5402 Problem Set 6

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Question I.

19.3.1

Show that

$$\sigma_{\text{Yukawa}} = 16\pi r_0^2 \left(\frac{g\mu r_0}{\hbar^2}\right)^2 \frac{1}{1+4k^2r_0^2}$$

where $r_0 = 1/\mu_0$ is the range. Compare σ to the geometrical cross section associated with this range.

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{4\mu^2 g^2}{\hbar^4(\mu_0^2 + 4k^2\sin^2(\theta/2))^2} \\
\sigma &= \int_0^\pi \int_0^{2\pi} \frac{4\mu^2 g^2 \sin\theta}{\hbar^4(\mu_0^2 + 4k^2\sin^2(\theta/2))^2}\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{8\pi\mu^2 g^2}{\hbar^4}\int_0^\pi \frac{\sin\theta}{(\mu_0^2 + 4k^2\sin^2(\theta/2))^2}\mathrm{d}\theta \\
&= \frac{16\pi^2 g^2}{\hbar^4\mu_0^2}\left(\frac{1}{\frac{4k^2}{\mu_0^2}+1}\right)
\end{aligned}$$

Using $r_0 = 1/\mu_0$ gives the desired answer.

Question II.

19.3.2

1. Show that if $V(r) = -V_0\theta(r_0 - r)$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = 4r_0^2\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 \frac{(\sin qr_0 - qr_0\cos qr_0)^2}{(qr_0)^6}$$

   Starting with

$$\begin{aligned}
f(\theta) &= -\frac{2\mu}{\hbar^2}\int \frac{\sin qr'}{q}V(r')r'\mathrm{d}r' \\
&= \frac{2\mu V_0}{\hbar^2 q}\left[\frac{\sin(qr_0) - r_0 q\cos(qr_0)}{q^2}\right]
\end{aligned}$$

   Then

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= |f(\theta)|^2 \\
&= 4r_0^2\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 \frac{(\sin qr_0 - qr_0\cos qr_0)^2}{(qr_0)^6}
\end{aligned}$$

2. Show that as $kr_0 \to 0$, the scattering becomes isotropic and

$$\sigma \simeq \frac{16\pi r_0^2}{9}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2$$

This requires an expansion of the term $\frac{(\sin qr_0 - qr_0\cos qr_0)^2}{(qr_0)^6}$ as

$$\begin{aligned}
\lim_{x\to 0}\frac{(\sin x - x\cos x)^2}{x^6} &= \lim_{x\to 0}\frac{1}{x^6}\left(x - \frac{x^3}{3!} + ... - x\left(1 - \frac{x^2}{x!}\right)\right)^2 \\
&= \lim_{x\to 0}\frac{\left(x^3\left(\frac{1}{2} - \frac{1}{6}\right)\right)^2}{x^6} \\
&= \frac{1}{9}
\end{aligned}$$

Including the 4π from the angle dependence we get the desired

$$\sigma \simeq \frac{16\pi r_0^2}{9}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2$$

Question III.

19.3.3

Show that for the Gaussian potential, $V(r) = V_0 e^{-r^2/r_0^2}$,

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} &= 4r_0^2\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 e^{-q^2r_0^2/2} \\
\sigma &= \frac{\pi^2}{2k^2}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 (1 - e^{-2k^2r_0^2})
\end{aligned}$$

Starting with

$$\begin{aligned}
f(\theta) &= -\frac{2\mu V_0}{\hbar^2 q}\int_0^\infty r'\sin qr' e^{-r'^2/r_0^2}\,\mathrm{d}r' \\
&= \left(\frac{-2\mu V_0}{\hbar^2}\right)\sqrt{\pi r_0^2}r_0^2 e^{-qr_0^2/4}
\end{aligned}$$

We then have

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= |f|^2 \\
&= \frac{\pi r_0^2}{4}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 e^{-q^2r_0^2/2}
\end{aligned}$$

So the total cross section is

$$\sigma = 2\pi\frac{\pi r_0^2}{4}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2\int_0^\pi \mathrm{d}\theta\sin\theta e^{-qr_0^2/2}$$

With the substitutions $q^2 = 2k^2(1-\cos\theta)$ and $\frac{\mathrm{d}(q^2)}{\mathrm{d}2k^2} = \sin\theta \mathrm{d}\theta$ we get the desired

$$\sigma = \frac{\pi^2}{2k^2}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 (1 - e^{-2k^2 r_0^2})$$

Question IV.

The differential cross section in a certain scattering process is known to be given by

$$\sigma(\theta) = \alpha + \beta\cos\theta + \gamma\cos^2\theta$$

(a) What is the scattering amplitude?

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f|^2$$
$$f = (\alpha + \beta\cos\theta + \gamma\cos^2\theta)^{1/2}$$

(b) Express α, β and γ in terms of the phase shifts δ_l

$$\begin{aligned}
\delta_l &= \tan^{-1}\left(\frac{j_l(kr_0)}{n_l(kr_0)}\right) \\
\delta_0 &= -kr_0 \\
f &= \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)e^{i\delta_l}\sin\delta_l \\
&= \frac{1}{k}[e^{-i\delta_0}\sin\delta_0 + 3\cos\theta e^{i\delta_l}\sin\delta_1] \\
\left|f^2\right| &= \frac{1}{k^2}(\sin^2\delta_0 + 3\cos\theta\sin\delta_0\sin\delta_1 e^{\delta_1-\delta_0} + 3\cos\theta\sin\delta_1\sin\delta_0 e^{i(\delta_0+\delta_1)} + 9\cos^2\theta\sin^2\delta_1) \\
\alpha
\end{aligned}$$

We have

$$\begin{aligned}
\alpha &= \frac{1}{k^2\sin^2\delta_0} \\
\beta &= \frac{6}{k^2}\sin\delta_0\sin\delta_1\cos(\delta_1-\delta_0) \\
\gamma &= \frac{9}{k^2}\sin^2\delta_1
\end{aligned}$$

(c) Are there any constraints on the magnitudes of α, β and γ if the scattering amplitude is not allowed to grow any faster that $\ln E$ as the energy E becomes very large?

$$\begin{aligned}
k &\propto \sqrt{E} \\
|f|^2 &\propto \frac{1}{k^2} \\
|f| &\propto \frac{1}{k}
\end{aligned}$$

(d) Deduce the total scattering cross-section and show that it is consistent with the optical theorem.

$$
\begin{aligned}
\sigma_{tot} &= \int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \\
&= 2\pi \int_{-1}^{1} \mathrm{d}x\,(\alpha + \beta x + \gamma x^2) \\
&= 4\pi \left(\alpha + \frac{\gamma}{3}\right) \\
&= \frac{4\pi}{k^2}\sin^2\delta_0 + \frac{12\pi}{k^2}\sin^2\delta_1
\end{aligned}
$$

Using the general form of $f(\theta, \phi)$

$$
\begin{aligned}
\sigma &= \frac{4\pi}{k}\,\mathrm{Im}(f) \\
&= \frac{4\pi}{k^2}(\sin^2\delta_0 + 3\sin^2\delta_1)
\end{aligned}
$$

Question V.

A beam of mono-energetic particles each with energy E and mass m is scattered by a spherically symmetric potential $U(r)$ that vanishes as $r \to \infty$. The scattering amplitude $f(\theta)$ can be expressed via the partial wave scattering amplitudes $f_l(\theta)$ and the scattering phase shifts $\delta_l(k)$

$$
\begin{aligned}
f(\theta) &= \sum_{l=0}^{\infty} f_l(\theta) \\
&= \frac{1}{2ik}\sum_{l=0}^{\theta}(2l + l)[e^{i\delta_l(k)} - 1]P_l(\cos\theta)
\end{aligned}
$$

where l is the angular momentum, θ is the scattering angle, $P_l(\cos\theta)$ are Legendre polynomials, and $k = (2mE\hbar^2)^{1/2}$ is the wave number.

(a) Show that the total scattering cross section σ can be calculated using the imaginary part of the forward scattering amplitude (the optical theorem) as:

$$
\sigma = \frac{4\pi}{k}\,\mathrm{Im}\{f(\theta = 0)\}
$$

$$
\begin{aligned}
f(\theta) &= \sum_{l=0}^{\infty} f_l(\theta) \\
&- \frac{1}{k}\sum_l (2l+1)e^{i\delta_l(k)}\sin\delta_l P_l(\cos\theta) \\
\sigma &= \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega \\
&= \int |f|^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{2\pi}{k^2}\sum_l\sum_n (2l+1)(2n+1)e^{i\delta_l(k)}\sin\delta_l e^{i\delta_n(k)}\sin\delta_n \int_{-1}^{1}\mathrm{d}x\,P_l(x)P_n(x) \\
&= \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l \\
\mathrm{Im}(f) &= \frac{1}{k}\sum_l (2l+1)\sin^2\delta_l \\
\sigma &= \frac{4\pi}{k}\,\mathrm{Im}(f(\theta=0))
\end{aligned}
$$

(b) Consider the potential well $U(r) = -U_0$ if $r < r_0$, and $U(r) = 0$ if $r > r_0$, where U_0 a positive constant.

(i) For this potential find the scattering phase shift $\delta_{l=0}(k)$ for s-wave scattering using the solution of the Schrodinger equation for the spherical wave with $l = 0$. At $r = r_0$ we have the boundary conditions

$$
\begin{aligned}
C\sin\big(k'r_0\big) &= \sin(kr_0+\delta_0) \\
Ck'\cos\big(k'r_0\big) &= k\cos(kr_0+\delta_0) \\
\frac{1}{k'}\tan\big(k'r_0\big) &= \frac{1}{k}\tan(kr_0+\delta_0) \\
\delta_0 &= \tan^{-1}\left(\frac{k}{k'}\tan k'r_0\right) - kr_0
\end{aligned}
$$

(ii) For this potential calculate the scattering cross section in the limit $k \to 0$ knowing that in this limit only s-wave scattering is important.
As $k \to 0$

$$
\begin{aligned}
\tan(kr_0+\delta_0) &= kr_0+\delta_0 \\
\delta_0 &= \frac{k}{k'}\tan k'r_0 - kr_0 \\
\sigma &= \frac{4\pi}{k^2}\sin^2\delta_0 \\
&= \frac{4\pi}{k^2}\delta_0^2 \\
&= 4\pi\left[\frac{\tan k'r_0}{k'} - r_0\right]^2
\end{aligned}
$$

Question VI.

19.5.3

For small kr_0 we only need to consider the $l = 0$ case, so

$$\begin{aligned}
\lim_{kr\to 0} \delta_l &= (kr_0)^{2l+1} \\
\delta_0 &= kr_0 \\
\sigma &\approx \sigma_0 \\
&= \frac{4\pi}{k^2}\sin^2\delta_0 \\
&\approx 4\pi r_0^2
\end{aligned}$$

which is 4 times the classical geometric cross section. We can then take

$$\begin{aligned}
\delta_l &= \arctan\left[\frac{j_l(kr_0)}{n_l(kr_0)}\right] \\
&= -(kr_0 - \pi l/2) \\
\sin^2\delta_l &\to \sin^2(kr_0 - \pi l/2)
\end{aligned}$$

gives the desired result.