ASTR 605 Problem Set 1

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Question I.

We have

$$F \sim m_1 m_2 r$$

so we calculate the equation of motion as

$$\frac{F_2}{m_2} - \frac{F_1}{m_1} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1$$

where we assume

$$-\vec{F}_1 = \vec{F}_2 = m_1 m_2 \vec{s}$$

so

$$m_1 \vec{s} + m_2 \vec{s} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1$$

Defining $M = m_1 + m_2$ we have

$$\begin{aligned} \vec{s}M &= \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = \ddot{\vec{s}} \\ \implies \ddot{\vec{s}} &= M\vec{s} \end{aligned}$$

which is a harmonic oscillator for motion of a centered ellipse.

Question II.

See Figure 1

Question III.

This describes a hyperbolic orbit. From the orbit equation

$$r = \frac{p}{1 + e\cos f}$$

where f is the true anomaly, e is the eccentricity, and p is the semi-latus rectum, which is $p = a(e^2 - 1)$ for a hyperbola. The energy is

$$E = \frac{v^2}{2}$$

For a hyperbolic orbit, the eccentricity can be expressed in terms of the distance of closest approach, which is $d - a$ (noting that $a < 0$ for a hyperbolic orbit). We have

$$e = 1 + \frac{(d-a)v^2}{\mu^2}$$

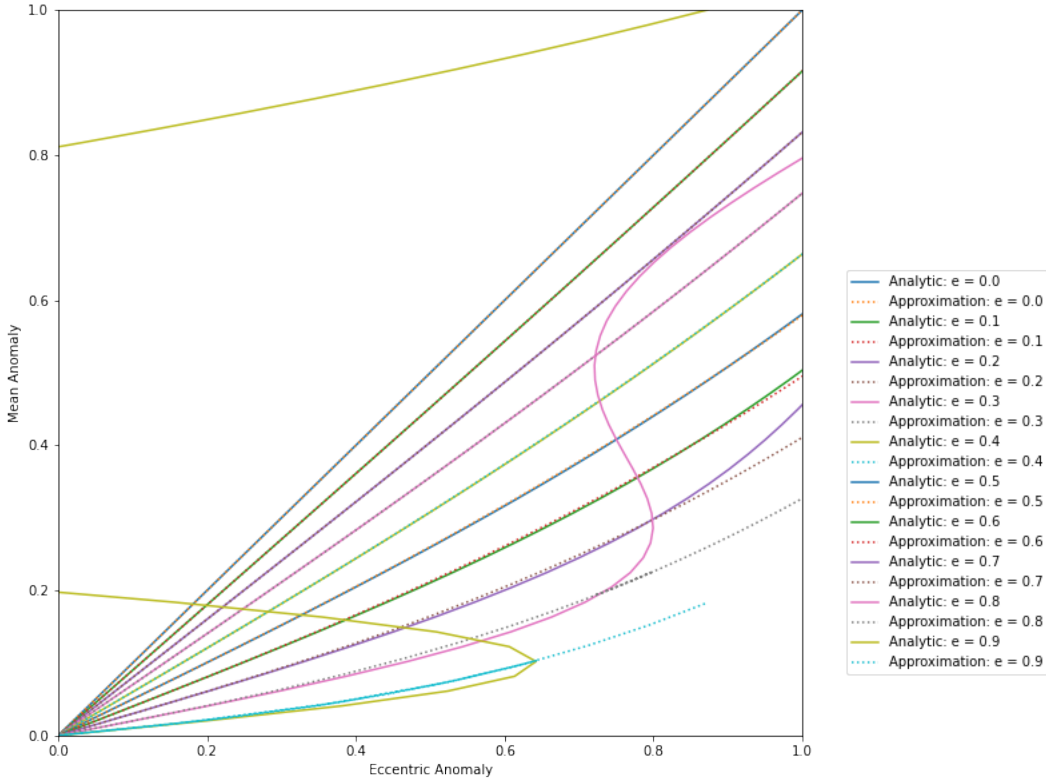

Figure 1: This solution to Kepler's equation breaks down at $e \approx 0.6$

the orbit equation becomes

$$r = \frac{p}{1 + \left(1 + \frac{(d-a)v^2}{\mu^2}\right)\cos f}$$

in terms of the true anomaly, the velocity at infinity, and the would-be closest approach d should there be no force.

Question IV.

For a circular orbit ($e = 0$), we have the following from the orbit equation

$$\begin{aligned} r &= \frac{p}{1 + e\cos f} \\ &= p \end{aligned}$$

where the semi-latus rectum p for a circular orbit is simply the semi major axis a. As a function of time, this is simply

$$r = a\cos\omega t$$

where ω is the angular frequency of the orbit.

For the parabolic orbit, we have the orbit equation yielding

$$\begin{aligned} r &= \frac{p}{1 + e\cos f} \\ &= \frac{2q}{1 + \cos f} \end{aligned}$$

where q is the distance to the central mass at closest approach.

Question V.

For $u = 1/r$, we take the solution of this differential equation to be

$$u \approx \frac{1 + e(1 + \alpha/2)\cos[\theta(1 - \alpha/2)]}{p(1 - \alpha/2)}$$

which is dominated by the term accounting for the precession of the perihelion. If we take the classical limit, this gives us back the familiar

$$u = \frac{1 + e\cos\theta}{p}$$

In the case of Mercury, which has semimajor axis $a \sim 6 \times 10^{10}$m, an eccentricity $e = 0.2056$, and relativistic correction $\alpha \approx 2.6 \times 10^{-8}$, we get that Mercury precesses through an angle

$$\begin{aligned} \delta\omega &= \frac{2\pi\alpha}{p} \\ &\approx 2 \times 10^{-7} \text{ rad} \end{aligned}$$

per revolution. In units of time, this is

$$\delta\omega \approx 1 \text{ arcsec/year}$$