STAT 630 Problem Set 8

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Question I.

(a) We begin by finding the log-likelihood function:

$$\begin{aligned} L(\theta|x_i, ...x_n) &= \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!} \\ &= e^{-\theta n} \prod_{i=1}^{n} \frac{\theta^{x_i}}{x_i!} \\ &= e^{-\theta n} \frac{\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \\ l &= \log L \\ &= -n\theta + \left[\sum_{i=1}^{n} x_i\right] \log\theta - \left[\sum_{j=1}^{n} \log x_j!\right] \end{aligned}$$

Take a derivative with respect to θ:

$$\begin{aligned} \frac{\partial l}{\partial \theta} &= -n + \left[\sum_{i=1}^{n} x_i\right] \frac{1}{\theta} \\ \hat{\theta} &= \frac{1}{n}\left[\sum_{i=1}^{n} x_i\right] \end{aligned}$$

(b) We can factor the likelihood function into

$$L = e^{-\theta n} \theta^{\sum_{i=1}^{n} x_i} \left(\frac{1}{\prod_{i=1}^{n} x_i!}\right)$$

so $\sum_{i=1}^{n} x_i$ is sufficient for θ.

(c)

$$\text{Bias}_\theta(\hat{\theta}) = E_\theta(\hat{\theta}) - \theta$$

$\hat{\theta}$ goes like Poisson(n, θ), so

$$\text{Bias}_\theta(\hat{\theta}) = \frac{n\theta}{n} - \theta = 0$$

The variance is that of $1/n^2$ Var Poisson$(n\theta)$, which is θ/n. The MSE is then

$$\begin{aligned} \text{MSE}_\theta(T) &= \text{Var}_\theta(T) + [\text{Bias}_\theta(T)]^2 \\ &= \text{Var}_\theta(T) = \frac{\theta}{n} \end{aligned}$$

(d) Using the plug-in estimate, the MLE for θ^2 is $\hat{\theta}^2$. We find the bias:

$$\begin{aligned}
\mathrm{Var}(\hat{\theta}) &= E(\hat{\theta}^2) - E(\hat{\theta})^2 \\
E(\hat{\theta}^2) &= \theta^2 + \frac{\theta}{n} \\
\mathrm{Bias}_\theta(\hat{\theta}^2) &= \theta^2 + \frac{\theta}{n} - \theta^2 \\
&= \frac{\theta}{n}
\end{aligned}$$

(e) The log-likelihood function for $x_i = 0$ is $l = -n\theta$, which is maximized when $\theta = 0$.

Question II.

(a) Determine the MLE of a for $\mathrm{Beta}(a, 1)$. Finding the likelihood function:

$$\begin{aligned}
\prod_{i=1}^{n} a x_i^{a-1} = a^n \prod_{i=1}^{n} x_i^{a-1} \\
l = n \log a + (a-1) \sum_{i=1}^{n} \log x_i
\end{aligned}$$

maximizing:

$$\begin{aligned}
\frac{\partial l}{\partial a} &= \frac{n}{a} + \sum_{i=1}^{n} \log x_i \\
&= 0 \\
\hat{a} &= -\frac{n}{\sum_{i=1}^{n} \log x_i}
\end{aligned}$$

(b) A sufficient statistic would be $\prod_{i=1}^{n} x_i$, where $h(s) = 1$ and $g(T(s)) = a^n \left(\prod_{i=1}^{n} x_i\right)^{a-1}$.

(c) The variance of $\mathrm{Beta}(a, 1)$ is $1/(a(a-1))$. Using the plugin estimate, the MLE for the variance is $1/(\hat{a}(\hat{a}-1))$.

(d) $E(X_i) = a/(a+1)$, so the method of moments estimator is

$$\hat{a} = -\frac{E(X_i)}{E(X_i) - 1}$$

Question III.

(a) The likelihood function is

$$
\begin{aligned}
L &= \prod_{i=1}^{n} \frac{3w_i^2}{\beta^3} I_{[0,\beta]}(\omega_i) \\
&= \frac{3^n}{\beta^{3n}} \prod_{i=1}^{n} w_i^2 I_{[0,\beta]}(\omega_i)
\end{aligned}
$$

so the MLE is $\hat{\beta} = \max(\{w_i\})$

(b) The expectation is

$$
\begin{aligned}
E(w) &= \int_0^\beta w \frac{3w_i^2}{\beta^3} \, \mathrm{d}w \\
&= \frac{3\beta}{4}
\end{aligned}
$$

Finding the expectation

$$
\begin{aligned}
E(w) &= \int_0^\beta \frac{3w^3}{\beta^2} \, \mathrm{d}w \\
&= \frac{3\beta}{4}
\end{aligned}
$$

so

$$
\begin{aligned}
\hat{\beta} &= \frac{4}{3}\bar{w} \\
&= \bar{w} = \frac{1}{n}\sum_{i=1}^{n} w_i
\end{aligned}
$$

If we model this sum as the sum of n random variables of $X_i \sim$ Beta(3,1), modulo a factor of $B(3,1)$

$$
\begin{aligned}
\mathrm{Bias}_\beta(\tilde{\beta}) &= \frac{3}{\beta^3} nE(X_i) - \beta \\
&= \beta^{-3} - \beta \neq 0
\end{aligned}
$$

So $\tilde{\beta}$ is biased.

Question IV.

(a) Finding the MLE:

$$\begin{aligned}
L &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x_i-\mu)^2}{2\sigma^2}} \\
l &= -\frac{n}{2}(\log 2\pi + \log \sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \\
\frac{\partial l}{\partial \sigma^2} &= \frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2 \\
\hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2
\end{aligned}$$

The plug in estimate gives

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

(b) We know that the expectation $E(\hat{\sigma}^2)$

$$E(\hat{\sigma}^2) = \sigma^2$$

since this is the definition of variance.

The bias

$$\mathrm{Bias}_{\sigma^2}(\hat{\sigma}^2) = \sigma^2 - \sigma^2 = 0$$

The variance is

$$\begin{aligned}
\mathrm{Var}_{\sigma^2}(\hat{\sigma}^2) &= \mathrm{Var}(\frac{\sigma^2}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2) \\
&= 2\frac{\sigma^4}{n^2}
\end{aligned}$$

The MSE is

$$\begin{aligned}
\mathrm{MSE}_{\sigma^2}(\hat{\sigma}^2) &= \mathrm{Var}_{\sigma^2}(\hat{\sigma}^2) + \mathrm{Bias}^2_{\hat{\sigma}^2} \\
&= 2\frac{\sigma^4}{n^2}
\end{aligned}$$

(c) Our $\hat{\sigma}^2$ has a smaller MSE than the S^2 from the notes, which is reasonable because we know the true value of μ_0.

Question V.

The likelihood function is

$$\begin{aligned} L &= \prod_{i=1}^{n} \beta x_i^{\beta-1} e^{-\beta x_i} \\ l &= n \log \beta + \sum_{i=1}^{n} (\beta - 1) \log x_i - \sum_{i=1}^{n} x_i^{\beta} \end{aligned}$$

Take the derivative with respect to β

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} x_i^{\beta} \log x_i$$

Question VI.

(a) This follows a multinomial distribution

$$\frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

(b) The likelihood function (using 6.1.2 from the text) is

$$\begin{aligned} L &= \theta^{2x_1} (2\theta(1-\theta))^{x_2} (1-\theta)^{2x_3} \\ l &= 2x_1 \log \theta + x_2 \log(2\theta(1-\theta)) + 2x_3 \log(1-\theta) \\ &= 2x_1 \log \theta + x_2 \log 2\theta + x_2 \log(1-\theta) + 2x_3 \log(1-\theta) \\ \frac{\partial l}{\partial \theta} &= \frac{2x_1}{\theta} + \frac{x_2}{\theta} - \frac{x_2}{1-\theta} - \frac{2x_3}{1-\theta} \\ &= \frac{2x_1 + x_2}{\theta} - \frac{x_2 + 2x_3}{1-\theta} \end{aligned}$$

(c) The MLE takes the form

$$\begin{aligned} \frac{\partial l}{\partial \theta} &= 0 \\ \hat{\theta} &= \frac{2x_1 + x_2}{2x_2 + 2x_1 - 2x_3} \end{aligned}$$

Question VII.

$T \sim \text{Gamma}(n\lambda)$, so in the transformation $Y = \frac{1}{T}$ we have

$$f(y) = \frac{\lambda^n}{\Gamma(n)} \frac{1}{y}^{n-3} e^{-\lambda/y}$$

so the expectation

$$
\begin{aligned}
E\left(\frac{1}{T}\right) &= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} \frac{1}{y}^{n-3} e^{-\lambda/y} y \, \mathrm{d}y \\
&= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} \frac{1}{y}^{n-2} e^{-\lambda/y} \, \mathrm{d}y
\end{aligned}
$$

we can factor this to reduce the integrand to Gamma$(n-1, \lambda)$

$$
\begin{aligned}
E\left(\frac{1}{T}\right) &= \frac{\lambda^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\lambda^{n-1}} \int_0^\infty \frac{\lambda^{n-1}}{\Gamma(n-1)} \frac{1}{y}^{n-2} e^{-\lambda/y} \, \mathrm{d}y \\
\implies E\left(\frac{1}{T}\right) &= \frac{\lambda}{n-\lambda}
\end{aligned}
$$

For the expectation $E\left(\frac{1}{T^2}\right)$

$$
\begin{aligned}
E\left(\frac{1}{T^2}\right) &= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} \frac{1}{y}^{n-3} e^{-\lambda/y} y^2 \, \mathrm{d}y \\
&= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} \frac{1}{y}^{n-1} e^{-\lambda/y} \, \mathrm{d}y \\
&= 1
\end{aligned}
$$

since this is simply a gamma distribution. Next we find

$$
\begin{aligned}
E\left[\left(\frac{a}{T} - \lambda\right)^2\right] &= E\left[\frac{a}{T^2} - \frac{2a\lambda}{T} + \lambda^2\right] \\
&= a^2 - \frac{2a\lambda^2}{n-1} + \lambda^2
\end{aligned}
$$

Taking the derivative to maximize with respect to a

$$
\begin{aligned}
\frac{\partial}{\partial a} E\left[\left(\frac{a}{T} - \lambda\right)^2\right] &= 2a - \frac{2\lambda^2}{n-1} \\
&= 0 \\
a &= \frac{2\lambda^2}{n-1}
\end{aligned}
$$

which gives

$$
L_a = \frac{\lambda^2}{(n-1)\sum_{i=1^n x_i}}
$$