STAT 630 Problem Set 10

Nikko Cleri

November 29, 2021

Question I.

7.1.1

(a) The joint distribution is $\pi(\theta)f_\theta(s)$:

	$s=1$	$s=2$
$\pi(1)f_1(s)$	$1/10$	$1/10$
$\pi(2)f_2(s)$	$2/15$	$4/15$
$\pi(3)f_3(s)$	$6/20$	$2/20$

we have $m(1) = 8/15$ and $m(2) = 7/15$, so

$$\pi(\theta|s) = \frac{\pi(\theta)f_\theta(s)}{m(s)}$$

gives

	$\theta=1$	$\theta=2$	$\theta=3$
$\pi(\theta\|s=1)$	$3/16$	$1/4$	$9/16$
$\pi(\theta\|s=2)$	$3/14$	$4/7$	$3/14$

For the four two-element cases, we have

	$\theta=1$	$\theta=2$	$\theta=3$
$\pi(\theta\|(1,1))$	$9/256$	$1/16$	$81/256$
$\pi(\theta\|(1,2))$	$9/224$	$4/28$	$27/224$
$\pi(\theta\|(2,1))$	$9/224$	$4/28$	$27/224$
$\pi(\theta\|(2,2))$	$9/196$	$16/49$	$9/196$

So the highest probability for θ for each are 3, 2, 2, and 2, respectively.

(b) We now have

	$\theta=1$	$\theta=2$	$\theta=3$
$\pi(\theta\|(1,1,1))$	$27/4096$	$1/64$	$729/4096$
$\pi(\theta\|(1,1,2))$	$27/3584$	$4/112$	$243/3584$
$\pi(\theta\|(1,2,1))$	$27/3584$	$4/112$	$243/3584$
$\pi(\theta\|(2,1,1))$	$27/3584$	$4/112$	$243/3584$
$\pi(\theta\|(1,2,2))$	$27/3136$	$16/196$	$81/3136$
$\pi(\theta\|(2,1,2))$	$27/3136$	$16/196$	$81/3136$
$\pi(\theta\|(2,2,1))$	$27/3136$	$16/196$	$81/3136$
$\pi(\theta\|(2,2,2))$	$27/2744$	$64/343$	$27/2744$

So the highest probability for θ for each are 3, 3, 3, 3, 2, 2, 2, and 2, respectively.

Question II.

7.1.2

(a) The posterior is of the form $\text{Beta}(n\bar{x}+\alpha, n(1-\bar{x}+\beta))$ which has

$$\begin{aligned} \mu &= \frac{n\bar{x}+\alpha}{n\bar{x}+\alpha+n(1-\bar{x}+\beta)} \\ \sigma^2 &= \frac{(n\bar{x}+\alpha)(n(1-\bar{x}+\beta))}{(n\bar{x}+\alpha+n(1-\bar{x}+\beta))^2(n\bar{x}+\alpha+n(1-\bar{x}+\beta)+1)} \end{aligned}$$

(b) For $\alpha=\beta=2$ and $n=50$, $\bar{x}=0.56$, we have $\text{Beta}(30,122)$ with

$$\begin{aligned} \mu &= 0.71 \\ \sigma^2 &= 0.001 \end{aligned}$$

(c) The mode of the distribution is given by the global maximum, so we take the derivative of the pdf and set equal to zero

$$\begin{aligned} \frac{\partial}{\partial x}\left[\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}\right] &= 0 \\ &= -\frac{(1-x)^b\,x^{a-2}\left((b+a-2)\,x-a+1\right)}{(x-1)^2} \\ \implies x &= \frac{a-1}{b+a-2} \end{aligned}$$

This gives the posterior mode 0.193.

(d) See the following plot

(e) The 90% and 95% credible intervals are (0.147, 0.252) and (0.138, 0.264), respectively.

Question III.

7.1.3

(a) Given $n=10$, $\bar{x}=1$ when $\sigma_0^2=1$, $\mu_0=0$ and $\tau_0^2=10$, we have the distribution

$$N\left[\left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\left(\frac{\mu_0}{\tau_0^2}+\frac{n\bar{x}}{\sigma_0^2}\right), \left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\right]$$

The probability that $\mu>0\approx 0.991$ from R.

(b) For $\sigma_0^2=4$, the probability that $\mu>0\approx 0.807$ from R.

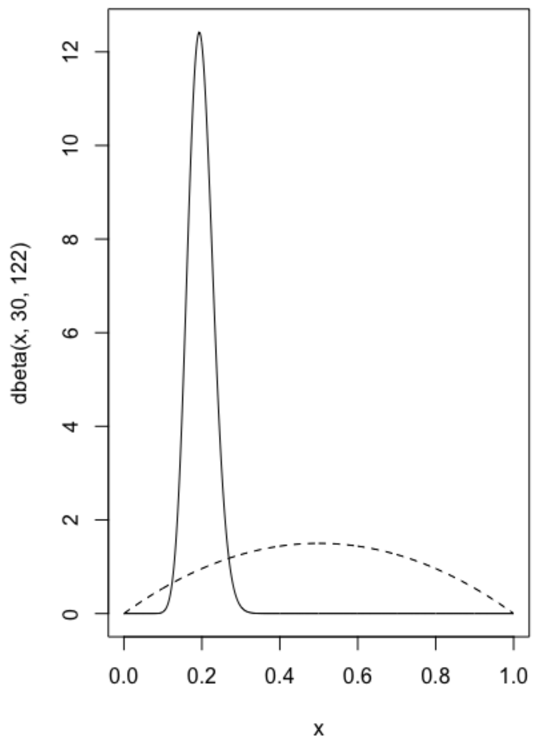

Figure 1: Prior and posterior distributions for question 2 part d.

(c) For $\sigma_0^2 = 4$ and $\tau_0^2 = 2$, the probability that $\mu > 0 \approx 0.756$ from R.

(d) From R, the 99% Bayes credible interval for the conditions in part c is (-0.654, 2.320).

Question IV.

7.1.4

(a) With a sample from Poisson(λ) and priors from Gamma(a, b), the posterior is Gamma$(a + \sum x_i, b + n)$.

(b) Assuming independence between the data and the priors, we have the posterior mean and variance as

$$\mu = \frac{a + \sum x_i}{n + b}$$
$$\sigma^2 = \frac{a + \sum x_i}{(n + b)^2}$$

(c) For the given conditions

$$\mu = \frac{2+27}{20+3} = \frac{29}{23}$$
$$\sigma^2 = \frac{2+27}{(20+3)^2} = \frac{29}{23^2}$$

(d) the 95% is (0.844, 1.759)

Question V.

7.1.9

(a) For a Uniform[0.4,0.6] prior with the likelihood Binomial(n, θ), we have

$$\pi(\theta|x) = \frac{5I_{[0.4,0.6]}\binom{n}{x}\theta^x(1-\theta)^{n-x}}{m(x)}$$

so for $x = n$, we have the normalization $m = 5\left(\frac{\theta^{1.6}}{1.6} + \frac{\theta^{1.4}}{1.4}\right)$ so the posterior is

$$\pi(\theta|x) = \frac{I_{[0.4,0.6]}\theta^x}{\frac{\theta^{1.6}}{1.6} + \frac{\theta^{1.4}}{1.4}}$$

(b) No, we have an indicator function $I_{[0.4,0.6]}$

(c) We should choose a prior which includes the possible values of the unknown parameter.

Question VI.

7.2.10

(a) Similarly to question 4, we have the prior as Gamma(a_0, b_0) and likelihood gives the posterior Gamma$(n + a_0,\ b_0 + \sum x_i)$. The mode of a Gamma distribution is $\frac{n+a_0-1}{b_0+\sum x_i}$. The posterior mean and variance are then

$$\mu = \frac{n + a_0}{b_0 + \sum x_i}$$
$$\sigma^2 = \frac{n + a_0}{(b_0 + \sum x_i)^2}$$

(b) For case one, we have $a_0 = 16$ and $b_0 = 4$. This gives the prior Gamma(16,4) For case 2, $a_0 = 1$ and $b_0 = 1/4$, which gives Gamma(1,1/4).

(c) The first case posterior becomes Gamma(36, 106), and the second case becomes Gamma(21, 102.25), with means 36/106 and 21/102.25, respectively. See the following figures.

(d) The respective 90% credible intervals are (0.252,0.438) and (0.138, 0.284).

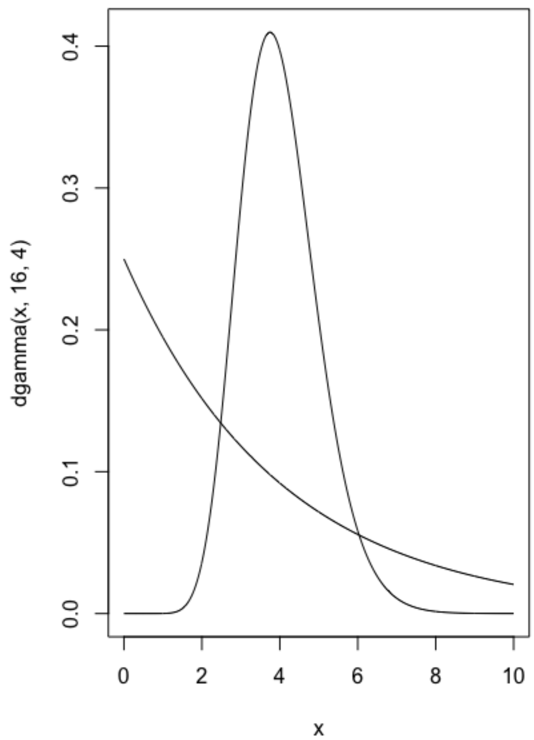

Figure 2: Posterior distributions for question 6 part b.

Question VII.

(a) H_0 is favored for $x = 1, 2, 3, 4$

(b) H_0 is favored for $x = 1, 2, 3, 4, 5$

(c) We have the following likelihood ratios

	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$	$x = 6$
$\frac{f_1(x)}{f_0(x)}$	0.5	0.31	0.287	0.514	2.077	20

so we rank the likelihood ratios $x = 6, 5, 4, 1, 2, 3$ decreasing.

(d) For $\alpha = 0.15$, the likelihood ratio test rejects if X$>$2.077. For $\alpha = 0.02$, the likelihood ratio test rejects if X$>$20.

(e) The power for $\alpha = 0.15$ is 0.67. The power for $\alpha = 0.15$ is 0.40.

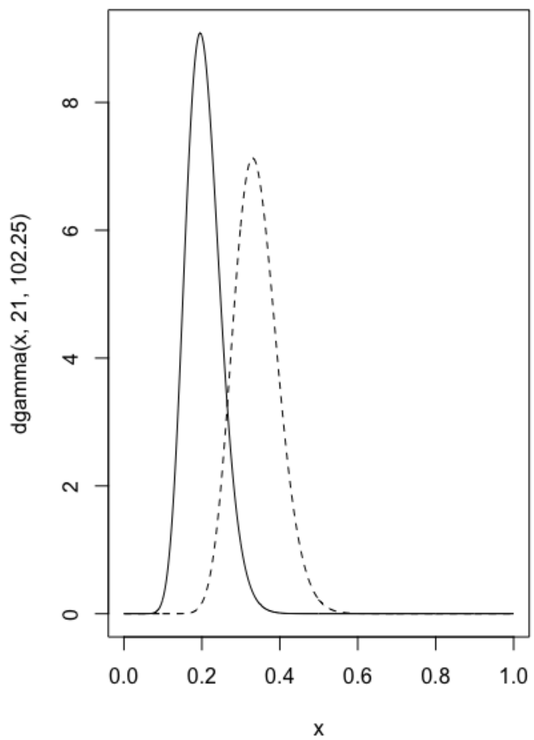

Figure 3: Posterior distributions for question 6 part c.