ASTR 606: Radiative Transfer

Problem Set 1

Nikko Cleri

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Question I.

Consider a two-dimensional phase space, $x - p_x$. How does the phase space density evolve given a force $F = -ap_x$, where a is a constant? What is the fractional change in area per unit time? Repeat the same analysis for a force $F = bx$, where b is a constant.

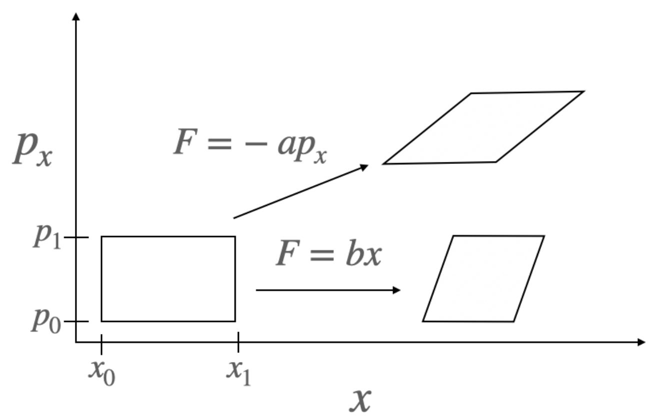

Figure 1: The evolution of the phase space density under the given forces.

We can take the initial area of the phase space density as

$$A(t_0) = (x_1 - x_0)(p_1 - p_0)$$

and the area at some evolved time

$$A(t + \Delta t) = (x_1 - x_0)\frac{\mathrm{d}x}{\mathrm{d}t}\Delta t(p_1 - p_0)\frac{\mathrm{d}p}{\mathrm{d}t}\Delta t$$

where $\frac{\mathrm{d}x}{\mathrm{d}t} \sim p$ and $\frac{\mathrm{d}p}{\mathrm{d}t} = F$. Taking the ratio gives

$$\begin{aligned}\frac{A(t + \Delta t)}{A(t_0)} &= \frac{(x_1 - x_0)\frac{\mathrm{d}x}{\mathrm{d}t}\Delta t(p_1 - p_0)\frac{\mathrm{d}p}{\mathrm{d}t}\Delta t}{(x_1 - x_0)(p_1 - p_0)} \\ &= \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}p}{\mathrm{d}t}\,(\Delta t)^2\end{aligned}$$

For $F = -ap_x$ we have

$$\frac{A(t + \Delta t)}{A(t_0)} = -ap_x^2(\Delta t)^2$$

For $F = bx$ we have

$$\frac{A(t + \Delta t)}{A(t_0)} = bxp_x(\Delta t)^2$$

Question II.

Define the luminosity as the rate at which an object radiates energy (units of energy per time).

(a) Assuming a uniform flux F_s from an object of radius R, derive an expression for the luminosity.

The luminosity is the integral of the surface flux over the surface area, so the luminosity can be written

$$\begin{aligned} L &= \int F_s \, \mathrm{d}A \\ &= 4\pi d^2 F_s \end{aligned}$$

where d is the luminosity distance. This is consistent with the known flux-luminosity relation if we take $F = F_s$ at the surface of the object.

(b) Consider a star with radius R_* and a planet with radius R_p which is a distance D from the star. Write an equation for the energy incident per unit time on one hemisphere of the planet as a function of the surface flux from the star.

We can use the relation from part (a) to find the fraction fo the energy per time incident on the planet's surface.

$$L = 4\pi \frac{R_p^2}{R_*^2} F_s$$

Question III.

Describe the meaning of isotropic radiation. For isotropic radiation, what is the flux integrated over a hemisphere of solid angle? What is the flux integrated over solid angle? How does the intensity relate to the mean intensity?

Isotropic radiation is radiation independent of direction (but not necessarily homogeneous). The flux integrated over a hemisphere of solid angle is

$$\begin{aligned} \int F \, \mathrm{d}\Omega &= \frac{1}{2} \int_0^{\pi} \int_0^{2\pi} F \, \mathrm{d}\phi \, \mathrm{d}\theta \\ &= 2\pi F \end{aligned}$$

Which is essentially half of the total luminosity.

The flux integrated over solid angle (after some charged discussion with Louie on the interpretation of this question) is the integral of vector flux over all possible angles, which is 0 (all vectors cancel).

The mean intensity is given by

$$\bar{u}_\nu = \frac{4\pi}{c} I_\nu$$

where I_ν is the energy specific intensity.

Question IV.

Given its luminosity is 3.845×10^{26} W, what is the effective temperature of the Sun? For a photospheric temperature of 6500 K, what is the energy density assuming blackbody radiation? How many photons reside in a volume with sides equal to the peak wavelength of the blackbody spectrum? Comment on the temperature dependence.

Effective temperature is

$$T_{eff} = \left(\frac{F}{\sigma}\right)^{\frac{1}{4}}$$
$$F_S = \frac{L_\odot}{4\pi R_\odot^2}$$

This gives an effective temperature of the Sun $\sim$ 5800K.

We also have

$$I = \frac{uc}{4\pi}$$

so we can get the energy density in terms of the flux as $\frac{\sigma}{c}T^4$.

The peak wavelength of the blackbody at 6500 K is $\sim$ 445 nm from Wien's law. We can manipulate the energy density to get the number of photons N in a box:

$$\begin{aligned} n &= \frac{u}{E_\gamma} \\ E_\gamma &= \frac{hc}{\lambda_{\text{peak}}} \\ N &= n\lambda_{\text{peak}}^3 \\ &= \frac{u}{hc}\lambda_{\text{peak}}^4 \end{aligned}$$

Question V.

Consider an observer inside a thin spherical shell of stars emitting radiation isotropically. Assume the luminosity (erg/s) of the star is L_0, the surface number density is Σ (cm^{-2}), r_a is the distance from the observed to the center of the shell, and r is the radius of the shell.

(a) Write down an equation for the energy specific intensity measured by the observed in terms of an angle measured from an axis passing through the center of the shell to the observer, θ. Make a plot of the intensity vs. $\cos\theta$ for some values of the ratio r_a/r. Comment on the symmetry of the solution and the direction from which the intensity peaks.

If the axis through the center of the circle from the observer is the x-axis, we define the distance from the observer to the point of observation on the shell to be D, and the distance from that point on the shell to be ρ. The distance from the intersection of ρ on the x-axis and the center of the sphere is x. See Figure 3 (I will also use R for the radius of the shell for most of this).

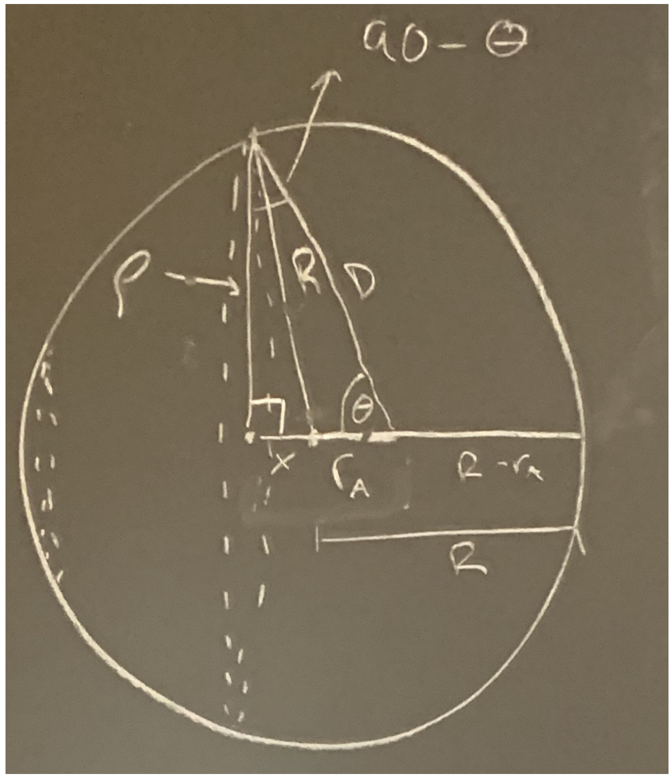

Figure 2: Diagram for question 5 (sorry for the poor drawing quality). Not to scale.

The intensity is dependent on the flux through a given area dA on the surface of the shell.

$$
\begin{aligned}
I &\sim \mathrm{d}F \\
&\sim L_0 \Sigma \, \mathrm{d}A
\end{aligned}
$$

This area is carved out depending on the angle θ. We begin with the following system of equations:

$$
\begin{aligned}
\rho^2 &= D^2 - (x + r_A)^2 \\
\rho^2 &= R^2 - x^2 \\
R^2 &= D^2 + r_A^2 - 2Dr_A \cos\theta
\end{aligned}
$$

We begin a litany of algebra and geometry to solve for the distance to the shell D as a function of the observational angle θ. The full extent of this algebra step by step can (at present) be seen on the walls of M310 if you are so inclined. Substituting R^2 as defined in the third equation gives ρ in terms of D, x and known quantities.

$$
\begin{aligned}
\rho^2 &= D^2 + r_A^2 - 2Dr_A \cos\theta - x^2 \\
&= D^2 - (x + r_A)^2
\end{aligned}
$$

Solving this gives x in terms of just D and known quantities:

$$
x = D\cos\theta - r_A
$$

Now putting this back into the second equation for ρ^2:

$$\begin{aligned}\rho^2 &= D^2 - [D\cos\theta - r_A]^2 \\ &= D^2 \sin^2\theta\end{aligned}$$

Now we can use these to get D in terms of known quantities, which gives a quadratic (we take the positive solution).

$$\begin{aligned}D &= r_A \cos\theta + \sqrt{r_A^2 \cos^2\theta - (R^2 - r_A^2)} \\ &= R\left[\frac{r_A}{R}\cos\theta + \sqrt{\left(\frac{r_A}{R}\cos\theta\right)^2 - \left(1 - \left(\frac{r_A}{R}\right)^2\right)}\right]\end{aligned}$$

(b) Write an equation for the total flux as a function of the observer position. Comment on how the flux changes depending on observer position.

The total flux will be the integral of the intensity from part (a) over the entire shell.

$$\begin{aligned}F &= \int I \,\mathrm{d}\Omega \\ &= \int L_0 \Sigma \,\mathrm{d}A\end{aligned}$$

The total flux will be constant throughout the shell (looking at all angles θ at the same time).

Question VI.

Consider an observer inside a spherical cavity in a black body with a temperature T.

(a) Write down an expression for the energy specific intensity as a function of an angle between the emission direction and the normal to a surface element.

The angle in this problem is defined differently, see Figure **??**. I introduce R, D, and r_A to indicate the same quantities as in question 5.

This takes very similar form to the previous problem, where we know that the intensity is the differential of the flux, $I \sim \mathrm{d}F$.

Unlike the previous problem, we know that the flux must be simply the flux we expect from a blackbody, $F = \sigma T^4$, where F is simply a constant assuming a constant T. Because of this, the intensity inside the cavity (the derivative of flux with respect to area) must be zero at all points.

(b) Then write an equation for the total flux. Comment on how the flux changes depending on observer position, and how your result would change for a non-spherical cavity.

And the flux is the integral of the intensity

$$F = \int I \,\mathrm{d}\Omega$$

simply goes like a blackbody (as stated in part (a)), so $F = \sigma T^4$. The flux is constant throughout the cavity, and should not change for a non-spherical cavity.

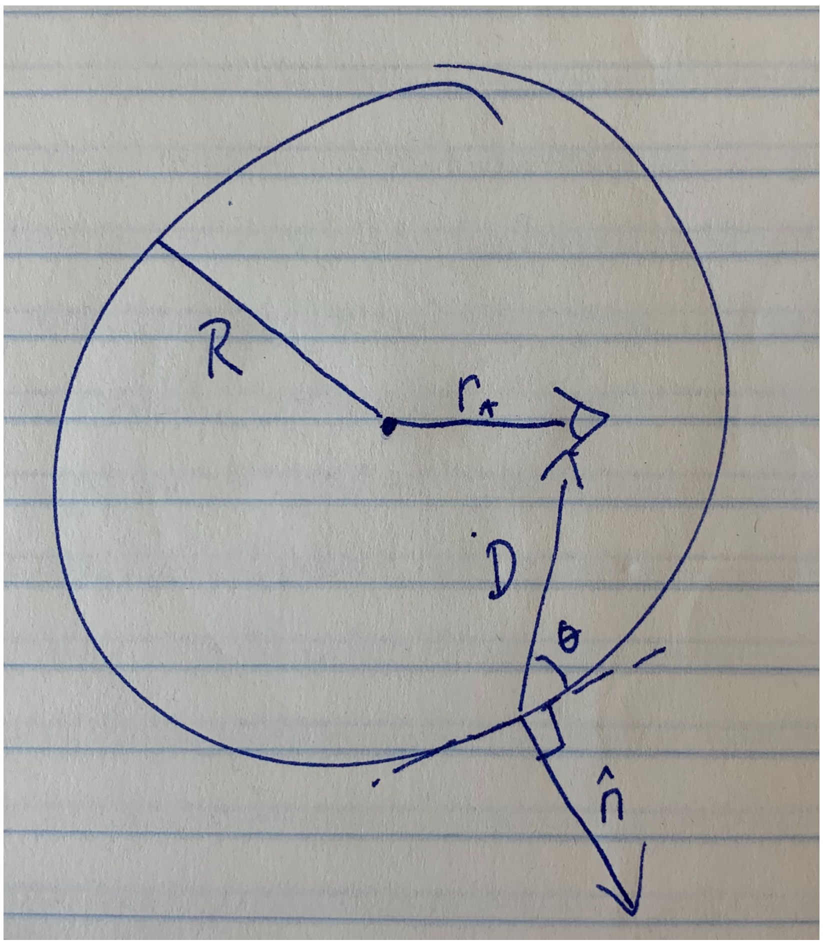

Figure 3: Diagram for question 6. Not to scale.