6720 Problem Set 3

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Resources: Carroll & Ostlie, Schneider
Time: ∼5 hours
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Question I.

The relation in question from Schneider Eq. 2.34 is

$$n(R,z) = n_0(e^{-|z|/h_{thin}} + 0.02e^{-|z|/h_{thick}})e^{-R/h_R}$$

where $h_{thin} \approx 325$ pc, $h_{thick} \approx 1.5$ kpc, and $h_R \approx 3.5$ kpc.

(a) To calculate the normalization n_0 we use the Sun's $z = 30$ pc and solve the above for n_0.

$$n(R,z) = n_0(e^{-|z|/h_{thin}} + 0.02e^{-|z|/h_{thick}})e^{-R/h_R}$$
$$n_0 = \frac{n(R,z)}{(e^{-|z|/h_{thin}} + 0.02e^{-|z|/h_{thick}})e^{-R/h_R}}$$

using $z = 30$ pc and $R = 8$ kpc, we find $n_0 = 2.1$ stars/pc^3.

(b) Performing this calculation for each z we get

$$n(R_0, 0 \text{ pc}) = 0.21 \text{ stars/pc}^3$$
$$n(R_0, 10 \text{ pc}) = 0.21 \text{ stars/pc}^3$$
$$n(R_0, 100 \text{ pc}) = 0.16 \text{ stars/pc}^3$$
$$n(R_0, 1000 \text{ pc}) = 0.01 \text{ stars/pc}^3$$

(c) Calculating for the central density we have $n(0,0) = 2.1$ stars/pc^3, so we can find the edge by solving for R here where $z = 0$ pc.

$$10^{-3}n(0,0) = n_0(e^{-|z|/h_{thin}} + 0.02e^{-|z|/h_{thick}})e^{-R/h_R}$$
$$e^{-R/h_R} = \frac{10^{-3}n(0,0)}{n_0(e^{-|z|/h_{thin}} + 0.02e^{-|z|/h_{thick}})}$$
$$R = -h_R \ln\left[\frac{10^{-3}n(0,0)}{n_0(e^{-|z|/h_{thin}} + 0.02e^{-|z|/h_{thick}})}\right]$$
$$z = 0$$
$$R = -h_R \ln\left[\frac{10^{-3}n(0,0)}{n_0}\right]$$

This gives the edge $R \approx 25$ kpc.

(d) in the same process, we take the following for when $R = 0$ to find the edge in height z.

$$10^{-3}n(0,0) = n_0(e^{-|z|/h_{thin}} + 0.02e^{-|z|/h_{thick}})$$
$$e^{-|z|/h_{thin}} + 0.02e^{-|z|/h_{thick}} = \frac{10^{-3}n(0,0)}{n_0}$$

This is a little rough to solve analytically, so numerically we have $|z| \approx 4.5$ kpc

(e) The total number of stars is given by the integral of the number density, so we have

$$N = \int_0^{2\pi}\int_{-z_{lim}}^{z_{lim}}\int_0^{R_{lim}} n(r,z)r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

Where the limits z_{lim} and R_lim are the limits from the previous questions. We can separate this into three integrals with the θ integral trivially giving 2π. We can also do the classic trick of splitting the absolute value to integrate from 0 to z_{lim} and picking up a factor of 2. We have

$$\begin{aligned}
N &= 4\pi\int_0^{z_{lim}}\int_0^{R_{lim}} n(r,z)r\,\mathrm{d}r\,\mathrm{d}z \\
&= 4\pi n_0\int_0^{z_{lim}}(e^{-z/h_{thin}} + 0.02e^{-z/h_{thick}})\,\mathrm{d}z\int_0^{R_{lim}} re^{-r/h_R}\,\mathrm{d}r \\
&= 4\pi n_0\left[-h_{thin}e^{-z/h_{thin}} - 0.02h_{thick}e^{-z/h_{thick}}\Big|_0^{z_{lim}}\right]\int_0^{R_{lim}} re^{-r/h_R}\,\mathrm{d}r \\
&= 4\pi n_0\left[-h_{thin}e^{-z/h_{thin}} - 0.02h_{thick}e^{-z/h_{thick}}\Big|_0^{z_{lim}}\right]\left(-h_R(h_R+r)e^{-r/h_R}\Big|_0^{R_{lim}}\right)
\end{aligned}$$

Evaluation with the limits from the previous questions gives $N \approx 5.4 \times 10^{10}$ stars, which is order of magnitude close to the expected 10^{11} stars.

(f) Assuming each star is .5 $M_\odot$, the total stellar mass is $M \approx 2.7 \times 10^{10}\ M_\odot$

Question II.

(a) (i) This is given by the integral of the surface density

$$\begin{aligned}
M &= \int_0^{2\pi}\int_0^{R_{lim}} r\Sigma_0 e^{-r/h}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\pi\Sigma_0\left(-h(h+r)e^{-r/h}\Big|_0^{R_{lim}}\right)
\end{aligned}$$

(ii) Solving for the normalization we have

$$6\times 10^{10} = 2\pi\Sigma_0\left(-h(h+r)e^{-r/h}\Big|_0^{25\text{ kpc}}\right)\quad \Sigma_0 = \frac{6\times 10^{10}}{2\pi\left(-h(h+r)e^{-r/h}\big|_0^{25\text{ kpc}}\right)}$$

This gives the normalization as $\Sigma_0 = 7.8 \times 10^8\ M_\odot/kpc^2$

(b) (i) We can use Kepler for this, where $a \to r$ is our radial variable:

$$T^2 = \frac{4\pi^2}{GM} a^3$$
$$\left(\frac{2\pi r}{v}\right)^2 = \frac{4\pi^2}{GM} r^3$$
$$v = \sqrt{\frac{GM}{r}}$$

we can use our expression from part a for M to find

$$v = \sqrt{\frac{-2\pi\Sigma_0 Gh(e^{-r/h}(r+h)-h)}{R}}$$

(ii) We need the conversion factor knowing that the units of Σ_0 are $M_\odot/kpc^2$, and assuming we use G in cgs with units of $cm^3 g^{-1} s^{-2}$ and R in kpc. This would mean the velocity as we have it is in units of

$$[v] - \frac{cm^{3/2} M_\odot^{1/2}}{kpc^{1/2} g^{1/2} s}$$

Which is very ugly but does reduce to length/time. Our best friend astropy.units has our back, and the conversion gives us a velocity in km/s with the factor

$$v = \left(7.5\left(\frac{cm^{3/2} M_\odot^{1/2}}{kpc^{1/2} g^{1/2} s}\right)\right)\sqrt{\frac{-2\pi\Sigma_0 Gh(e^{-r/h}(r+h)-h)}{R}}[km/s]$$

(iii) See the shared colab file

(iv) The radial velocity from this model is ≈ 140 km/s at R_0, which undershoots the known value of 220 km/s due most likely to the dark matter (missing mass) not accounted for in this model.

(c) (i) From the previous problem set we had the results

$$\rho(r) = \frac{\rho_0}{1+(r/r_c)^2}$$
$$M(r) = 4\pi\rho_0 r_c \left[r - r_c \arctan(r/rc)\right]$$

(ii) The velocity $V(r)$ for the dark matter halo only will be the same as the stellar rotational velocity but a different mass profile, so we have

$$v = \left(7.5\left(\frac{cm^{3/2} M_\odot^{1/2}}{kpc^{1/2} g^{1/2} s}\right)\right)\sqrt{\frac{4\pi G\rho_0 r_c^2(R - r_c \arctan(R/r_c))}{R}}[km/s]$$

(iii) See the shared colab

(d) (i) We can show this by the following

$$\begin{aligned}
V &= \sqrt{\frac{GM}{r}} \\
&= \sqrt{\frac{G(M_{disk} + M_{halo})}{r}} \\
&= \sqrt{\frac{G(M_{disk}}{r} + \frac{GM_{halo})}{r}} \\
V_{tot}^2 &= \frac{G(M_{disk}}{r} + \frac{GM_{halo})}{r} \\
&= V_{disk}^2 + V_{halo}^2
\end{aligned}$$

(ii) See the shared colab

(iii) From the colab file we see the ratio of $\frac{V_{halo}}{V_{disk}} \approx 1.4$ at R_0, for $V_{halo} \approx 200$ km/s.

(iv) As seen in the colab, the disk component only dominates at the smallest radii ($<$ 1 kpc).

(e) We find the masses $M_{halo}(50\ kpc) \approx 7 \times 10^{11}$ and $M_{disk}(50\ kpc) \approx 4 \times 10^{10}$, so the disk-to-halo mass ratio is $\frac{M_{disk}}{M_{halo}} \approx .05$ at this radius.

(f) The bulge contribution will dominate at small radius (since that is where the bulge is), and the halo will dominate at large radius (or else we would see that the disk velocity is the same as the known 220 km/s in part b iv). They also may both contribute as a function of scale height z.