PROBLEMS

Problem 1 (Book Problem 7.5)

Let $V : E \subset \mathbb{R}^n \to \mathbb{R}$ be a positive definite function on the open and connected set $E \subset \mathbb{R}^n$. Show that $r > 0$ such that $B_r(0) \subset E$, there exists class $\mathcal{K}$ functions $\alpha_1, \alpha_2 \in \mathcal{K}$ defined on $[0, r]$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

For all $x \in B_r(0)$. (Hint: consider the functions:

$$\psi_1(s) = \inf_{s \leq \|x\| < r} V(x) \quad \psi_2(s) = \sup_{\|x\| < s} V(x)$$

for $s \in [0, r)$ coupled with the previous problem (Book problem 7.4).)

Problem 2 (Book Problem 7.7)

Consider the two dynamical systems

$$\dot{x} = f(x) \quad \dot{y} = \frac{f(y)}{1 + \|f(y)\|}$$

for $x, y \in \mathbb{R}^n$ and $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable. Show that stability is preserved under this transformation. That is, $V : E \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for the nominal system if and only if it is a Lyapunov function for the transformed system. Is this also true for asymptotic stability?

Problem 3 (Book Problem 8.5)

Consider the linear control system

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$$

Assume that $P = P^\top \succ 0$ solves the Continuous Time Algebraic Ricatti Equation:

$$PA + A^\top P + Q - PBB^\top P = 0$$

for some $Q = Q^\top \succ 0$. show that applying the feedback control law:

$$u(x) = -\frac{1}{2} B^\top P x$$

results in the origin being globally exponentially stable.

Problem 4 (Book Problem 8.6)

Rational Stability: Let $\dot{x} = f(x)$ where $f : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on the open and connected set $E \subseteq \mathbb{R}^n$. The equilibrium point $x^* = 0 \in E$ is rationally stable if there exists $M\delta, k, \eta \in \mathbb{R}_{>0}$ with $\eta < 1$ such that:

$$\|x(t_0)\| < \delta \quad \implies \quad \|x(t)\| \leq M \left(1 + \|x(t_0)\|^k t\right)^{-\frac{1}{k}} \|x(t_0)\|^\eta$$

Problem 4 continued on next page. . .

Consider a continuously differentiable function $V : E \to \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned} k_1\|x\|^{c_1} \leq V(x) \leq& k_2\|x\|^{c_2} \\ \dot{V}(x) \leq& -k_3\|x\|^{c_3} \end{aligned}$$

for $k_1, k_2, k_3, c_1, c_2, c_3 \in \mathbb{R}_{>0}$. Show that

$$\frac{c_3}{c_2} > 1 \quad \Longrightarrow \quad x^* = 0 \text{ is rationally stable.}$$

Problem 5 (Book Problem 9.5)

Let $\dot{x} = f(x)$ with $f : E = B_r(0) \to \mathbb{R}^n$ and the some conditions on f and E as in the statement of Theorem 9.3. The goal is to prove 9.15 and 9.16 using the V given in 9.18, i.e.

$$V(x) = \int_0^\infty \|\varphi_\tau(x)\|^2 d\tau$$

Specifically, show that there exists a $r_0 > 0$ with $r_0 < r$ such that V satisfies

$$\begin{aligned} k_1\|x\|^2 \leq V(x) \leq& k_2\|x\|^2 \\ \dot{V}(x) \leq& -k_3\|x\|^2 \end{aligned}$$

for $k_1, k_2, k_3 > 0$.

PROJECT

This homework includes no projection portion, in order to facilitate studying for the midterm, which will cover material through Lecture 9.

OPTIONAL PROBLEMS

Optional problems will not be graded (and solutions will not be released). They may be of increased difficulty, or be problems which are omitted from the homework to keep it from being too long. These problems may make for good study material or qualifying exam preparation. Since solutions will not be released, feel free to ask questions about these questions in office hours (at then end, after required homework problem questions have been covered).
Note, Lectures 7 and 8 are central chapters to this entire course, and have excellent problems - nearly every problem in these chapter is good (and testable) material. Instead of writing down all the problems, I have listed a couple highlights.

Problem 6 (Book Problem 7.6)

Consider a nonlinear system $\dot{x} = f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^n$, and assume the existence of a function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying $\alpha_1(\|x\|) \leq W(x) \leq \alpha_2(\|x\|)$ and $\dot{W} \leq \lambda W$ with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. By drawing inspiration from the proof of Lyapunov's theorem, show that the solutions of $\dot{x} = f(x)$ exist for all $t \geq 0$.

Problem 7 (Book Problem 7.10)

Consider the discrete-time dynamical system with $x \in \mathbb{R}^2$ given by:

$$x_{k+1} = F(x_k) \quad k \in \mathbb{Z}_{\geq 0}$$

for $F : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable. Furthermore, assume that $x^* = 0$ is an equilibrium point (fixed point) $F(0) = 0$. For a function $V : \mathbb{R}^n \to \mathbb{R}$, the rate of change along solutions of the discrete time dynamical system is given by:

$$\Delta V(x) \triangleq V(F(x)) - V(x)$$

In the context of discrete-time dynamical systems:

(a) Restate the definitions of stability and asymptotic stability.

(b) Show that the origin is stable if V is a positive definite function and ΔV is negative semidefinite.

(c) Show that the origin of is asymptotically stable if V is a positive definite function and $\Delta V(x)$ is negative definite.

Problem 8 (Book Problem 8.3)

Consider the linear system:

$$\dot{x} = Ax \quad x \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n}$$

where A is stable. Hence, for $A = A^\top > 0$ there exists $P = P^\top > 0$ solving the CTLE. Using this P we define the function μ by:

$$\mu(Q) = \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$$

Show that $\mu(Q)$ has the following properties:

- $\mu(Q) = \mu(kQ)$ for all real constants $k > 0$.

Problem 8 continued on next page. . .

- $\mu(I) \geq \mu(Q)$ for all $Q = Q^\top > 0$ such that $\lambda_{min}(Q) = 1$.
- $\mu(I) \geq \mu(Q)$ for all $Q = Q^\top > 0$.

As a result of the above properties, determine which Q certifies the fastest rate of convergence for the linear system.

Problem 9 (Book Problem 8.8)

Consider again the discrete-time dynamical system given in Problem P7.10, i.e.:

$$x_{k+1} = F(x_k) \quad k \in \mathbb{N}$$

for $F : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable. Furthermore, assume that $x^* = 0$ is an equilibrium point $F(0) = 0$. We say this system is exponentially stable if $\exists M, \delta > 0$ and $0 < \lambda < 1$ such that

$$\|x_0 - x^*\| < \delta \quad \Longrightarrow \quad \|x_k - x^*\| \leq M\lambda^k \|x_0 - x^*\|$$

Find conditions on $V : \mathbb{R}^n \to \mathbb{R}$, along with conditions on the rate of change along solutions to the discrete time dynamical system:

$$\Delta V(x) \triangleq V(F(x)) - V(x)$$

such that it guarantees exponential stability of the system.

Problem 10 (Book Problem 9.7)

Consider a nonlinear system that can be decomposed into two subsystems:

$$\begin{aligned} \dot{x} &= f(x, t) \\ \dot{y} &= g(x, y) \end{aligned}$$

where $x, y \in \mathbb{R}^n$, and f and g are locally Lipschitz continuous in the x and y arguments and f piecewise continuous in t. Assume that the system $\dot{x} = f(x, t)$ is locally exponentially stable at the equilibrium point $x^* = 0$ with a corresponding Lyapunov function V_x satisfying (9.2) and (9.3) wherein $\alpha_i(r) = k_i r^c$ for $k_i, c \in \mathbb{R}_{>0}$. Additionally, assume that

$$\dot{y} = g(0, y)$$

is locally exponentially stable at the equilibrium point $y^* = 0$.

(a) Using Lyapunov methods, show that under these conditions the original system is locally exponentially stable at $(x^*, y^*) = (0, 0)$.

(b) Suppose that the first dynamical system experiences disturbances that are only a function of the second system, i.e., the system dynamics become:

$$\begin{aligned} \dot{x} &= f(x) + d(y) \quad \|d(y)\| \leq \mu \|y\| \\ \dot{y} &= g(x, y) \end{aligned}$$

for some $\mu > 0$. Find an upper bound on μ so that the overall system is still exponentially stable. Explain the intuition behind this upper bound.