STAT 630 Problem Set 6

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Question I.

4.2.12

For $n = 20$ and $N = 10^5$, .17554 of sample means lie within $0.19 < M_n < 0.21$. For $n = 50$ and $N = 10^5$, .27548 of sample means lie within $0.19 < M_n < 0.21$. This demonstrates the weak law of large numbers.

Question II.

4.4.4

$$\begin{aligned} F(x) &= \int_0^x \frac{1 + x/n}{1 + 1/2n} \, \mathrm{d}x \\ &= x \frac{1 + x/2n}{1 + 1/2n} \\ \lim_{n \to \infty} F(x) &= x \end{aligned}$$

for the uniform distribution Uniform(1,0)

$$F(w) = \frac{w - 0}{1 - 0} = w$$

so

$$\{W_n\} \xrightarrow{D} W$$

Question III.

4.4.12

(a)

$$\begin{aligned} \bar{X} &\sim \left(2, \frac{4}{16}\right) \\ P(\bar{X} \leq 2.5) &= \int_{-\infty}^{2.5} f(\bar{x}) \, \mathrm{d}\bar{x} \\ &\sim 0.84 \end{aligned}$$

(b)

$$
\begin{aligned}
\bar{X} &\sim \left(2, \frac{4}{36}\right) \\
P(\bar{X} \leq 2.5) &= \int_{-\infty}^{2.5} f(\bar{x})\,\mathrm{d}\bar{x} \\
&\sim 0.933
\end{aligned}
$$

(c)

$$
\begin{aligned}
\bar{X} &\sim \left(2, \frac{4}{100}\right) \\
P(\bar{X} \leq 2.5) &= \int_{-\infty}^{2.5} f(\bar{x})\,\mathrm{d}\bar{x} \\
&\sim 0.994
\end{aligned}
$$

(d) For the mgf of the sum of the random variables from the exponential distribution:

$$
\begin{aligned}
M(t) &= \prod_{-=1}^{n} \left(\frac{\lambda}{\lambda - t}\right) \\
&= \left(\frac{\lambda}{\lambda - t}\right)^{n}
\end{aligned}
$$

which is the mgf of gamma(n, λ). $\bar{X}$ then goes as gamma$(n, n\lambda)$. This gives the actual probability (from R) for $n = 16,\ \lambda = 0.5$ of 0.843.

(e) Similarly, for $n = 36$ gives probability 0.926.

(f) Similarly, for $n = 100$ gives probability 0.991.

Question IV.

4.4.16

From R: $P(M_{30} \leq -5) = 0.502$. The central limit theorem gives the approximation of $\bar{X} \sim N\left(-5, \frac{\frac{900}{12}}{10^5}\right)$, which has probability ≈ 0.50.

Question V.

4.6.1

$$
\begin{aligned}
X_1 &\sim N(3, 2^2) \\
X_2 &\sim N(-8, 5^2) \\
U &= X_1 - 5X_2 \\
&\sim N(43, 629) \\
f(u) &= \frac{1}{\sqrt{2\pi(629)}} e^{-\frac{(u-43)^2}{1268}} \\
V &= -6X_1 + CX_2 \\
&\sim N(-18 - 8C, 144 + 25C) \\
f(u) &= \frac{1}{\sqrt{2\pi(144+25C)}} e^{-\frac{(v-(-18-8C))^2}{2(144+25C)}}
\end{aligned}
$$

For independence, $1 * -6 * 2^2 + C * -5 * 5^2 = 0$ so $C = -24/125$.

4.6.2

$$
\begin{aligned}
X &\sim N(3, 5) \\
Y &\sim N(-7, 2) \\
Z &= 4X - Y/3 \\
&\sim N(43/3, 722/9) \\
\text{Cov}(X, Z) &= 1(4)(5) = 20
\end{aligned}
$$

4.6.7

$$
\begin{aligned}
C\frac{X_1}{X_2^2 + ... + X_n + 1^2} &\sim t(n) \\
C &= \sqrt{n}
\end{aligned}
$$

because there are n degrees of freedom.

Question VI.

4.6.10

- **(a)** $\chi^2(1)$
- **(b)** $\chi^2(2)$
- **(c)** $t(3)$
- **(d)** $t^2(3)$
- **(e)** $F(30, 70)$

Question VII.

$$
\begin{aligned}
f(x_1, ...x_n) &= \prod_{i=1}^{n} \lambda e^{\lambda x_i} \\
&= \lambda^n e^{\lambda \sum_{i=1}^{n} x_i}
\end{aligned}
$$

Question VIII.

$$
\begin{aligned}
f(t_1, ...t_n) &= \prod_{i=1}^{n} \binom{4}{x_i} \theta_i^t (1-\theta)^{4-t_i} \\
&= \prod_{i=1}^{n} (1-\theta)^4 \binom{4}{x_i} \left(\frac{\theta}{1-\theta}\right)^{t_i} \\
&= (1-\theta)^{4n} \left(\frac{\theta}{1-\theta}\right)^{\sum_{i=1}^{n} t_i} \prod_{j=1}^{n} \binom{4}{x_j}
\end{aligned}
$$

Question IX.

The joint distribution is

$$
\begin{aligned}
f(\{x_i\}) &= \prod_{i=1}^{n} \frac{1}{2\beta} e^{-|x_i-\mu|/\beta} \\
&= \left(\frac{1}{2\beta}\right)^n \prod_{i=1}^{n} e^{-|x_i-\mu|/\beta} \\
&= \left(\frac{1}{2\beta}\right)^n e^{-\sum_{i=1}^{n} |x_i-\mu|/\beta}
\end{aligned}
$$

Question X.

See Figures 1 and 2

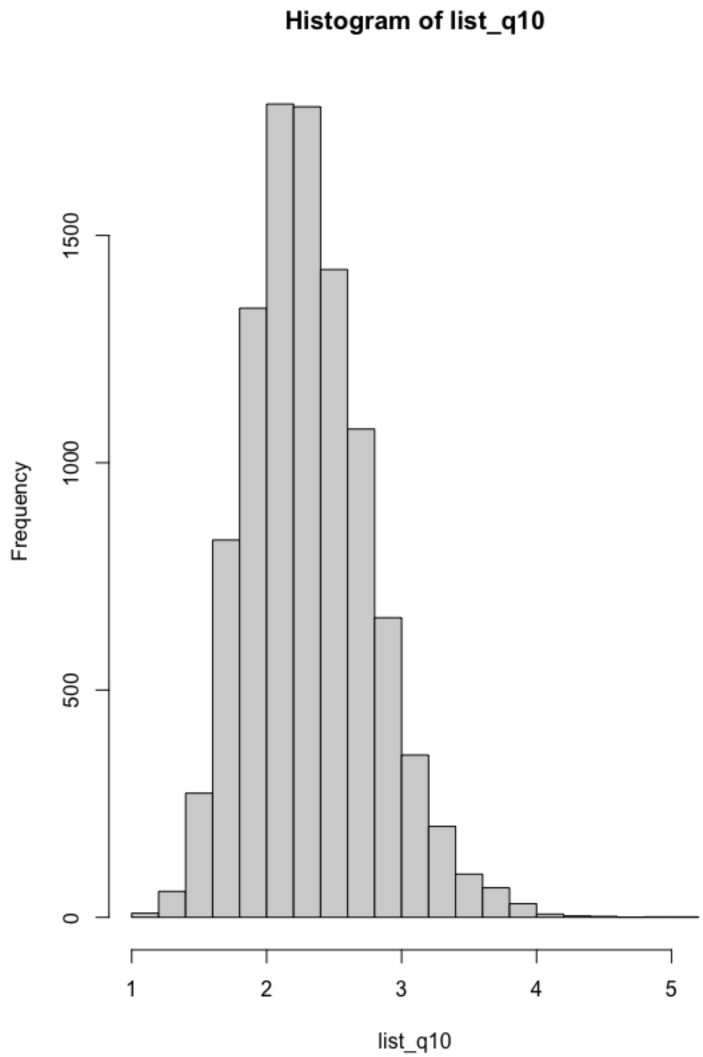

Figure 1: Histogram for question 10. We see a right-skewed distribution peaked around 2.2.

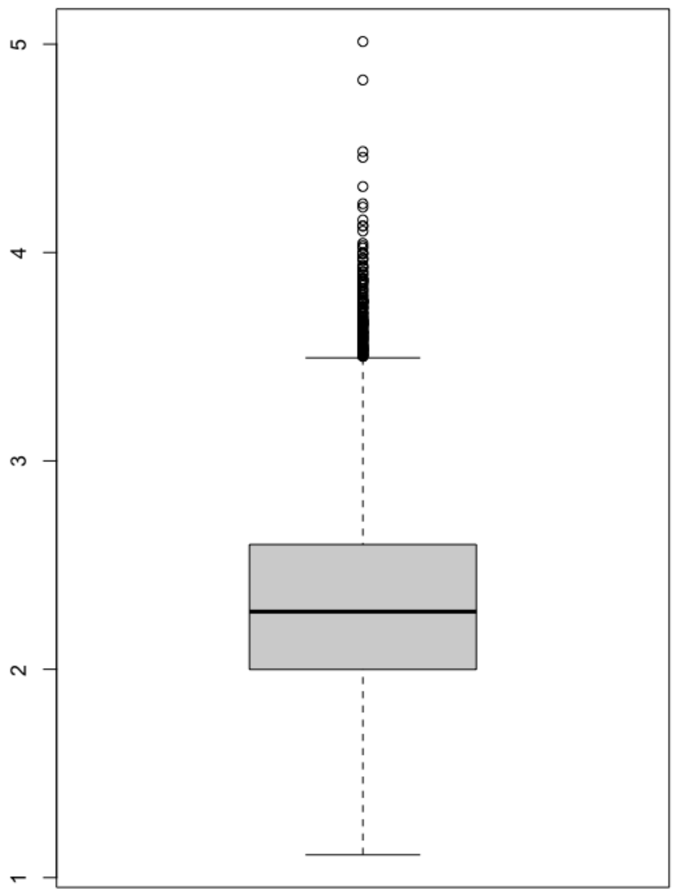

Figure 2: Box plot for question 10. We see a right-skewed distribution peaked around 2.2.