CS-225: Discrete Structures in CS
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Extra Notes on Conditionals

There are several tricky nuances to conditional statements that are common areas of frustration for students. While each person must make their own peace/understanding out of the situations and wordings, I thought it might be somewhat helpful if I share my own ways of thinking through these ideas.

For me, it generally comes down to my parenting challenges (I'm supposed to call them "opportunities for personal growth" ...) with my second daughter. We will call her "Asa." She's a spunky kid, which is to her great credit. However, this often leads to frustrations for her loving and doting father, especially around meals.

Some people enjoy salads. I have also been known to enjoy a salad when out at a restaurant. However, at our home, salads are best seen as necessary nutrient supplements for the good stuff: pizza, pasta, tacos, etc. But, still, to remain healthy and at our best, we must eat salad.

Conversely, I understand that some people don't much care for dessert. I have also been known to have a disappointing dessert at a restaurant. However, at our home, desserts are enjoyable occurances that one always looks forward to.

You will hear a lot about Asa's food battles with her loving father as we explore conditionals.

Why is $p \to q$ true when p is false?

A topic that frequently comes up when students are learning conditionals ("if...then ...") is why the statement $p \to q$ is true when the hypothesis (p) is false. This question most often comes when people want to know why $F \to T$ is a true statement.

At my house, I often tell Asa "if you eat your salad, then you can have dessert." In this situation, the hypothesis is $p =$ "you eat your salad" and the conclusion is $q =$"you can have dessert". Let's go through the possible scenarios:

- Salad is eaten, dessert is had: In this case, p and q are both true. We have $T \to T$. Asa has vegetables and dessert. Happiness is had. My statement to her was true.
- Salad is eaten, dessert is not had: In this case, p is true but q is false. We have $T \to F$. Asa has vegetables, but dessert has been withheld. I am a liar and correctly subjected to all sorts of mean, nasty comments about my failure as a reasonable parent. My statement to her was false.
- Salad is not eaten, dessert is had: In this case, p is false and q is true. We have $F \to T$. Asa did not eat her vegetables, but she was given dessert. She is far, far too content to say anything mean to me. My statement to her was true.
- Salad is not eaten, dessert is not had: In this case, both p and q are false. We have $F \to F$. Asa is not particularly happy with me, since she did not get dessert. But she's also smart enough to know that any complaining would just be met with an exceedingly annoying and long-winded explanation about her failure to meet her end of the bargain. She's a spirited kid, but she's not a dumb kid. She instead acknowledges that I didn't break my word. My statement to her was true.

Different Wording Options

Necessary

The word "necessary" often causes much confusion. At dinner, I often tell Asa "To get dessert it is necessary to eat your salad." What I really mean is that "If you don't eat your salad, then you will not get dessert." So, if p = "you eat your salad" and q ="you get dessert", then the sentence "For q it is necessary that p" is the same as saying "if not p then not q". Since contrapositives are logically identical to their original conditional, we could also change that from "if not p then not q" to "if q then p".

What's key here is that the placement of the word "necessary" isn't what's important. Rather, it's the phrase that is deemed necessary that is important. That is, each of the following sentences are equivalent:

- To get dessert it is necessary to eat your salad.
- Eating your salad is necessary to get dessert.
- To get dessert, eating your salad is necessary.
- It is necessary to eat salad to get dessert.
- If you don't eat your salad, then you will not get dessert.
- If you get dessert then you ate your salad.

Sufficient

The word "sufficient" is often (correctly) seen as the opposite pairing with necessary. At dinner, I might tell Asa "To get dessert it is sufficient to eat your salad." This would not be identical to the statement in the previous section, since in this case the statement is equivalent to "If you eat your salad then you get dessert." So, if p ="you get dessert" and q = "you eat your salad", then the statement "For p it is sufficient that q" is the same as saying "if q then p."

Once again, what's key here is that the placement of the word "sufficient" isn't what's important. Rather, it's the phrase that is deemed sufficient that is important. That is, each of the following sentences are equivalent:

- To get dessert it is sufficient to eat your salad.
- Eating your salad is sufficient to get dessert.
- To get dessert, eating your salad is sufficient.
- It is sufficient to eat salad to get dessert.
- If you eat your salad, then you get dessert.
- If you don't get dessert, then you didn't eat your salad.

Unless

The word "unless" is the variation that I personally find most tricky. However, I am fond of telling Asa "No dessert for you unless you eat your salad!", so we should figure out what that actuallly means. What that sentence means is "If you don't eat your salad then you get no dessert." So, if p = "no dessert for you" and q = "you eat your salad", then the statement "p unless q" means "If not q then p".

As before, we can't rely on the placement of the word "unless" in the sentence. Instead, we must rely on what is being claimed "unless" the other thing happens. That is, each of the following sentences are equivalent:

- No dessert for you unless you eat your salad.
- Unless you eat your salad, no dessert for you!
- If you do not eat your salad, then you get no dessert.
- If you get dessert, then you ate your salad.

Only If

The phrase "only if" is so deceptive. At first glance, it looks like an "if ... then ..." but backwards. And, well, it's generally put in a different part of a sentence than the "if" goes. Since we have two "backwards" constructions, it turns out that the basic use of "only if" is actually normal and forwards!

Here's what I mean. Suppose I tell Asa "you get dessert only if you eat salad". This is actually the same as saying "If you get dessert then you ate salad." So, if p = "you get dessert" and q ="you eat salad", then the phrase "p only if q" is the same as "if p then q".

As always, we can't rely on the placement of the phrase "only if" in the sentence. Instead, we must rely on what the "only if" is applied to. That is, each of the following sentences are equivalent:

- You get dessert only if you eat salad.
- Only if you eat salad do you get dessert.
- If you get dessert then you ate salad.
- If you did not eat salad then you do not get dessert.

Dude, Really? Just say the rules!

Fair enough.

- p is necessary for q is denoted $q \to p$
- p is sufficient for q is denoted $p \to q$
- p unless q is denoted $\sim q \to p$
- p only if q is denoted $p \to q$