STAT 630 Problem Set 5

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Question I.

2.9.7

$$p_X(x) = \begin{cases} 1/3 & x = 0 \\ 1/2 & x = 2 \\ 1/6 & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$p_Y(y) = \begin{cases} 1/6 & y = 2 \\ 1/12 & y = 5 \\ 3/4 & y = 9 \\ 0 & \text{otherwise} \end{cases}$$

For the convolution $Z = X + Y$, we have

$$\begin{aligned} p_Z(z) &= P(X + Y = z) \\ &= \begin{cases} 1/18 & z = 2 \\ 1/12 & z = 4 \\ 1/18 & z = 5 \\ 1/24 & z = 7 \\ 1/72 & z = 8 \\ 1/4 & z = 9 \\ 3/8 & z = 11 \\ 1/8 & z = 12 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Question II.

2.9.14

Theorem 2.9.3 (b) states: If X and Y are jointly absolutely continuous, with density functions f_X and f_Y, then $Z = X + Y$ is also absolutely continuous, with density function f_Z given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - w) f_Y(w) \, \mathrm{d}w$$

For our $X, Y \sim N(0,1)$, we have

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-w) f_Y(w)\,\mathrm{d}w \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\frac{1}{2}(-y^2-(z-y)^2)}\,\mathrm{d}y \\
&= \frac{1}{2\sqrt{\pi}} e^{\frac{-z^2}{4}}
\end{aligned}$$

is the desired $Z \sim N(0,2)$.

Question III.

3.1.2(a-f)

$$p_{X,Y}(x,y) = \begin{cases} 1/7 & x=5, y=0 \\ 1/7 & x=5, y=3 \\ 1/7 & x=5, y=4 \\ 3/7 & x=8, y=0 \\ 1/7 & x=8, y=4 \\ 0 & \text{otherwise} \end{cases}$$

(a)

$$\begin{aligned}
E(X) &= 5(3/7) + 8(4/7) \\
&= 6.7
\end{aligned}$$

(b)

$$\begin{aligned}
E(Y) &= 0(4/7) + 3(1/7) + 4(2/7) \\
&= 1.6
\end{aligned}$$

(c)

$$\begin{aligned}
E(3X + 7Y) &= 3E(X) + 7E(Y) \\
&= 31.3
\end{aligned}$$

(d)

$$\begin{aligned}
E(X^2) &= 25(3/7) + 64(4/7) \\
&= 47.3
\end{aligned}$$

(e)

$$\begin{aligned}
E(Y^2) &= 0(4/7) + 9(1/7) + 16(2/7) \\
&= 5.9
\end{aligned}$$

(f)

$$
\begin{aligned}
E(XY) &= 0(1/7) + 15(1/7) + 20(1/7) + 0(3/7) + 32(1/7) \\
&= 9.6
\end{aligned}
$$

Question IV.

3.1.5

$X \sim \text{Geometric}(\theta)$, $Y \sim \text{Poisson}(\lambda)$

$$
\begin{aligned}
E(8X - Y + 12) &= 12 + 8E(X) - E(Y) \\
&= 12 + 8\frac{1-\theta}{\theta} - \lambda
\end{aligned}
$$

3.1.6

$Y \sim \text{Binomial}(100, 0.3)$, $Z \sim \text{Poisson}(7)$

$$
\begin{aligned}
E(Y + Z) &= E(Y) + E(Z) \\
&= 30 + 7 \\
&= 37
\end{aligned}
$$

3.1.11

(a)

$$
\begin{aligned}
E(Z) = {}& 2(1/36) + 3(2/36) + 4(3/36) + 5(4/36) + 6(5/36) + \\
& 7(6/36) + 8(5/36) + 9(4/36) + 10(3/36) + 11(2/36) + 12(1/36) \\
& = 7
\end{aligned}
$$

(b) We know the sample space is

$$
\begin{aligned}
S = \{&(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\
&(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\
&(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\
&(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\
&(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\
&(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}
\end{aligned}
$$

so we have

$$
\begin{aligned}
E(W) = {}& \frac{1}{36}(1 + 2 + 3 + 4 + 5 + 6 + 2 + 4 + 6 + 8 + 10 + 12 + 3 + 6 + 9 + 12 + 15 + 18 + \\
& 4 + 8 + 12 + 16 + 20 + 24 + 5 + 10 + 15 + 20 + 25 + 30 + 6 + 12 + 18 + 24 + 30 + 36) \\
& = 12.25
\end{aligned}
$$

Question V.

3.1.23

$(X_1, X_2, X_3) \sim$Multinomial$(n, \theta_1, \theta_2, \theta_3)$. We have

$$p_{(X_1,X_2,X_3)}(x_1, x_2, x_3) = \binom{n}{x_1 x_2 x_3} \theta_1^{x_1} \theta_2^{x_2} \theta_3^{x_3}$$

It follows that each $X_i \sim$Binomial(n, θ_i), which each have the expectation $E(X_i) = n\theta_i$.

Question VI.

3.2.2(a-f)

$$f_{X,Y}(x, y) = \begin{cases} 4x^2y + 2y^5 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a)

$$\begin{aligned} E(X) &= \int_0^1 \int_0^1 x f_{X,Y}(x, y) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 \int_0^1 4x^3y + 2y^5 x \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 y + y^5 \, \mathrm{d}y \\ &= \frac{2}{3} \end{aligned}$$

(b)

$$\begin{aligned} E(Y) &= \int_0^1 \int_0^1 y f_{X,Y}(x, y) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 \int_0^1 4x^2y^2 + 2y^6 \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 \frac{4}{3}y^2 + 2y^6 \, \mathrm{d}y \\ &= \frac{46}{63} \end{aligned}$$

(c)

$$\begin{aligned} E(3X + 7Y) &= 3E(X) + 7E(Y) \\ &= \frac{64}{9} \end{aligned}$$

(d)

$$\begin{aligned}E(X^2) &= \int_0^1 \int_0^1 x^2 f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^1 4x^4y + 2y^5x^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \frac{4}{5}y + \frac{2}{3}y^5\,\mathrm{d}y \\ &= \frac{23}{45}\end{aligned}$$

(e)

$$\begin{aligned}E(Y^2) &= \int_0^1 \int_0^1 y^2 f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^1 4x^2y^3 + 2y^7\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \frac{4}{3}y^3 + 2y^7\,\mathrm{d}y \\ &= \frac{7}{12}\end{aligned}$$

(f)

$$\begin{aligned}E(XY) &= \int_0^1 \int_0^1 xy f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^1 4x^3y^2 + 2y^6x\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 y^2 + y^6\,\mathrm{d}y \\ &= \frac{10}{21}\end{aligned}$$

Question VII.

3.2.6

$X \sim$ Uniform[-12,9], $Y \sim N$(-8,9)

$$\begin{aligned}E(11X + 14Y + 3) &= 3 + 11E(X) + 14E(Y) \\ &= 3 + 11\left(\frac{1}{2}(9-12)\right) + 14(-8) \\ &= -125.5\end{aligned}$$

Question VIII.

3.2.19

$X \sim \text{Pareto}(a),\ a > 1$. Prove $E(X) = 1/(a-1)$.

$$\begin{aligned}
f(x) &= a(1+x)^{-a-1} \quad 0 < x < \infty \\
E(X) &= \int_0^\infty xa(1+x)^{-a-1}\,\mathrm{d}x \\
&= -x(1+x)^{-a}\Big|_0^\infty + \int_0^\infty a(1+x)^{-a}\,\mathrm{d}x \\
&= \int_0^\infty a(1+x)^{-a}\,\mathrm{d}x \\
&= \frac{1}{-a+1}(1+x)^{-a+1}\Big|_0^\infty \\
&= \frac{1}{a-1}
\end{aligned}$$

When $0 < a \leq 1$, the expectation diverges to infinity.

3.2.22

$X \sim \text{Beta}(a, b)$. Prove $E(X) = a/(a+b)$.

$$\begin{aligned}
f(x) &= \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)^{-1} x^{a-1}(1-x)^{b-1} \quad 0 < x < 1 \\
E(X) &= \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)^{-1} \int_0^1 xx^{a-1}(1-x)^{b-1}\,\mathrm{d}x \\
&= \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)^{-1} \int_0^1 x^{a}(1-x)^{b-1}\,\mathrm{d}x \\
&= \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)^{-1} \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+1+b)} \\
&= \frac{a}{a+b}
\end{aligned}$$

Question IX.

3.3.2(b-d)

(b)

$$\begin{aligned}
\text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
&= 9.6 - 1.6 * 6.7 \\
&= -1.12
\end{aligned}$$

(c)

$$\begin{aligned}
\text{Var}(X) &= E(X^2) - E(X)^2 \\
&= 47.2 - 6.7^2 \\
&= 2.31 \\
\text{Var}(Y) &= E(Y^2) - E(Y)^2 \\
&= 5.9 - 1.6^2 \\
&= 3.34
\end{aligned}$$

(d)

$$\begin{aligned}
\text{Corr}(X,Y) &= \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} \\
&= -.40
\end{aligned}$$

3.3.3

$$\begin{aligned}
\text{Corr}(X,Y) &= \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} \\
&= \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - E(X)^2}\sqrt{E(Y^2) - E(Y)^2}} \\
&= -0.18
\end{aligned}$$

Question X.

3.2.8

$Y \sim$ Exponential(9),$Z \sim$ Gamma(5,4)

$$\begin{aligned}
E(Y+Z) &= E(Y) + E(Z) \\
&= \frac{1}{9} + \frac{5}{4} = \frac{49}{36}
\end{aligned}$$

3.3.20 Also find $\text{Var}(Y+Z)$ assuming Y and Z are independent.

$X \sim \text{Gamma}(a, \lambda)$. We know the expectations

$$\begin{aligned}
E(X) &= \frac{a}{\lambda} \\
E(X^2) &= \frac{a(a+1)}{\lambda}
\end{aligned}$$

so

$$\begin{aligned}
\text{Var}(X) &= E(X^2) - E(X)^2 \\
&= \frac{a^2}{\lambda}
\end{aligned}$$

For the variance of the independent random variables Y and Z is

$$\begin{aligned}
\mathrm{Var}(Y+Z) &= \mathrm{Var}(Y) + \mathrm{Var}(Z) \\
&= \frac{1}{9^2} + \frac{5^2}{4} \\
&= \frac{2029}{324}
\end{aligned}$$

Question XI.

3.3.6

Random variables X, Y, Z, where X, Z are independent. Prove

$$\mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(Y, Z)$$

Commutators of linear combinations of random variables act such that

$$\mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$$

We know immediately that $\mathrm{Cov}(X, Z) = 0$ because of independence, so we have the desired

$$\mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(Y, Z)$$

Question XII.

Let X, Y, and Z be uncorrelated random variables with variances σ_X^2, σ_Y^2, and σ_Z^2, respectively. Let $U = X + Z$ and $V = Y + Z$. Find $\mathrm{Cov}(U, V)$ and $\mathrm{Corr}(U, V)$.

$$\begin{aligned}
\mathrm{Cov}(U, V) &= \mathrm{Cov}(X+Z, Y+Z) \\
&= \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z) + \mathrm{Cov}(Z, Y) + \mathrm{Cov}(Z, Z) \\
&= \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z) + \mathrm{Cov}(Z, Y) + \sigma_Z^2
\end{aligned}$$

Since the random variables are uncorrelated, each of the first three terms are zero, so

$$\begin{aligned}
\mathrm{Cov}(U, V) &= \mathrm{Cov}(X+Z, Y+Z) \\
&= \sigma_Z^2
\end{aligned}$$

For the correlation $\mathrm{Corr}(U, V)$, we have

$$\begin{aligned}
\mathrm{Corr}(U, V) &= \frac{\mathrm{Cov}(U, V)}{\sigma_U^2 \sigma_V^2} \\
&= \frac{\sigma_Z^2}{\sigma_U^2 \sigma_V^2}
\end{aligned}$$

Finding σ_U^2 and σ_V^2 in terms of known quantities,

$$\begin{aligned}
\mathrm{Var}(U) &= \mathrm{Var}(X) + \mathrm{Var}(Z) + 2\mathrm{Cov}(X, Z) \\
&= \mathrm{Var}(X) + \mathrm{Var}(Z) \\
\mathrm{Var}(V) &= \mathrm{Var}(Y) + \mathrm{Var}(Z) + 2\mathrm{Cov}(Y, Z) \\
&= \mathrm{Var}(Y) + \mathrm{Var}(Z)
\end{aligned}$$

So we have

$$\mathrm{Corr}(U, V) = \frac{\sigma_Z^2}{(\sigma_X^2 + \sigma_Z^2)(\sigma_Y^2 + \sigma_Z^2)}$$