STAT 630 Problem Set 11

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Question I.

(a) We can relate c and the size α by the following

$$c = Z_{1-\alpha} \frac{\sigma_0}{\sqrt{n}}$$

where

$$\sigma_0 = \left(\frac{1}{12} \theta^2 \right)^{1/2}$$

so

$$\begin{aligned} & M_n < c \\ \implies & M_n < Z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{n}} \end{aligned}$$

(b) The power is

$$\beta(\mu) = 1 - \Phi \left(\frac{\hat{\theta} - \theta_1}{\sigma_0/\sqrt{n}} + Z_{1-\alpha} \right)$$

where $\hat{\theta} = M_n$ and σ_0 is as it was in part a.

(c) For a likelihood ratio test we have

$$\begin{aligned} \frac{L(\theta_1)}{L(\theta_0)} &= \frac{M_n^n}{\theta_1^n} \\ \frac{M_n^n}{\theta_1^n} &\leq M_n^n < c^n \end{aligned}$$

so for any alternative hypothesis $\theta = \theta_1$ this is the UMP.

Question II.

6.3.26

The one-sided rejection region is

$$\frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}} > Z_{1+a}$$

which yields the P-value

$$P = 1 - \Phi\left(\frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}}\right)$$

6.3.27

The power function is of the form

$$\beta(\mu') = \Phi\left(\frac{\mu - \mu'}{\sigma/\sqrt{n}} + Z_{1-\alpha}\right)$$

This yields the same rejection region as the test which rejects when $\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}} > Z_{1-a}$ because it is also one sided rejecting the higher tail.

Question III.

8.2.3

We conduct a likelihood ratio test for the hypotheses $H_0: \ \mu = 1$ and $H_0: \ \mu = \mu' > 1$.

$$\begin{aligned} \frac{L(\mu')}{L(\mu_0)} &< Z_{1-\alpha}\frac{\sigma_0}{\sqrt{n}} \\ &< \frac{Z_{0.99}}{\sqrt{5}} \end{aligned}$$

Question IV.

For the sample mean of 4.88 and standard deviation 0.696 we get the P-value

$$\begin{aligned} P &= 2\left(1 - \Phi\left(\frac{\bar{x} - \mu}{\sigma_0/\sqrt{n}}\right)\right) \\ &= 0.591 \end{aligned}$$

For the case where the variance is unknown, we have

$$\begin{aligned} P &= 2\left(1 - \Phi\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right)\right) \\ &= 0.585 \end{aligned}$$

Question V.

The Wald statistic is

$$W = \frac{n(\hat{\theta} - \theta_0)^2}{\hat{\theta}(1 - \hat{\theta})}$$

and the score statistic is

$$S = \frac{n(\hat{\theta} - \theta_0)^2}{\theta_0(1 - \theta_0)}$$

where

$$\hat{\theta} = 0.65$$
$$\theta_0 = 0.62$$

which gives

$$W = 0.989$$
$$S = 0.955$$

each of these are equal to their respective Z^2. The P-values are then

$$\begin{aligned} P(\sqrt{W}) &= 2\left(1 - \Phi(\sqrt{W}\right) \\ &= 0.333 \\ P(\sqrt{S}) &= 2\left(1 - \Phi(\sqrt{S}\right) \\ &= 0.340 \end{aligned}$$

Question VI.

(a) The test rejects when $|X - 50| > 10$ for a binomial(100,θ) distribution with the hypotheses $H_0: \ \theta = 0.5$ and $H_0: \ \theta \neq 0.5$. This rejection region implies that

$$Z_{1-\alpha/2} = \frac{10}{n\theta(1-\theta)/\sqrt{n}}$$

so alpha can be drawn from the relevant table.

(b) the power function is

$$\beta(\mu) = 1 - \Phi\left(\frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} + Z_{1-\alpha/2}\right) + \Phi\left(\frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} - Z_{1-\alpha/2}\right)$$

where $\sigma_0 = n\theta(1-\theta)$.

Question VII.

8.2.4

(a) For a .975 confidence interval, we have

$$0.975 = 1 - \frac{\alpha}{2}$$
$$\alpha = 0.05$$

(b) The power function is

$$\begin{aligned}\beta(\mu) &= 1 - \Phi\left(\frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} + Z_{1-\alpha/2}\right) + \Phi\left(\frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} - Z_{1-\alpha/2}\right) \\ &= 1 - \Phi\left(\frac{\mu}{1/\sqrt{20}} + Z_{0.975}\right) + \Phi\left(\frac{\mu}{1/\sqrt{20}} + Z_{0.975}\right)\end{aligned}$$

Question VIII.

8.2.6

It is not reasonable to accept that H_0 is true. The test may have passed with the size 0.05, but the power of 0.10 suggests that there is 0.9 probability that they accept the null hypothesis but the alternative is true.

Question IX.

The Wald statistic is

$$W = \frac{n(\hat{\theta} - \theta_0)^2}{\hat{\theta}(1 - \hat{\theta})}$$

and the score statistic is

$$S = \frac{n(\hat{\theta} - \theta_0)^2}{\theta_0(1 - \theta_0)}$$

where

$$\hat{\theta} = \frac{2x_1 + x_2}{2(x_1 + x_2 + x_3)}$$
$$\theta_0 = 0.5$$

Question X.

8.2.20

(a) The Likelihood ratio test yields

$$
\begin{aligned}
\frac{L(\lambda_1)}{\lambda_0} &= \frac{\prod_{i=1}^{n} \frac{\lambda_1^{x_i} e^{\lambda_1}}{x_i!}}{\prod_{j=1}^{n} \frac{\lambda_0^{x_i} e^{\lambda_0}}{x_i!}} \\
&= \frac{\lambda_1^{\sum_{i=1}^{n} x_i}}{\lambda_0^{\sum_{i=1}^{n} x_i}} \left(e^{n(\lambda_0 - \lambda_1)}\right) \\
&< Z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{n}}
\end{aligned}
$$

(b) This test is not the uniformly most powerful; Imposing the Neyman Pearson Lemma we know that this test has size less than or equal to α, which implies that it has power less than or equal to the likelihood ratio in part a. to that of the likelihood ratio test.

(c) For the generalized likelihood ratio test, we do the same calculation as in part a with the MLE for λ.

$$
\begin{aligned}
\frac{L(\hat{\lambda})}{\lambda_0} &= \frac{\prod_{i=1}^{n} \frac{\hat{\lambda}^{x_i} e^{\hat{\lambda}}}{x_i!}}{\prod_{j=1}^{n} \frac{\lambda_0^{x_i} e^{\lambda_0}}{x_i!}} \\
&= \frac{\hat{\lambda}^{\sum_{i=1}^{n} x_i}}{\lambda_0^{\sum_{i=1}^{n} x_i}} \left(e^{n(\lambda_0 - \hat{\lambda})}\right) \\
&< Z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{n}}
\end{aligned}
$$

(d) The mean of the data from assignment 9 is $\bar{x} \approx 1.56$, so with $\alpha = 0.01$ we have the P-value

$$
\begin{aligned}
P &= 2\left(1 - \Phi\left(\frac{1.88 - 2}{1.3/\sqrt{50}}\right)\right) \\
&= 3.39 \times 10^{-5}
\end{aligned}
$$

so we reject the null hypothesis.