5201 Problem Set 5

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Question I.

A uniform rigid rod of mass M and length l at inclination α from the vertical with lower end in contact with smooth horizontal plane, find angular velocity when rod becomes horizontal. Conservation of energy:

$$\begin{aligned}
E &= T + V \\
V &= \frac{l}{2} Mg\cos\alpha \\
T &= \frac{1}{2}Mv^2 + \frac{1}{2}I\dot{\alpha}^2 \\
v &= \frac{l}{2}\sin\alpha\dot{\alpha} \\
T &\rightarrow \frac{1}{2}M(\frac{l}{2}\sin\alpha\dot{\alpha})^2 + \frac{1}{2}I\dot{\alpha}^2
\end{aligned}$$

Using the boundary condition where $\alpha = \frac{\pi}{2}$ when the rod hits the plane and $\alpha = \alpha_0$ initially:

$$\begin{aligned}
\frac{Ml^2}{8}\dot{\alpha_f}^2 + \frac{I}{2}\dot{\alpha_f}^2 &= \frac{l}{2}Mg\cos\alpha_0 \\
\dot{\alpha_f} &= \sqrt{\frac{\frac{l}{2}Mg\cos\alpha_0}{\frac{Ml^2}{8} + \frac{I}{2}}}
\end{aligned}$$

For a rod rotating about its center of mass, $I = \frac{1}{12}Ml^2$, so:

$$\begin{aligned}
\dot{\alpha_f} &= \sqrt{\frac{\frac{l}{2}Mg\cos\alpha_0}{\frac{Ml^2}{8} + \frac{Ml^2}{24}}} \\
&= \sqrt{\frac{\frac{1}{2}g\cos\alpha_0}{\frac{l^2}{8} + \frac{l^2}{24}}} \\
&= \sqrt{\frac{3g\cos\alpha_0}{l^2}}
\end{aligned}$$

Question II.

Find deviation from northerly direction is $3\tan\lambda - \alpha$ where λ is latitude and α is angle of projection with respect to the horizontal.

$$\begin{aligned}
\vec{\omega} &= |\omega|(0, \cos\lambda, \sin\lambda) \\
\vec{v} &= (\dot{x}, \dot{y}, \dot{z}) \\
\vec{\omega}\times\vec{v} &= |\omega|(\dot{z}\cos\lambda - \dot{y}\sin\lambda, \dot{x}\sin\lambda, -\dot{x}\cos\lambda) \\
(\ddot{x}, \ddot{y}, \ddot{z}) &= (0, 0, -g) - 2|\omega|\vec{\omega}\times\vec{v}
\end{aligned}$$

To the zeroth order:

$$\begin{aligned}
(\ddot{x}, \ddot{y}, \ddot{z}) &= (0, 0, -g) \\
x &= x_0 + v_{x_0}t = 0 \\
y &= y_0 + v_{y_0}t = v_0 t\cos\alpha \\
z &= z_0 + v_{z_0}t = v_0 t\sin\alpha - \frac{1}{2}gt^2
\end{aligned}$$

Solving for time:

$$\begin{aligned}
z &= 0 = t(v_0\sin\alpha - \frac{1}{2}gt) \\
t &= \frac{2v_0\sin\alpha}{g}
\end{aligned}$$

And now to solve for the deflection to the east or west, we can return to the first order approximation and insert $(\dot{x}, \dot{y}, \dot{z})$ from the zeroth order:

$$\begin{aligned}
\ddot{x} &= -2|\omega|((v_0\sin\alpha - gt)\cos\lambda - v_0\cos\alpha\sin\lambda) \\
\ddot{x} &= -2|\omega|(v_0\sin\alpha\cos\lambda - gt\cos\lambda - v_0\cos\alpha\sin\lambda)
\end{aligned}$$

Integrate twice and input t to find x:

$$\begin{aligned}
\dot{x} &= -2|\omega|(v_0 t\sin\alpha\cos\lambda - \frac{1}{2}gt^2\cos\lambda - v_0 t\cos\alpha\sin\lambda) \\
x &= -2|\omega|(\frac{1}{2}v_0 t^2\sin\alpha\cos\lambda - \frac{1}{6}gt^3\cos\lambda - \frac{1}{2}v_0 t^2\cos\alpha\sin\lambda) \\
&= -|\omega|(v_0 t^2\sin\alpha\cos\lambda - \frac{1}{3}gt^3\cos\lambda - v_0 t^2\cos\alpha\sin\lambda) \\
&= -|\omega|(\frac{4v_0^3}{g^2}\sin^3\alpha\cos\lambda - \frac{8v_0^3}{3g^2}\sin^3\alpha\cos\lambda - \frac{4v_0^3}{g^2}\sin^2\alpha\cos\alpha\sin\lambda) \\
&= -\frac{4|\omega|v_0^3\sin^2\alpha}{g^2}(\sin\alpha\cos\lambda - \frac{2}{3}\sin\alpha\cos\lambda - \cos\alpha\sin\lambda) \\
&= \frac{12|\omega|v_0^3\sin^2\alpha\cos\alpha\cos\lambda}{g^2}(3\tan\lambda - \tan\alpha)
\end{aligned}$$

This shows that the deflection east or west is dependent on the sign of the quantity $(3\tan\lambda - \tan\alpha)$ where a positive value denotes an eastward deflection.

Question III.

Goldstein 5.13

(a) If θ is the angle between one of the rods and the x axis, $\theta(0) = \frac{\pi}{6}$, and we take the symmetry into account, we can say that the hinge is constrained to move only along the y axis. If l' is the position of a point on the rod and we define $x' = (l - l')\cos\theta$ and $y' = l'\sin\theta$. Finding the energy of the system:

$$
\begin{aligned}
T &= \int_0^l \frac{m}{l}(x'^2 + y'^2)\dot{\theta}^2\, dl' = \frac{ml^2}{6}\dot{\theta}^2 \\
V &= \frac{mgl}{2}\sin\theta
\end{aligned}
$$

So we now apply conservation of energy imposing the boundary conditions on θ and $\dot{\theta}$, and from the expressions from T and V we can solve for $\dot{\theta}$:

$$
\begin{aligned}
E &= T + V = \frac{mgl}{4} \\
\dot{\theta} &= \sqrt{\frac{3g}{2l}(1 - 2\sin\theta)}
\end{aligned}
$$

The speed at which the rod hits the floor is $v = l|\dot{\theta}(\theta = 0)|$

$$
v = l\sqrt{\frac{3g}{2l}}
$$

(b) Finding the time it takes for the rods to hit the floor by separating the expression for $\dot{\theta}$ and integrating over the angle:

$$
\begin{aligned}
\int_{\frac{\pi}{6}}^{0} \frac{d\theta}{\sqrt{1 - 2\sin\theta}} &= -t\sqrt{\frac{3g}{2l}} \\
\sqrt{\frac{2l}{3g}} \int_0^{\frac{\pi}{6}} \frac{d\theta}{\sqrt{1 - 2\sin\theta}} &= t
\end{aligned}
$$

Question IV.

(a) Show magnitude of $\vec{\omega}$ is constant. Using Euler's equations for an object with two equal moments (such as this frisbee, in this case $\lambda_1 = \lambda_2$), the Euler equations have the simplification:

$$
\begin{aligned}
\lambda_3\dot{\omega}_3 &= (\lambda_1 = \lambda_2)\omega_1\omega_2 \\
\lambda_3\dot{\omega}_3 &= 0 \\
\dot{\omega}_3 &= 0 \\
\omega_3 &= constant
\end{aligned}
$$

(b) Show that as seen by me the disc's axis precesses around the fixed direction of angular momentum with angular speed $\Omega = \omega\sqrt{4 - 3\sin^2(\alpha)}$. Noting that the frisbee undergoes a free precession (no external torques), and is axially symmetric, we know that, from a result in section 10.8 in Taylor, the outside observer sees a precessional angular velocity of:

$$\Omega = \omega \frac{\sqrt{\lambda_3^2 + (\lambda_1^2 - \lambda_3^2)\sin^2\alpha}}{\lambda_1}$$

We also use the result that if $\lambda_1 = \lambda_2 = \lambda$ then $\lambda_3 = 2\lambda$, we get:

$$\begin{aligned} \Omega &= \omega \frac{\sqrt{4\lambda^2 + (\lambda^2 - 4\lambda^2)\sin^2\alpha}}{\lambda} \\ \Omega &\to \omega\sqrt{4 - 3\sin^2\alpha} \end{aligned}$$

Question V.

A top is spinning with angular speed ω about its axis of symmetry under gravity with its lowest point O fixed. The axis of symmetry precesses about the vertical through O. Defining a suitable frame of reference, find the angular velocity of the top in terms of spherical coordinates of the center of mass of the top.

We will start with defining the axis of symmetry of the object as the z axis, and we will use the Euler angles and their time derivatives: $\dot{\psi}$, which defines the rotation of the top about the z axis, $\dot{\phi}$, which defines the precession of the top about the axis vertical from the surface z', and $\dot{\theta}$, which defines the nutation of the z axis relative to z'. Here(x, y, z) and (x', y', z') are the coordinates in the body frame with $\hat{x}$ along the line of nodes, and the outside frame respectively. This gives us an angular velocity in terms of these Euler angles of:

$$\begin{aligned} \vec{\omega} &= \dot{\theta}\hat{x} + (\dot{\phi}\sin\theta)\hat{y} + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{z} \\ &= \dot{\theta}\hat{x} + (\dot{\phi}\sin\theta)\hat{y} + (\omega + \dot{\phi}\cos\theta)\hat{z} \end{aligned}$$

This describes the the rotations of this object entirely.

If we assume the precession is not steady, $\dot{\theta} \neq 0$, we can find the nutation. First we find the angular momentum, then the angular momenta around the body axis z and the space axis z':

$$\begin{aligned} \vec{L} &= (-\lambda_1\dot{\phi}\sin\theta)\hat{x}' + \lambda_1\dot{\theta}\hat{y}' + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{z} \\ L_z &= \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) \\ L_{z'} &= \lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) \\ L_{z'} &= \lambda_1\dot{\phi}\sin^2\theta + L_z\cos\theta \end{aligned}$$

Solving this gives the precessional angular velocity $\dot{\phi}$:

$$\dot{\phi} = \frac{L_{z'} - L_z\cos\theta}{\lambda_1\sin^2\theta}$$

Question VI.

A uniform sphere rolls without slipping on a horizontal plane which is kept rotating with angular speed Ω about a fixed vertical axis. Show that the motion of the center of the sphere describes a circle fixed in space and find the period for a constant Ω.

Taking into account the Coriolis and centrifugal terms in the total forces on the sphere, we can write a system of equations for the center of mass of the sphere through Newton's second law and solve to show the periodicity of the motion and find the time.

$$\begin{aligned} m\ddot{\vec{r}} &= F + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} \\ m\ddot{x} &= F_x + 2m\Omega\dot{y} + m\Omega^2 x \\ m\ddot{y} &= F_y + 2m\Omega\dot{x} + m\Omega^2 y \end{aligned}$$

If we introduce $\eta = x + iy$, we can combine these coupled equations to solve as a function of time:

$$\begin{aligned} m\ddot{\eta} &= -2im\Omega\dot{\eta} + m\Omega^2\eta \\ \ddot{\eta} &= -2i\Omega\dot{\eta} + \Omega^2\eta = 0 \\ \eta(t) &= Ae^{-i\Omega t} + Bte^{-i\Omega t} \end{aligned}$$

This solution to the differential equation describes a circular motion with a period $t = \frac{2\pi}{\Omega}$