5201 Problem Set 7

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Question I.

January 2019 Classical Mechanics Prelim Problem

A simple pendulum of mass m and length l hangs from a trolley with total mass M running on smooth horizontal rails. Ignore the moment of inertia of the wheels. The pendulum swings in a plane parallel to the rails; see Figure 1.

(a) Using the position of the trolley X and the angle of inclination θ as your generalized coordinates, write down the Lagrangian and Lagrange's equations.

Let's begin by defining our coordinates:

$$\begin{aligned} X &= X \\ x &= X + l\sin\theta \\ y &= l - l\cos\theta \end{aligned}$$

Where θ is the angle from the vertical of the pendulum. Now we have to find the Lagrangian:

$$\begin{aligned} T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}l\cos\theta\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 \\ V &= mgy \\ &= mg(l - l\cos\theta) \end{aligned}$$

Lagrange's Equations give:

$$\begin{aligned} \mathcal{L} &= T - V \\ \frac{\partial\mathcal{L}}{\partial X} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}} \\ \frac{\partial\mathcal{L}}{\partial X} &= 0 \\ \frac{d}{dt}\left[(M+m)\dot{X} + ml\cos\theta\dot{\theta}\right] &= 0 \\ (M+m)\dot{X} + ml\cos\theta\dot{\theta} &= \text{constant} \end{aligned}$$

So X is an ignorable coordinate. The θ equation gives:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \theta} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \\ -m\dot{X}l\sin\theta\dot{\theta} - mgl\sin\theta &= \frac{d}{dt}(m\dot{X}l\cos\theta + ml^2\dot{\theta}) \\ -mgl\sin\theta &= m\ddot{X}l\cos\theta + ml^2\ddot{\theta}\end{aligned}$$

These give the equations of motions via the Lagrangian.

(b) Suppose the whole setup is first at rest with the pendulum held at an angle θ_0, and the pendulum is then released. What is the angular velocity of the pendulum when it points straight down?

Using conservation of energy since no nonconservative forces are acting on the system:

$$\begin{aligned}V_i &= T_f \\ mgl - mgl\cos\theta_0 &= \frac{1}{2}(M+m)\dot{X}_f^2 + m\dot{X}_f l\cos\theta\dot{\theta}_f + \frac{1}{2}ml^2\dot{\theta}_f^2\end{aligned}$$

Since the system starts from rest and we already determined that X is an ignorable coordinate, we are justified in the following:

$$\begin{aligned}(M+m)\dot{X} + ml\cos\theta\dot{\theta} &= \text{constant} \\ (M+m)\dot{X}_f + ml\dot{\theta}_f &= 0\end{aligned}$$

Solving the result from conservation of energy for $\dot{X}_f$ and substituting into this result:

$$\begin{aligned}\dot{X}_f &= -\frac{ml\dot{\theta}_f}{M+m} \pm \frac{1}{M+m}\sqrt{m^2l^2\dot{\theta}_f^2 - 2(M+m)\left(\frac{1}{2}ml^2\dot{\theta}_f^2 + mgl(\cos\theta_0 - 1)\right)} \\ ml\dot{\theta}_f &= (M+m)\left[\frac{ml\dot{\theta}_f}{M+m} \pm \frac{1}{M+m}\sqrt{m^2l^2\dot{\theta}_f^2 - 2(M+m)\left(\frac{1}{2}ml^2\dot{\theta}_f^2 + mgl(\cos\theta_0 - 1)\right)}\right] \\ 0 &= \sqrt{m^2l^2\dot{\theta}_f^2 - 2(M+m)\left(\frac{1}{2}ml^2\dot{\theta}_f^2 + mgl(\cos\theta_0 - 1)\right)} \\ 0 &= m^2l^2\dot{\theta}_f^2 - 2(M+m)\left(\frac{1}{2}ml^2\dot{\theta}_f^2 + mgl(\cos\theta_0 - 1)\right) \\ \dot{\theta}_f &= \sqrt{\frac{2(M+m)mg(cos\theta_0 - 1)}{Mml}}\end{aligned}$$

Hint: One way to solve this problem is to find two constants of the motion.

Question II.

August 2017 Classical Mechanics Prelim Problem

A solid uniform circular cylinder of mass m and radius r rolls (under gravity) inside a fixed hollow cylinder of radius $R(> r)$, the axes of the cylinders being parallel to each other and the horizontal. At any time t during the motion, the plane containing the axes of the cylinders makes an angle θ with the vertical. Assume that the smaller cylinder rolls without slipping inside the larger one and that the amplitude of motion is such that it does not fall off at any point(see Fig. 2). Take the earth's gravity to be a uniform acceleration g in the downward vertical.

(a) Express the angular speed of the small cylinder in terms of $\dot{\theta}$.

Here we have defined two angular coordinates: θ, the angle from the vertical to the center of mass of the small cylinder, and ϕ, the rotation of the small cylinder about its own center of mass axis. We can use the rolling condition to write a relation between the two and find $\dot{\phi}$, the angular speed of the small cylinder:

$$\begin{aligned}(R-r)\theta &= r\phi \\ \phi &= \theta\frac{R-r}{r} \\ \dot{\phi} &= \dot{\theta}\frac{R-r}{r}\end{aligned}$$

(b) Write down the Lagrangian for the motion described above.

We define some coordinates with the origin at the center of the large cylinder:

$$\begin{aligned}x &= (R-r)\sin\theta \\ y &= R-(R-r)\cos\theta\end{aligned}$$

Finding the energies:

$$\begin{aligned}T &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}I\dot{\phi}^2 \\ &= \frac{1}{2}(R-r)^2\dot{\theta}^2+\frac{1}{4}m(R-r)^2\dot{\theta}^2 \\ &= \frac{3}{4}m(R-r)^2\dot{\theta}^2 \\ U &= mg(R-(R-r)\cos\theta) \\ \mathcal{L} &= T-U \\ &= \frac{3}{4}m(R-r)^2\dot{\theta}^2-mg(R-(R-r)\cos\theta)\end{aligned}$$

(c) Find the period of small oscillations of the small cylinder.

Solving the Lagrange equation of motion for this system:

$$\frac{3}{2}m(R-r)^2\ddot{\theta} + mg(R-r)\sin\theta = 0$$
$$\ddot{\theta} + \frac{2g}{3(R-r)}\sin\theta = 0$$
$$\omega^2 = \frac{2\pi}{T} = \frac{2g}{3(R-r)}$$
$$T = \frac{2\pi}{\sqrt{\frac{2g}{3(R-r)}}}$$

Question III.

August 2019 Classical Mechanics Prelim Problem

A disk of mass M and radius R rotates in a horizontal plane about its center. A point mass m can slide freely along one of the radii of the disk, and is attached to the center of the disk by a massless spring of natural length l and force constant k, as shown in the figure.

(a) When the mass is at distance r from the center of the disk find the moment of inertia of the system (disk, spring and mass combined) about an axis through the center of the disk perpendicular to the disk.

The moment of inertia for a nonconstant r is given by:

$$\begin{aligned} I &= I_{disk} + I_m \\ &= \frac{1}{2}MR^2 + mr^2 \end{aligned}$$

(b) When the mass is at a distance r from the center of the disk to find an expression for the energy of the system in terms of r, $\dot{r}$ and the total angular momentum J of the system.

$$\begin{aligned} J &= I\dot{\theta} \\ &= \frac{1}{2}MR^2\dot{\theta} + mr^2\dot{\theta} \\ T &= \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \\ V &= \frac{1}{2}k(r-l)^2 \\ T + V &= \frac{J^2}{2(\frac{1}{2}MR^2 + mr^2)} + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}k(r-l)^2 \end{aligned}$$

(c) Derive Lagrange's equations.

$$
\begin{aligned}
\mathcal{L} &= T - V \\
&= \frac{1}{4}MR^2\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}k(r-l)^2
\end{aligned}
$$

The θ equation shows that θ is an ignorable coordinate:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= 0 \\
\frac{1}{2}MR^2\dot{\theta} + mr^2\dot{\theta} &= \text{constant} = J
\end{aligned}
$$

The r equation gives:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\
m\ddot{r} - mr\dot{\theta}^2 + k(r-l) &= 0
\end{aligned}
$$

(d) Suppose the disk is rotating at a constant angular velocity Ω_0 with the spring having a constant extension $r = r_0$ during this rotation. Find r_0 as a function of Ω_0.

Using the initial conditions, and knowing that constant angular velocity means $\ddot{r} = 0$:

$$
\begin{aligned}
-mr_0\Omega_0^2 + k(r_0 - l) &= 0 \\
r_0 &= \frac{kl}{k - m\Omega_0^2}
\end{aligned}
$$

(e) Find the frequency of small oscillations around the equilibrium configuration.

Solving the r equation:

$$
\begin{aligned}
m\ddot{r} - mr\dot{\theta}^2 + k(r-l) &= 0 \\
\ddot{r} + r\left(\frac{k}{m} - \dot{\theta}^2\right) - \frac{kl}{m} &= 0
\end{aligned}
$$

From this we have the coefficient of r is ω^2:

$$\omega^2 = \frac{k}{m} - \dot{\theta}^2$$
$$\omega = \sqrt{\frac{k}{m} - \dot{\theta}^2}$$

Question IV.

August 2019 Classical Mechanics Prelim Problem

A point particle of mass m moves in one dimension subject to the potential:

$$V(x) = \frac{a}{\sin^2 \frac{x}{x_0}}$$

with constant a and x_0.

(a) Solve for Hamilton's characteristic function $W(x, a)$ for this potential, obtaining $W(x, a)$ as an indefinite integral that you do not need to evaluate. Hint: W is the solution to the Hamilton-Jacobi equation for fixed total energy α, written as $H(x, \partial W/\partial x) - a = 0$.

We have the Hamiltonian via the Hamilton-Jacobi equation:

$$\frac{1}{2m}\left(\frac{\partial W}{\partial \alpha}\right)^2 + \frac{a}{\sin^2 \frac{x}{x_0}} = \alpha$$

where

$$\mathcal{H} + \frac{\partial S}{\partial t} = 0$$
$$S(q, \alpha; t) = W(q, \alpha) - \alpha t$$
$$p = \frac{\partial S}{\partial \alpha}$$

Solving for W:

$$W = \int \sqrt{2m\alpha - \frac{2ma}{\sin^2 \frac{x}{x_0}}} dx$$

(b) In the angle-action variable method, a new constant of the motion J is introduced which measures the change in W, viz. $\int dW$, over a single period of the motion. Under what conditions is this method applicable? Justify its use for this potential.

We are justified in using the angle-action method since the Hamiltonian is separable and periodic, so we have an action:

$$J = \frac{1}{2\pi} \oint p dx$$

(c) Evaluate J for this potential, given total energy α.

Noting that one quarter of the period is $(0, \pi x_0)$:

$$
\begin{aligned}
J &= \frac{\sqrt{8m\alpha}}{\pi}\int_0^{\pi x_0}\sqrt{1-\frac{a}{\alpha}\csc^2\left(\frac{x}{x_0}\right)}dx \\
&= x_0\sqrt{2m\alpha}\left(1-\sqrt{\frac{a}{\alpha}}\right) \\
&= x_0\sqrt{2m\alpha} - x_0\sqrt{2ma} \\
\alpha &= \frac{1}{2m}\left(\frac{J}{x_0}+\sqrt{2ma}\right)^2 \\
&= \frac{J^2}{2mx_0^2}+\frac{J}{x_0}\sqrt{\frac{a}{2m}}+a
\end{aligned}
$$

Hint: you may find the following definite integral to be useful

$$
\int_{-\phi_0}^{\phi_0}\sqrt{b^2-\sec^2\phi}\,d\phi = \pi(b-1)
$$

where ϕ_0 are the zeros of the integrand.

(d) Based on your result from part (c), what is cycle frequency? Check your results in the limit of small oscillations.

$$
\begin{aligned}
\omega &= \frac{\partial \mathcal{H}}{\partial J} \\
&= \frac{1}{mx_0^2}+\frac{1}{x_0}\sqrt{\frac{a}{2m}} \\
&= \frac{1}{mx_0^2}\left(1+x_0\sqrt{\frac{a}{2m}}\right)
\end{aligned}
$$

Question V.

January 2019 Classical Mechanics Prelim Problem

Imagine the system shown in Figure 4 a pointlike mass m, hanging on a string. The string is fixed on a spring with spring constant k on the other end. The spring, with zero equilibrium length (i.e., the length that it would have if lying unattached on a table), is attached to the wall. The string of length L is hanging over a small metal peg at a distance $L/2$ from the wall (small circle in the figure) with no friction. The system exists in normal gravity. Assume that the string is taut at all times.

(a) Treating this as a 2D problem (as in the figure), find the Lagrangian and the Lagrange equations of motion.

Defining our coordinates:

$$x = r\cos\phi$$
$$y = -r\sin\phi$$

If we call the extension of the spring d:

$$L = r + \frac{L}{2} - d$$
$$r - d = \frac{L}{2}$$

Finding our energies:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$
$$U = -mgr\sin\phi + \frac{1}{2}kd^2$$
$$= -mgr\sin\phi + \frac{1}{2}k\left(r - \frac{L}{2}\right)^2$$
$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + mgr\sin\phi - \frac{1}{2}k\left(r - \frac{L}{2}\right)^2$$

Our Lagrange Equations give:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0$$
$$mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} - mgr\cos\phi = 0$$
$$r\ddot{\phi} + 2\dot{r}\dot{\phi} - g\cos\phi = 0$$
$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}}$$
$$m\ddot{r} - mg\sin\phi - mr^2\dot{\phi}^2 + k\left(r - \frac{L}{2}\right) = 0$$
$$\ddot{r} - g\sin\phi - r^2\dot{\phi}^2 + \frac{k}{m}\left(r - \frac{L}{2}\right) = 0$$

(b) There is an equilibrium point. What is it? show that small oscillations around this point give independent oscillations of ϕ and r. What are the frequencies?

The equilibrium point exists at a point where the sum of the forces is zero($\ddot{r} = 0 = \ddot{\phi}$), so $k(r - \frac{L}{2}) = mg$, and we argue that $\phi = \frac{\pi}{2}$ is the only point where there would be no pendulum-like restoring force. To find the small oscillation frequency about ϕ and r, we assume small perturbations in each coordinate:

$$\ddot{r} - g + \frac{k}{m}\left(r - \frac{L}{2}\right) = 0$$
$$\omega_r = \sqrt{\frac{k}{m}}$$
$$r\ddot{\phi} - g\cos\phi = 0$$
$$\omega_\phi = \sqrt{\frac{g}{r}}$$

(c) if $k/m >> g/L$, show that the initial condition in the right side of Figure 2 leads to an approximately periodic motion. Show that the period of the ϕ-part is given by

$$\Delta t = \sqrt{\frac{L}{g}} \int_0^\pi \frac{d\phi}{\sqrt{\sin\phi}}$$

This limit says that the spring will not oscillate compared to the pendulum-like oscillation of the mass. We can use conservation of energy since there are no nonconservative forces on the system, and we expect something like a pendulum:

$$mg\frac{L}{2}\sin\phi_f = \frac{1}{2}\left(\frac{L}{2}\right)^2 \dot{\phi}^2$$
$$\frac{4g}{L}\sin\phi_f = \dot{\phi}^2$$
$$t = \sqrt{\frac{g}{L}} \int_0^\pi \frac{d\phi}{\sqrt{\sin\phi}}$$

This is the given period.

Question VI.

January 2019 Classical Mechanics Prelim Problem

A central potential in a 3D (Euclidean) space is given by:

$$U = \begin{cases} 0 & \text{for} \quad r \le r_1 \\ \epsilon > 0 & \text{for} \quad r_1 < r \le r_2 \\ \infty & \text{for} \quad r_2 < r \end{cases}$$

(a) For a point mass m moving in this potential, write the equivalent one-dimensional problem (with variable r). What types of curves does the mass follow for $r \neq r_1, r_2$?

Following similarly to the conservation of energy in a central force problem conducted in Taylor and Goldstein:

$$
\begin{aligned}
U_{eff} &= U(r) + \frac{l^2}{2mr^2} \\
E &= \frac{1}{2}m\dot{r}^2 + U_{eff} \\
&= \frac{1}{2}m\dot{r}^2 + U(r) + \frac{l^2}{2mr^2}
\end{aligned}
$$

We also have that the potential can be thought of as concentric spherical wells, with delta function forces at r_1 and r_2 shown from $F = \nabla U$. Since the force acts only at $r \neq r_1, r_2$, the particle will not accelerate on the path between these radii, thus the particle will follow straight lines.

(b) Show that for $0 \leq E < \epsilon$ (with total energy E), one possible periodic orbit is a square. What, in this case, is the relationship between energy E and angular momentum l?

For this total energy $0 \leq E < \epsilon$ we know that the radius of the orbit must live in $r < r_1$. Using the result from before, where we know that the path of the particle is a straight line due to the fact that there is no force acting on the mass. Since the particle has a total energy $0 \leq E < \epsilon$, the particle "reflects" off of the potential barrier at r_1. If we define coordinates such that y is perpendicular to the spherical well and x tangent, we can use arguments of conservation of energy and momentum showing that when the particle hits a barrier we have:

$$
\begin{aligned}
|p_i| &= |p_f| \\
v_{x_i} &= v_{x_f} \\
E_i &= E_f \\
\frac{1}{2}m(v_{x_i}^2 + v_{y_i}^2) &= \frac{1}{2}m(v_{x_f}^2 + v_{y_f}^2) \\
v_{y_i} &= \pm v_{y_f}
\end{aligned}
$$

This gives us a result analogous to Snell's Law in optics, noting that it follows directly that $\theta_i = \theta_f$. We can arbitrarily set the incident angle to $\pi/4$, and from this a particle will reflect at $\pi/4$, giving a right angle. Since the sum of external angles of a polygon is 2π, we know that this is a four sided polygonal orbit. We can eliminate all non-square quadrilaterals noting that if $\theta_i = \theta_f$, no quadrilateral exists inscribed in a circle where this is the case other than a square. With this energy we can show the relation between E and l:

$$
\begin{aligned}
U_{eff} &= U(r) + \frac{l^2}{2mr^2} \\
E &= \frac{1}{2}m\dot{r}^2 + U_{eff} \\
&= \frac{1}{2}m\dot{r}^2 + U(r) + \frac{l^2}{2mr^2} \\
&= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2}
\end{aligned}
$$

(c) Are there circular orbits? Which? If yes, what is the relationship between E and l?

We know that there can be at a circular orbit at r_1 for $E < \epsilon$and r_2 for $E = \epsilon$ if we set the incident angle to be $\theta_i \approx \pi$, so the particle will "ride" the potential wall in an infinigon orbit, reflecting infinitely many times on the barrier, approximating a circle. This gives a relation between E and l similar to the previous part. Also for a circular orbit $\dot{r} = 0$For the orbit at r_1:

$$
\begin{aligned}
E &= \frac{1}{2}m\dot{r}^2 + U(r) + \frac{l^2}{2mr^2} \\
&= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} \\
&= \frac{l^2}{2mr^2}
\end{aligned}
$$

And at r_2:

$$
\begin{aligned}
E &= \frac{1}{2}m\dot{r}^2 + U(r) + \frac{l^2}{2mr^2} \\
&= \frac{1}{2}m\dot{r}^2 + \epsilon + \frac{l^2}{2mr^2} \\
&= \epsilon + \frac{l^2}{2mr^2}
\end{aligned}
$$

(d) What happens if, in the case of $E > \epsilon$, the particle moves "over the edge" from $r < r_1$ to $r > r_1$? Thus, sketch what types of orbits can happen in this energy regime.

The particle will change the angle of its orbit to accommodate for the overcoming of the energy threshold, similar to light refracting when crossing an interface of different indices of refraction. The orbits will still be straight lines, but change angle at each crossing at r_1 (Fig. 1).

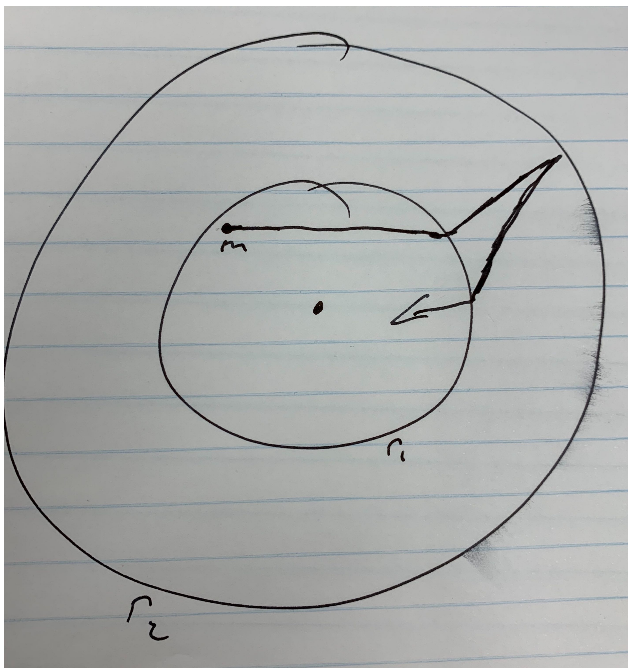

Figure 1: Part D: Orbits 'refract' at potential barrier