ASTR 605 Problem Set 3

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Question I.

For the NFW profile we have

$$\rho(r) = \frac{\rho_0}{\frac{r}{r_0}\left(1 + \frac{r}{r_0}\right)^2}$$

which gives the mass profile and potential

$$M(r) = 4\pi\rho_0 r_0^3 \left(\ln(1 + r/r_0) - \frac{r/r_0}{1 + r/r_0}\right)$$
$$\Phi(r) = -4\pi G\rho_0 r_0^2 \frac{\ln(1 + r/r_0)}{r/r_0}$$

From the Lagrangian of an axisymmetric potential we have the conservation equations

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mathrm{d}\Phi}{\mathrm{d}r} = 0$$
$$\frac{\mathrm{d}r^2\dot{\theta}}{\mathrm{d}t} = 0$$

taking the first equation of motion we have the $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ going to

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -4\pi G\rho_0 r_0^2 \left(\frac{r_0^2}{1 - r/r_0}\frac{1}{r} - \frac{r_0^3 \ln(1 + r/r_0)}{r^2}\right)$$

so the equation of motion becomes

$$\ddot{r} - r\dot{\theta}^2 - 4\pi G\rho_0 r_0^2 \left(\frac{r_0^2}{1 - r/r_0}\frac{1}{r} - \frac{r_0^3 \ln(1 + r/r_0)}{r^2}\right) = 0$$

This gives the motion of an elliptical harmonic oscillator. The Burkert profile

$$\rho(r) = \frac{\rho_0}{\frac{r}{r_0}\left(1 + \frac{r^2}{r_0^2}\right)}$$

follows similarly, where we have

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mathrm{d}\Phi}{\mathrm{d}r} = 0$$
$$\frac{\mathrm{d}r^2\dot{\theta}}{\mathrm{d}t} = 0$$

where

$$\Phi(r) = \frac{G}{r}\int \rho(r)\,\mathrm{d}r$$

which has the derivative

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{-G}{r^2} \int \rho(r)\,\mathrm{d}r + \frac{G\rho_0}{r})$$

and the result follows similarly.

Question II.

For the generating function

$$F = qQ$$

we have the transformations

$$p_i = \frac{\mathrm{d}F}{\mathrm{d}q_i} = Q$$
$$P_i = -\frac{\mathrm{d}F}{\mathrm{d}Q_i} = -q$$

so the transformation effectively switches the roles of the variables.

Question III.

If we start with the $u \equiv 1/r$ substitution, we have

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \left(1 - \frac{2\beta}{L^2}\right) u = \frac{\alpha}{L^2}$$

which has the general solution

$$u = C \cos\left(\frac{\theta - \theta_0}{K}\right) + \frac{\alpha K^2}{L}$$

where

$$K \equiv \left(1 - \frac{2\beta}{L^2}\right)$$
$$C = \frac{K}{L}\sqrt{2E + \left(\frac{\alpha K}{L}\right)^2}$$

solving for the θ_0 parameter we have

$$\theta_0 = \theta - K \arccos\left[\frac{1}{C}\left(u - \frac{\alpha K^2}{L^2}\right)\right]$$

and returning from the u substitution we have

$$\theta_0 = \theta - K \arccos\left[\frac{1}{C}\left(\frac{1}{r} - \frac{\alpha K^2}{L^2}\right)\right]$$

which is a function only of integrals of motion, and is such an integral of motion itself.