5500 Problem Set 10

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November 9, 2020

Question I.

(a)

$$\hat{\rho} = \frac{1}{Z} e^{-\beta(\hat{H} - \mathbf{v}\cdot\hat{\mathbf{P}})}$$
$$\mu = \frac{m\mathbf{v}^2}{2} + kT \ln n\lambda^3$$

If we use the following form of the exponent

$$\hat{H} - \mathbf{v}\cdot\hat{\mathbf{P}}) = \sum_i \left[\frac{(\hat{\mathbf{p}} - m\mathbf{v})^2}{2m} - \frac{m\mathbf{v}^2}{2}\right]$$

We find that for the density operator to be normalized the partition function must be

$$Z = \frac{1}{N!h^{3N}} \left[\int d^3\mathbf{p} d^3\mathbf{r} e^{-\beta(\hat{H} - \mathbf{v}\cdot\hat{\mathbf{P}})}\right]^N$$
$$= \frac{1}{N!} \left[\frac{V}{\lambda^3} e^{\frac{m\mathbf{v}^2}{2kT}}\right]^N$$

The Helmholtz free energy is given by

$$F = -kT \ln Z$$
$$= kTN \left[\ln \frac{V}{\lambda^3} - \ln N + 1 + \frac{m\mathbf{v}^2}{2kT}\right]$$

Taking $n = N/V$, the chemical potential is

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$$
$$= \frac{m\mathbf{v}^2}{2} + kT \ln n\lambda^3$$

is the desired result.

(b) In thermal equilibrium the chemical potential is the same everywhere.

$$\mu_v = \mu_0$$
$$\frac{m\mathbf{v}^2}{2} + kT \ln n_v\lambda^3 = kT \ln n_0\lambda^3$$
$$n_0 = ne^{-\frac{m\mathbf{v}^2}{2kT}}$$

Question II.

We start with

$$f_E(E)dE = f_p(\mathbf{p})d^3\mathbf{p}$$

and we argue that

$$E = \frac{|\mathbf{p}|^2}{2m}$$
$$d^3\mathbf{p} = 4\pi m\sqrt{2mE}dE$$

In terms of energy we have

$$\begin{aligned} f_E(E)dE &= \frac{1}{(2\pi mkT)^{3/2}}e^{-E/kT}4\pi m\sqrt{2mE}dE \\ &= 2\sqrt{\frac{E}{\pi}}\left(\frac{1}{kT}\right)^{3/2}e^{-E/kT}dE \end{aligned}$$

gives the Maxwell-Boltzmann energy distribution

$$f_E(E) = 2\sqrt{\frac{E}{\pi}}\left(\frac{1}{kT}\right)^{3/2}e^{-E/kT}$$

Question III.

(a) The Boltzmann transport equation with the relaxation time equation becomes

$$v_x\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{f-f_0}{\tau}$$
$$f_1 \simeq f_0 - v_x\tau_c\frac{\mathrm{d}f_0}{\mathrm{d}x}$$

We can take the distribution function in the classical limit to be

$$f_0 = e^{(\mu-\epsilon)\tau}$$

(b) We now want to find the coefficient of diffusion. To first order the nonequilibrium distribution is

$$f = f_0 - \frac{v_x\tau_c f_0}{\tau}\frac{\mathrm{d}\mu}{\mathrm{d}x}$$

We can then take

$$\begin{aligned} \mathbf{j} &= \int vfd^3\mathbf{p} \\ &= \int v\left(f_0 - \frac{v_x\tau_c f_0}{\tau}\frac{\mathrm{d}\mu}{\mathrm{d}x}\right)d^3\mathbf{p} \\ &= -\frac{\tau_c\tau}{M}\frac{\mathrm{d}n}{\mathrm{d}x} \end{aligned}$$

This is the diffusion equation with the diffusivity

$$\begin{aligned} D &= \frac{\tau_c \tau}{M} \\ &= \frac{1}{3} \left\langle v^2 \right\rangle \tau_c \end{aligned}$$