5500 Problem Set 12

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Question I.

(a) We start with

$$(\Delta x)^2 = \int \mathrm{d}\omega\, \frac{\chi_{xx}(\omega)}{2\pi}$$
$$\chi_{xx} = \chi' + i\chi''$$
$$(\Delta x)^2 = \int \mathrm{d}\omega\, \frac{\chi' + i\chi''}{2\pi}$$

In the classical limit we have $\langle x^2 \rangle = kT\chi'(0)$

$$(\Delta x)^2 = \int \mathrm{d}\omega\, \frac{2\gamma kT}{\pi m[(\omega_0^2 - \omega^2)^2 - 2i\gamma\omega]}$$

(b)

$$(\Delta x)^2 = \int \mathrm{d}\omega\, \frac{2\gamma kT}{\pi m[(\omega_0^2 - \omega^2)^2 - 2i\gamma\omega]}$$

We can abuse to Kramers-Kronig relations

$$\chi'(\omega = 0) = \frac{1}{\pi} \int \mathrm{d}\omega\, \frac{\chi''}{\omega}$$

This trivializes the integral to give $kT/(m\omega_0)^2$, which is independent of the parameter γ.

Question II.

$$\chi(\omega = 0) = \frac{k}{2\gamma} \exp\left\{ -\frac{|x - x'|}{\kappa} \right\}$$
$$\kappa = \left(\frac{2\alpha}{\gamma}(T - T_c) \right)^{-1/2}$$

(a) We can start by looking at the fluctuation-dissipation theorem

$$\chi'' = \frac{\omega}{2kT} S$$

At $\omega = 0$

$$\chi''(\omega = 0) = \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left[\frac{k}{2\gamma}\exp\left\{-\frac{|x - x'|}{\kappa}\right\}\right]\Bigg|_{\omega=0}$$

We will also need

$$S(t = 0) = \int \frac{S(\omega)}{2\pi}$$

Doing this derivative and imposing the above condition along with Kramers-Kronig give

$$S(t = 0) = kT\chi(\omega = 0)$$

(b) We can solve this graphically by finding where the fluctuations are greater than T.

Question III.

(a) Using the Liouville von Neumann equation and equation 10.12 the density operator can be expressed by

$$\tilde{\rho}_{n+1}(t) = \tilde{\rho}_0 + \frac{i}{\hbar}\int_{-\infty}^{t} h(t')\left[A, \tilde{\rho}_n(t')\right]\mathrm{d}t'$$

where n is the order of the perturbation. To second order perturbation

$$\tilde{\rho}_3(t) = \tilde{\rho}_0 + \frac{i}{\hbar}\int_{-\infty}^{t} h(t')\left[A, \tilde{\rho}_2(t')\right]\mathrm{d}t'$$

(b) For a thermal density operator,

$$\rho = \frac{1}{Z}e^{\beta H}$$

We can also impose that the rate of change of the expectation value of the unperturbed Hamiltonian is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle H_0\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\{\rho H_0\}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{Z}\operatorname{Tr}\left\{e^{\beta H}H_0\right\}\end{aligned}$$

After a litany of calculus we will get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle H_0\rangle = \frac{\hbar^2\omega}{2}\chi(\omega)''$$

(c) χ'' is the dissipative part of the response function χ because it represents the energy dissipated from the system.