STAT 630 Problem Set 4

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Question I.

2.7.3 Suppose

$$p_{X,Y}(x,y) = \begin{cases} 1/5 & x=2,\ y=3 \\ 1/5 & x=3,\ y=2 \\ 1/5 & x=-3,\ y=-2 \\ 1/5 & x=-2,\ y=-3 \\ 1/5 & x=17,\ y=19 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute p_X

$$p_X(x) = \begin{cases} 1/5 & x=2 \\ 1/5 & x=3 \\ 1/5 & x=-3 \\ 1/5 & x=-2 \\ 1/5 & x=17 \\ 0 & \text{otherwise} \end{cases}$$

(b) Compute p_Y

$$p_Y(y) = \begin{cases} 1/5 & y=3 \\ 1/5 & y=2 \\ 1/5 & y=-2 \\ 1/5 & y=-3 \\ 1/5 & y=19 \\ 0 & \text{otherwise} \end{cases}$$

(c) Compute $P(Y > X)$

$$\begin{aligned} P(Y > X) &= 3(1/5) \\ &= \frac{3}{5} \end{aligned}$$

(d) Compute $P(Y = X)$

$$P(Y = X) = 0$$

(e) Compute $P(XY < 0)$

$$P(XY < 0) = 0$$

2.7.4 (a,d - it suffices to express C as a fraction)

For each of the following joint density functions $f_{X,Y}$, find the value of C and compute $f_X(x)$, $f_Y(y)$, and $P(X \leq 0.8, Y \leq 0.6)$.

(a)

$$f_{X,Y}(x, y) = \begin{cases} 2x^2y + Cy^5 & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

To find the constant C we integrate

$$\int_0^1 \int_0^1 2x^2y + Cy^5 \, \mathrm{d}y \, \mathrm{d}x = 1$$

$$= \int_0^1 x^2y^2 + \frac{C}{6}y^6 \Big|_{y=0}^{y=1} \mathrm{d}x$$

$$= \int_0^1 x^2 + \frac{C}{6} \, \mathrm{d}x$$

$$= \frac{x^3}{3} + \frac{Cx}{6} \Big|_0^1$$

$$= \frac{1}{3} + \frac{C}{6}$$

So $C = 4$. We find f_X as

$$\begin{aligned} f_X(x) &= \int_0^1 2x^2y + 4y^5 \, \mathrm{d}y \\ &= x^2y^2 + \frac{4}{6}y^6 \Big|_{y=0}^{y=1} \\ &= x^2 + \frac{4}{6} \end{aligned}$$

Similarly, we find f_Y

$$\begin{aligned} f_Y(y) &= \int_0^1 2x^2y + 4y^5 \, \mathrm{d}x \\ &= \frac{x^3}{3}y + 4y^5 \Big|_{x=0}^{x=1} \\ &= \frac{y}{3} + 4y^5 \end{aligned}$$

Now to find $P(X \leq 0.8, Y \leq 0.6)$:

$$\begin{aligned} P(X \leq 0.8, Y \leq 0.6) &= \int_0^{0.8} \int_0^{0.6} 2x^2 y + 4y^5 \, \mathrm{d}y \, \mathrm{d}x \\ &= 0.086 \end{aligned}$$

(d)

$$f_{X,Y}(x,y) = \begin{cases} Cx^5y^5 & 0 \leq x \leq 4, \ 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Similarly to the previous part, we integrate to find C:

$$\begin{aligned} 1 &= \int_0^4 \int_0^{10} Cx^5y^5 \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^4 \frac{Cx^5y^6}{6} \bigg|_{y=0}^{y=10} \mathrm{d}x \\ &= \int_0^4 \frac{10^6 C x^5}{6} \, \mathrm{d}x \\ &= \frac{10^6 C x^6}{36} \bigg|_0^4 \end{aligned}$$

so $C = 8.8 \times 10^{-9}$. Finding f_X:

$$\begin{aligned} f_X(x) &= \int_0^{10} Cx^5y^5 \, \mathrm{d}y \\ &= \frac{10^6 C x^5}{6} \quad 0 < x < 4 \end{aligned}$$

and f_Y:

$$\begin{aligned} f_Y(y) &= \int_0^{4} Cx^5y^5 \, \mathrm{d}x \\ &= \frac{4^6 C y^5}{6} \quad 0 < y < 10 \end{aligned}$$

and finally:

$$\begin{aligned} P(X \leq 0.8, Y \leq 0.6) &= \int_0^{0.8} \int_0^{0.6} Cx^5y^5 \, \mathrm{d}y \, \mathrm{d}x \\ &\approx 3 \times 10^{-12} \end{aligned}$$

Question II.

2.7.8

Let X and Y have joint density $f_{X,Y}(x,y) = (x^2 + y)/36$ for $-2 < x < 1$ and $0 < y < 4$, otherwise $f_{X,Y}(x,y) = 0$. Compute each of the following.

(a) The marginal density $f_X(x)$ for all $x \in R^1$

$$\begin{aligned}
f_X(x) &= \int_0^4 \frac{x^2+y}{36}\,\mathrm{d}y \\
&= \frac{1}{36}\left[x^2 y + \frac{y^2}{2}\right]\Big|_0^4 \\
&= \frac{1}{9}\left[x^2+2\right] \quad -2 < x < 1
\end{aligned}$$

(b) The marginal density $f_Y(y)$ for all $y \in R^1$

$$\begin{aligned}
f_Y(y) &= \int_{-2}^1 \frac{x^2+y}{36}\,\mathrm{d}x \\
&= \frac{1}{36}\left[\frac{x^3}{3} + xy\right]\Big|_{-2}^1 \\
&= \frac{1}{36}\left[\frac{x^3}{3} + xy\right] \\
&= \frac{1}{12}\left[1+y\right] \quad 0 < y < 4
\end{aligned}$$

(c) $P(Y < 1)$

$$\begin{aligned}
P(Y<1) &= \int_0^1 \frac{1}{12}\left[1+y\right]\mathrm{d}y \\
&= \frac{1}{8}
\end{aligned}$$

(d) The joint cdf $F_{X,Y}(x,y)$ for all $x, y \in R^1$

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_{-\infty}^x \int_{-\infty}^y \frac{x'^2+y'}{36}\,\mathrm{d}y'\,\mathrm{d}x' \\
&= \int_{-2}^x \int_0^y \frac{x'^2+y'}{36}\,\mathrm{d}y'\,\mathrm{d}x' \\
&= \frac{1}{36}\int_{-2}^x x'^2 y' + \frac{y'^2}{2}\Big|_0^y \,\mathrm{d}x' \\
&= \frac{1}{36}\int_{-2}^x x'^2 y + \frac{y^2}{2}\,\mathrm{d}x' \\
&= \frac{1}{36}\left[\frac{x'^3 y}{3} + \frac{y^2 x'}{2}\Big|_{-2}^x\right] \\
&= \frac{1}{36}\left[\frac{x^3 y}{3} + \frac{y^2 x}{2} + \frac{8y}{3} + \frac{2y^2}{2}\right]
\end{aligned}$$

2.7.9

Let X and Y have joint density $f_{X,Y}(x,y) = (x^2+y)/4$ for $0 < x < y < 2$, otherwise $f_{X,Y}(x,y) = 0$. Compute each of the following.

(a) The marginal density $f_X(x)$ for all $x \in R^1$

$$\begin{aligned}
f_X(x) &= \frac{1}{4}\int_x^2 x^2 + y \, \mathrm{d}y \\
&= \frac{1}{4}\ x^2 y + \left.\frac{y^2}{2}\right|_x^2 \\
&= \frac{1}{8}\left[-2x^3 + 3x^2 + 4\right]
\end{aligned}$$

(b) The marginal density $f_Y(y)$ for all $y \in R^1$

$$\begin{aligned}
f_Y(y) &= \frac{1}{4}\int_0^y x^2 + y \, \mathrm{d}x \\
&= \frac{1}{4}\left[\frac{y^3}{3} + y^2\right] \quad 0 < y < 2
\end{aligned}$$

(c) $P(Y < 1)$

$$\begin{aligned}
P(Y < 1) &= \frac{1}{4}\int_0^1 \frac{y^3}{3} + y^2 \, \mathrm{d}y \\
&= \frac{5}{48}
\end{aligned}$$

2.7.16

Suppose that the joint density $f_{X,Y}$ is given by $f_{X,Y}(x, y) = Ce^{-(x+y)}$ for $0 < x < y < \infty$ and is 0 otherwise.

(a) Determine C so that $f_{X,Y}$ is a density.

$$\begin{aligned}
1 &= \int_0^\infty \int_0^y Ce^{-(x+y)} \, \mathrm{d}x \, \mathrm{d}y \\
&= C\int_0^\infty e^{-y} \int_0^y e^{-x} \, \mathrm{d}x \, \mathrm{d}y \\
&= -C\int_0^\infty e^{-y} \left(\left. e^{-x} \right|_{x=0}^{x=y}\right) \mathrm{d}y \\
&= -C\int_0^\infty e^{-y}\left[e^{-y} - 1\right] \mathrm{d}y \\
&= -C\int_0^\infty e^{-2y} - e^{-y} \, \mathrm{d}y \\
&= -C\left(\frac{-1}{2}e^{-2y} + \left. e^{-y} \right|_0^\infty\right) \\
&= \frac{C}{2}
\end{aligned}$$

So $C = 2$.

(b) Compute the marginal densities of X and Y.

To find the marginal density of X we integrate over y:

$$\begin{aligned} f_X(x) &= \int_x^{\infty} 2e^{-(x+y)}\,\mathrm{d}y \\ &= 2e^{-2x} \quad 0 < x < \infty \end{aligned}$$

To find f_Y

$$\begin{aligned} f_Y(y) &= \int_0^{\infty} 2e^{-(x+y)}\,\mathrm{d}x \\ &= 2e^{-y} \quad 0 < y < \infty \end{aligned}$$

Question III.

2.7.10 You may use the result (without proof) in Exercise 2.7.13

Let X and Y have the Bivariate-Normal(3,5,2,4,1/2) distribution.

(a) Specify the marginal distribution of X.

The joint pdf is

$$f_{X,Y}(x,y) = N \exp\left\{ \frac{-1}{2(1-\rho^2)\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)}{\sigma_X}\frac{(y-\mu_Y)}{\sigma_Y} + \frac{(x-\mu_Y)^2}{\sigma_Y^2}\right]} \right\}$$

Where N is the normalization factor

$$N = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

Integrate over y to get the marginal pdf of X. Using the result from question 2.7.13, we have

$$\begin{aligned} f_X(x) &= \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\} \\ &= \frac{1}{2\sqrt{2\pi}} \exp\left\{-\frac{(x-3)^2}{8}\right\} \end{aligned}$$

Integrate again over x to get the cdf:

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{2\sqrt{2\pi}} \exp\left\{-\frac{(x'-3)^2}{8}\right\} \mathrm{d}x'$$

which will give the cdf in terms of the error function.

(b) Specify the marginal distribution of Y

Integrate over x to get the marginal distribution of Y. Similarly, this gives

$$\begin{aligned} f_Y(y) &= \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right\} \\ &= \frac{1}{4\sqrt{2\pi}} \exp\left\{-\frac{(y-5)^2}{32}\right\} \end{aligned}$$

Integrate again over y to get the cdf:

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{4\sqrt{2\pi}} \exp\left\{-\frac{(y'-5)^2}{32}\right\} \mathrm{d}y'$$

(c) Are X and Y? Why or why not?

X and Y cannot be independent because the covariance parameter $\rho \neq 0$.

Question IV.

2.7.17. [This is a continuous analogue to the multinomial distribution.]

(*Dirichlet*(a_1, a_2, a_3) *distribution*) Let (X_1, X_2) have the joint density

$$f_{X_1,X_2}(x_1, x_2) = \frac{\Gamma(a_1 + a_2 + a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} x_1^{a_1-1} x_2^{a_2-1} (1 - x_1 - x_2)^{a_3-1}$$

for $x_1 \geq 0$, $x_2 \geq 0$, and $0 \leq x_1 + x_2 \leq 1$. A Dirichlet distribution is often applicable when X_1, X_2 and $1 - X_1 - X_2$ correspond to random proportions.

(a) Prove that $f_{X_1,X_2}(x_1, x_2)$ is a density.

Integrating the joint density we have

$$\int_0^1 \int_0^{1-x2} f_{X_1,X_2}(x_1, x_2) = \int_0^1 \int_0^{1-x2} \frac{\Gamma(a_1 + a_2 + a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} x_1^{a_1-1} x_2^{a_2-1} (1 - x_1 - x_2)^{a_3-1} \, \mathrm{d}x_1 \, \mathrm{d}x_2$$

For notation purposes, let

$$\gamma = \frac{\Gamma(a_1 + a_2 + a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$

Now we perform the substitution $u = \frac{x_1}{1-x_2}$, which gives:

$$\begin{aligned} \int_0^1 \int_0^{1-x2} f_{X_1,X_2}(x_1, x_2) &= \int_0^1 \int_0^1 \gamma (1-x_2)^{a_1-1} u^{a_1-1} x_2^{a_2-1} (1-x_2)^{a_3-1} (1-u)^{a_3-1} (1-x_2) \, \mathrm{d}u \, \mathrm{d}x_2 \\ &= \int_0^1 \gamma x^{a_2-1} (1-x_2)^{a_1+a_3-1} B(a_1, a_3) \, \mathrm{d}x_2 \\ &= \gamma B(a_1, a_3) B(a_2, a_1 + a_3) \\ &= 1 \end{aligned}$$

(b) Prove that $X_1 \sim \text{Beta}(a_1, a_2 + a_3)$and $X_2 \sim \text{Beta}(a_2, a_1 + a_3)$.

Integrating over x_2 we have

$$\int_0^1 f_{X_1,X_2}(x_1, x_2) = \gamma \int_0^{1-x1} x_2^{a_2-1}(1 - x_1 - x_2)^{a_3-1} \, \mathrm{d}x_2$$

Performing the same substitution as the previous part we get

$$\gamma \int_0^{1-x1} x_1^{a_1-1} x_2^{a_2-1}(1 - x_1 - x_2)^{a_3-1} \, \mathrm{d}x_2 = \gamma \int_0^1 x_1^{a_1-1}(1 - x_1)^{a_2-a_3-1} u^{a_2-1}(1 - u)^{a_3-1} \, \mathrm{d}u$$

where the u terms integrate away to get the desired $X_1 \sim \text{Beta}(a_1, a_2 + a_3)$. The X_2 marginal distribution follows very similarly, integrating over x_1.

$$\int_0^{1-x2} f_{X_1,X_2}(x_1, x_2) = \int_0^{1-x2} \gamma x_1^{a_1-1} x_2^{a_2-1}(1 - x_1 - x_2)^{a_3-1} \, \mathrm{d}x_1$$

The substitution $u = \frac{x_1}{1-x_2}$ gives the desired $X_2 \sim \text{Beta}(a_2, a_1 + a_3)$ in the same manner.

Question V.

2.8.2

Suppose X and Y have joint probability function:

$$p_{X,Y}(x, y) = \begin{cases} 1/16 & x = -2, \ y = 3 \\ 1/4 & x = -2, \ y = 5 \\ 1/2 & x = 9, \ y = 3 \\ 1/16 & x = 9, \ y = 5 \\ 1/16 & x = 13, \ y = 3 \\ 1/16 & x = 13, \ y = 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $p_X(x)$ for all $x \in R^1$

$$p_X(x) = \begin{cases} 5/16 & x = -2 \\ 9/16 & x = 9 \\ 2/16 & x = 13 \\ 0 & \text{otherwise} \end{cases}$$

(b) Compute $p_Y(y)$ for all $y \in R^1$

$$p_Y(y) = \begin{cases} 6/16 & y = 5 \\ 10/16 & y = 3 \\ 0 & \text{otherwise} \end{cases}$$

(c) Determine whether or not X and Y are independent.

X and Y are not independent, in general $P(X|Y) \neq P(X)$. Example: $P(X = -2|Y = 3) = 1/16 \neq P(X = -2) = 5/16$.

2.8.3

Suppose X and Y have joint probability function:

$$p_{X,Y}(x,y) = \begin{cases} \frac{12}{49}(2 + x + xy + 4y^2) & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $p_X(x)$ for all $x \in R^1$

$$p_X(x) = \begin{cases} \frac{12}{49}\int_0^1 (2 + x + xy + 4y^2)\,\mathrm{d}y & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{12}{49}\left(\frac{3}{2}x + \frac{10}{3}\right) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Compute $p_Y(y)$ for all $y \in R^1$

$$p_Y(y) = \begin{cases} \frac{12}{49}\int_0^1 (2 + x + xy + 4y^2)\,\mathrm{d}y & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{12}{49}\left(\frac{3}{2} + \frac{y}{2} + 4y^2\right) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Determine whether or not X and Y are independent.

X and Y are not independent. $p_X(x)p_Y(y) \neq p_{X,Y}(x,y)$

X and Y are not independent, in general $P(X|Y) \neq P(X)$. Example: $P(X = -2|Y = 3) = 1/16 \neq P(X = -2) = 5/16$.

2.8.5

Suppose X and Y have joint probability function:

$$p_{X,Y}(x,y) = \begin{cases} 1/9 & x = -4,\ y = 2 \\ 2/9 & x = -5,\ y = -2 \\ 3/9 & x = 9,\ y = -2 \\ 2/9 & x = 9,\ y = 0 \\ 1/9 & x = 9,\ y = 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $P(Y = 4|X = 9)$

$$P(Y = 4|X = 9) = 1/6$$

(b) Compute $P(Y = -2|X = 9)$

$$P(Y = -2|X = 9) = 1/2$$

(c) Compute $P(Y = 0|X = -4)$

$$P(Y = 0|X = -4) = 0$$

(d) Compute $P(Y = -2|X = 5)$

$$P(Y = -2|X = 5) = 1$$

(e) Compute $P(X = 5|Y = -2)$

$$P(X = 5|Y = -2) = 1/3$$

2.8.7(a,d) **(a)**

$$\begin{aligned} f_{X,Y}(x,y) &= \begin{cases} 2x^2y + Cy^5 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \\ f_X(x) &= x^2 + \frac{2}{3} \\ f_Y(y) &= \frac{y}{3} + 4y^5 \end{aligned}$$

for $\quad 0 \le x \le 1,\ 0 \le y \le 1$. We previously found that $C = 4$. The conditional density $f_{Y|X}(y|x)$ is

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{2x^2y + 4y^5}{x^2 + \frac{2}{3}} \end{aligned}$$

X and Y are not independent.

(b)

$$\begin{aligned} f_{X,Y}(x,y) &= \begin{cases} Cx^5y^5 & 0 \le x \le 4,\ 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases} \\ f_X(x) &= \int_0^{10} Cx^5y^5 \,\mathrm{d}y \\ &= \frac{10^6Cx^5}{6} \quad 0 < x < 4 \\ f_Y(y) &= \int_0^4 Cx^5y^5 \,\mathrm{d}x \\ &= \frac{4^6Cy^5}{6} \quad 0 < y < 10 \end{aligned}$$

So the conditional

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{6Cx^5y^5}{10^6Cx^5} \\ &= \frac{6y^5}{10^6} \end{aligned}$$

So X and Y are independent.

2.8.14

Let X and Y have joint density $f_{X,Y}(x,y) = (x^2+y)/36$ for $-2 < x < 1$ and $0 < y < 4$, otherwise $f_{X,Y}(x,y) = 0$.

(a) Compute the conditional density $f_{Y|X}(y|x)$ for all $x, y \in R^1$ with $f_X(x) > 0$.

We found the marginal densities previously:

$$\begin{aligned} f_X(x) &= \frac{1}{9}\left[x^2+2\right] \quad -2 < x < 1 \\ f_Y(y) &= \frac{1}{12}\left[1+y\right] \quad 0 < y < 4 \end{aligned}$$

so the conditional density is

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{x^2+y}{4(x^2+2)} \end{aligned}$$

(b) Compute the conditional density $f_{X|Y}(x|y)$ for all $x, y \in R^1$ with $f_y(y) > 0$.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{x^2+y}{3(1+y)} \end{aligned}$$

(c) Are X and Y independent? Why or why not?

X and Y are not independent because the conditional densities do not equal the marginal densities.

2.8.15

Let X and Y have joint density $f_{X,Y}(x,y) = (x^2+y)/4$ for $0 < x < y < 2$, otherwise $f_{X,Y}(x,y) = 0$.

(a) Compute the conditional density $f_{Y|X}(y|x)$ for all $x, y \in R^1$ with $f_X(x) > 0$.

We found the marginal densities previously:

$$\begin{aligned} f_X(x) &= \frac{1}{8}\left[-2x^3+3x^2+4\right] \\ f_Y(y) &= \frac{1}{4}\left[\frac{y^3}{3}+y^2\right] \quad 0 < y < 2 \end{aligned}$$

so the conditional density is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$
$$= \frac{x^2 + y}{\frac{y^3}{3} + y^2}$$

(b) Compute the conditional density $f_{X|Y}(x|y)$ for all $x, y \in R^1$ with $f_y(y) > 0$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$
$$= \frac{2(x^2 + y)}{-2x^3 + 3x^2 + 4}$$

(c) Are X and Y independent? Why or why not?

X and Y are not independent because the conditional densities do not equal the marginal densities.

Question VI.

Suppose $Y_1, ..., Y_n$ is a random variable from the beta$(\alpha, 1)$ distribution. Find the cdfs and pdfs for $Y_{(1)} = \min(Y_1, ..., Y_n)$ and $Y_{(n)} = \max(Y_1, ..., Y_n)$. Are either of these beta random variables?

The cdf of the minimum $Y_{(1)}$ is given by

$$F_{Y_{(1)}}(y) = P(Y_{(1)} \leq y)$$
$$= 1 - (1 - F_{Y_1}(y))^n$$

where

$$F_{Y_1}(y) = \frac{\int_0^y y'^{a-1}(1-y')^{b-1}}{\int_0^1 y'^{a-1}(1-y')^{b-1}}$$

where $b = 1$, so we have

$$F_{Y_1}(y) = \frac{\frac{y^a}{a}}{\frac{1}{a}}$$
$$= y^a$$

so the cdf for the sample minimum is

$$F_{Y_{(1)}}(y) = 1 - (1 - F_{Y_1}(y))^n$$
$$= 1 - (1 - y^a)^n$$

The pdf is then the derivative of the cdf:

$$\begin{aligned} f_{Y_{(1)}} &= \frac{\mathrm{d}}{\mathrm{d}y} F_{Y_{(1)}}(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y} 1 - (1 - y^a)^n \\ &= nay^{a-1}(1 - y^a)^{n-1} \end{aligned}$$

which has the form of beta(a, n).

For the sample maximum, we have the cdf:

$$\begin{aligned} F_{Y_{(n)}}(y) &= P(Y_{(n)} \leq y) \\ &= (F_{Y_1}(y))^n \end{aligned}$$

which is simply

$$F_{Y_{(n)}}(y) = y^{an}$$

and we have the pdf

$$\begin{aligned} f_{Y_{(n)}} &= \frac{\mathrm{d}}{\mathrm{d}y} F_{Y_{(n)}}(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y} y^{an} \\ &= nay^{an-1} \end{aligned}$$

which does not take the form of a beta distribution.

Question VII.

Suppose $Y_1, ..., Y_n$ is a random variable from the Weibull$(\alpha, 1)$ distribution (recall Exercises 2.4.19 and 2.5.21). Find the cdf and pdf for $Y_{(1)} = \min(Y_1, ..., Y_n)$. Show that this is another Weibull distribution.

The cdf of the minimum $Y_{(1)}$ is given by

$$\begin{aligned} F_{Y_{(1)}}(y) &= P(Y_{(1)} \leq y) \\ &= 1 - (1 - F_{Y_1}(y))^n \end{aligned}$$

where the cdf of the Weibull(α, 1) distribution is

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^\alpha} & x \geq 0 \end{cases}$$

which we determined in homework 3. We have the cdf of the minimum distribution

$$\begin{aligned} F_{Y_{(1)}}(y) &= 1 - \left[1 - \left(1 - e^{-x^\alpha}\right)\right]^n \\ &= \left(e^{-x^\alpha}\right)^n \\ &= e^{-nx^\alpha} \end{aligned}$$

And the pdf is the derivative:

$$\begin{aligned} f_{Y_{(1)}}(y) &= \frac{\mathrm{d}}{\mathrm{d}x} e^{-nx^\alpha} \\ &= -\alpha n x^{\alpha-1} e^{-nx^\alpha} \end{aligned}$$

Which is the pdf of the Weibull(α, n) distribution.