6710 Final

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Question I.

Let's star with some conceptual questions about the equations of stellar structure.

(a) What are the two initial assumptions for the equations of stellar structure?

We assume spherical symmetry and local thermal equilibrium (LTE).

(b) What is the difference between the Eulerian and Lagrangian frameworks for these equations?

Eulerian models deal with space differentials (integrate over $\mathrm{d}r$), while Lagrangian models deal with mass differentials (integrate over $\mathrm{d}m$)

(c) What equation of state governs most regions of most stars?

Ideal gas law is good for most regions of most stars (except where there is degeneracy or radiation dominance)

(d) What kinds of stars (or regions of stars) are governed by the other equations of state?

Radiation dominates near the surfaces of stars, while electron degeneracy dominates in degenerate matter (most importantly in white dwarfs, and massive stellar cores). Neutron degeneracy pressure becomes important when we consider neutron stars.

(e) What energy transport mechanism dominates most regions of stars?

Most regions of stars are dominated by energy transport by radiation.

(f) What elements are fused in main sequence stars?

The main sequence is defined by H fusion, by p-p chain or CNO.

(g) How does the main sequence fusion process differ for Sun-like ($< 1.5\ M_\odot$) and bigger ($> 1.5\ M_\odot$) stars on the main sequence?

Sun-like stars are dominated by p-p chain fusion, while higher mass stars are dominated by CNO fusion on the main sequence.

(h) What is the final product of a massive ($> 8\ M_\odot$) star before it explodes as a supernova?

These massive stars will fuse up to $^{56}_{26}$Fe (really Ni but it is unstable).

Question II.

Sketch a "pseudo-HR" diagram that plots stellar radius (y-axis) versus stellar mass (x-axis). Label the axes, including units. (I suggest plotting each axis as a logarithm.) Draw the main sequence and indicate which stars are most common. Also draw the evolutionary path of a Sun-like star after it leaves the main sequence.

We use the main sequence radius-mass relation $R \propto M^{0.7}$, with scatter being higher at higher mass. Lower mass, smaller stars are more common. The evolutionary path of the Sun on the radius-mass diagram follows:

(i) SGB: radius increases to $3R_{MS}$

(ii) RGB: mass loss, He flash

(iii) Horizontal Branch: envelope contracts, radius decreases

(iv) AGB: core contracts, envelope expands $\rightarrow$ PN

(v) Remnant: WD

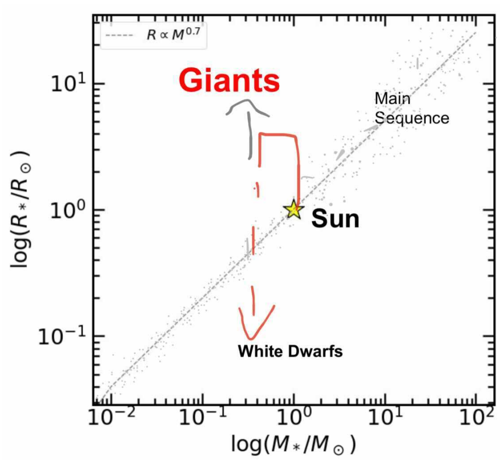

Figure 1: Modified H-R diagram for question 2. The red line represents the evolutionary path of the Sun, with the final destination as a WD after blowing off mass via to a PN (gray line). Lower mass stars are the most common.

(My annotations are not the greatest here but hopefully it is good enough to follow.)

Question III.

Derive the equation for the Eddington luminosity for neutrinos in a degenerate star. (Recall that the opacity κ is related to cross-section σ as $\kappa = \frac{\sigma}{\mu m_H}$, and you will find the neutrino cross-section σ_ν in your notes). Do any celestial objects reach this luminosity?

Starting with the cross section for neutrino interactions

$$\sigma_\nu \approx 10^{-45} A^2 \left(\frac{E_\nu}{m_e c^2}\right)^2 \text{ cm}^2$$
$$\approx 10^{-49} A^2 \left(\frac{\rho}{\mu_e}\right)^{2/3} \text{ cm}^2$$

The Eddington limit is defined to be when radiation pressure exceeds gravity, so we have the familiar

$$L_{Edd} = \frac{4\pi c G m}{\kappa}$$

Inserting the opacity we have

$$\begin{aligned} L_{Edd} &= \frac{4\pi c G \mu m_H m}{\sigma} \\ &= 10^{49} \frac{4\pi c G \mu m_H}{A^2 \left(\frac{\rho}{\mu_e}\right)^{2/3}} m \\ &= 10^{49} \left(4\pi c G\right) \left[\frac{\mu m_H m}{A^2 \left(\frac{\rho}{\mu_e}\right)^{2/3}}\right] \end{aligned}$$

If we assume a fully degenerate C/O White Dwarf where $\mu \approx 2$, $\mu_e \approx 2$, $\rho \approx 10^6$ g/cm^3, and the neutrinos are scattering off of ^{16}O nuclei ($A = 16$), we have an Eddington luminosity of

$$L_{Edd} = 1.04 \times 10^{53} \left(\frac{M}{M_\odot}\right) \text{ erg/s}$$

This is $\sim 10^6$ times brighter than a quasar for one solar mass, so there are no stable sources in the sky which reach this luminosity.

Question IV.

Algol, also known as the "demon star" and the "ghoul star", is a multiple-star system in the constellation Perseus. the eclipsing close binary at the center of the system (the eponym for the "Algol Paradox" and the class of "Algol variables") has a main sequence star Aa1 with a mass of 4.5 $M_\odot$ and a post-MS red giant Aa2 with a mass of 1 $M_\odot$, with a binary period of 2.87 days and an age of 570 Myr.

(a) How long will Aa1 remain on the main sequence?

Main sequence lifetime is given by

$$\begin{aligned} \tau &= (10^{10} \text{ yr}) \left(\frac{M_*}{M_\odot}\right)^{-2.8} \\ &\approx 1.5 \times 10^8 \text{ yr} \\ &\approx 150 \text{ Myr} \end{aligned}$$

(b) Now let's work backwards to infer the properties of Aa1 and Aa2 before mass transfer occurred. What was the minimum mass of Aa2 before it became a giant star and lost mass?

(I'm still not totally clear on which age this 570 Myr refers to, so I am going to assume that it is the main sequence lifetime of Aa2, under the assumption that it just became a giant.)

Using a similar method as before we have

$$\tau = (10^{10}\ \text{yr}) \left(\frac{M_*}{M_\odot}\right)^{-2.8}$$
$$\approx 570\ \text{Myr}$$
$$M_{min} \approx 2.8\ M_\odot$$

(c) Assuming the orbital separation has remained constant, and using the minimum mass of star Aa2 in part (b), what was the radius of Aa2 when mass transfer started?

If we assume that the total mass in the system has remained constant 5.5 $M_\odot$, we have the Roche limit

$$|a_g| < |a_t|$$
$$\left|\frac{GM_{Aa2}}{R_{Aa2}^2}\right| < \left|\frac{2M_{Aa1}}{d^3} R_{Aa2}\right|$$
$$R = \left|\left[\frac{M_{Aa2} d^3}{2(M_{tot} - M_{Aa2})}\right]^{1/3}\right|$$

We can approximate the separation with Kepler's third law as

$$d^3 = \frac{GM_{tot}}{4\pi^2}$$
$$\approx 15\ R_\odot$$

This yields a radius of $R_{Aa2} \approx 9\ R_\odot$. (This feels wrong to me but I am pretty strapped for time, or else I would go back to try to find a different solution.)

Question V.

The first generation of stars in the early Universe ("Population III") have no metals, just 75% hydrogen and 25% helium.

(a) How does the lack of metals ($Z = 0$) make a Population III star different from a subsequent, metal-enriched "Population I" star like the Sun? (*Hint:* Start from the stellar model and assumptions in your Midterm project, and then consider which terms are affected by composition.)

For Sun-like stars we assume an ideal gas equation of state with energy transport dominated by radiative diffusion by electron scattering. The ideal gas law reads

$$P = \frac{\rho(r)}{\mu m_H} kT(r)$$

which is immediately composition dependent in μ. We also have energy transport by radiative diffusion which reads

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{-3\kappa}{64\pi\sigma}\frac{\rho L}{T^3 r^2}$$

The electron scattering opacity $\kappa_{es} = 0.20(1+X)$, so energy transport is also composition dependent.

The sun has composition $X = 0.73$, $Y = 0.25$, $Z = 0.02$. If the sun is representative of Pop I stars then we have

$$\frac{\mu_{PopIII}}{\mu_{PopI}} \approx 1.03$$
$$\frac{\kappa_{PopIII}}{\kappa_{PopI}} = \frac{0.20(1+0.75)}{0.20(1+0.70)}$$
$$\approx 1.03$$

These relations indicate that the ideal gas pressure will be $\approx 3\%$ greater for Population I stars, and the energy transport rate by radiative diffusion will be $\approx 3\%$ greater for Population II stars.

(b) It may be possible for collapsing primordial molecular clouds to continue accreting mass beyond the Jeans mass, collapsing so quickly that they form a seed black hole "quasi-star". how quickly would a $10^5\ M_\odot$ cloud collapse after meeting the Jeans condition? (*Hint:* You can assume the cloud is optically thin to radiation, such that there is no heat transfer.)

Under these assumptions the cloud collapses according to freefall, so if we assume $M_J = 10^5\ M_\odot$ we have

$$t_{ff} = \left(\frac{3\pi}{32G\rho_0}\right)^{1/2}$$

and we can solve for ρ_0 using the Jeans mass

$$M_J \simeq \left(\frac{5kT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho_0}\right)^{1/2}$$
$$\rho_0 = \left[\frac{4\pi}{3}\left(M_J\left[\frac{5kT}{G\mu m_H}\right]^{-3/2}\right)^2\right]^{-1}$$

where $\mu = \frac{1}{2X+\frac{3}{4}Y+\frac{1}{2}Z}$. If we assume a temperature of 10^4 K this gives a density of $\rho_0 = 6.8\times10^{-18}$ g cm^{-3}. Putting this density into our free fall time we have

$$t_{ff} \approx 8.0\times10^{11}\ \mathrm{s}$$
$$\approx 25000\ \mathrm{yr}$$