6710 Problem Set 4

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Question I.

Assume that the p-p chain is responsible for all of the Sun's luminosity. (In your last homework, you showed that more of the solar luminosity comes from the p-p chain than the CNO cycle, and we will ignore the CNO fraction here.) At the core temperature of the Sun, the pp3 chain can be ignored (it happens «1% of the time).

(a) How many neutrinos are generated by the Sun each second? Write your answer in terms of the relative rate of the pp1 and pp2 chains.

Assuming the relative rate of the pp chains is 70% pp1 and 30% pp2, and we assume that all of the Sun's luminosity is from pp chain reactions, we can say

$$\begin{aligned}\text{counts/s} &= 0.7\frac{2L_\odot}{E_{\nu_{pp1}}} + 0.3\frac{4L_\odot}{E_{\nu_{pp2}}}\\ &= 2.4\times 10^{38}\text{ neutrinos per second}\end{aligned}$$

since the pp chain requires 2 cycles to complete, either producing 2 pp2 neutrinos or 4 pp2 neutrinos. This number is reasonable given the literature.

(b) What is the neutrino flux (number/time/area) incident on the Earth?

If we assume that the neutrino luminosity is 70% pp1 and 30% p2 as in part a, we can use the flux-luminosity relation to find

$$\begin{aligned}F_\nu &= \frac{L_\nu}{4\pi d^2}\\ &= \frac{2L_\odot}{E_{\nu_{pp1}}}\frac{.7}{4\pi d^2} + \frac{4L_\odot}{E_{\nu_{pp2}}}\frac{.3}{4\pi d^2}\\ &= 8.6\times 10^{10}\text{ cm}^{-2}\text{ s}^{-1}\end{aligned}$$

This gives us the neutrino flux on Earth in cgs units. This number is also reasonable given literature predictions.

(c) Assuming Poisson statistics, how many neutrinos would you need to measure the relative fractions of the pp1 and pp2 chains to a precision of 1%?

Poisson statistics say that the signal to noise ratio $S/N = \sqrt{S}$, so a 1% detection implies a $S/N = 100$, which implies that we require 10^4 neutrino detections per process.

Question II.

Let's compare the rate of energy released from mass accreting around a supermassive black hole to the energy released by fusion in stars.

(a) What is the gravitational potential energy change for a particle with mass m that starts at 100 pc and falls to the innermost stable circular orbit (ISCO) around a nonspinning black hole?

The change in gravitational potential energy is given by

$$\begin{aligned}\Delta U &= -\frac{GMm}{r_f} + \frac{GMm}{r_i} \\ &= \frac{GMm}{3R_s} + \frac{GMm}{r_i} \\ &= -\frac{mc^2}{6} + \frac{GMm}{r_i}\end{aligned}$$

where R_s is the Schwarzchild radius, and we know that for a nonspinning object the ISCO is $3R_S = 6GM/c^2$. This is clearly dominated by the potential at the ISCO, so we can reduce this to

$$\Delta U = -\frac{mc^2}{6}$$

(b) Divide your answer from par (a) by the rest energy of the particle. We call this quantity the radiative efficiency η, where $L = \eta \dot{M} c^2$, where $\dot{M}$ is the mass infall rate.

The radiative efficiency is $\eta = \Delta U/mc^2 = 1/6$.

(c) What is the radiative efficiency of thermonuclear fusion via the p-p chain? Consider the mass deficit per proton in the complete reaction ($4\text{p} \rightarrow^4_2 \text{He} + 2\text{e}^+ + 2\nu_e$) and divide this by the rest energy of the proton.

We can find the mass defect of this reaction by taking the masses on both sides and multiplying by c^2 (we can safely ignore the neutrino masses). We find a mass defect of 5.9 MeV per proton for an efficiency of $\eta = .006$ dividing by the rest energy of the proton.

(d) At what mass accretion rate will an AGN exceed the light from fusion of an entire galaxy of Sun-like stars?

Assuming a galaxy of 10^8 Sun-like stars we have

$$\begin{aligned}10^{11} L_\odot &= \eta \dot{M} c^2 \\ \dot{M} &= \frac{10^{11} L_\odot}{\eta c^2} \\ &= 1.14 \times 10^{24} \text{ g/s}\end{aligned}$$

If we assume an efficiency of $\eta = 1/6$ from part (a), this gives us a mass accretion rate which would cause and AGN to exceed the light from fusion of an entire galaxy of $\dot{M} = 1.14{\times}10^{24}$ g/s.

Question III.

As we discussed in class, quantum tunneling plays a key role in nuclear fusion in the centers of stars. It is also crucial for the alpha decay of heavy elements like $^{235}_{92}$U and $^{239}_{94}$Pu. Here you can think of the alpha particle as "trapped" within the nucleus by a potential like the Coulomb barrier, which it then tunnels through to an escape and cause alpha decay. The rate of decay λ is proportional to this tunneling probability, and the half-life $\tau_{1/2} = \ln 10/\lambda$. The alpha decay of $^{235}_{92}$U releases an energy of $E = 4.68$ MeV with a half life $\tau_{1/2} = 7.1 \times 10^8$ yr. The energy released by alpha decay in $^{239}_{94}$Pu. is 5.24 MeV: what is it's half-life?

We know that the decay parameter λ is proportional to the tunneling probability, so

$$\begin{aligned}
\lambda &\propto P_{tunneling} \\
&\propto e^{-b/\sqrt{E}} \\
b &= 31 Z_1 Z_2 \sqrt{A} \\
\frac{\lambda_{Pu}}{\lambda_U} &= \frac{e^{-b_{Pu}/\sqrt{E_{Pu}}}}{e^{-b_U/\sqrt{E_U}}} \\
&= \frac{e^{-11560/72.4}}{e^{-11312/68.4}} \\
&\approx 300 \\
\lambda_{Pu} &\approx 300 \lambda_U
\end{aligned}$$

We know also that the half-life is given by

$$\begin{aligned}
\tau_{1/2} &= \frac{\ln 2}{\lambda} \\
\frac{\tau_{1/2_{Pu}}}{\tau_{1/2_U}} &= \frac{\lambda_U}{\lambda_{Pu}} \\
\tau_{1/2_{Pu}} &= \frac{\tau_{1/2_U}}{300} \\
&= 2.4 \times 10^6 \text{ yr}
\end{aligned}$$

This is exactly 100 times the known half life of $^{239}_{94}$Pu.