5402 Problem Set 5

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Question I.

With the Hamiltonian

$$H = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2| + V |2\rangle\langle 1| + V^* |1\rangle\langle 2|$$

(a) The energy eigenvalues can be found by first finding the first order change in energy as

$$\begin{aligned}
E'_n &= \langle n^0 | H' | n^0 \rangle \\
E'_1 & \langle 1| V |2\rangle\langle 1| + V^* |1\rangle\langle 2| |1\rangle \\
&= \langle 1| V |2\rangle\langle 1| |1\rangle + \langle 1| V^* |1\rangle\langle 2| |1\rangle \\
&= \langle 1| V |2\rangle \\
E'_2 &= \langle 2| V |2\rangle\langle 1| + V^* |1\rangle\langle 2| |2\rangle \\
&= \langle 2| V |2\rangle\langle 1| |2\rangle + \langle 2| V^* |1\rangle\langle 2| |2\rangle \\
&= \langle 2| V^* |1\rangle
\end{aligned}$$

So the energy eigenvalues will be $E'_1 = E_1 + \langle 1|V|2\rangle$ and $E'_2 = E_2 + \langle 2|V^*|1\rangle$. The eigenstates are given by

$$\begin{aligned}
|n\rangle &= |n^0\rangle + \frac{|m^0\rangle \langle m^0|H'|n^0\rangle}{E_n^0 - E_m^0} \\
|1'\rangle &= |1\rangle + \frac{|2\rangle \langle 2|H'|1\rangle}{E_1 - E_2} \\
&= |1\rangle + \frac{|2\rangle \langle 2|V|2\rangle}{E_1 - E_2} \\
|2'\rangle &= |2\rangle + \frac{|1\rangle \langle 1|V^*|1\rangle}{E_2 - E_1}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle 2' | D | 1 \rangle &= \langle 2|D|g\rangle + \frac{V}{E_2 - E_1} \langle 1|D|g\rangle \\
&= \frac{dV}{E_2 - E_1}
\end{aligned}$$

Question II.

With the unperturbed Hamiltonian

$$H_0 = E_1^0 \left|1\middle\rangle\middle\langle 1\right| + E_2^0 \left|2\middle\rangle\middle\langle 2\right|$$

and the time dependent perturbation

$$V(t) = \lambda \cos \omega t \left|1\middle\rangle\middle\langle 2\right| + \lambda \cos \omega t \left|2\middle\rangle\middle\langle 1\right|$$

(a)

$$\begin{aligned}
P(|2\rangle) &= |c_1(t)|^2 \\
c_1(t) &= \delta_{21} - \frac{i}{\hbar}\int \langle 2|H_1(t')|1\rangle\, e^{i\omega_{21}t'}\,\mathrm{d}t' \\
\omega_{fi} &= \frac{E_f^0 - E_i^0}{\hbar}
\end{aligned}$$

This coefficient becomes

$$\begin{aligned}
c_2(t) &= -\frac{i}{\hbar}\int \lambda \cos \omega t' e^{i\omega_{21}t'}\,\mathrm{d}t' \\
&= -\frac{i\lambda}{2\hbar}\int \left(e^{i\omega t'} + e^{-i\omega t}\right) e^{i\omega_{21}t'}\,\mathrm{d}t' \\
&= -\frac{i\lambda}{2\hbar}\int e^{i\omega t'}e^{i\omega_{21}t'} + e^{-i\omega t'}e^{i\omega_{21}t'}\,\mathrm{d}t' \\
&= -\frac{i\lambda}{2\hbar}\int e^{i(\omega+\omega_{21})t'} + e^{i(\omega_{21}-\omega)t'}\,\mathrm{d}t' \\
&= -\frac{i\lambda}{2\hbar}\left(\frac{-i}{\omega+\omega_{21}}e^{i(\omega+\omega_{21})t'} - \frac{i}{\omega_{21}-\omega}e^{i(\omega_{21}-\omega)t'}\right)\Bigg|_0^t \\
&= -\frac{i\lambda}{2\hbar}\left(\frac{-i}{\omega+\omega_{21}}e^{i(\omega+\omega_{21})t'} - \frac{i}{\omega_{21}-\omega}e^{i(\omega_{21}-\omega)t'}\right)\Bigg|_0^t \\
c_2(t) &= -\frac{i\lambda}{2\hbar}\left(\frac{-i}{\omega+\omega_{21}}\left[e^{i(\omega+\omega_{21})t} - 1\right] - \frac{i}{\omega_{21}-\omega}\left[e^{i(\omega_{21}-\omega)t} - 1\right]\right)
\end{aligned}$$

Taking $|c_2(t)|^2$ gives the probability.

(b) In the limit as $E_1 - E_2 \to \pm\hbar\omega$ we have $\omega_{21} \to \omega$ which gives an infinite transmission coefficient and by proxy an infinite probability.

Question III.

(a) The energies are given by

$$E = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\begin{aligned} E_1 &= \frac{\pi^2\hbar^2}{2ma^2} \\ E_2 &= \frac{2\pi^2\hbar^2}{ma^2} \\ E_3 &= \frac{9\pi^2\hbar^2}{2ma^2} \end{aligned}$$

and the wavefunctions are given by

$$\begin{aligned} \phi_1 &= \sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right) \\ \phi_2 &= \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right) \\ \phi_3 &= \sqrt{\frac{2}{a}}\cos\left(\frac{3\pi x}{a}\right) \end{aligned}$$

(b) From the time dependent Schrodinger equation

$$\begin{aligned} i\hbar\left|\dot{\psi(t)}\right\rangle &= H\left|\psi(t)\right\rangle \\ \left|\psi(t)\right\rangle &= \sum_n c_n(t)\left|n^0\right\rangle \\ c_n(t) &= c_n(0)e^{-iE_n^0 t/\hbar} \\ \left|\psi(t)\right\rangle &= \sum_n d_n(t)e^{-iE_n^0 t/\hbar}\left|n^0\right\rangle \\ &= e^{-iE_n^0 t/\hbar}\left|\psi(0)\right\rangle \end{aligned}$$

In the Schrodinger equation:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle &= H\left|\psi(t)\right\rangle \\ &= i\hbar\frac{\partial}{\partial t}e^{-iE_n^0 t/\hbar}\left|\psi(t)\right\rangle \\ &= i\hbar\left[\frac{iE_n^0}{\hbar}e^{-iE_n^0 t/\hbar}\left|\psi(t)\right\rangle + e^{-iE_n^0 t/\hbar}\frac{\partial}{\partial t}\left|\psi(t)\right\rangle\right] \\ &= e^{iE_n^0 t/\hbar}V(t)e^{-iE_n^0 t/\hbar}\left|\psi(t)\right\rangle \\ &= V_n(t)\left|\psi(t)\right\rangle \end{aligned}$$

We then have

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = V_n(t)\left|\psi(t)\right\rangle$$

and plugging in the expansion for $|\psi\rangle$ in the Schrodinger equation gives

$$i\hbar \sum_n \dot{c}_n(t) |n\rangle = \sum_n c_m(t) e^{iE_n^0 t/\hbar} V(t) e^{-iE_n^0 t/\hbar} |n\rangle$$

Taking the inner product with $\langle m|$ gives

$$i\hbar \sum_n \dot{c}_n(t) \langle m|n\rangle = \sum_n c_m($$

$$\dot{c}_m(t) = -\frac{i}{\hbar} \sum_n c_n(t) e^{i\hbar(\omega_m - \omega_n)} V_{mn}(t)$$

The desired result follows directly.

(c) As we know, the probability is the square of the amplitude $|c_n(t)|^2$.

$$c_n(t) = -\frac{i}{\hbar} \int_0^t \mathrm{d}t'\, e^{i(\omega_n - \omega_0)t'} V_{n0(t')} c_0(0)$$

$$|c_n(t)|^2 = \left| -\frac{i}{\hbar} \int_0^t \mathrm{d}t'\, e^{i(\omega_n - \omega_0)t'} V_n 0(t') c_0(0) \right|^2$$

$$= \left| -\frac{i}{\hbar} \int_0^t \mathrm{d}t'\, \langle \phi_2 | V \delta(x) \sin(\omega_0 t) | \phi_1 \rangle\, e^{i(\omega_n - \omega_0)t'} c_0(0) \right|^2$$

After expanding the exponential and simplifying we find a term goes to zero in the integrand, so $|c_n(t)|^2 = 0$

Question IV.

18.4.4

Under the transformation

$$A' \to A + \boldsymbol{\nabla}\Lambda$$

$$\phi' \to \phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda$$

we have the Hamiltonian for the potentials A and ϕ

$$H = \frac{1}{2m}\left(p - \frac{q}{c}A\right)^2 + q\phi$$

and the Hamiltonian obtained under the transformation

$$H_\Lambda = \frac{1}{2m}\left(p - \frac{q}{c}(A - \boldsymbol{\nabla}\Lambda)\right)^2 + q\left(\phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda\right)$$

The change in the wavefunction under the gauge transformation is

$$\psi_\Lambda = e^{-q\Lambda/\hbar c}\psi$$

Inserting into the time dependent Schrodinger equation gives

$$\begin{aligned}
H_\Lambda \psi_\Lambda &= \left[\frac{1}{2m}\left(p - \frac{q}{c}(A - \boldsymbol{\nabla}\Lambda)\right)^2 + q\left(\phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda\right)\right] e^{-q\Lambda/\hbar c}\psi \\
&= \frac{1}{2m}\left(p - \frac{q}{c}(A - \boldsymbol{\nabla}\Lambda)\right)^2 e^{-q\Lambda/\hbar c}\psi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda e^{-q\Lambda/\hbar c}\psi \\
&= \frac{1}{2}e^{-q\Lambda/\hbar c}\left[i\hbar\boldsymbol{\nabla}\psi - \frac{q}{c}A\psi\right]^2 + q\left(\phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda\right) e^{-q\Lambda/\hbar c}\psi \\
&= e^{-q\Lambda/\hbar c}\left[\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - \frac{q}{c}A\right)^2 + q\phi\right]\psi
\end{aligned}$$

We then have

$$\begin{aligned}
i\hbar\frac{\partial}{\partial \psi} &= \left[\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - \frac{q}{c}A\right)^2 + q\phi\right]\psi + \frac{q}{c}\left[\psi\frac{\partial}{\partial t}\Lambda - \psi\frac{\partial}{\partial t}\Lambda\right] \\
&= \left[\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - \frac{q}{c}A\right)^2 + q\phi\right]\psi
\end{aligned}$$

So

$$\begin{aligned}
-i\hbar\frac{\partial}{\partial t}\psi_\Lambda &= H_\Lambda\psi_\Lambda \\
&= H\psi \\
&= -i\hbar\frac{\partial}{\partial t}\psi
\end{aligned}$$