$

for $\beta \in \mathcal{KL}$ defined on $[0,a) \times [0,\infty)$.

As this lemma is central to establishing Lyapunov's theorem through class $\mathcal{K}$ functions, we will prove it by proving two claims under the assumptions of the Lemma. First, however, we make a note on these assumptions:

***Remark* 6.3.** The assumption that the domain of α can be extended to the interval $[-\varepsilon, a)$ for some $\varepsilon > 0$ follows from the fact that we take y_0 in a (left-side) closed interval: $y_0 \in [0,a)$. The extension of α is done so that $\dot{y}(t)$ with $y(0) = 0$ is well-defined at $t = 0$; without this extension, non-uniqueness of solutions can result at $y_0 = 0$ (see Example 12.2). Alternatively, if one considers $y_0 \in (0,a)$ then the extension of α is not necessary; this is the case with the classic version of the result.

Claim 6.1 (Peano's Uniqueness Theorem)**.** *Under the assumptions of Lemma 6.1, the solutions of $\dot{y} = -\alpha(y)$ with $y(t_0) = y_0 \in [0,a)$ are unique for all $t \ge t_0 \in \mathbb{R}_{\ge 0}$.*

Proof. By way of contradiction, assume the existence of two different solutions $y_1(t)$ and $y_2(t)$ with $y_1(t_0) = y_2(t_0) = y_0$ for $t \in I(y_0) = [t_0, t_0 + \tau)$, with $\tau \in \mathbb{R}_{>0} \cup \{\infty\}$, i.e., assume non-uniqueness of solutions. Without loss of generality, assume that $y_2(t) > y_1(t)$ for $t \in (t_1, t_1 + \varepsilon)$ with $t_0 < t_1$ and $t_1 + \varepsilon \le t_0 + \tau$ and $y_2(t) = y_1(t)$ for $t \in [t_0, t_1]$. By the strictly increasing property of class $\mathcal{K}$ functions:

$$y_2(t) \ge y_1(t) \quad \Longrightarrow \quad -\alpha(y_2(t)) \le -\alpha(y_1(t)) \quad \Longrightarrow \quad \dot{y}_2(t) \le \dot{y}_1(t),$$

for all $t \in [t_1, t_1 + \varepsilon)$. Define the function:

$$\varphi(t) \triangleq y_2(t) - y_1(t) \quad \Longrightarrow \quad \dot{\varphi}(t) = \dot{y}_2(t) - \dot{y}_1(t) \le 0 \quad \Longrightarrow \quad \varphi(t) \text{ is decreasing.}$$

[2]To be precise, Lemma 6.1 is just an implication. A converse follows from the converse Lyapunov theorem discussed in Section 7.3.

Noting that $\varphi(t_1) = 0$ and using the decreasing nature of φ, for all $t \in [t_1, t_1 + \varepsilon)$:

$$\varphi(t_1) = 0 \text{ and } \dot{\varphi}(t) \leq 0 \quad \xrightarrow{\text{by Lem. 3.1}} \quad \varphi(t) = y_2(t) - y_1(t) \leq 0 \quad \Longrightarrow \quad y_2(t) \leq y_1(t),$$

yielding a contradiction. Repeating the same argument as above, under the assumption that $y_2(t) < y_1(t)$ for $t \in (t_1, t_1 + \varepsilon)$, we obtain a similar contradiction. Thus $y_1(t) = y_2(t)$ for all $t \in I(y_0) = [t_0, t_0 + \tau)$.

To conclude that solutions exist for all time, note that $\alpha(0) = 0$ and therefore if $y_0 = 0$ the solution is the constant curve $y(t) \equiv 0$. It then follows by uniqueness of solutions, as shown above, that for $y_0 \geq 0$ the Comparison Lemma implies $y(t) \geq 0$ for all $t \in I(y_0)$. Additionally, $\dot{y}(t) = -\alpha(y(t)) \leq 0$ since $y(t) \geq 0$ which leads to $y(t) \leq y_0$ for all $t \in I(y_0)$. Therefore, $y(t) \in [0, y_0]$, and since $[0, y_0]$ is a compact set it implies that $I(y_0) = \mathbb{R}_{\geq 0}$, i.e., solutions exist and are unique for all time $t \in \mathbb{R}_{\geq 0}$. ☐

Claim 6.2. *Under the assumptions of Lemma 6.1, define β as the solution to $\dot{y} = -\alpha(y)$ with initial condition y_0 which is thus given by:*

$$\beta(y_0, t - t_0) \triangleq y_0 + \int_{t_0}^{t} -\alpha(y(\tau))d\tau.$$

The function β is of class $\mathcal{KL}$.

Proof. We will examine the properties of β argument by argument (and take $t_0 = 0$ for simplicity):

<u>First argument</u>: Fix $s \in [0, \infty)$, then we need to show that $\beta(\cdot, s)$ is of class $\mathcal{K}$.

Maps zero to zero: For $y_0 = 0$, $y(s) = \beta(0, s) = 0$ because $y_0 = 0$ is an equilibrium point: $-\alpha(0) = 0$ (since $\alpha \in \mathcal{K}$), i.e., $y(t) \equiv 0$. Hence, it follows that $y(s) = 0$.

Strictly increasing: Let $r_1 < r_2$ with $r_1, r_2 \in [0, a)$, and thus by the Comparison Lemma $y_1(s) = \beta(r_1, s) \leq y_2(s) = \beta(r_2, s)$. But if $y_1(s) = y_2(s)$ then it would violate the uniqueness of solutions given that they have different initial conditions; namely $r_1 < r_2$. Therefore, we conclude that $y_1(s) = \beta(r_1, s) < y_2(s) = \beta(r_2, s)$ as desired.

<u>Second argument</u>: Fix $r \in [0, a)$, then we need to show that $\beta(r, \cdot)$ is:

Decreasing: Since β is just the solution, taking the derivative yields:

$$\frac{d}{dt}\beta(r, t) = \dot{y}(t) = -\alpha(y(t)) \leq 0,$$

because $y(t) \geq 0$ and $\alpha \in \mathcal{K}$.

Limiting behavior: Since β is decreasing and contained in the compact set $[0, y_0]$:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \beta(r, t) = c \in [0, y_0].$$

Furthermore, $y(t) \in [c, y_0]$ because β is decreasing. Assume by way of contradiction that $c \neq 0$, wherein we can pick $\gamma = \min_{y\in[c,y_0]} \alpha(y)$. But we would then have:

$$y(t) = \beta(r, t) = y_0 + \int_0^t -\alpha(y(\tau))dt \leq y_0 - \gamma t \leq c, \quad \forall t \geq \frac{y_0 - c}{\gamma},$$

contradicting $y(t) \geq c > 0$. Therefore, $c = 0$ as desired.

☐

Example 6.8. For the class $\mathcal{K}$ function $\alpha(r) = kr$, we have:

$$\dot{y} = -ky,$$

with solution (for $t_0 = 0$):

$$y(t) = e^{-kt}y_0 \qquad \Longrightarrow \qquad \beta(r,s) = e^{-ks}r,$$

where $\beta \in \mathcal{KL}$.

6.3 Revisiting Stability

We now connect stability to class $\mathcal{K}$ and class $\mathcal{KL}$ functions—that is, we revisit stability from a modern perspective. After establishing that this modern formulation implies the classical notion, we will return again to the simple motivating example: the pendulum. We will see that, in this simple example, both the "classic" and "modern" notions of stability can be directly inferred from the Lyapunov function—this intuition will be formalized in subsequent lectures.

Modern Stability Formulation. We now revisit the formalization of stability (Definition 6.1) by using class $\mathcal{K}$ functions to replace $\varepsilon - \delta$ arguments. While this is stated as a proposition, it could equally be considered the "modern" definition of stability.

Proposition 6.1. *An equilibrium point x^* of $\dot{x} = f(x)$ is:*

Stable if there exist $a \in \mathbb{R}_{>0} \cup \{\infty\}$ and a class $\mathcal{K}$ function $\alpha : [0,a) \to [0,\infty)$ such that:

$$\|x(t_0) - x^*\| \leq a \implies \|x(t) - x^*\| \leq \alpha(\|x(t_0) - x^*\|), \quad \forall\ t \geq t_0; \tag{6.5}$$

Asymptotically Stable if there exist $a \in \mathbb{R}_{>0} \cup \{\infty\}$ and a class $\mathcal{KL}$ function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ such that:

$$\|x(t_0) - x^*\| \leq a \implies \|x(t) - x^*\| \leq \beta(\|x(t_0) - x^*\|, t - t_0), \quad \forall\ t \geq t_0; \tag{6.6}$$

Exponentially Stable if it is asymptotically stable for a $\mathcal{KL}$ function β of the form $\beta(r,s) = Me^{-\lambda s}r$ for some $M, \lambda \in \mathbb{R}_{>0}$.

Proof. We can easily see that (6.5) implies (6.1). Given a desired ε we can pick $a < \delta$ in (6.5) and choose[3] $\delta = \alpha^{-1}(\varepsilon)$ since it then follows from (6.5) that:

$$\|x(t) - x^*\| \leq \alpha(\|x(t_0) - x^*\|) \leq \alpha(a) < \alpha(\delta) = \alpha(\alpha^{-1}(\varepsilon)) = \varepsilon.$$

Consider now inequality (6.6). Since β is decreasing in its second argument, we always have the bound $\beta(r,s) \leq \beta(r,0)$ which, when combined with (6.6), provides:

$$\|x(t) - x^*\| \leq \beta(\|x(t_0) - x^*\|, t - t_0) \leq \beta(\|x(t_0) - x^*\|, 0).$$

The preceding inequality directly shows stability since $\beta(r,0)$ is a class $\mathcal{K}$ function of r. Moreover, $\lim_{s\to\infty} \beta(r,s) = 0$ leads to:

$$\lim_{t\to\infty} \|x(t) - x^*\| \leq \lim_{t\to\infty} \beta(\|x(t_0) - x^*\|, t - t_0) = 0,$$

[3] Note the use of inequalities in Proposition 6.1 and strict inequalities in Definition 6.1. This was done for consistency with the Comparison Lemma, but strict inequalities could be used if desired.

from which we conclude that (6.2) holds, i.e., (6.6) implies (6.2).

Finally, the claim on exponential stability follows trivially from the choice of the $\mathcal{KL}$ function β, i.e., the definition in (6.3) is equivalent to the form given in the proposition. $\square$

***Remark* 6.4.** As previously mentioned, Proposition 6.1 could be equally considered as the definition of stability. In particular, the conditions are necessary and sufficient, i.e., (6.1) $\Leftrightarrow$ (6.5) and (6.2) $\Leftrightarrow$ (6.6). In its proof we established the $\Leftarrow$ direction. Showing (6.1) $\Rightarrow$ (6.5), however, requires more work as we need to construct α from the knowledge that a δ enforcing (6.1) always exists for any ε. References which give an elementary proof of this fact can be found in Additional Reading. Similarly, in the context of asymptotic stability, we proved the $\Leftarrow$ direction and showing the reverse implication, i.e., that (6.2) $\Rightarrow$ (6.6), again requires the construction of β and will not be addressed in this book.

Motivating Example. Stability—in both the classic and modern formulations—is defined using solutions and, as we mentioned several times, solutions of nonlinear systems can hardly be computed in closed-form. This motivates the use of Lyapunov functions as a means to certify the stability of nonlinear systems without the need to compute their solutions. This connection will be formally established in Lecture 7, but we hint toward this connection by returning to our canonical example of a mechanical system: the pendulum, as first introduced in Lecture 1. We will study this system, first, using its local linear approximation (linearization) and then from a nonlinear perspective. In both cases, this example will serve to motivate the use of "energy-like" functions, i.e., Lyapunov functions, to establish stability.

Consider the downward pendulum (see Section 1.1) with $x_1 = \theta$ the angular position and $x_2 = \dot{\theta}$ the angular velocity of the pendulum, where here we add damping as in (1.26):

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) - \gamma x_2 \end{bmatrix}. \tag{6.7}$$

Importantly, these dynamics have the following associated energy function:

$$\mathtt{E}(x) = \underbrace{\frac{1}{2}x_2^2}_{\text{Kinetic Energy}} - \underbrace{\cos(x_1)}_{\text{Potential Energy}}.$$

Recall that, to stabilize the inverted pendulum in Lecture 1 (specifically Section 1.2) we considered the energy minus the energy at the equilibrium point of interest (1.22) to obtain a candidate Lyapunov function. We can consider the exact same function in this case:

$$\mathtt{V}(x) \triangleq \mathtt{E}(x) - \mathtt{E}(0) = \underbrace{\frac{1}{2}x_2^2}_{\text{Kinetic Energy}} - \underbrace{\cos(x_1)}_{\text{Potential Energy}} - \underbrace{(-1)}_{\mathtt{E}(0)}. \tag{6.8}$$

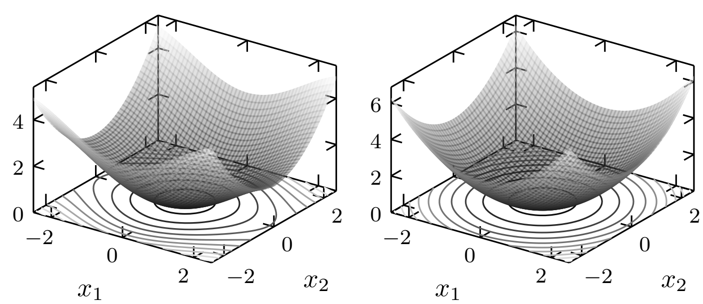


Figure 6.5. (Left) Level sets of the energy-based Lyapunov function (6.8), and (right) level sets of a quadratic approximation of this Lyapunov function (6.9).

Finally, we established that this function is locally positive definite by considering its local approximation via the small angle approximation, $\cos(x_1) \approx 1 - \frac{x_1^2}{2}$, wherein:

$$\mathtt{V}(x) \approx \frac{1}{2}\left(x_1^2 + x_2^2\right), \tag{6.9}$$

and this function is clearly positive except at $(0,0)$ where it is zero. A quadratic approximation of the Lyapunov function will be considered in the context of the linearized dynamics for the inverted pendulum:

$$\dot{x} = Df(0)x = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix} x. \tag{6.10}$$

Importantly, this is just the dynamics considered in (1.27) and Example 3.1.

Example 6.9 (Analyzing the Linearized Dynamics). Consider the quadratic Lyapunov function (6.9), wherein we are interested in studying its <u>sublevel sets</u>:

$$\Omega_c = \left\{ x \in \mathbb{R}^2 : \mathtt{V}(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) \leq c \right\},$$

for $c > 0$. The boundary of this set:

$$\partial\Omega_c = \left\{ x \in \mathbb{R}^2 : \mathtt{V}(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) = c \right\},$$

is the <u>Lyapunov level set</u> (which is a manifold per Example 5.10 in Lecture 5). We are interested in understanding how the dynamics evolve with respect to these sets, i.e., if these sets are *forward invariant*[4] with respect to the dynamics in (6.10). Note that if the solution $x(t)$ forever remains in Ω_c, we must have $\mathtt{V}(x(t)) \leq c$ for all $t \geq t_0$. As we cannot compute $x(t)$, we cannot directly check the inequality $\mathtt{V}(x(t)) \leq c$. We can, however, compute its time derivative that will provide information on how the value of $\mathtt{V}$ changes over time along the solution $x(t)$. This is achieved by computing the directional derivative (or *Lie* derivative) of $\mathtt{V}$ along solutions of (6.10):

$$\begin{aligned} \dot{\mathtt{V}}(x) &= \frac{\partial \mathtt{V}}{\partial x}\Big|_x \dot{x} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} x_2 \\ -x_1 - \gamma x_2 \end{bmatrix}}_{Df(0)x} \\ &= -\gamma x_2^2, \end{aligned}$$

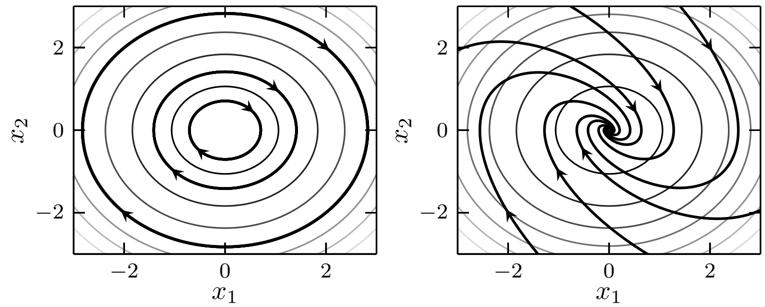
where x denotes the solution, hence a function of time, which requires the chain rule to calculate $\dot{\mathtt{V}}$. Concretely, it is calculated via:

$$\dot{\mathtt{V}}(x(t)) = \frac{d}{dt}\mathtt{V}(x(t)) = \frac{\partial \mathtt{V}}{\partial x}\Big|_{x(t)} \dot{x}(t) \qquad \Longrightarrow \qquad \dot{\mathtt{V}}(x) = \frac{\partial \mathtt{V}}{\partial x}\Big|_x \dot{x}.$$

The behavior of $\mathtt{V}$ with respect to the dynamics allows us to understand, geometrically, the behavior of the system $\dot{x} = Df(0)x$ with respect to the set Ω_c. That is, we have the following

[4]A set $\mathcal{S}$ is forward invariant if $x(t_0) \in \mathcal{S}$ implies $x(t) \in \mathcal{S}$ for all $t \geq t_0$. See also Definition 4.2.

Figure 6.6. Two examples of the evolution of the linear system (6.10) relative to level sets of the Lyapunov function (6.9). On the left, $\gamma = 0$ and on the right $\gamma = 1$.



characterizations:

$$
\begin{array}{lcl}
\dot{\mathtt{V}}(x) = 0 \quad \forall\, x \in \partial\Omega_c & \Leftrightarrow & \text{trajectories evolve tangentially to } \mathtt{V}(x) = c \\
& \Leftrightarrow & x(t) \in \partial\Omega_c \quad \forall\, t \geq 0; \\
\dot{\mathtt{V}}(x) < 0 \quad \forall\, x \in \partial\Omega_c & \Leftrightarrow & \text{trajectories evolve in the direction of decreasing } \mathtt{V}(x) \\
& \Leftrightarrow & x(t) \in \Omega_c \quad \forall\, t \geq 0; \\
\dot{\mathtt{V}}(x) > 0 \quad \forall\, x \in \partial\Omega_c & \Leftrightarrow & \text{trajectories evolve in the direction of increasing } \mathtt{V}(x) \\
& \Leftrightarrow & x(t) \notin \Omega_c \quad \forall\, t > 0.
\end{array}
$$

Therefore, for the linearized dynamics we find that for $\gamma = 0$, trajectories evolve tangentially to $\mathtt{V}(x) = c$, i.e., tangentially to a circle of radius $\sqrt{2c}$. When $\gamma > 0$, $\dot{\mathtt{V}} \leq 0$ so trajectories either evolve tangentially to or in the decreasing direction of $\mathtt{V}$. In either case, this will imply stability of the system since its trajectories remain in the bounded set Ω_c. This is illustrated in Figure 6.6.

The classical notion of stability can be establish concretely by noting that given $\varepsilon > 0$, we can pick $\delta = \varepsilon$ wherein:

$$
\|x_0\| < \delta \quad \Rightarrow \quad \mathtt{V}(x_0) \leq \frac{1}{2}\delta^2 \quad \Rightarrow \quad \mathtt{V}(x(t)) \leq \frac{1}{2}\delta^2 \quad \Rightarrow \quad \|x(t)\| \leq \delta = \varepsilon, \tag{6.11}
$$

for all $t \geq 0$, thus establishing stability under the condition that $\dot{\mathtt{V}}(x) \leq 0$. This same inequality can be leveraged to obtain the modern formulation of stability. In particular:

$$
\dot{\mathtt{V}}(x(t)) \leq 0 \quad \xRightarrow{\text{by Lem. 3.1}} \quad \mathtt{V}(x(t)) \leq \mathtt{V}(x_0) \quad \Rightarrow \quad \|x(t)\| \leq \|x_0\| \tag{6.12}
$$

Therefore, picking $\alpha(r) = r$ trivially establishes stability via Proposition 6.1.

Example 6.10 (**Analyzing the Nonlinear Dynamics).** The power of Lyapunov methods is that the analysis performed in the linear case directly generalizes to the nonlinear case with appropriate modifications of the Lyapunov function. In particular, we consider the nonlinear Lyapunov function (6.8) and the corresponding sublevel set:

$$
\Omega_c = \left\{ x \in \mathbb{R}^2 : \mathtt{V}(x) = \frac{1}{2}x_2^2 - \cos(x_1) + 1 \leq c \right\}.
$$

Analyzing $\mathtt{V}$ along solutions of the nonlinear system yields:

$$
\begin{aligned}
\dot{\mathtt{V}}(x) &= \frac{\partial \mathtt{V}}{\partial x}\Big|_x \dot{x} \\
&= \begin{bmatrix} \sin(x_1) & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} x_2 \\ -\sin(x_1) - \gamma x_2 \end{bmatrix}}_{f(x)} \\
&= -\gamma x_2^2.
\end{aligned}
$$

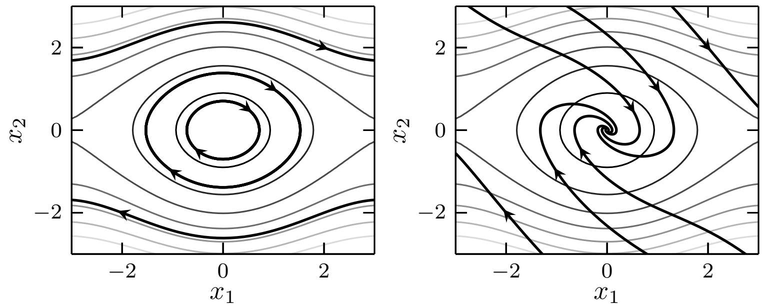


Figure 6.7. Two examples of the evolution of the nonlinear system (6.7) relative to level sets of the Lyapunov function (6.9). On the left, $\gamma = 0$ and on the right $\gamma = 1$.

Hence, with the nonlinear system, when $\gamma = 0$ the system evolves tangentially to level sets of $\mathtt{V}$, and when $\gamma < 0$, $\dot{\mathtt{V}} \leq 0$, so trajectories evolve tangentially to or in the decreasing direction of $\mathtt{V}$. The behavior of this system relative to level sets of $\mathtt{V}$ is shown in Figure 6.7. Note that, in this case, the behavior of the nonlinear system mirrors that of the linear system except now level sets of the nonlinear Lyapunov function $\mathtt{V}$ are no longer perfect circles and, importantly, are only bounded for c small. Yet the end result is the same: we can establish stability for the nonlinear system through the same logic used in (6.11) applied to the nonlinear system. This simple example will serve as motivation for the formal introduction of Lyapunov's method in the next lecture.

Additional Reading

A more in-depth discussion of $\mathcal{K}$ and $\mathcal{KL}$ functions can be found in [108, 110]. The use of $\mathcal{K}$ and $\mathcal{KL}$ functions to define stability has its origins in the work of Hahn [87, 88]. As noted in Remark 6.4, one can prove the converse direction via elementary methods and thus Proposition 6.1 can be viewed as a definition; the reader is referred to [32] for more on the connection between the classical definition of stability and the definition using class $\mathcal{K}$ functions.

Note that an alternative proof of Lemma 6.1 can be found in [110], but we note that in [110] α was required to be a locally Lipschitz continuous class $\mathcal{K}$ function. The proof presented in this lecture does not rely on this assumption. The proof of Claim 6.1 is the standard proof of Peano's Uniqueness Theorem extended to a closed interval, as motivated by Theorem 1.31 in [2] (see also Theorem 6.2 in [92]). The proof of Claim 6.2 is different from existing proofs of this claim and invokes standard results in real analysis (e.g., Theorem 12.4 in [42]).

Problems for Lecture 6

[P6.1] Prove the properties of class $\mathcal{K}$ functions presented in Section 6.2 and, specifically, *invertibility* and *composability*.

[P6.2] Show that topologically, $\alpha \in \mathcal{K}_\infty$ is a proper function, i.e., the pre-image of a compact set is compact. That is, if $C \subset \mathbb{R}_{\geq 0}$ is a compact set, then $\alpha^{-1}(C)$ is a compact set.

[P6.3] [**Advanced Problem**] The goal of this problem is to establish that Proposition 6.1 gives an alternative definition of stability. In particular, that proposition proved that (6.1) $\Leftarrow$ (6.5). Establish that this is necessary and sufficient by proving that (6.1) $\Rightarrow$ (6.5). To establish this:

(a) Use (6.1) to construct a function $\gamma \in \mathcal{K}$ such that:

$$\forall\, \varepsilon > 0, \quad \|x(t_0) - x_0\| < \gamma(\varepsilon) \qquad \Longrightarrow \qquad \|x(t) - x^*\| < \varepsilon.$$

(b) Use Part (a) to conclude that (6.1) $\Rightarrow$ (6.5).

[P6.4] A group, G, is a set (finite or infinite) together with a group operation, denoted by $a \cdot b$, and defined for all $a, b \in G$ while satisfying:

Closure: For $a, b \in G$, $a \cdot b \in G$.

Associativity: For $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Identity: There exists an element $1 \in G$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in G$.

Inverse: For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Show that the set of class $\mathcal{K}$ functions is a group. Similarly, show that the set of class $\mathcal{K}_\infty$ functions is also a group, i.e., a subgroup of the group $\mathcal{K}$ (since it is closed under the group operation and inverse operation).

[P6.5] ([110]) Consider the second order differential equation:

$$\ddot{\theta} + p(\theta) = 0,$$

for $\theta \in \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$ continuously differentiable with $p(0) = 0$ and $\frac{\partial p}{\partial \theta}(\theta) \neq 0$. This can be converted to an ODE as follows:

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -p(x_1) \end{bmatrix}. \tag{6.13}$$

with $x_1 = \theta$ and $x_2 = \dot{\theta}$. Utilize the corresponding energy of the system given by:

$$\mathtt{E}(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} p(t)dt,$$

to construct a Lyapunov function and give conditions on p for which the system is stable. What can you say about the asymptotic stability of the system?

[P6.6] ([110]) Consider the differential equation:

$$\dot{x} = J(x)\nabla H(x),$$

where $J(x)$ is a skew-symmetric matrix for every $x \in \mathbb{R}^n$, i.e., $J^T(x) = -J(x)$, and $\nabla H(x) = (DH(x))^T$. This is a generalization of Hamilton's equations of motion since we can always take J to be:

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

to recover said equations. Using this interpreation we can regard H as the energy of this system. Show that solutions starting in a level-set of H remain in that level set for all future time, i.e, energy is preserved along solutions.

Consider now the differential equation:

$$\dot{x} = J(x)\nabla H(x) + R(x)\nabla H(x),$$

where $R(x)$ is negative semi-definite, i.e., $x^T R(x) x \leq 0$ for all $x \in \mathbb{R}^n$. This can be interpreted as a generalization of Hamilton's equations with dissipation. Show that solutions starting in a sublevel set of H remain in that sublevel set forever, i.e, energy does not increase (and may decrease) along solutions.

[P6.7] [**Advanced Problem**] The goal of this problem is to verify that the vector field defined on the sphere in Example 5.16 is locally represented by the dynamics of a point dipole as given in (6.4). To this end:

(a) Consider the stereographic projection in (5.13) given by $\psi : \mathbb{S}^2 \backslash \{N\} \to \mathbb{R}^2$ where $N = (0, 0, 1)$ is the north pole. Verify that $(\mathbb{S}^2 \backslash \{N\}, \psi)$ is a local coordinate chart (as defined in Proposition 5.1).

(b) Prove that the vector field obtained by pushing forward the constant vector field, i.e., the vector field (5.15), has a single equilibrium point at the north pole. Show that all solutions converge to this point in the limit and yet the system is not stable. [Hint: one can explicitly solve the vector field in (5.14) and map these solutions to solutions of (5.15) via the stereographic coordinate chart.]

(c) We can construct a different local coordinate chart—one that is akin to the charts constructed for the circle in Example 5.12. This is termed the arctic coordinate chart and given by:

$$\psi_a(x_1, x_2, x_3) = (x_1, x_2), \qquad \psi_a^{-1}(z_1, z_2) = (z_1, z_2, \sqrt{1 - z_1^2 - z_2^2}).$$

Find the open set U_a such that $\psi_a : \mathbb{S}^2 \cap U_a \to W_a$ is a diffeomrophsim (with $W_a = \psi_a(\mathbb{S}^2 \cap U_a)$) and, hence (U_a, ψ_a) is a coordinate chart on the sphere.

(d) For the vector field on $\mathbb{S}^2$ defined in (5.15), use the arctic coordinate chart from Part (b) to show that locally on $\mathbb{R}^2$ the corresponding vector field is given by:

$$Y(z) = T\psi_a \circ X \circ \psi_a^{-1}(z) = (z, f_a(z)), \qquad f_a(z) = \begin{bmatrix} 1 - z_2^2 - \sqrt{1 - z_1^2 - z_2^2} \\ 2z_1 z_2 \end{bmatrix} \tag{6.14}$$

Finally, show that locally this vector field can be expressed as the dynamical system given in (6.4). [Hint: use the binomial approximation.]

[P6.8] [**Advanced Problem**] Show that stability can be understood as a form of continuity.

(a) Consider the map $F : \mathbb{R}^n \to C([0, \infty), \mathbb{R}^n)$ sending an initial condition $x_0 \in \mathbb{R}^n$ to the solution $x : [0, \infty) \to \mathbb{R}^n$. If you regard x_0 as an element of the normed vector space $\mathbb{R}^n$ equipped with the Euclidean norm, and the solution $x(t)$ as an element of the normed vector space $C([0, \infty), \mathbb{R}^n)$ equipped with the infinity norm (see (A.2) in Appendix A), show that stability can be framed as continuity of F at the equilibrium $x^* \in \mathbb{R}^n$.

(b) Utilize the connection between stability and continuity to state the definition of continuity utilizing class $\mathcal{K}$ functions. Show that this alternative notion of continuity using class $\mathcal{K}$ functions implies the classic definition (as was done in Proposition 6.1 for stability).

[P6.9] [**Advanced Problem**] In the proof of Theorem 3.3, we utilized the implication in (3.19) wherein the corresponding footnote noted that β was actually a function of x. That is, using the notation in that proof, consider the function $\beta : U \to \mathbb{R}_{>0}$ defined by the implication:

$$\|\lambda - \lambda_0\| < \beta(x) \implies \|f(x, \lambda) - f(x, \lambda_0)\| < a. \tag{6.15}$$

Using the results of Problem P6.8, and specifically the class $\mathcal{K}$ formulation of continuity, prove that β is continuous.

[P6.10] [**Advanced Problem**] Consider a manifold M and a vector field $X : M \to TM$, on which we wish to define stability in a manner that does not require the Euclidean norm (since the notion of distance on manifolds is not Euclidean in nature). In this content, define and show the following:

(a) Define the notion of an equilibrium point, x^*, for a vector field: $X : M \to TM$.

(b) Let U and V denote open, connected, and bounded neighborhoods of x^*. An equilibrium point $x^* \in M$ of a vector field $X : M \to TM$ on a manifold is *stable* if:

$$\forall U \subset M,\ \exists V \subset U \text{ with } x_0 \in V \ \text{ s.t. } \ x(t_0) \in V \cap M \quad \Longrightarrow \quad x(t) \in U \cap M, \quad \forall\, t \geq t_0, \tag{6.16}$$

Show that when $M = \mathbb{R}^n$, this is equivalent to the definition of stability given in Definition 6.1.

(c) Formulate asymptotic in a similar fashion, and without the use of norms.

(d) Repeat parts (b) and (c) but through the use of class $\mathcal{K}$ functions as in Proposition 6.1