5402 Problem Set 1

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Question I.

Shankar 14.3.5

With some manipulation of M

$$
\begin{aligned}
M &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\
&= \begin{pmatrix} \frac{\alpha+\delta}{2} + \frac{\alpha-\delta}{2} & \frac{\beta+\gamma}{2} - i\left(i\frac{\beta-\gamma}{2}\right) \\ \frac{\beta+\gamma}{2} - i\left(i\frac{\beta-\gamma}{2}\right) & \frac{\alpha+\delta}{2} + \frac{\alpha-\delta}{2} \end{pmatrix} \\
&= \left(\frac{\alpha+\delta}{2}\right) I + \left(\frac{\beta+\gamma}{2}\right)\sigma_x + i\left(\frac{\beta-\gamma}{2}\right)\sigma_y + \left(\frac{\alpha-\delta}{2}\right)\sigma_z
\end{aligned}
$$

This gives us M in terms of the identity and the Pauli matrices.

Question II.

Shankar 14.3.7

$$
\begin{aligned}
I + i\sigma_x &= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\sigma_x\right) \\
&= \sqrt{2}(\cos\frac{\pi}{4} + i\sigma_x)\sin\frac{\pi}{4}) \\
&= \sqrt{2}e^{i\sigma_x\pi/4}
\end{aligned}
$$

So we get

$$
\begin{aligned}
(I + i\sigma_x)^{\frac{1}{2}} &= 2^{\frac{1}{4}}e^{i\sigma_x\pi/8} \\
&= 2^{\frac{1}{4}}(\cos\frac{\pi}{8} + i\sigma_x\sin\frac{\pi}{8})
\end{aligned}
$$

Next we have

$$
\begin{aligned}
2I + \sigma_x &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\
(2I + \sigma_x) &= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\
&= \frac{1}{3}(2I - \sigma_x)
\end{aligned}
$$

And very simply

$$\sigma_x^{-1} = \sigma_x$$
$$\because \sigma_x^2 = 1$$

Question III.

Shankar 14.4.6

We know that any 2x2 operator can be written as a linear combination of the Pauli matrices and the identity operator:

$$\begin{aligned}
\rho &= a_0 I + \mathbf{A} \cdot \sigma \\
\operatorname{Tr} \rho &= 2a_0 \\
\rho &= \frac{1}{2}(I + \mathbf{a} \cdot \sigma)
\end{aligned}$$

To find the expectation of the polarization we do

$$\begin{aligned}
\langle \sigma_x \rangle &= \operatorname{Tr}\{\sigma - x\rho\} \\
&= \frac{1}{2} \operatorname{Tr}\left\{\sigma_x I + a_x \sigma_x^2 + a_y \sigma_x \sigma_y + a_z \sigma_x \sigma_z\right\} \\
&= \frac{1}{2}(0 + 2a_x + 0 + 0) \\
&= a_x
\end{aligned}$$

The other components follow similarly to show $\langle \sigma \rangle = \mathbf{a}$.

In the case where we have an ensemble of electrons in a magnetic field we have the following two states in a Boltzmann distribution:

$$\begin{aligned}
p_\uparrow &= \frac{1}{P} e^{\gamma B\hbar/2kT} \\
p_\downarrow &= \frac{1}{P} e^{-\gamma B\hbar/2kT} \\
P &= e^{\gamma B\hbar/2kT} + e^{-\gamma B\hbar/2kT}
\end{aligned}$$

Question IV.

Shankar 14.5.3

Leaving the first filter we have states $|x-\rangle$ and $|x+\rangle$. The second filter allows only $|x+\rangle$ states, keeping half of the particles that entered the second filter.

If the third filter allows only $|z+\rangle$ particles we have $1/4$ of the particles that left the first filter exiting the third, since the particles entering the third are an evenly mixed state of $|z+\rangle$ and $|z-\rangle$.

If the second filter blocks nothing and the third filter transmits only $|z-\rangle$ then we will have no transmission through the third filter, since all particles entering the third filter will have $|z+\rangle$.

Question V.

A beam of massive spin-1 particles passes through a Stern-Gerlach apparatus and splits into three output beams, each one corresponding to one of the allowed projections of the particle spins onto a direction defined by the magnetic field inside the apparatus. Unless stated otherwise, all directions in this problem are with respect to a fixed laboratory coordinate system.

(a) Using the standard raising and lowering operators of angular momentum $J_\pm = J_x \pm iJ_y$ such that $J_\pm |jm\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|jm\pm 1\rangle$ show that the three matrices below form a valid representation of the spin operators for these particles

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};\ S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix};\ S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We can construct the raising and lowering operators of spin $S_\pm = S_x \pm iS_y$ with matrix elements given by $\langle 1, m_i | S_\pm | 1, m_j\rangle$ giving

$$S_- = \hbar\sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$S_+ = \hbar\sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

S_x and S_y follow directly, and we get S_z by taking the commutator $[S_x, S_y] = S_z$.

(b) Suppose the incoming beam is initially completely unpolarized, so its density operator in the matrix representation given in part (a) is $\rho_0 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The density operator may be written in terms of the three eigenstates of the spin in the z direction as

$$\rho_0 = \frac{1}{3}[|+\rangle\langle +| + |0\rangle\langle 0| + |-\rangle\langle -|]$$

not only in the laboratory frame, but in any Cartesian coordinate system obtained from the laboratory frame with an arbitrary rotation. Why?

the density operator is invariant under simple rotations in the laboratory frame as follows:

$$\begin{aligned} \rho_0 &= \frac{1}{3}\mathbb{I} \\ \rho_0' &= R\rho_0 R^\dagger \\ [R, \rho_0] &= \frac{1}{3}[R, \mathbb{I}] \\ &= 0 \\ \rho_0' &= \rho_0 \end{aligned}$$

Thus the density operator is invariant under rotations in the laboratory frame.

(c) What fraction of the particles in the unpolarized beam will pass through a Stern-Gerlach filter set up in such a way that only the particles with the component of the spin equal to 0 in the x direction are selected?

The beam is unpolarized, so the particles have no preference of spin. The probability must be 1/3, shown by $\mathrm{Tr}\{\rho P_{x0}\} = 1/3$.

Question VI.

An electron in a Coulomb field of a proton is in the state

$$|\psi\rangle = \frac{4}{5}|1,0,0\rangle + \frac{3i}{5}|2,1,1\rangle$$

where $|n,l,m\rangle$ are the standard energy eigenstates of hydrogen.

(a) What is the expectation value of the energy in this eigenstate? What are $\langle \mathbf{L}^2 \rangle$ and $\langle L_z \rangle$?

H acts on the system as $H|psi\rangle = \frac{k}{n^2}|\psi\rangle$ gives the expectation value $\langle H \rangle = \mathrm{Tr}\{H\rho\} = -\frac{16}{25}\frac{kqq}{1} + \frac{9}{25}\frac{kqq}{4}$.

The expectation value of L^2 comes from $L^2|\psi\rangle = \hbar^2 l(l+1)|\psi\rangle$.

$$\begin{aligned}\mathrm{Tr}\{^2\rho\} &= 2\frac{16}{25}\hbar^2 \\ &= \langle L^2 \rangle\end{aligned}$$

We also have $\langle L_z \rangle = 0$

(b) What is $|\psi(t)\rangle$? What, if any of the expectation values in (a), vary with time?

$|\psi(t)\rangle = e^{iHt}|\psi\rangle$, and we expect that the expectation value of L_z will be time variant.