ASTR 606: Radiative Transfer

Problem Set 3

Nikko Cleri

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Question I.

Most main-sequence stars are in hydrostatic equilibrium, have an equation of state dominated by the ideal gas law, and have energy transport dominated by radiation.

Simplify the energy transport equation as $dT/dr \sim -T_e/R$ and with $T_{eff} \propto T_c$. using the appropriate opacity for a hot ionized star, what is the proportionality of the central temperature T_c with total mass M, luminosity L, and radius R?

$$\begin{aligned}\frac{\partial T}{\partial r} &\to -\frac{T_c}{R} \\ &\propto -\kappa \frac{L\rho}{R^2 T_c^3} \\ \frac{T_c}{R} &\propto -\kappa \frac{ML}{R^5 T_c^3} \\ T_c &\propto -\kappa \frac{ML}{R^4 T_c^3} \\ T_c^4 &\propto -\kappa \frac{ML}{R^4} \\ T_c &\propto -\left(\kappa \frac{ML}{R^4}\right)^{1/4}\end{aligned}$$

where $\kappa = \kappa_{es} = \frac{\sigma_{SB}}{\mu_e m_h} = 0.20(1+X)\ \mathrm{cm^2 g^{-1}}$ Now derive the proportionality of T_c with M and R using the hydrostatic equilibrium equation and the ideal gas law. Again, you can simplify by assuming that $dP/dr \sim -P_c/R$ and that the central density $\rho_c \propto \bar{\rho}$.

$$\begin{aligned}\frac{\partial P}{\partial r} &\to -\frac{P_c}{R} \\ &\propto -\frac{GM\rho}{R^2} \\ \frac{\rho k T}{\mu m_h R} &\propto -\kappa \frac{GM\rho}{R^2} \\ T &\propto \frac{M}{R}\end{aligned}$$

ignoring the constants. Combining gives

$$\begin{aligned}\frac{M}{R} &\propto \left(\frac{ML}{R^4}\right)^{1/4} \\ L &\propto M^3\end{aligned}$$

Question II.

$$
\begin{aligned}
B_\nu &= \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT}-1\right)^{-1} \\
\frac{\partial B_\nu}{\partial T} &= \frac{2h^2\nu^4}{T^2kc^2}\frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}\frac{\partial\frac{\partial B_\nu}{\partial T}}{\partial\nu} = \frac{\partial}{\partial\nu}\left[\frac{2h^2\nu^4}{T^2kc^2}\frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}\right] \\
&= \frac{2h^2}{kc^2T^2}\left[\frac{4\nu^3e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}+\frac{h\nu^4e^{h\nu/kT}}{kT\left(e^{h\nu/kT}-1\right)^2}-\frac{2h\nu^4e^{2h\nu/kT}}{kT\left(e^{h\nu/kT}-1\right)^3}\right]
\end{aligned}
$$

Setting this to zero we get

$$
0 = 4+\frac{h\nu}{kT}-\frac{2h\nu}{kT}\frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)}
$$

letting the exponential in the third term go to one, this gives

$$
\nu = \frac{4kT}{h}
$$

Question III.

We can derive the Stokes parameters by starting with the following representation:

$$
\begin{aligned}
\mathbf{E} &= (\hat{x}E_1+\hat{y}E_2)e^{-i\omega t} \\
&\equiv \mathbf{E}_0e^{-i\omega t}
\end{aligned}
$$

where

$$
\begin{aligned}
E_x &= \mathcal{E}_l\cos(\omega t-\phi_l) \\
E_x &= \mathcal{E}_r\cos(\omega t-\phi_r)
\end{aligned}
$$

For a general elliptical polalrization

$$
\begin{aligned}
E'_x &= \mathcal{E}_0\cos\beta\cos\omega t \\
E'_y &= \mathcal{E}_0\sin\beta\sin\omega t
\end{aligned}
$$

Transforming the electric field components by rotating through an angle χ:

$$
\begin{aligned}
E_x &= \mathcal{E}_0(\cos\beta\cos\chi\cos\omega t+\sin\beta\sin\chi\sin\omega t) \\
E_y &= \mathcal{E}_0(\cos\beta\cos\chi\cos\omega t+\sin\beta\sin\chi\sin\omega t)
\end{aligned}
$$

Which is identical to the second step, taking

$$
\begin{aligned}
\mathcal{E}_l\cos\phi_l &= \mathcal{E}_0\cos\beta\cos\chi \\
\mathcal{E}_l\sin\phi_l &= \mathcal{E}_0\sin\beta\sin\chi \\
\mathcal{E}_r\cos\phi_r &= \mathcal{E}_0\cos\beta\sin\chi \\
\mathcal{E}_r\sin\phi_r &= -\mathcal{E}_0\sin\beta\cos\chi
\end{aligned}
$$

(For notation, we will call these equations a, b, c, and d, respectively). We now begin a litany of algebra to derive the Stokes parameters in the (β, χ) representation. Beginning with I:

$$I \equiv \mathcal{E}_l^2 + \mathcal{E}_r^2$$

by taking the squares of a and b to solve for $\mathcal{E}_l^2$ and the squares of c and d to solve for $\mathcal{E}_l^2$, we have

$$\begin{aligned}
\mathcal{E}_l^2 &= \mathcal{E}_0^2 \left[\cos^2\beta\cos^2\chi + \sin^2\beta\sin^2\chi\right] \\
\mathcal{E}_r^2 &= \mathcal{E}_0^2 \left[\cos^2\beta\sin^2\chi + \sin^2\beta\cos^2\chi\right]
\end{aligned}$$

we now have

$$\begin{aligned}
I &\equiv \mathcal{E}_l^2 + \mathcal{E}_r^2 \\
&= \mathcal{E}_0^2 \left[\cos^2\beta\cos^2\chi + \sin^2\beta\sin^2\chi + \cos^2\beta\sin^2\chi + \sin^2\beta\cos^2\chi\right] \\
&= \mathcal{E}_0^2
\end{aligned}$$

Q follows similarly

$$\begin{aligned}
Q &\equiv \mathcal{E}_l^2 + \mathcal{E}_r^2 \\
&= \mathcal{E}_0^2 \left[\cos^2\beta\cos^2\chi + \sin^2\beta\sin^2\chi - (\cos^2\beta\sin^2\chi + \sin^2\beta\cos^2\chi)\right] \\
&= \mathcal{E}_0^2 \cos 2\beta \sin 2\chi
\end{aligned}$$

For V we have

$$\begin{aligned}
V &\equiv 2\mathcal{E}_l\mathcal{E}_r \sin(\phi_l - \phi_r) \\
&= 2\mathcal{E}_l\mathcal{E}_r(\sin\phi_l\cos\phi_r - \cos\phi_l\sin\phi_r) \\
&= 2(\mathcal{E}_0\sin\beta\sin\chi\mathcal{E}_0\cos\beta\sin\chi + \mathcal{E}_0\cos\beta\cos\chi\mathcal{E}_0\sin\beta\cos\chi) \\
&= \mathcal{E}_0^2 \sin 2\beta
\end{aligned}$$

Finally, U follows similarly to give

$$\begin{aligned}
U &\equiv 2\mathcal{E}_l\mathcal{E}_r \cos(\phi_l - \phi_r) \\
&= \mathcal{E}_0^2 \cos 2\beta \sin 2\chi
\end{aligned}$$

Question IV.

For natural light:

$$\begin{aligned}
I &= I_0 \\
Q = U = V &= 0
\end{aligned}$$

For two beams

$$\begin{aligned}
X &= X_1 + X_2 \quad X \in \{I, Q, U, V\} \\
Q_i &= I_i \cos(2\beta_i)\cos(2\chi_i) \\
U_i &= I_i \cos(2\beta_i)\sin(2\chi_i) \\
V_i &= I_i \sin(2\beta_i)
\end{aligned}$$

The equation for V gives

$$\begin{aligned} V &= V_1 + V_2 \\ 0 &= I_1 \sin\sin(2\beta_1) + I_1 \sin\sin(2\beta_1) \\ \implies -\sin(2\beta_1) &= \sin(2\beta_1) \\ \implies -\beta_2 &= \beta_1 \end{aligned}$$

For Q and U

$$\begin{aligned} 0 &= I_1 \cos(2\beta_1)\cos(2\chi_1) + I_i \cos(2\beta_2)\cos(2\chi_2) \\ \implies \frac{\cos(2\chi_1)}{\cos(2\chi 2)} &= -1 \\ 0 &= I_1 \cos(2\beta_1)\sin(2\chi_1) + I_i \cos(2\beta_2)\sin(2\chi_2) \\ \implies \frac{\sin(2\chi_1)}{\sin(2\chi 2)} &= -1 \\ \implies \frac{\cos(2\chi_1)}{\cos(2\chi 2)} &= \frac{\sin(2\chi_1)}{\sin(2\chi 2)} \end{aligned}$$

which is valid only for the given condition $|x_1| = |x_2| = \frac{\pi}{2}$

Question V.

(a) The magnetic field of two point magnetic charges in Gaussian units is

$$\begin{aligned} B &= \frac{Q}{r^2} - \frac{Q}{(r+a)^2} \\ &= Q\left(\frac{2ra + a^2}{r^2(r+a)^2}\right) \end{aligned}$$

in the limit $a << r$, this gives the desired

$$\vec{B} = \frac{Q\vec{a}}{r^3}$$

Question VI.

Electric monopoles do not radiate because they are stationary. The monopole and dipole radiation of gravitational radiaton are not permitted because total mass energy of the system is conserved (monopole does not radiate) and momentum is conserved (dipole does not radiate). The quadrupole moment has a nonvanishing second derivative, therefore radiation due to the quadrupole moment is physically permitted.

Gravitational radiation is much smaller than electromagnetic radiation because the relevant constants (G and $4\pi\epsilon_0$) are of drastically different orders of magnitude. For two objects of mass M

and charge $|Q|$, the ratio of the power of gravitational to electromagnetic radiation emitted during the oscillation of the two goes like

$$\frac{P_G}{P_{EM}} \propto \frac{GM^2}{Q^2}\frac{L^2}{\lambda^2}$$

Where the first term is very small except for very large masses, and the second term is proportional to the square of the ratio of the speeds of the objects, so massive, rapidly accelerating objects are the best gravitational wave sources.