PROBLEMS

Problem 1 (Book Problem 2.1)

For each of the functions $f(x)$ given below, demonstrate whether f is continuous, locally Lipschitz continuous, Lipschitz continuous, or continuously differentiable on $\mathbb{R}$.

(a) $f(x) = x^2 + |x|$

(b) $f(x) = -x + a\sin(x)$

(c) $f(x) = -x + 2|x|$

(d) $f(x) = \tan(x)$

Problem 2 (Book Problem 2.2)

Let $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous on $\mathbb{R}$. Show that $f_1 + f_2$, $f_1 f_2$, and $f_1 \circ f_2$ are locally Lipschitz continuous. $f_1 \circ f_2$ denotes function composition, i.e., $f_1 \circ f_2(x) = f_1(f_2(x))$

Problem 3

Give a brief summary of your research interests, particularly (if applicable) what type of dynamical systems you work with, and what properties of those dynamical systems you care about.

Problem 4

Briefly discuss your motivation for taking this class.

Problem 5 (Book Problem 2.10)

Consider the system in Problem P1.7, consider an iron ball levitating within a magnetic field created by a single electromagnet. A simplified model describing the vertical position of the ball is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{2m}\lambda^2 - g \\ -\frac{R}{c}(1 - x_1)\lambda + u \end{bmatrix}$$

where x_1 denotes the ball's height, x_2 the ball's velocity, m is the ball's mass, g is gravity's constant, λ is the magnetic flux linkage in the electromagnet, R is the resistance of the electromagnet's coil, c is a positive constant modeling the electromagnet's geometry and construction, and u is the voltage applied at the coil's terminals constituting the input to this system. This simplified model describes, e.g., a magnetically levitated train where the train mass has been lumped into the ball.
Show that solutions exist and are unique by showing that the system is locally Lipschitz continuous. Is it Lipschitz continuous? (Assume u is constant in time. What if u is a continuous function of time?)

PROJECT

To begin the course project, we would like you to choose a nonlinear dynamical system that you are interested in exploring further this term. Please include the nonlinear differential equations, a description of the system, and the motivation behind your choice. Feel free to choose something related to your research, or simply something you find interesting.
Some notes/guidelines:

- Your system should have at least 2-3 states. Feel free to choose larger numbers of degrees of freedom or simplify a complex system if it has too many degrees of freedom (the full 3D quadrotor to a planar model for instance). Keep in mind that the analysis will get more tedious the larger the number of states – we will for example ask you to calculate the Lipschitz constant of your system and run Picard Iteration, which becomes unwieldy in higher dimensions.
- Your system may need some initial form of control to stabilize it around an equilibrium point for future project analysis - for instance, take the planar quadcoptor, which falls downwards with acceleration g under no control action. Feel free to use an appropriate control technique to achieve this (linear feedback, LQR, etc...). If this is necessary, note that simplicity in the control law chosen will simplify analysis done later in the project.
- You are not constrained to mechanical systems. To give some examples, interesting projects in the past have been: the double or triple pendulum, the Lorentz system, optimization programs as discrete time dynamical systems, models of memory, a biological clock, neural networks as dynamical systems, a modification to the Lotka–Volterra equations, etc...
- If you have questions about the project please come to office hours, we are happy to discuss further.

OPTIONAL PROBLEMS

Optional problems will not be graded (and solutions will not be released). They may be of increased difficulty, or be problems which are omitted from the homework to keep it from being too long. These problems may make for good study material or qualifying exam preparation. Since solutions will not be released, feel free to ask questions about these questions in office hours (at then end, after required homework problem questions have been covered).

Problem 6 (Book Problem 2.5)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable for all $y \in \mathbb{R}^n$ and define $g(y)$ by:

$$g(y) = \frac{f(y)}{1 + \|f(y)\|}$$

Show that $\dot{y} = g(y)$, with $y(0) = y_0$ has a *unique* solution defined for all $t \geq 0$.

Problem 7 (Book Problem 2.8)

There are nonlinear systems whose solutions exist for all $t \geq 0$. Consider the following nonlinear system:

$$\begin{aligned}
\dot{x}_1 &= x_1 x_2 - x_2 x_3 \\
\dot{x}_2 &= -x_1^2 - x_3^2 \\
\dot{x}_3 &= x_1 x_2 + x_2 x_3
\end{aligned}$$

and the function $H(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$. Show that H is constant along the solutions of the nonlinear system, i.e., $H(x(t)) = H(x(0))$. Use the fact that the manifold $H(x) = H(x(0))$ is compact to conclude existence of solutions for all $t \geq 0$.