STAT 630 Problem Set 6

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Question I.

3.3.25

(a) We know that each $X_i \sim \text{binomial}(n, \theta_i)$, so the first result follows immediately:

$$\text{Var}(X_i) = n\theta_i(1 - \theta_i)$$

We then have

$$\text{Var}(X_i + X_j) = \text{Var}(X_i) + \text{Var}(X_j) + 2\text{Cov}(X_i, X_j)$$

but we also know

$$\text{Var}(X_i + X_j) = n(\theta_i + \theta_j)(1 - (\theta_i + \theta_j))$$

so

$$\begin{aligned}
\text{Cov}(X_i, X_j) &= \frac{1}{2}(\text{Var}(X_i + X_j) - \text{Var}(X_i) - \text{Var}(X_j)) \\
&= n(\theta_i + \theta_j)(1 - (\theta_i + \theta_j)) - n\theta_i(1 - \theta_i) - n\theta_j(1 - \theta_j)
\end{aligned}$$

gives the desired

$$\text{Cov}(X_i, X_j) = -n\theta_i\theta_j$$

(b) The correlation is then

$$\begin{aligned}
\text{Corr}(X_i, X_j) &= \frac{\text{Cov}(X_i, X_j)}{\sqrt{\text{Var}(X_i)}\sqrt{\text{Var}(X_i)}} \\
&= \frac{-n\theta_i\theta_j}{n\sqrt{\theta_i(1 - \theta_i)}\sqrt{\theta_i(1 - \theta_i)}} \\
&= \frac{-\theta_i\theta_j}{\sqrt{\theta_i(1 - \theta_i)}\sqrt{\theta_i(1 - \theta_i)}}
\end{aligned}$$

which is independent of n.

Question II.

(a) We know the expectation of the beta(a, b) distribution is

$$E(X) = \frac{a}{a + b}$$

The variance is

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

so we need

$$\begin{aligned}
E(X^2) &= \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)^{-1} \int_0^1 x^2 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x \\
&= \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)^{-1} \int_0^1 x^{a+1}(1-x)^{b-1}\,\mathrm{d}x \\
&= \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)^{-1} \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+2+b)} \\
&= \frac{a(a+1)}{(a+b)(a+b+1)}
\end{aligned}$$

so

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - E(X)^2 \\
&= \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2 \\
&= \frac{ab}{(a+b)^2(a+b+1)}
\end{aligned}$$

(b) We know

$$\begin{aligned}
\mathrm{Var}(X_i + X_j) &= \mathrm{Var}(X_i) + \mathrm{Var}(X_j) + 2\mathrm{Cov}(X_i, X_j) \\
&= \frac{(a_1+a_2)a_3}{((a_1+a_2)+a_3)^2((a_1+a_2)+a_3+1)} \\
\mathrm{Cov}(X_i, X_j) &= \frac{1}{2}(\mathrm{Var}(X_i+X_j) - \mathrm{Var}(X_i) - \mathrm{Var}(X_j)) \\
&= \frac{(a_1+a_2)a_3}{2((a_1+a_2)+a_3)^2((a_1+a_2)+a_3+1)} \\
&\quad - \frac{a_1(a_2+a_3)}{2(a_1+(a_2+a_3))^2(a_1+(a_2+a_3)+1)} \\
&\quad - \frac{a_2(a_1+a_3)}{2(a_2+(a_1+a_3))^2(a_2+(a_1+a_3)+1)} \\
&= -\frac{a_1 a_2}{(a_1+a_2+a_3)^2(a_1+a_2+a_3+1)}
\end{aligned}$$

Question III.

3.4.5 $Y = 3X + 4$. The moment generating function of Y is

$$\begin{aligned}
m_Y(s) &= E(e^{(3X+4)t}) \\
&= e^{4t} E(e^{3Xt}) \\
&= e^{4t} m_X(3t)
\end{aligned}$$

3.4.12 $X \sim \mathrm{Geometric}(\theta)$

(a) Finding the mgf

$$\begin{aligned}
m_X(s) &= E(e^{sX}) \\
&= \sum_{X=0}^{\infty} e^{sx}\theta(1-\theta)^x \\
&= \theta \sum_{X=0}^{\infty} (e^s(1-\theta))^x \\
&= \frac{\theta}{1-(1-\theta)e^s}
\end{aligned}$$

(b)

$$\begin{aligned}
E(X) &= m_X'(0) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\theta}{1-(1-\theta)e^s}\bigg|_{s=0} \\
&= \frac{-\theta(\theta-1)}{(\theta^2)} \\
&= \frac{1-\theta}{\theta}
\end{aligned}$$

(c) For the variance we need

$$\begin{aligned}
E(X^2) &= m_X''(0) \\
&= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{\theta}{1-(1-\theta)e^s}\bigg|_{s=0} \\
&= \frac{(\theta^2-\theta)\theta)}{\theta^3} \\
&= \frac{\theta^2-3\theta+2}{\theta^2}
\end{aligned}$$

which gives

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - E(X)^2 \\
&= \frac{\theta^2-3\theta+2}{\theta^2} - \left(\frac{1-\theta}{\theta}\right)^2 \\
&= \frac{1-\theta}{\theta^2}
\end{aligned}$$

3.4.16

(a) Finding the mgf

$$\begin{aligned}
m_Y(s) &= E(e^{sY}) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{sy}e^{-|y|}\,\mathrm{d}y \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{sy}e^{y}\,\mathrm{d}y + \int_{0}^{\infty} e^{sy}e^{-y}\,\mathrm{d}y \\
&= \frac{1}{2}\left(\frac{1}{1-s} + \frac{1}{1+s}\right) \\
&= \frac{1}{1-s^2}
\end{aligned}$$

(b)

$$\begin{aligned}
E(Y) &= m_Y'(0) \\
&= \left.\frac{2s}{(1-s^2)^2}\right|_{s=0} \\
&= 0
\end{aligned}$$

(c) For the variance we need

$$\begin{aligned}
E(Y^2) &= m_Y''(0) \\
&= \left.\frac{3s^2+1}{(1-s^2)^3}\right|_{s=0} \\
&= 2
\end{aligned}$$

which gives

$$\begin{aligned}
\mathrm{Var}(Y) &= E(Y^2) - E(Y)^2 \\
&= 2 - 0 = 2
\end{aligned}$$

Question IV.

3.4.20

$$m_X(t) = \frac{\lambda^a}{\Gamma(a)}\int_0^{\infty} e^{tx}x^{a-1}e^{-\lambda x}\,\mathrm{d}x$$

Let $u = x(\lambda - t)$

$$\begin{aligned}
m_X(t) &= \frac{\lambda^a}{\Gamma(a)}\frac{1}{(\lambda-t)^a}\int_0^{\infty} u^{a-1}e^{-u}\,\mathrm{d}u \\
&= \frac{\lambda^a}{(\lambda-t)^a}
\end{aligned}$$

3.4.23 The mgf of the sum is

$$\begin{aligned} M(s) &= \prod_{i=1}^{n} M_i(s) \\ &= \frac{\lambda^{an}}{(\lambda - t)^{an}} \end{aligned}$$

So Y ~Gamma(an, λ)

Question V.

3.5.4

$$p_{X,Y}(x,y) = \begin{cases} 1/11 & x=-4, y=2 \\ 2/11 & x=-4, y=3 \\ 4/11 & x=-4, y=7 \\ 1/11 & x=6, y=2 \\ 1/11 & x=6, y=3 \\ 1/11 & x=6, y=7 \\ 1/11 & x=6, y=13 \\ 0 & \text{otherwise} \end{cases}$$

(a) The conditional distribution $p_{X|Y=2}$ is then

$$p_{X|Y=2}(x|y=2) = \begin{cases} 1/2 & x=-4 \\ 1/2 & x=6 \\ 0 & \text{otherwise} \end{cases}$$

so $E(X|Y=2) = -4(1/2) + 6(1/2) = 1$.

(b)

$$p_{X|Y=3}(x|y=3) = \begin{cases} 2/3 & x=-4 \\ 1/3 & x=6 \\ 0 & \text{otherwise} \end{cases}$$

so $E(X|Y=3) = -4(2/3) + 6(1/3) = -2/3$.

(c)

$$p_{X|Y=7}(x|y=7) = \begin{cases} 4/5 & x=-4 \\ 1/5 & x=6 \\ 0 & \text{otherwise} \end{cases}$$

so $E(X|Y=7) = -4(4/5) + 6(1/5) = -2$

(d)

$$p_{X|Y=13}(x|y=13) = \begin{cases} 1 & x = 6 \\ 0 & \text{otherwise} \end{cases}$$

so $E(X|Y=13) = 6$

(e)

$$E(X|Y) = \begin{cases} 1 & y = 2 \\ -2/3 & y = 3 \\ 0 & \text{otherwise} \end{cases}$$

3.5.11

$$f_{X,Y}(x,y) = \frac{6}{19}(x^2 + y^3) \quad 0 < x < 2, 0 < y < 1$$

(a)

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{6}{19}(x^2 + y^3)\,\mathrm{d}y \\ &= \frac{6}{19}\left(x^2 + \frac{1}{3}\right) \\ E(X) &= \frac{6}{19}\int_0^2 x\left(x^2 + \frac{1}{3}\right)\mathrm{d}x \\ &= \frac{6}{19}\frac{26}{3} \end{aligned}$$

(b)

$$\begin{aligned} f_Y(y) &= \int_0^2 \frac{6}{19}(x^2 + y^3)\,\mathrm{d}x \\ &= \frac{6}{19}\left(\frac{4}{3} + y^3\right) \\ E(Y) &= \frac{6}{19}\int_0^2 y\left(\frac{4}{3} + y^3\right)\mathrm{d}y \\ &= \frac{6}{19}\frac{13}{15} \end{aligned}$$

(c)

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{\frac{6}{19}(x^2+y^3)}{\frac{6}{19}\left(\frac{4}{3}+y^3\right)} \\
E(X|Y) &= \int_0^2 x f_{X|Y}(x|y)\,\mathrm{d}x \\
&= \left(\frac{1}{y^3+\frac{4}{3}}\right)\left(2y^3+\frac{8}{3}\right)
\end{aligned}$$

(d)

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{\frac{6}{19}(x^2+y^3)}{\frac{6}{19}\left(x^2+\frac{1}{3}\right)} \\
E(Y|X) &= \int_0^1 y f_{Y|X}(y|x)\,\mathrm{d}y \\
&= \left(\frac{1}{x^2+\frac{1}{3}}\right)\left(\frac{x^2}{2}+\frac{1}{5}\right)
\end{aligned}$$

(e)

$$\begin{aligned}
E(E(X|Y)) &= \int_0^2 \frac{6}{19}\left(x^2+\frac{1}{3}\right)\left(\frac{1}{x^2+\frac{1}{3}}\right)\left(\frac{x^2}{2}+\frac{1}{5}\right)\mathrm{d}x \\
&= \int_0^2 \frac{6}{19}\left(\frac{x^2}{2}+\frac{1}{5}\right)\mathrm{d}x \\
&= E(X)
\end{aligned}$$

(f)

$$\begin{aligned}
E(E(Y|X)) &= \int_0^1 \frac{6}{19}\left(\frac{4}{3}+y^3\right)\left(\frac{1}{y^3+\frac{4}{3}}\right)\left(2y^3+\frac{8}{3}\right)\mathrm{d}y \\
&= \int_0^1 \frac{6}{19}\left(2y^3+\frac{8}{3}\right)\mathrm{d}y \\
&= E(Y)
\end{aligned}$$

3.5.16

$$\begin{aligned}
E(E(X|Y)) &= E(X) \\
&= E\left(\frac{a}{y}\right) \\
&= aE\left(\frac{1}{y}\right) \\
&= \frac{a}{\lambda}
\end{aligned}$$

Question VI.

Let T have an exponential(λ) distribution, and conditional on T, let U be uniform on $[0, T]$. Find the unconditional mean and variance of U.

$$\begin{aligned}
E(U) &= E(E(U|T)) \\
&= E\left(\frac{T}{2}\right) \\
&= \frac{1}{2\lambda}
\end{aligned}$$

for the variance

$$\begin{aligned}
\mathrm{Var}(U) &= \mathrm{Var}[E(U|T)] + E[\mathrm{Var}(U|T)] \\
&= \mathrm{Var}\left(\frac{T}{2}\right) + E\left(\frac{T^2}{12}\right) \\
&= \frac{1}{4\lambda^2} + \frac{1}{6\lambda^2} \\
&= \frac{5}{12\lambda^2}
\end{aligned}$$

Question VII.

3.6.10 $f(w) = 3w^2,\ 0 < w < 1$

(a) $E(W)$

$$\begin{aligned}
E(W) &= \int w f(w)\,\mathrm{d}w \\
&= \frac{3}{4}
\end{aligned}$$

(b) Chebyshev's inequality gives the bound

$$P(|W - E(W)| \geq 1/4) \leq \frac{\sigma_W^2}{1/4^2}$$

The variance is

$$\begin{aligned}
\operatorname{Var}(W) &= E(W^2) - E(W)^2 \\
E(W^2) &= \int w^2 f(w)\,\mathrm{d}w \\
&= \frac{3}{5} \\
\operatorname{Var}(W) &= \frac{3}{80}
\end{aligned}$$

The bound is $\frac{\sigma_W^2}{1/4^2} = \frac{\frac{3}{80}}{\frac{1}{16}} = \frac{3}{5}$

(c) Compare the bound in part (b) to the exact probability

The exact probability is that W is between $1/2$ and 1 is

$$\int_{1/2}^{1} 3w^2\,\mathrm{d}w = \frac{7}{8}$$

The Chebyshev bound shows the upper limit for the probability outside $1/4$ from the mean is $3/5$.

Question VIII.

4.2.10

$$\begin{aligned}
\frac{Z_n}{n} &= \frac{\sum_{i=1}^{n} X_i^2}{n} \\
&\xrightarrow{P} E(X_i^2) \\
&\xrightarrow{P} \frac{91}{6}
\end{aligned}$$

4.2.11

$$\begin{aligned}
\frac{X_n}{n} &= \frac{\sum_{i=1}^{n} 4Y_i + 5Z_i}{n} \\
&\xrightarrow{P} E(4Y_i + 5Z_i) \\
&\xrightarrow{P} \frac{9}{2}
\end{aligned}$$