5500 Problem Set 7

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Question I.

Show (for instance by starting from the grand partition function) that the entropies of Bose-Einstein and Fermi-Dirac ideal gases are

$$S = -k\sum_i [\langle n_i\rangle \ln\langle n_i\rangle \mp (1 \pm \langle n_i\rangle)\ln(1 \pm \langle n_i\rangle)]$$

starting with the substitution $y \equiv e^{\beta(\mu-\epsilon)}$, we then have the expectation of the number density is

$$\begin{aligned} &\frac{1}{y^{-1}+1} \\ &= \frac{y}{1+y} \end{aligned}$$

so then we get

$$\begin{aligned} \langle n_i\rangle \ln\langle n_i\rangle + (1-\langle n_i\rangle)\ln(1-\langle n_i\rangle) &= \frac{y}{1+y}\ln\frac{y}{1+y} + \frac{1}{1+y}\ln\frac{1}{1+y} \\ &= \frac{y}{1+y}\ln y - \frac{y}{1+y}\ln(1+y) \\ &= \frac{y}{1+y}\ln y - \ln(1+y) \end{aligned}$$

This gives a grand potential

$$\Omega - \frac{1}{\beta}\sum_i \ln(1+y_i)$$

So we find the entropy

$$\begin{aligned} S &= -\left(\frac{\partial\Omega}{\partial T}\right) \\ &= k\beta^2\left(\frac{\partial\Omega}{\partial\beta}\right) \\ &= k\beta^2\sum_i\left(\frac{1}{\beta^2}\ln(1+y_i) - -\frac{1}{\beta}\frac{1}{1+y_i}\left(\frac{\partial y_i}{\partial\beta}\right)\right) \\ &= k\sum_i\left(\ln 1+y_i - \frac{y_i}{1+y_i}\ln y_i\right) \end{aligned}$$

From this we simply plug in the corresponding number density expectation for fermions and bosons. Respectively we have

$$\begin{aligned} \langle n_f\rangle &= \frac{y}{1-y} \\ \langle n_b\rangle &= \frac{y}{1+y} \end{aligned}$$

From this the given relation follows immediately.

Question II.

Find the entropy $S = Ns(n,T)$ and the chemical potential $\mu = \mu(n,T)$ of an ideal Bose gas in the limit of a classical ideal gas ($n \to 0$ and/or $T \to \infty$). There are situations in which the absolute value of the entropy matters; this is one way of getting it right.

In the ideal limit the fugacity goes to $z_0 = \frac{n\lambda^3}{g} << 1$. For the chemical potential we have

$$\mu = kT \ln \frac{n\lambda^3}{g} + ...$$

The entropy from Gibbs is

$$\begin{aligned} G &= \mu N \\ &= U - TS + pV \\ S &= \frac{U + pV - \mu N}{T} \end{aligned}$$

Using what we know for U, pV and μ we have

$$\begin{aligned} U &= \frac{3}{2} NkT \\ pV &= NkT \\ \mu &= kT \ln \frac{n\lambda^3}{g} \\ S &= \left(\frac{5}{2} - \ln \frac{n\lambda^3}{g} \right) Nk + \mathcal{O}(z_0) \end{aligned}$$

Question III.

By comparing in two and three dimensions the expressions of the density of the ideal, massive, free Bose gas as a function of fugacity and temperature, argue that a free two dimensional gas is not likely to undergo Bose-Einstein condensation at any nonzero temperature.

With inspiration from Bagnato & Kleppner (1991) and Hohenberg (1967), we consider a Bose gas in an isotropic power law potential $U(r) = U_0(r/a)^\eta$. This gives us a density

$$\begin{aligned} \rho(\epsilon) &= \frac{M}{\hbar^2} \int_0^{r'} r \, \mathrm{d}r \\ &= \frac{Ma^2}{2\hbar^2} \left(\frac{\epsilon}{U_0} \right)^{2/\eta} \end{aligned}$$

where $r' = (\epsilon/U_0)^{1/\eta}$. The two dimensional Bose gas state density is given by

$$\begin{aligned} g_2(\eta, x) &= \int_0^\infty \frac{y^{2/\eta}}{e^{y-x} - 1} \, \mathrm{d}y \\ g_2(\eta, 0) &= \Gamma(2/\eta + 1)\zeta(2/\eta + 1) \end{aligned}$$

in the $\mu = 0$ case. A two dimensional trap with rigid walls corresponds to the infinite η limit, and since $g(\infty, 0)$ diverges, a Bose-Einstein condensate does not occur.

Question IV.

Show by direct (and unnecessarily clumsy) calculation that energy density E and pressure p of a blackbody satisfy $p = \frac{1}{3}\mathrm{E}$.

The partition function for all modes of a photon gas in a volume V is given by

$$\begin{aligned} Z &= \prod \left[\frac{1}{1 - e^{-\beta\hbar\omega}}\right] \\ \ln Z &= \sum \ln \left[\frac{1}{1 - e^{-\beta\hbar\omega}}\right] \end{aligned}$$

Representing in terms of an integral with the state density we have

$$\begin{aligned} \ln Z &= \frac{V}{\pi^2 c^3} \int_0^\infty \omega^2 \ln\Big(1 - e^{-\hbar\omega\beta}\Big)\, \mathrm{d}\omega \\ &= \frac{V\pi^2 (kT)^3}{45\hbar^3 c^3} \end{aligned}$$

From this we can find the Helmholtz free energy F, from which we can determine the pressure as a function of the energy density.

$$\begin{aligned} F &= -kT \ln Z \\ &= -\frac{V\pi^2 (kT)^2}{45\hbar^3 c^3} \\ &= -\frac{4\sigma V T^4}{3c} \\ S &= -\left(\frac{\partial F}{\partial T}\right)_{V,N} \\ &= \frac{16\sigma V T^3}{3c} \end{aligned}$$

Where σ is the Stefan-Boltzmann constant. We now find the internal energy to then find the energy density relation with pressure.

$$\begin{aligned} U &= F + TS \\ &= -\frac{4\sigma V T^4}{3c} - \frac{16\sigma V T^4}{3c} \\ &= \frac{4\sigma V T^4}{c} \end{aligned}$$

Now we find the pressure as

$$
\begin{aligned}
p &= -\left(\frac{\partial F}{\partial V}\right)_{T,N} \\
&= \frac{4\sigma T^4}{3c} \\
&= \frac{1}{3}\mathrm{E}
\end{aligned}
$$

where E is the energy density $\frac{U}{V}$.