STAT 630 Problem Set 9

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Question I.

6.5.1

(a) We have shown previously that

$$
\begin{aligned}
L &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x_i-\mu)^2}{2\sigma^2}} \\
l &= -\frac{n}{2}(\log 2\pi + \log \sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \\
\frac{\partial l}{\partial \sigma^2} &= \frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2 \\
\hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2
\end{aligned}
$$

The Fisher information is

$$I(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log(f_\theta(X))\right]$$

so we have for

$$
\begin{aligned}
nI(\sigma^2) &= -nE_\theta\left[\frac{\partial}{\partial\sigma^2}\left(\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2\right)\right] \\
&= \frac{-n}{2\sigma^4} + \frac{n\sigma^2}{\sigma^6} \\
&= \frac{n}{2\sigma^4}
\end{aligned}
$$

(b) The asymptotic distribution is

$$\hat{\sigma}^2 \sim N(\sigma^2, 2\frac{\sigma^4}{n})$$

which does agree with the CLT.

Question II.

(a)

$$Y = -\log(X_i) \sim \; Exp(a)$$

The sum of exponentially distributed random variables of parameter a goes like Gamma(n, a). The expectation of $\hat{a}$ is

$$\int -\frac{n}{y} f_Y(y)\,\mathrm{d}y$$
$$= \frac{n}{n-1}a$$

so the Bias is

$$\mathrm{Bias}_\alpha(\hat{\alpha} = \frac{n}{n-1}a - a$$

which is 0 in the large n limit.

(b) The Fisher information is

$$\begin{aligned} I(\hat{a}) &= -E_{\hat{a}}\left[\frac{\partial^2}{\partial a^2}\log f\right] \\ &= \frac{n}{a^2} \end{aligned}$$

and the asymptotic normal distribution is $N(a, \frac{a^2}{n})$

Question III.

(a) The Fisher information is

$$\begin{aligned} I &= -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log(f_\theta(X))\right] \\ &= -E_\theta\left(\frac{-T_i}{\lambda^2}\right) \\ &= \frac{1}{\lambda} \\ nI &= \frac{n}{\lambda} \end{aligned}$$

The asymptotic distribution is $N(\lambda, \frac{\lambda}{n})$. The variance of $\bar{T}$ is $\frac{n\lambda}{n^2} = \frac{\lambda}{n}$

(b) The log likelihood function is

$$l = \sum_{i=1}^{n} x_i \log\left(\lambda^{-1/2}\right)^{-2} - n((\lambda^{-1/2})^{-2}) + \sum_{i=1}^{n} x_i!$$

The MLE is

$$\hat{\lambda^{-1/2}} = \sqrt{\frac{n}{\sum_{i=1}^{n} x_i}}$$

The delta method gives

$$\sqrt{n}\left[\hat{\lambda^{-1/2}} - \lambda^{-1/2}\right] \to N(0, \frac{1}{4\lambda^2 n})$$

(c) The results from R give $\lambda \approx 1.56$ and $\theta = 0.800$, with a standard error of 0.026.

Question IV.

6.3.1

The mean of the data is $\bar{x} = 4.88$. From the appendix,

$$\begin{aligned} c &= Z_{0.975} \\ &= 1.96 \end{aligned}$$

so we have

$$P(4.88 - 1.96\sqrt{\frac{0.5}{10}} < \mu < 4.88 + 1.96\sqrt{\frac{0.5}{10}}) = 0.95 \tag{1}$$

gives the 95 percent confidence interval (4.44, 5.32). The observed mean lies within the interval.

6.3.2

If we now drop the knowledge of the variance, we now have a t-test

$$c = t_{0.975}(9) = 2.262$$

and

$$\begin{aligned} S &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \\ &= 0.69 \end{aligned}$$

which gives the confidence interval (4.38, 5.38), which contains the sample mean.

Question V.

6.3.8

The Wald confidence interval is (for $Z_0.95 = 1.645$)

$$0.62 \pm 1.645\left(\frac{0.62(1-0.62)}{250}\right)^{1/2}$$

gives the interval (0.5695, 0.6705). The score interval is

$$\left(\hat{\theta} + \frac{Z^2}{2n} \pm Z\left(\frac{\hat{\theta}(1-\hat{\theta})}{n} + \frac{Z^2}{4n^2}\right)^{1/2}\right)\frac{1}{1+\frac{Z^2}{n}}$$

which gives the interval (0.5685, 0.6690). Both the Wald and score intervals include the sample mean.

Question VI.

6.5.4

(a) The sample mean is $\hat{\lambda} = \bar{x} = 9.65$. The Wald interval for 95 percent confidence is

$$\bar{x} \pm Z\sqrt{\frac{\bar{x}}{n}}$$

which gives the interval (8.288, 11.012).

(b) The asymptotic pivot $\frac{\hat{\lambda}-\lambda}{\sqrt{\lambda/n}} \sim N(0,1)$, from the notes, this gives the interval (6.71, 17.18).

(c) From R: The Wald interval had 246 of 10000 sample means lie within the interval, while the score interval had all 10000 sample means lie within the interval.

Question VII.

6.5.5

From R we find the mean of the sample as $\bar{x} = 1627$ and has variance $S^2 = 672476$. Using the MOME $\hat{\theta} = 2/\bar{x}$, we have the interval using the score method

$$\frac{\hat{\theta}}{1 \pm \frac{Z_{0.95}}{\sqrt{n}}}$$

to give the interval (0.00093, 0.0018)

6.5.6 This gives the same result except with $\hat{\theta} = 1/\bar{x}$. This gives the interval (0.00047, 0.00090)