Lecture 4

Linearization

This lecture presents more advanced concepts, and at a faster pace, than previous lectures. Pointers to several good sources for the material discussed in this lecture are provided in the Additional Reading section. Concepts from this lecture will be utilized in subsequent lectures, including: Lyapunov methods for establishing exponential stability (Lecture 8), set invariance for establishing safety (Lecture 12), and the lectures on periodic orbits (Lectures 13-15)

The goal of this lecture is to relate the behavior of linear systems, for which we have explicit (closed-form) solutions, to the behavior of nonlinear systems. This is done through *linearization*—wherein nonlinear systems are studied as linear systems:

$$\dot{x} = f(x) \qquad \mapsto \qquad \dot{x} = Df(x^*)x,$$

for an equilibrium point x^*, i.e., a point satisfying $f(x^*) = 0$. Studying the behavior of nonlinear systems through linearization must be done judiciously. On the one hand, linearization lets us use the rich theory of linear systems. On the other hand, it fails to capture the complexity of nonlinear systems. In this lecture we discuss which properties of a nonlinear system can be analyzed via linearization and which ones do not.

4.1 Flows

The solution of a linear system $\dot{x} = Ax$ has the closed form expression $x(t) = e^{At}x_0$ for the initial condition x_0 at time $t = 0$. Rather than considering the solution for a given initial condition, we could instead consider the *flow*, $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$, which maps an initial condition to the state reached a time t by the solution starting at said initial condition. In other words, solutions $x : I \to \mathbb{R}^n$ are regarded as functions of time with the initial condition as a parameter whereas flows $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ are regarded as functions of the initial condition with time as a parameter. Therefore, defining the map $\varphi_t \triangleq e^{At}$ we can observe it satisfies the following properties:

(i) Identity: $\varphi_0(x) = x$,

(ii) Composability: $\varphi_s(\varphi_t(x)) = \varphi_{s+t}(x)$ for all $t, s \in \mathbb{R}$,

(iii) Reversibility: $\varphi_{-t}(\varphi_t(x)) = \varphi_t(\varphi_{-t}(x)) = x$ for all $t \in \mathbb{R}$.

Solutions of nonlinear systems also result in flows and these will have similar properties.

Definition 4.1. Let $E \subseteq \mathbb{R}^n$ be an open and connected set and consider a locally Lipschitz continuous function $f : E \to \mathbb{R}^n$. Consider the IVP[1]

$$\dot{x} = f(x), \qquad x(0) = x_0.$$

with solution $x(t)$ defined on the interval $I(x_0)$; this is the maximum interval of existence if for every other interval $(-a, b)$, $a, b \in \mathbb{R}_{>0}$, on which $x(t)$ is a solution to the IVP we have the inclusion $(-a, b) \subseteq I(x_0)$.

The flow φ_t of the differential equation $\dot{x} = f(x)$ is defined by:

$$\varphi_t(x_0) \triangleq x(t).$$

***Note* 4.1.** The flow φ_t is only locally defined. If $t \in I(x_0)$, then there exists an open set $E_t \subset E \subset \mathbb{R}^n$ containing x_0 so that φ_t is defined for all $x \in E_t$. When the corresponding differential equation is forward complete, the open set E_t no longer depends on t and becomes $\mathbb{R}^n$. This is the case for linear differential equations.

Properties of the flow. The flow for nonlinear systems has similar properties to the flow of linear systems provided one is careful about its domain of existence. Before illustrating this point, recall that given a map $h : X \to Y$ and a set $S \subseteq X$ we denote by $h(S)$ the set:

$$h(S) \triangleq \bigcup_{s \in S} \{h(s)\} = \{y \in Y \mid y = h(s) \text{ for some s} \in S\}.$$

This leads to the following, the proof of which is left to the reader.

Proposition 4.1. *Consider the nonlinear differential equation $\dot{x} = f(x)$, where $f : E \to \mathbb{R}^n$ is a locally Lipschitz function defined on the open and connected set $E \subseteq \mathbb{R}^n$, and let φ_t be its flow. Then for all $x_0 \in E$ the flow φ_t satisfies the following properties:*

(i) Identity: $\varphi_0(x_0) = x_0$.

(ii) Composability: If $t \in I(x_0)$ and $s \in I(\varphi_t(x_0))$, then $s + t \in I(x_0)$ and:

$$\varphi_s(\varphi_t(x_0)) = \varphi_{s+t}(x_0).$$

(iii) Reversibility: For $t \in I(x_0)$, there exists a neighborhood $E_t \subset E$ with $x_0 \in E_t$ such that $\varphi_t : E_t \to E$ is defined ($t \in I(x)$ for all $x \in E_t$), and for all $x \in E_t$ and $y \in \varphi_t(E_t)$:

$$\varphi_{-t}(\varphi_t(x)) = x, \qquad \varphi_t(\varphi_{-t}(y)) = y.$$

Set Invariance. The concept of flows offers a simple way to express how solutions interact with subsets of the state space.

Definition 4.2. Consider the nonlinear differential equation $\dot{x} = f(x)$, where $f : E \to \mathbb{R}^n$ is a locally Lipschitz function defined on the open and connected set $E \subseteq \mathbb{R}^n$, and let φ_t be its flow. A set $S \subseteq E$ is said to be:

invariant if for all $x \in S$, $\varphi_t(x) \in S$ for all $t \in I(x)$;

[1] Here we take the initial time $t_0 = 0$ for simplicity of exposition. All results can be framed for $t_0 \neq 0$, e.g., $I(x_0) \subset \mathbb{R}$ is the maximal interval of existence if for every other interval $(t_0 - a, t_0 + b)$, $a, b \in \mathbb{R}_{>0}$, $(t_0 - a, t_0 + b) \subseteq I(x_0)$.

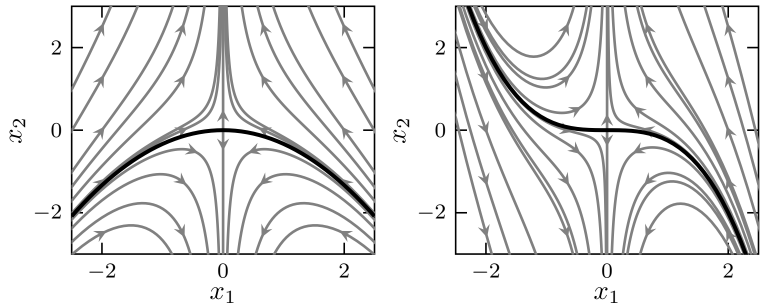

Figure 4.1. Phase portrait showing the invariant set S for Example 4.1 with $k = 2$, i.e., S is defined by $3x_2 = -x_1^2$ (left) and $k = 3$, i.e., S is defined by $4x_2 = -x_1^3$ (right).

forward invariant if for all $x \in S$, $\varphi_t(x) \in S$ for all $t \in I(x) \cap \mathbb{R}_{\geq 0}$;

backward invariant if for all $x \in S$, $\varphi_t(x) \in S$ for all $t \in I(x) \cap \mathbb{R}_{\leq 0}$.

Example 4.1. Let $x \in \mathbb{R}^2$ and consider the ODE:

$$\dot{x} = f(x) = \begin{bmatrix} -x_1 \\ x_2 + x_1^k \end{bmatrix}, \tag{4.1}$$

for $k \in \mathbb{N}_{\geq 1} = \{1, 2, \ldots\}$. Note that the system has a single polynomial nonlinearity of degree k. It can be verified that the corresponding flow is given by:

$$\varphi_t(x_0) = \begin{bmatrix} x_{01}e^{-t} \\ x_{02}e^t + \frac{x_{01}^k}{k+1}\left(e^t - e^{-kt}\right) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix},$$

where x_0 is the initial condition. For any x_0 in the set:

$$S = \left\{ x \in \mathbb{R}^2 \mid (k+1)x_2 = -x_1^k \right\},$$

we have $(k+1)x_{02} = -x_{01}^k$. Hence, the flow φ_t satisfies:

$$x_0 \in S \qquad \Longrightarrow \qquad \varphi_t(x_0) = \begin{bmatrix} x_{01}e^{-t} \\ -\frac{x_{01}^k}{k+1}e^{-kt} \end{bmatrix} \in S \qquad \Longrightarrow \qquad \varphi_t(S) \subseteq S.$$

Thus the set S, shown in Figure 4.1, is forward invariant with respect to φ_t.

Example 4.2. Consider the ODE:

$$\dot{x} = f(x) = \begin{bmatrix} -x_2 + x_1(1 - x_1^2 - x_2^2) \\ x_1 + x_2(1 - x_1^2 - x_2^2) \end{bmatrix}, \tag{4.2}$$

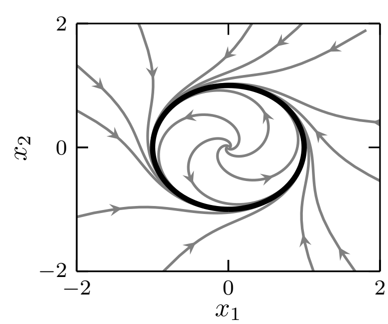

Figure 4.2. Phase portrait for the nonlinear system in Example 4.2 illustrating the invariant set consisting of the unit circle, i.e., $S = \mathbb{S}^1$.

and the set defined by:

$$\mathbb{S}^1 \triangleq \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\},$$

that is, $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$. It can be verified that $\mathbb{S}^1$ is an invariant set with respect to the flow of f by noting that, for $x_0 \in \mathbb{S}^1$, the flow is given by:

$$\varphi_t(x_0) = \begin{bmatrix} x_{01}\cos(t) - x_{02}\sin(t) \\ x_{01}\sin(t) + x_{02}\cos(t) \end{bmatrix} \quad \text{with} \quad x_{01}^2 + x_{02}^2 = 1. \tag{4.3}$$

This can be verified by directly checking that φ_t satisfies (4.2). Note that this is *not* the solution to the ODE for every initial condition but rather for initial conditions on the unit circle. From (4.3) we conclude that for $x_0 \in \mathbb{S}^1$, $\varphi_t(x_0) \in \mathbb{S}^1$ for all $t \in \mathbb{R}$. Therefore, for this system the unit circle is an invariant set. We will see in Lecture 13 that invariant sets of this form are termed *periodic orbits*.

4.2 Equivalence of Spaces and Flows

In this section, we introduce equivalences first between spaces and then between by flows. These notions will be essential in understanding the relationships between linear and nonlinear systems. We start by recalling that a map $h : X \to Y$ from a set X to a set Y is said to be:

injective if for any $x_1, x_2 \in X$ the equality $h(x_1) = h(x_2)$ implies $x_1 = x_2$,

surjective if for every $y \in Y$ there exists $x \in X$ such that $h(x) = y$,

bijective if h is injective and surjective, in which case there exists a unique inverse $h^{-1} : Y \to X$ defined by $h^{-1}(y) = x$ for any x such that $h(x) = y$ (which is well-defined because h is bijective),

an homeomorphism if X and Y are topological spaces, h is bijective, and h and h^{-1} are continuous, i.e., $h \in C(X, Y)$ and $h^{-1} \in C(Y, X)$,

a diffeomorphism if $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are open sets, h is bijective, and h and h^{-1} are continuously differentiable[2], i.e., $h \in C^1(X, Y)$ and $h^{-1} \in C^1(Y, X)$.

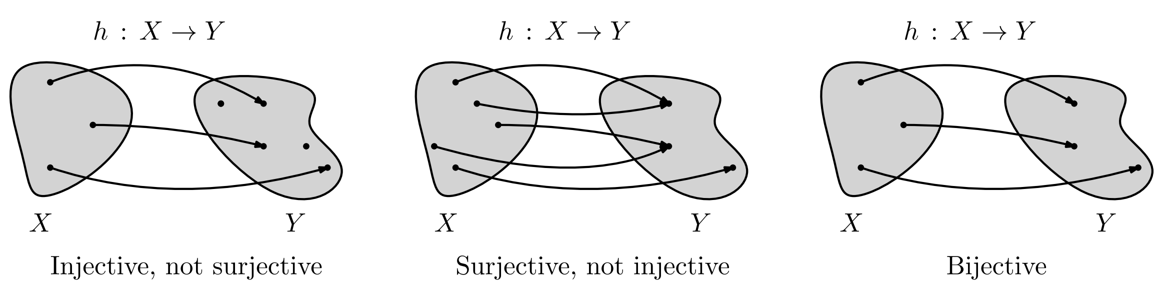

Figure 4.3. Illustration of properties of a mapping $h : X \to Y$.

[2]Note that the term "diffeomorophism" can be applied for different levels of smoothness, i.e., continuously differentiable, twice continuously differentiable, ..., infinitely continuously differentiable, and analytic. Often terms like "sufficiently smooth" are used to indicate that the differentiability of a given function should be inferred from context. In this book, if additional smoothness is needed, we will explicitly note such requirements.

Equivalences between sets. The simplest equivalence between two sets X and Y is given by a bijection $h : X \to Y$. We can understand this equivalence as a relabeling of the elements of X by the elements of Y, i.e., each element $x \in X$ is relabeled as $h(x) \in Y$. This equivalence preserves the cardinality of sets, i.e., two equivalent sets have the same cardinality. Although an apparently simple observation, it has important applications.

Example 4.3. We can establish that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality by constructing a bijection $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ using Cantor's celebrated diagonal argument. The definition of h is given in Figure 4.4 where we show to which elements of $\mathbb{N} \times \mathbb{N}$ the elements of $\mathbb{N}$ are mapped to. The reader can directly observe that h is injective. Surjectivity of h follows from the fact the image of h will cover $\mathbb{N} \times \mathbb{N}$ by successively covering each diagonal: the first covered diagonal is $(0, 0)$, the second is $(0, 1)$, $(1, 0)$, the third is $(0, 2)$, $(1, 1)$, $(2, 0)$, etc.

$$
\begin{array}{lllll}
\vdots & & & & \\
6 \mapsto (0,3) & \ldots & & & \\
3 \mapsto (0,2) & 7 \mapsto (1,2) & \ldots & & \\
1 \mapsto (0,1) & 4 \mapsto (1,1) & 8 \mapsto (2,1) & \ldots & \\
0 \mapsto (0,0) & 2 \mapsto (1,0) & 5 \mapsto (2,0) & 9 \mapsto (3,0) & \ldots
\end{array}
$$

Figure 4.4. Cantor's diagonal bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$.

The next equivalence addresses topological spaces.

Definition 4.3. The topological spaces X and Y are homeomorphic or topologically equivalent, denoted by $X \cong Y$, if there exists an homeomorphism $h : X \to Y$.

Since topological spaces have more structure than sets, we expect that, in addition to cardinality, other properties will be preserved by topological equivalence. This is summarized by the following fact, and illustrated in the following examples.

Fact 4.1. *Let $X \cong Y$. If X is connected, open or compact, then Y is connected, open or compact, respectively. That is, if $h : X \to Y$ is an homeomophism then it preserves connectedness, openness and compactness: $S \subseteq X$ is open iff $h(S) \subseteq Y$ is open, S is connected iff $h(S)$ is connected, and S is compact iff $h(S)$ compact*

Example 4.4. Any open interval on $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. Mathematically, we have $(a, b) \cong \mathbb{R}$ for any $a, b \in \mathbb{R}$, $b > a$. For example, if $a = -\pi/2$ and $b = \pi/2$ then $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ provides an example of such an homeomorphism. We can observe that both (a, b) and $\mathbb{R}$ are open sets and connected—properties that are preserved by homeomorphisms.

Example 4.5. Homeomorphisms preserve compactness per Fact 4.1. We can use this fact to quickly establish that the unit circle:

$$\mathbb{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\},$$

is not homeomorphic to an open interval (a, b). For the sake of contradiction, assume they are homeomorphic. Then, since the unit circle is compact, that would imply that (a, b) is compact and we thus reach a contradiction.

Although we already know that $\mathbb{S}^1$ is not homeomorphic to (a, b) it is still instructive to try to construct an homeomorphism. We can parameterize points in $\mathbb{S}^1$ using the map $h(\theta) =$

$(\sin(\theta), -\cos(\theta))$. It is simple to see that each point in the image of h belongs to $\mathbb{S}^1$ since $\sin^2(\theta) + \cos^2(\theta) = 1$. In order for h to be an homeomorphism, it must be injective. Hence, we choose the domain of h to be $(0, 2\pi)$, i.e., $a = 0$ and $b = 2\pi$. Note, however, that h is not surjective since $(0, -1) \in \mathbb{S}^2$ is not the image under h of any point in $(0, 2\pi)$. If we extend the domain of h to be an open set containing 0, e.g., $(-\varepsilon, 2\pi)$ for some $\varepsilon > 0$, then h is not longer injective since $h(-\varepsilon/2) = h(2\pi - \varepsilon/2)$. Similarly, if we extend the domain of h to be an open set containing 2π, e.g., $(0, 2\pi + \varepsilon)$ for some $\varepsilon > 0$, then h is not longer injective since $h(\varepsilon/2) = h(2\pi + \varepsilon/2)$. Although insightful, this discussion does not constitute a proof that $\mathbb{S}^1$ is not homeomorphic to (a, b) since we made several choices such as the choice of parameterization, the choice of its domain, etc. The reader can appreciate how the preservation of properties, such as compactness, by homeomorphims can lead to very simple proofs, such as the one provided in the first paragraph of this example.

Equivalences between flows. Conceptually similar to equivalences between spaces, we can study the equivalence of solutions to dynamical systems. However, for dynamical systems we have several different notions of equivalence. Consider two dynamical systems:

$$\dot{x} = f(x), \qquad \dot{y} = g(y), \tag{4.4}$$

with corresponding flows φ_t^f and φ_t^g where $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^n$ are locally Lipschitz continuous functions defined on the open and connected sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$. We denote the maximum interval of existence for solutions of the dynamical systems defined by f and g by $I_f(x)$ for $x \in X$ and $I_g(y)$ for $y \in Y$, respectively.

Definition 4.4. The dynamical systems defined by f and g in (4.4) are topologically conjugate if there exists an homeomorphism $h : X \to Y$ such that:

$$h(\varphi_t^f(x)) = \varphi_t^g(h(x)), \quad \forall t \in I_f(x) \cap I_g(h(x)).$$

If, in addition, there exists a strictly increasing homeomorphism $\tau_x : I_f(x) \to I_g(h(x))$ (i.e., τ_x is a parameterization of time due to the strictly increasing property since $t_1 < t_2$ implies that $\tau_x(t_1) < \tau_x(t_2)$), then f and g are topologically equivalent if:

$$h(\varphi_t^f(x)) = \varphi_{\tau_x(t)}^g(h(x)).$$

Finally, if h is a diffeomorphism, and τ_x is a diffeomorphism satisfying $\dot{\tau}_x > 0$ then f and g are orbitally equivalent.

Example 4.6. An illustrative example of orbital equivalence is given in Figure 4.5. In particular, two orbitally equivalent sinks shown on the left, orbitally equivalent unstable equilibrium given by "flattening" a space, and finally, orbitally equivalent trajectories given by taking a local coordinate chart (see Lecture 5 where charts are defined and used).

Example 4.7. The simplest example of orbital equivalence is given by a change of coordinates in the state space. Consider the differential equation $\dot{x} = f(x)$ and the diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$. Rewriting $\dot{x} = f(x)$ in the coordinates $y = h(x)$ provides:

$$\dot{y} = Dh(x)f(x)|_{x=h^{-1}(y)} \triangleq g(y).$$

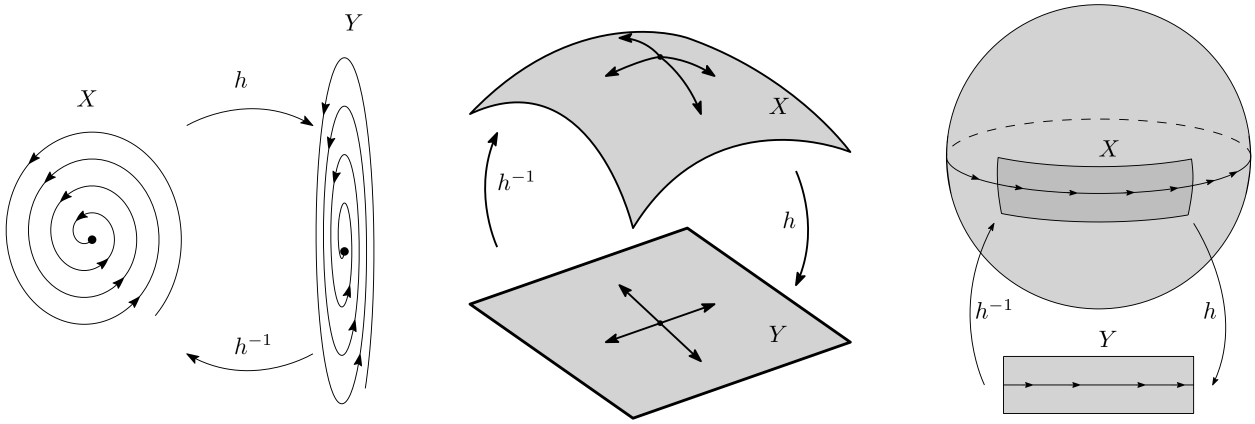

Figure 4.5. Illustration of three examples of orbital equivalence.

We claim that $\dot{x} = f(x)$ is orbitally equivalent to $\dot{y} = g(y)$ by taking $\tau_x(t) = t$. To verify this, let $\varphi_t^f(x)$ be a solution to $\dot{x} = f(x)$ where we can confirm that $h(\varphi_t^f(x))$ is a solution of $\dot{y} = g(y)$ with initial condition $h(x)$:

$$\begin{aligned}\frac{d}{dt}h(\varphi_t^f(x)) &= Dh(\varphi_t^f(x))\dot{\varphi}_t^f(x) = Dh(\varphi_t^f(x))f(\varphi_t^f(x)) \\ &= g(h(\varphi_t^f(x))).\end{aligned}$$

The converse direction can be checked in the same fashion, i.e., starting with a solution to $\dot{y} = g(y)$ and showing the result is a solution to $\dot{x} = f(x)$, completing the verification.

If we take this idea one step further and introduce a change of coordinates in the set of states $h : \mathbb{R}^n \to \mathbb{R}^n$ and a parameterization of time $\tau : \mathbb{R} \to \mathbb{R}$ we obtain the relation:

$$h(\varphi_t^f(x)) = \varphi_{\tau(t)}^g(h(x)).$$

Since we want to preserve the (increasing) direction of time, we require τ to be an increasing function, i.e., $\dot{\tau} > 0$. Finally, we arrive at the definition of orbital equivalence by considering a more general parameterization of time $\tau : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ that is state dependent, i.e., $\tau(x,t) = \tau_x(t)$.

Example 4.8. Every asymptotically stable 1-dimensional linear system is orbitally equivalent. To see this, consider two such systems, $\dot{x} = f(x) = -\alpha_1 x$ and $\dot{y} = g(y) = -\alpha_2 y$ with $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$. We will study stability in Lecture 6 but for now it suffices to note that $\varphi_t^f(x) = e^{-\alpha_1 t}x \to 0$ and $\varphi_t^g(y) = e^{-\alpha_2 t}y \to 0$ as $t \to \infty$. To establish orbital equivalence we consider $h(x) = x$ and $\tau_x(t) = \frac{\alpha_1}{\alpha_2}t$ wherein $\varphi_t^f(x) = e^{-\alpha_1 t}x = e^{-\alpha_2 \tau_x(t)}x = \varphi_{\tau_x(t)}^g(x)$. The maps h and τ_x are clearly diffeomorphism and $\dot{\tau}_x = \frac{\alpha_1}{\alpha_2} > 0$.

Similarly, one can show that every 1-dimensional unstable linear system is orbitally equivalent (since $\alpha_1, \alpha_2 < 0$).

As with the equivalence of spaces, equivalence of dynamical systems also preserves properties. This is illustrated in the next result. Its proof provides another justification for the requirement that the parameterization of time, τ_x, in the definition of topological equivalence is strictly increasing ($\dot{\tau}_x > 0$ for orbital equivalence).

Proposition 4.2. *Let $\dot{x} = f(x)$ and $\dot{y} = g(y)$ be nonlinear differential equations where $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^n$ are locally Lipschitz continuous functions defined on the open and connected sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$, let $h : X \to Y$ be an homeomorphism, and consider the set $S \subseteq X$. The following statements hold:*

(i) *If f and g are topologically equivalent and S is invariant with respect to φ_t^f, then $h(S)$ is invariant with respect to φ_t^g.*

(ii) *If f and g are topologically equivalent and S is forward (resp. backward) invariant with respect to φ_t^f, then $h(S)$ is forward (resp. backward) invariant with respect to φ_t^g.*

Proof. The proof of this result essentially follows from the definition of topological equivalence. For example, for part **(i)**, let $y \in h(S)$ wherein the goal is to show that $\varphi_{t'}^g(y) \in h(S)$ for all $t' \in I_g(y)$. Since $\tau_{h^{-1}(y)}$ is an homeomorphism between $I_f(h^{-1}(y))$ and $I_g(y)$, it is a bijection and there exists a $t \in I_f(h^{-1}(y))$ such that $\tau_{h^{-1}(y)}(t) = t'$. It follows that:

$$h^{-1}(y) \in S \quad \Longrightarrow \quad \varphi_t^f(h^{-1}(y)) \in S \quad \xrightarrow{x=h^{-1}(y)} \quad \varphi^g_{t'=\tau_{h^{-1}(y)}(t)}(y) = h(\varphi_t^f(h^{-1}(y))) \in h(S),$$

as desired. Part **(ii)** follows in a similar fashion where, for forward invariance, $t > 0$ implies $t' = \tau_{h^{-1}(y)}(t) > 0$ since τ_x is strictly increasing. □

4.3 Linearization and Hartman-Grobman Theorem

Having introduced notions of equivalence between dynamical systems, we now address the question: when is a nonlinear system equivalent to its linearization around an equilibrium point? The main result answering this question is the *Hartman-Grobman* Theorem.

We start by recalling the notion of equilibrium point (plural is equilibria).

Definition 4.5. Consider the nonlinear differential equation:

$$\dot{x} = f(x), \tag{4.5}$$

where $f : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on the open and connected set $E \subseteq \mathbb{R}^n$. A point $x^* \in \mathbb{R}^n$ is an equilibrium point for $\dot{x} = f(x)$ if $f(x^*) = 0$.

Equilibria define a special type of invariant sets. At an equilibrium point x^*, the solution to (4.5) is given by $x(t) = x^*$ for all $t \in I_f(x^*) = \mathbb{R}$, i.e., $x(t) \equiv x^*$. Therefore:

Fact 4.2. *If x^* is an equilibrium point for* (4.5)*, then the set $S = \{x^*\}$ is invariant.*

We now recall the notion of linearization at an equilibrium.

Definition 4.6. The linearization of the nonlinear differential equation (4.5) at an equilibrium point $x^* \in E$ is given by the linear differential equation:

$$\dot{x} = \underbrace{Df(x^*)}_{A}\, x. \tag{Linearization}$$

To understand how the behavior of the linear system relates to the nonlinear system, we need to consider a specific class of equilibria; these are characterized by the eigenvalues of the matrix $A = Df(x^*)$. Recall that eigenvalues, $\lambda \in \mathbb{C}$, are (complex) solutions of the characteristic equation: $\det(\lambda I - A) = 0$. We let $\sigma(A) \subset \mathbb{C}$ denote the spectrum of A consisting of all eigenvalues. Finally, $\Re(\lambda) \in \mathbb{R}$ is the real part of a given eigenvalue $\lambda \in \sigma(A)$.

Definition 4.7. An equilibrium point $x^* \in E$ of (4.5) is hyperbolic if all of the eigenvalues of $A = Df(x^*)$ have nonzero real part: $\Re(\lambda) \neq 0$ for all $\lambda \in \sigma(A)$.

***Remark* 4.1.** The term hyperbolic stems from the observation that trajectories of a linear differential equation (whose matrix A has positive and negative eigenvalues) not starting or converging to the origin resemble hyperbolas. See, for example, Figure 2.1 illustrating the trajectories of the linear differential equation in Example 2.3.

Example 4.9. Returning to Example 4.1, $\dot{x} = f(x)$ has an equilibrium point at $0 \in \mathbb{R}^2$ and the linearization of f (for $k \geq 2$) is given by:

$$Df(0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, the eigenvalues are 1 and -1, and thus $x^* = 0$ is hyperbolic.

Hartman-Grobman Theorem. The Hartman-Grobman theorem provides one of the most fundamental results in nonlinear systems; it states that near a hyperbolic equilibrium every nonlinear system is equivalent to a linear system.

Theorem 4.1. *Consider a nonlinear differential equation $\dot{x} = f(x)$, where $f : E \to \mathbb{R}^n$ is a continuously differentiable map defined on the open and connected set $E \subseteq \mathbb{R}^n$, and let $x^* \in E$ be an equilibrium point. If x^* is hyperbolic then there exist a neighborhood X of x^* (i.e., $x^* \in X \subseteq E$ with X an open set) and an homeomorphism $h : X \to Y \triangleq h(X)$ mapping $x^* \in X$ to $h(x^*) = 0 \in Y$, satisfying the following implication (implying topological conjugacy):*

$$\forall x_0 \in X \;\; \exists a(x_0) \in \mathbb{R}_{>0} \qquad \text{s.t.} \qquad h(\varphi_t(x_0)) = e^{At} h(x_0), \qquad \forall t \in (-a(x_0), a(x_0)), \tag{4.6}$$

where $A = Df(x^*)$.

This result, quite striking at first, asserts that nonlinear systems are topologically conjugate to their linearization around hyperbolic equilibria. There are some important caveats to this result:

- It is only valid in a neighborhood of equilibria. Hence, it provides no information about the behavior of the nonlinear system away from equilibria.
- It is fundamentally *local*, and it is not easy to obtain a precise characterization of how local, i.e., how small the neighborhood X needs to be.
- It does not account for more complex (and thus interesting) types of invariant sets, e.g., periodic orbits.

Example 4.10. Returning to Example 4.1 (with the corresponding linearization given in Example 4.9), we can explicitly construct the homeomorphism h given by Theorem 4.1 since we can explicitly calculate the flow of the system $\dot{x} = f(x)$ in (4.1) and the flow of its linearization:

$$\underbrace{\varphi_t(x_0) = \begin{bmatrix} x_{01} e^{-t} \\ x_{02} e^t + \frac{x_{01}^k}{k+1}\left(e^t - e^{-kt}\right) \end{bmatrix}}_{\text{flow of nonlinear system (4.1)}}, \qquad \underbrace{e^{Df(0)t} x_0 = \begin{bmatrix} x_{01} e^{-t} \\ x_{02} e^t \end{bmatrix}}_{\text{flow of linearization } \dot{x} = Df(0)x}.$$

By judiciously comparing φ_t with $e^{Df(0)t}$ we arrive at the map:

$$h(x) = \begin{bmatrix} x_1 \\ x_2 + \frac{x_1^k}{k+1} \end{bmatrix},$$

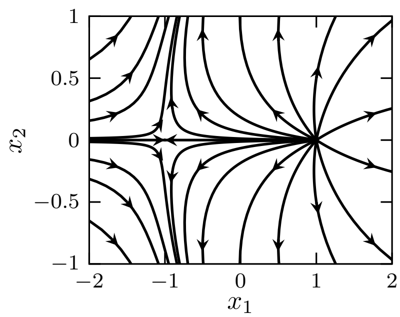

Figure 4.6. Phase portrait for Example 4.11 showing how the local behavior of the linearization at each equilibria can be "pieced" together to get a sense for the global behavior of the nonlinear system.

and can confirm it is a bijection with the continuous inverse given by $h^{-1}(y) = (y_1, y_2 - y_1^k/(k+1))$. We now verify that h defines a topological conjugacy between φ_t and $e^{Df(0)t}$ as asserted in Theorem 4.1:

$$\begin{aligned} h(\varphi_t(x_0)) &= \begin{bmatrix} x_{01}e^{-t} \\ \left(x_{02} + \frac{x_{01}^k}{k+1}\right) e^t \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} + \frac{x_{01}^k}{k+1} \end{bmatrix} \\ &= e^{Df(0)t} h(x_0). \end{aligned}$$

This example is quite special since we have a closed-form expression for the flow of the nonlinear system. This will not be the case, in general, and thus we won't be able to use the flow to obtain the homeomorphism h.

Example 4.11. By considering the linearization of a nonlinear system at multiple equilibria, we can in some cases build a more "global" picture of the system's behavior. To see this, let:

$$\dot{x} = f(x) = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ 2x_2 \end{bmatrix}.$$

We have two equilibria:

$$x_1^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad x_2^* = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

and the matrices defining the linearization at these equilibria are:

$$Df(x_1^*) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad Df(x_2^*) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore, by stitching together the local behavior provided by the linearizations we can attempt to build a global picture of the nonlinear dynamics as shown in Figure 4.6.

Example 4.12. This example highlights the limits of linearization. Let $\varepsilon \neq 0$ and consider the system:

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -x_1 - \varepsilon x_1^2 x_2 \end{bmatrix}.$$

Linearizing at the equilibrium point located at the origin yields:

$$Df(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad e^{Df(0)t}x_0 = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} x_0,$$

by Example A.7 in Appendix A. Since the eigenvalues of $Df(x^*)$ are $\pm i$, i.e., purely imaginary, we cannot apply the Hartman-Grobman theorem. In reality, the nonlinear system behaves very different from its linearization (see Figure 4.7). The linearization consists of marginally stable periodic orbits while the nonlinear system has a stable equilibrium point at the origin (this will be verified in Lecture 10 and specifically Example 10.4). Importantly, even simulation of the nonlinear system is deceptive in this case: it appears as though the nonlinear system may have a stable periodic orbit near the origin when it does not (this is verified in Example 13.4 in Lecture 13).

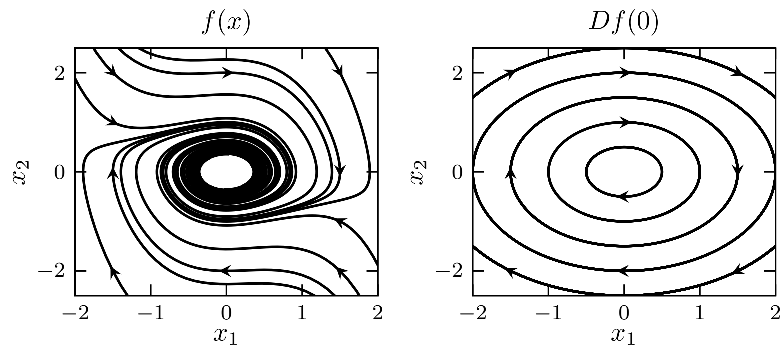

Figure 4.7. The phase portrait for the nonlinear system in Example 4.12 is shown on the left and the phase portrait for its linearization at the origin on the right. In this case, it is not clear (even from simulations) if the system has a stable sink at the origin or it is marginally stable. The Hartman-Grobman theorem does not give anymore insight on the behavior of the nonlinear system since the equilibrium point at the origin is not hyperbolic, yielding the behavior shown on the left. Thus the linear and nonlinear system display qualitatively different behavior.

Example 4.13. Returning to the system introduced in Example 4.2, we note there is a single equilibrium point at the origin. The linearization at this point is given by:

$$Df(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

and referring to Example A.7 in Appendix A we conclude the flow of the linearization is given by:

$$e^{Df(0)t}x_0 = e^t \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} x_0.$$

We conclude the existence of an *unstable source* at the origin which, interestingly enough, shows the limits of linearization. We know, via Example 4.2, of the existence of a periodic orbit on the unit circle and no information about this periodic orbit is provided by the linearization. In other words, the linearization fails to identify the most interesting behavior of the system: a periodic orbit that is "almost" globally asymptotically stable, i.e., all the trajectories, except for the trajectory starting at the origin, converge to the periodic orbit.

Additional Reading

While not within the scope of this book, a proof of the Hartman-Grobman Theorem can be found in [166]; additionally some of the examples used in this lecture follow from this text. Further details on orbitally and topologically equivalent flows can be found in [182].

Problems for Lecture 4

[P4.1] Prove Proposition 4.1.

[P4.2] Consider a function $h : X \to Y$ with $X, Y \subset \mathcal{V}$ with $\mathcal{V}$ a normed vector space. Prove that if h is a bijection, then h^{-1} is well-defined, and that h^{-1} is also a bijection.

[P4.3] [**Advanced Problem**] Prove Fact 4.1, i.e., that homeomorphisms preserves connectedness, openness and compactness.

[P4.4] Consider the Cantor map defined in Example 4.3: $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Prove that this map is bijective.

[P4.5] Prove that any open ball in $\mathbb{R}^n$ is homeomorphic to all of $\mathbb{R}^n$. You can first establish that any two open balls are homemorphic and then prove the result for the open ball of radius 1 centered at the origin.

[P4.6] As will be seen in Lecture 13, a solution $\varphi_t(x)$ to $\dot{x} = f(x)$, for $x \in \mathbb{R}^n$ is periodic if $\varphi_{t+T}(x) = \varphi_t(x)$ for some $T > 0$. The corresponding periodic orbit is given by:

$$\mathcal{O} \triangleq \{\varphi_t(x) \in \mathbb{R}^n \ : \ t \in \mathbb{R}\} = \{\varphi_t(x) \in \mathbb{R}^n \ : \ t \in [0, T)\}.$$

Show that every periodic orbit is diffeomorphic to the unit circle.

[P4.7] Suppose that $\dot{x} = f(x)$ has a forward invariant set $S \subset \mathbb{R}^n$ and that:

$$\lim_{t\to\infty} \varphi_t^f(x) = 0 \qquad \forall\, x \in S.$$

Show that for the dynamical system $\dot{y} = g(y)$, if f and g are topologically equivalent, then the dynamics dictated by g have a forward invariant set such that any flow with an initial condition in such set converge to a single point also in this set.

[P4.8] Consider the system in Example 1.3. What does the linearization say about the system dynamics? What behavior does the linearization fail to capture?

[P4.9] Returning to Problem P2.5 in Lecture 2, show that for a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$, the dynamical system $\dot{x} = f(x)$ is orbitally equivalent to

$$\dot{y} = g(y) \triangleq \frac{f(y)}{1 + \|f(y)\|}.$$

Hint: pick the parametrization of time

$$\tau(t, x_0) = \int_0^t \Big(1 + \|f(\varphi_s^f(x_0))\|\Big)\, ds.$$

[P4.10] [**Advanced Problem**] Show that a flow is a group homomorphism from the group $(\mathbb{R}, +)$ to the group of automorphims of $\mathbb{R}^n$. (Note: groups are defined in Problem 4. The automorphism group of $\mathbb{R}^n$ consists of all the maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ that have a unique inverse and where the product operation is function composition.)