6720 Problem Set 1

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Resources: Carroll & Ostlie
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Question I.

(a)

$$
\begin{aligned}
1 \text{ radian} &= \frac{360^\circ}{2\pi} \\
1^\circ &= 3600'' \\
1 \text{ radian} &= \frac{(360 * 3600)''}{2\pi} \\
&= 206265''
\end{aligned}
$$

This gives our beloved 1 radian = 206265″

(b) *HST* observes 30,000 galaxies in 100 arcmin^2 (300 per square arcmin). The area of the sky is 4π radian^2, which is $4\pi\left(\frac{360}{2\pi}60\right)^2$ arcmin$^2 \approx 1.49 \times 10^8$ arcmin^2.

If we assume uniform density of galaxies over all sky, we find that *HST* can observe an incredible 4.5×10^{10} galaxies.

Question II.

The event horizon is realized as the Schwarzchild radius, which (assuming a well-behaved, nonspinning BH) is

$$R_S = \frac{2GM}{c^2}$$

For a Pop III seed of mass $M \sim 10^9\ M_\odot$, we have a Schwarzchild radius of $R_S \sim 2.9 \times 10^{12}$ m, or ~ 20 AU (about half the semimajor axis of Pluto's orbit).

For a direct collapse black hole of mass $M \sim 10^5\ M_\odot$, we have a Schwarzchild radius of $R_S \sim 2.9 \times 10^8$ m, or $\sim .42\ R_\odot$.

Question III.

(a) We can solve van Maanen's rotation speed as

$$
\begin{aligned}
v &= r\omega \\
r &= 50 \text{ kpc} \\
\omega &= .02 \text{ arcsec/year}
\end{aligned}
$$

where v is the (linear) rotation speed, ω is the angular speed, and r is the radius of M101 Converting to desired units gives a rotation speed of 4740 km/s, an order of magnitude greater than the rotation speed of the Milky Way.

(b) In a similar fashion we can calculate the angular rotation ω of M101 with $v = r\omega$, where $r = 22.7$ kpc and $v = 220$ km/s.

$$\begin{aligned} v &= r\omega \\ \omega &= \frac{v}{r} \\ &= 2.0 \times 10^{-3} \text{ arcsec/year} \end{aligned}$$

This is a much more reasonable angular speed. M101 will rotate through 1 arcsecond in ~ 500 years. Astronomers in the 1920s would not have been able to resolve this rotation, as it would take 250 years for a measurable rotation to be resolved in M101 with modern ground-based instruments.

Question IV.

Magnitudes do not add, but luminosities (and fluxes) do. We can start with our definition of apparent magnitude in some band as

$$m = -2.5 \log_{10} \left(\frac{F}{F_0} \right)$$

and the flux ratio of two objects is defined as

$$\frac{F_2}{F_1} = -2.5 \log_{10} \left(\frac{F_1}{F_2} \right)$$

If we add two objects together we have the resultant magnitude

$$\begin{aligned} m_f &= -2.5 \log_{10} \left(10^{-0.4 m_1} + 10^{-0.4 m_2} \right) \\ &= -2.5 \log_{10} \left(10^{-0.4 m_{gal}} + 10^{-0.4 m_{gal}} \right) \\ &= 11.25 \end{aligned}$$

Which is brighter, as expected.

Question V.

If the sun has an absolute magnitude of $M_{sun} = 4.74$, we can find the apparent magnitude of a sun-like star at a distance of 0.9 Mpc using the distance modulus

$$\begin{aligned} m - M &= 5 \log_{10} \left(\frac{d}{10 \text{ pc}} \right) \\ m &= 5 \log_{10} \left(\frac{d}{10 \text{ pc}} \right) + M \\ &= 29.51 \end{aligned}$$

so a sun-like star in Andromeda would have an apparent magnitude of 29.51.

Using a similar methodology to question 4, we can say

$$m_{gal} = -2.5 \log_{10} \left(N_* 10^{-0.4*29.51} \right)$$

where N_* is the number of stars in Andromeda and m_{gal} is the apparent magnitude of Andromeda. Solving for N_* we have

$$\begin{aligned}
-\frac{m_{gal}}{2.5} &= \log_{10} N_* - 0.4 * 29.51 \\
\log_{10} N_* &= 11.8 - \frac{m_{gal}}{2.5} \\
&= 11.8 - 1.4 \\
\log_{10} N_* &= 10.4
\end{aligned}$$

so there are $10^{10.4}$ stars in a sun-like star dominated Andromeda. This is reasonable if we assume Andromeda is vaguely Milky Way-like in mass.

Question VI.

The potential energy of a sphere is given as

$$\begin{aligned}
U &= -\int_0^R \frac{GM(r)}{r} \rho(r) 4\pi r^2 \, \mathrm{d}r \\
\rho(r) &= \frac{M}{\frac{4}{3}\pi R^3} \\
M(r) &= \rho(r) V(r)
\end{aligned}$$

where V is the volume of the sphere and ρ is it's density. We can now continue with the potential energy as

$$\begin{aligned}
U &= -\int_0^R \frac{GM(r)}{r} \rho(r) 4\pi r^2 \, \mathrm{d}r \\
&= -\int_0^R \frac{G\rho V}{r} \rho 4\pi r^2 \, \mathrm{d}r \\
V(r) &= \frac{4}{3}\pi r^3
\end{aligned}$$

This simplifies to

$$\begin{aligned}
U &= -\frac{16}{3}\pi^2 \rho^2 G \int_0^R r^4 \, \mathrm{d}r \\
&= -\frac{16}{15}\pi^2 \rho^2 G R^5
\end{aligned}$$

Substituting in ρ gives the desired

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

Question VII.

(a) The total mass of the cluster is simply $N_* M_*$ where $N_* = 20000$ stars and $M_* = 0.5\ M_\odot$. This yields a mass of $10^5\ M_\odot$

(b) The average kinetic energy is simply

$$\begin{aligned}\langle K \rangle &= \frac{1}{2} M \langle v \rangle^2 \\ &\approx 5 \times 10^{44} \text{ ergs}\end{aligned}$$

so the total kinetic energy is $K = N_* \langle K \rangle \approx 10^{49}$ ergs. The virial theorem states that $\langle K \rangle = -\frac{1}{2}\langle U \rangle \implies \langle U \rangle \approx -10^{45}$ ergs. We then have the total potential energy as $U = N_* \langle U \rangle \approx -2 \times 10^{49}$ ergs.

(c) The physical size can be found assuming the cluster is a uniform density sphere where

$$\begin{aligned}U &= -\frac{3}{5}\frac{GM^2}{R} \\ R &= -\frac{3}{5}\frac{GM^2}{U} \\ &\approx 0.25 \text{ pc}\end{aligned}$$

(d) Using the classic $S = R\theta$ where $S = 0.5$ pc is the diameter of the cluster, $\theta = 3'$. Solving for R, the distance to the cluster is $R = S/\theta \approx .25$ kpc