5201 Problem Set 6

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Question I.

August 2019 Classical Mechanics Prelim Problem

The orbit of a particle moving under the influence of a central force potential $V(r)$ is $r\theta = \text{constant}$. Assume that the motion is non-relativistic and that the potential is zero for infinitely large r.

(a) Using Newton's Laws of motion derive the equations of motion in polar coordinates.

For a particle of mass m under a central potential we arrive at Newton's second law for the $\hat{r}$ and $\hat{\theta}$ directions.

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= -\frac{\partial V}{\partial r} = -\frac{dV}{dr} \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= 0 \end{aligned}$$

Where the first equation is Newton's second law in the $\hat{r}$ direction and the second in the $\hat{\theta}$ direction. Since $V(r)$ is purely a function of r, we can write the partial in the first equation as a total derivative. Solving for the second derivatives:

$$\begin{aligned} \ddot{r} &= -\frac{1}{m}\frac{dV}{dr} + r\dot{\theta}^2 \\ \ddot{\theta} &= -\frac{2}{r}\dot{r}\dot{\theta} \end{aligned}$$

(b) Prove that the angular momentum and the total energy are conserved.

The angular momentum about the origin O:

$$\begin{aligned} \vec{L} &= m\vec{r} \times \vec{v} \\ &= (mr^2\dot{\theta})\hat{z} \\ &= l\hat{z} \\ \frac{dL}{dt} &= \frac{dl}{dt} = \frac{d}{dt}(mr^2\dot{\theta}) \\ &= mr(2\dot{r}\dot{\theta} + r\ddot{\theta}) \\ &= 0 \end{aligned}$$

From the result in part a, we see that angular momentum is conserved. For the total energy:

$$
\begin{aligned}
E &= T + V \\
&= \frac{1}{2}m(\dot{r} + (r\dot{\theta})^2) + V(r) \\
\frac{dE}{dt} &= m\dot{r}\ddot{r} + mr\dot{r}\dot{\theta}^2 + mr^2\dot{\theta}\ddot{\theta} + \frac{\partial V}{\partial t} \\
\frac{\partial V}{\partial t} &= \dot{r}\frac{\partial V}{\partial r} \\
\frac{dE}{dt} &= m\dot{r}\ddot{r} + mr\dot{r}\dot{\theta}^2 + mr^2\dot{\theta}\ddot{\theta} - m\dot{r}(\ddot{r} - r\dot{\theta}^2) \\
&= mr\dot{r}\dot{\theta}^2 + mr^2\dot{\theta}\ddot{\theta} + mr\dot{r}\dot{\theta}^2 \\
&= 2mr\dot{r}\dot{\theta}^2 + mr^2\dot{\theta}\ddot{\theta} \\
\ddot{\theta} &= \frac{2\dot{r}\dot{\theta}}{r} \\
\frac{dE}{dt} &= 2mr\dot{r}\dot{\theta}^2 + mr^2\dot{\theta}\ddot{\theta} \\
&= 2mr\dot{r}\dot{\theta}^2 + mr^2\dot{\theta}\frac{2\dot{r}\dot{\theta}}{r} \\
&= 2mr\dot{r}\dot{\theta}^2 - 2mr\dot{r}\dot{\theta}^2 \\
&= 0
\end{aligned}
$$

(c) Determine the most general form for the potential $V(r)$ that can produce this orbit subject to its approaching zero at large r.

With the substitution $u = \frac{1}{r}$, the equation of the orbit of the particle becomes:

$$
\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= \frac{m}{u^2l^2}F \\
r\theta &= k \\
u &= \theta k
\end{aligned}
$$

If we let $k \to \frac{1}{k}$ since k is some arbitrary constant:

$$
\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= \frac{m}{u^2 l^2} F \\
\frac{d^2u}{d\theta^2} &= 0 \\
\frac{1}{r} &= \frac{mr^2}{l^2} F \\
F &= \frac{l^2}{mr^3} \\
V(r) &= \int_{\infty}^{r} F dr \\
&= \frac{l^2}{2mr^2}
\end{aligned}
$$

Question II.

August 2019 Classical Mechanics Prelim Problem

A one dimensional anharmonic oscillator consists of a mass m in a potential given by

$$U(x) = \frac{1}{2}kx^2 - \frac{1}{3}m\lambda x^3 + Ex\cos(\omega t)$$

where λ is small, i.e. the second term in the potential is small compared to the first term in the potential. All of k, λ and E are constants. Assume initial conditions $x = a$ at $t = 0$ and $\frac{dx}{dt} = 0$ at $t = 0$.

(a) Solve the equation of motion for $\lambda = 0$ and $E = 0$.

Finding the Hamiltonian, assume $T = \frac{1}{2}m\dot{x}^2$ and our canonical coordinates are $q = x$ and $p = m\dot{x}$:

$$
\begin{aligned}
H &= T + U = \frac{p^2}{2m} + \frac{1}{2}kq^2 \\
\dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\
\dot{p} &= -\frac{\partial H}{\partial q} = -kq \\
\ddot{q} &= -\frac{k}{m}q \\
q &= ce^{-i\omega t}
\end{aligned}
$$

Where $\omega = \sqrt{\frac{k}{m}}$.

Solving for the constants with the initial conditions given:

$$x = a\cos\omega t$$

(b) Solve the equation of motion to first order in λ for $E = 0$ and the above initial conditions. Hint: use a first order solution of the form $x(\lambda) = x(\lambda = 0) + \lambda x_1$ where $x(\lambda = 0)$ is the solution from part (a) and substitute this in the equation of motion and solve for x_1. Note that in the final solution the equilibrium value of x_1 is not zero.

If we use the solution of the form $q(\lambda) = q_0 + \lambda q_1$ where q_0 is our solution from part A we can implement this:

$$\begin{aligned} x &= a\cos\omega t + \lambda x_1 \\ \dot{x} &= -\omega\sin\omega t + \lambda\dot{x}_1 \\ \ddot{x} &= -\omega^2 a\cos\omega t + \lambda\ddot{x}_1 \end{aligned}$$

Now:

$$\begin{aligned} -\omega^2 a\cos\omega t + \lambda\ddot{x}_1 + \omega^2(a\cos\omega t + \lambda x_1) &= \lambda(a\cos\omega t + \lambda x_1)^2 \\ \lambda\ddot{x}_1 + \omega^2\lambda x_1 &= \lambda(a^2\cos^2\omega t + 2a\lambda x_1\cos\omega t + \lambda^2 x_1^2) \\ \ddot{x}_1 + \omega^2 x_1 &= a^2\cos^2\omega t + 2a\lambda x_1\cos\omega t + \lambda^2 x_1^2 \end{aligned}$$

To first order in λ:

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 &= a^2\cos^2\omega t + 2a\lambda x_1\cos\omega t \\ \ddot{x}_1 &= a^2\cos^2\omega t - (\omega^2 - 2a\lambda\cos\omega t)x_1 \end{aligned}$$

Which can be solved for x_1.

(c) Solve the equation of motion for $\lambda = 0$, when E is not zero.

Taking the derivative of the potential yields:

$$\begin{aligned} m\ddot{x} &= -kx - E\cos\omega t \\ \ddot{x} &= -\frac{k}{m}x - \frac{E}{m}\cos\omega t \\ \ddot{x} + \omega_0^2 x &= -\frac{E}{m}\omega t \end{aligned}$$

Where $\omega_0^2 = \frac{k}{m}$. We can try the following solution, where D is some currently unknown constant:

$$\begin{aligned}
x &= a\cos\omega_0 t + D\cos\omega t \\
-\omega^2 D\cos\omega t(\omega_0^2 - \omega^2) &= \frac{E}{m}\cos\omega t \\
D &= \frac{E}{m(\omega^2 - \omega_0^2)} \\
x &= a\cos\omega_0 t + \frac{E}{m(\omega^2 - \omega_0^2)}\cos\omega t
\end{aligned}$$

Question III.

January 2018 Classical Mechanics Prelim Problem

Consider a uniform, rigid rod of mass m that is placed with one end against a vertical wall and the other (lower) end on a horizontal floor. Both the wall and the floor are frictionless and the rod is free to move in a vertical plane containing it, perpendicular to the wall. The initial angle of inclination of the rod to the horizontal is α.

(a) Write down an equation or equations of motion when it is released from rest, while the rod is in contact with both the wall and the floor. Defining the origin of our coordinates to be at the corner of the wall and the floor, where L is half the length of the rod and θ is the generalized inclination of the rod:

$$\begin{aligned}
x &= 2L\cos\theta - L\cos\theta = L\cos\theta \\
y &= L\sin\theta \\
\dot{x} &= -L\sin\theta\dot{\theta} \\
\dot{y} &= L\cos\theta\dot{\theta} \\
\mathcal{L} &= T - V \\
T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 \\
&= \frac{m}{2}(L^2\dot{\theta}^2\sin^2\theta + L^2\dot{\theta}^2\cos^2\theta) + \frac{1}{6}mL^2\dot{\theta} \\
&= \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{6}mL^2\dot{\theta}^2 \\
&= \frac{2}{3}mL^2\dot{\theta}^2 \\
V &= mgL\sin\theta \\
\mathcal{L} &= \frac{2}{3}mL^2\dot{\theta}^2 - mgL\sin\theta
\end{aligned}$$

Solving this Lagrangian for the equations of motion:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \\
-mgL\cos\theta &= \frac{4}{3}mL^2\ddot{\theta} \\
\ddot{\theta} &= -\frac{3g}{4L}\cos\theta
\end{aligned}
$$

(b) Find the angle (to the horizontal) at which the rod loses contact with the vertical wall.

The constraint here is that $m\ddot{x} = 0$, so

$$
\begin{aligned}
\ddot{x} &= 0 \\
&= -L\cos\theta\dot{\theta}^2 - L\sin\theta\ddot{\theta}
\end{aligned}
$$

We can use the result from part a and the result from conservation of energy here to solve this for θ_c the angle at which the rod loses contact with the wall. I will be using this notation to also denote ω_c as $\dot{\theta}$ evaluated at the angle where the rod loses contact with the wall. We can use conservation of energy here since the only forces acting on the system are the gravitational force and normal forces, which are both conservative.

$$
\begin{aligned}
mgL\sin\alpha &= mgL\sin\theta_c + \frac{2}{3}mL^2\omega_c^2 \\
\omega_c^2 &= \frac{3g}{2L}(sin\alpha - \sin\theta_c) \\
\ddot{\theta} &= -\frac{3g}{4L}\cos\theta \\
\ddot{x} &\rightarrow -\frac{3g}{2L}\sin\alpha\cos\theta_c + \frac{3g}{2L}\sin\theta_c\cos\theta_c + \frac{3g}{4L}\sin\theta_c\cos\theta_c = 0 \\
&\Rightarrow \frac{3}{4}\cos\theta_c(3\sin\theta_c - 2\sin\alpha) = 0 \\
&\Rightarrow \theta_c = \arcsin\left(\frac{2}{3}\sin\alpha\right)
\end{aligned}
$$

(c) Write down an equation or equations of motion after the rod has lost contact with the wall and deduce the horizontal velocity component of the center of mass of the rod when it is about to hit the floor.

A note on the notation here: I will be using the subscript c to denote a coordinate at the separation of the rod from the wall, as I did with θ_c and ω_c in part b. I will also be using the coordinate θ to be the angle from the horizontal after the rod loses contact. Beginning with conservation of energy in a similar way as before:

$$
\begin{aligned}
mgL\sin\theta_c + \frac{2}{3}mL^2\omega_c^2 &= mgl\sin\theta + \frac{1}{2}m\dot{x}_c^2 + \frac{1}{2}m\dot{y}_c^2 + \frac{1}{12}(2L)^2\dot{\theta}^2 \\
mgL(\sin\theta_c - \sin\theta) + \frac{1}{2}m\dot{y}_c^2 + \frac{1}{3}L^2\omega_c^2 &= \frac{1}{2}m\dot{y}^2 + \frac{1}{3}L^2\dot{\theta}^2 \\
mgL(\sin\theta_c - \sin\theta) + \frac{1}{2}mL^2\cos^2\theta_c\omega_c^2 + \frac{1}{3}L^2\omega_c^2 &= \frac{1}{2}mL^2\dot{\theta}^2\cos^2\theta + \frac{1}{3}L^2\dot{\theta}^2
\end{aligned}
$$

Solving this for $\dot{\theta}$, gives us an equation of motion for the rod after losing contact with the wall:

$$
\dot{\theta}^2 = \frac{1}{3\cos^2\theta + 2}\left[\left(\frac{6g}{L}\sin\theta_c + 3\omega_c^2\cos^2\theta_c + 2\right) - \frac{6g}{L}\sin\theta\right]
$$

We showed that after the rod loses contact with the wall, the horizontal component of velocity stays constant, given by $\dot{x} = -L\sin\theta_c\omega_c$.

Question IV.

January 2018 Classical Mechanics Prelim Problem

A uniform rigid rod of length $2a$ and mass m hangs in a horizontal position being supported by two vertical strings, each of length l attached to its ends. The other ends of the strings are attached to a ceiling so that the rod and the strings are all in a vertical plane. The rod is given an angular velocity ω about the vertical axis through its center. Assume that the strings are massless and remain tight during the rotation.

(a) Write down the kinetic and potential energy of the system at a twist angle θ through which the rod has turned.

On the notation of this problem: θ will be used to denote the twist angle of the rod, meaning the angle of rotation about its center, and ϕ will be used to denote the angle from the vertical made by one of the strings. This yields a constraint from which we can get ϕ in terms of θ such that we only have this variable to use.

$$
\begin{aligned}
a\sin\theta &= l\sin\phi \\
\phi &= \arcsin\left(\frac{a}{l}\sin\theta\right)
\end{aligned}
$$

The initial energy of the system is given by the following expression, where we define the zero of potential energy to be at the top of the strings (the ceiling):

$$
\begin{aligned}
V_i &= -mgl \\
T_i &= \frac{1}{6}ma^2\omega^2
\end{aligned}
$$

We can define a new coordinate $z = -l\cos\phi$ which denotes the height of the rod at any ϕ. From this we can write the energy at any angle:

$$\begin{aligned} V &= -mgl\cos\phi \\ T &= \frac{1}{6}ma^2\dot{\theta}^2 + \frac{1}{2}l^2\sin^2\phi\dot{\phi}^2 \end{aligned}$$

Rewriting this in terms of θ:

$$\begin{aligned} E &= \frac{1}{6}ma^2\dot{\theta}^2 + \frac{1}{2}l^2\sin^2\phi\dot{\phi}^2 - mgl\cos\phi \\ \phi &= \arcsin\left(\frac{a}{l}\sin\theta\right) \\ \dot{\phi} &= \frac{\frac{a}{l}}{\sqrt{1-(\frac{a}{l}\sin\theta)^2}}\cos\theta\dot{\theta} \\ E &= \frac{1}{6}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\frac{\frac{a^2}{l^2}\cos^2\theta\dot{\theta}^2}{1-\frac{a^2}{l^2}\sin^2\theta} - mgl\cos\left[\arcsin\left(\frac{a}{l}\sin\theta\right)\right] \end{aligned}$$

This rather ugly expression gives the total energy of the system at any twist angle θ.

(b) Find its angular velocity $\dot{\theta}$ when it has turned through any twist angle θ.

We can use conservation of energy here since there are no nonconservative forces acting on the system:

$$\begin{aligned} \frac{1}{6}ma^2\omega^2 - mgl &= \frac{1}{6}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\frac{\frac{a^2}{l^2}\cos^2\theta\dot{\theta}^2}{1-\frac{a^2}{l^2}\sin^2\theta} - mgl\cos\left[\arcsin\left(\frac{a}{l}\sin\theta\right)\right] \\ \frac{1}{6}ma^2\omega^2 + mgl\left(\cos\left[\arcsin\left(\frac{a}{l}\sin\theta\right)\right]-1\right) &= \frac{1}{6}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\frac{\frac{a^2}{l^2}\cos^2\theta\dot{\theta}^2}{1-\frac{a^2}{l^2}\sin^2\theta} \\ \frac{2gl\left(\cos\left[\arcsin\left(\frac{a}{l}\sin\theta\right)\right]-1\right)}{a^2} + \frac{\omega^2}{3} &= \dot{\theta}^2\left[\frac{1}{3} + \frac{\sin^2\theta\cos^2\theta}{\frac{l^2}{a^2}-\sin^2\theta}\right] \\ \dot{\theta} &= \sqrt{\frac{\frac{2gl\left(\cos\left[\arcsin\left(\frac{a}{l}\sin\theta\right)\right]-1\right)}{a^2} + \frac{\omega^2}{3}}{\left[\frac{1}{3} + \frac{\sin^2\theta\cos^2\theta}{\frac{l^2}{a^2}-\sin^2\theta}\right]}} \end{aligned}$$

(c) Show that its center will rise through a distance $a^2\omega^2/6g$ before coming to instantaneous rest.

We can use the same conservation of energy to show this, setting $\dot{\theta} = 0$ at the turning point:

$$-mgl + \frac{1}{6}ma^2\omega^2 = mgl\cos\left[\arcsin\left(\frac{a}{l}\sin\theta\right)\right]$$

If we introduce the notation $d = l\cos\left[\arcsin\left(\frac{a}{l}\sin\theta\right)\right]$, which denotes the distance of the rod from the ceiling, we solve this for d.

$$mgd = -mgl + \frac{1}{6}ma^2\omega^2$$
$$d = l - \frac{a^2\omega^2}{6g}$$

And this result yields the given answer taking into account our origin, so the rod rises $\frac{a^2\omega^2}{6g}$ before coming to instantaneous rest.

(d) Find an upper bound on ω for such a classical turning point to exist.

To do this, we set the rise of the rod found in the previous part equal to the length of the strings:

$$l = \frac{a^2\omega_{max}^2}{6g}$$
$$\omega_{max} = \sqrt{\frac{6gl}{a^2}}$$

Question V.

January 2017 Classical Mechanics Prelim Problem

A uniform density billiard ball of radius R and mass M is struck with a horizontal cue stick at a height h above a horizontal billiard table. The moment of inertia I of a uniform density sphere is $I = 2MR^2/5$ where M is the mass of the sphere and R is its radius. The following three points all lie in a common vertical plane: the center of the ball, the point of contact between the ball and the cue stick, and the point of contact between the ball and the table. Take the gravity of the earth to produce a uniform acceleration g acting in the downward vertical.

(a) Find the value of h for which the ball will roll without slipping no matter how small the coefficient of friction between the ball and the table might be.

Consider the force applied by the cue with magnitude F. If we take the linear and rotational impulse on the cue ball:

$$
\begin{aligned}
M\Delta v &= F\Delta t \\
I\Delta\omega &= F(h-R)\Delta t \\
F &= \frac{I\Delta\omega}{(h-R)\Delta t} \\
M\Delta v &= \frac{I\Delta\omega}{(h-R)\Delta t}\Delta t \\
&= \frac{2MR^2\Delta\omega}{5(h-R)} \\
\Delta v &= \frac{2R^2\Delta\omega}{5(h-R)}
\end{aligned}
$$

And taking the initial state of the ball to be rest, we can write:

$$
v = \frac{2R^2\omega}{5(h-R)}
$$

Now we solve for the height knowing that the rolling condition implies $v = R\omega$:

$$
\begin{aligned}
R\omega &= \frac{2R^2\omega}{5(h-R)} \\
h-R &= \frac{2R}{5} \\
h &= \frac{7}{5}R
\end{aligned}
$$

(b) Consider the case where the ball is rolling with a velocity $\mathbf{v}$ on a level horizontal surface and collides with a step of height $d < R$ as shown in the figure. Take the inelastic limit of the collision. Assume that the coefficient of static friction is sufficient to prevent slipping throughout the process. For a given d and R determine the minimum velocity needed for which the ball will be able to jump over the step.

Let us consider the quantities L and ω to be the magnitudes of the angular momentum and angular velocity before the collision, and L' and ω' to be the magnitudes of the angular momentum and angular velocity after the collision.Taking the angular momenta about the protruding corner of the step:

$$
\begin{aligned}
L &= mv(R-d) + \frac{2}{5}MR^2\omega \\
&= \frac{7}{5}MvR^2 - Mvd \\
L' &= \frac{2}{5}MR^2\omega' + MR^2\omega' \\
&= \frac{7}{5}MR^2\omega'
\end{aligned}
$$

To solve for ω' we conserve the angular momentum about the corner of the step. We can do this since there are no external net torques on the system:

$$\begin{aligned}
\frac{7}{5}MvR^2 - Mvd &= \frac{7}{5}MR^2\omega' \\
\omega' &= \frac{5}{7R}(\frac{7}{5}vR - vd) \\
&= \frac{v}{R}(1 - \frac{5d}{7R})
\end{aligned}$$

Now we assume that since the kinetic energy of the ball is just sufficient to make it over the step:

$$\begin{aligned}
\frac{1}{2}(\frac{2}{5}MR^2 + MR^2)\omega'^2 &= Mgd \\
\frac{1}{2}(\frac{7}{5}MR^2)\omega'^2 &= Mgd \\
\frac{1}{2}\frac{7}{5}MR^2\frac{v^2}{R^2}(1 - \frac{5d}{R})^2 &= Mgd \\
\sqrt{\frac{10gd}{7(1-\frac{5d}{7R})^2}} &= v
\end{aligned}$$

Question VI.

August 2018 Classical Mechanics Prelim Problem

Two cylinders of mass m each, with radius r and height h are connected by a massless bar as in the figure. the centers of the cylinders are separated by L.

(a) What are the principal axes of this setup through the center of mass of the dumbbell? (You can describe them or make a simple drawing.) Calculate the moment of inertia tensor through the center of mass. (On the way, show that the moment of inertia through the center of a cylinder around an axis perpendicular to the symmetry axis is $m(3r^2 + h^2)/12$.)

The principle axes of rotation are: along the center of the massless bar, which we will call the z axis, then along two axes orthogonal to the z axis, and the x and y axes perpendicular to the z axis with the y parallel to the page pointing up and x into the page. Calculating the inertia tensor, we have to find the moments of inertia for each of the cylinders individually about their centers of mass then use the Parallel Axis Theorem to move the axes of rotation to the center of the massless bar connecting them. For one of the cylinders, where $\vec{a}$ is the vector from the center of mass of the cylinder to the axis of rotation, in this case the center of the massless bar:

$$\begin{aligned}
I'_{ij} &= I_{ij} + m(\delta_{ij}a^2 - a_i a_j) \\
I_{ii} &= \int \lambda(x_j^2 + x_k^2)dV \\
I_{33} &= \int \lambda(x_1^2 + x_2^2)dV
\end{aligned}$$

Where I_{ij} is the center of mass inertia tensor and I'_{ij} is the inertia tensor about the transformed axis of rotation, λ is the mass density of the cylinder $\frac{M}{\pi r^2 h}$. In cylindrical coordinates:

$$\begin{aligned}
I_{33} &= \frac{M}{\pi r^2 h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi}\int_0^r \rho^2 \rho d\rho d\theta dz \\
I_{33} &= \frac{1}{2}Mr^2 \\
I_{22} &= \int \lambda(x_1^2 + x_3^2)dV \\
I_{22} &= \frac{M}{\pi r^2 h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi}\int_0^r (z^2 + \rho^2\cos^2\theta)d\rho d\theta dz \\
\cos^2\theta &= \frac{1}{2} + \frac{1}{2}\cos(2\theta) \\
I_{22} &= \frac{M}{\pi r^2 h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{r}\int_0^{2\pi} (z^2 + \rho^2\cos^2\theta)d\theta d\rho dz \\
I_{22} &= \frac{M}{\pi r^2 h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{r}(2\pi z^2\rho + \pi\rho^3 - \frac{\rho^3}{4}\sin 2\theta|_0^{2\pi})d\rho dz \\
I_{22} &= \frac{M}{\pi r^2 h}\int_{-\frac{h}{2}}^{\frac{h}{2}}(2\pi z^2 r^2 + \frac{\pi}{4}r^3)dz \\
I_{22} &= \frac{1}{4}Mr^2 + \frac{1}{12}Mh^2
\end{aligned}$$

By symmetry, $I_{22} = I_{11}$. We now will take the system of both of the cylinders after the Parallel Axis theorem:

$$\begin{aligned}
I'_{ij} &= I_{ij} + m(\delta_{ij}a^2 - a_i a_j) \\
a_1 &= a_2 = 0 \\
a_3 &= \frac{L}{2} \\
I_{33} &= mr^2 \\
I_{11} &= \frac{1}{2}mr^2 + \frac{1}{6}mh^2 + \frac{1}{2}mL^2 \\
I_{22} &= I_{11}
\end{aligned}$$

(b) What is the moment of inertia through the point marked with an "x" on the figure (assume the mark is on the upper outer edge of one of the cylinders), around the horizontal axis that marks the tangent to the cylinder rim?

We do Parallel Axis Theorem again to find:

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= r \\
a_3 &= \frac{L}{2} + \frac{h}{2} \\
I_{11} &= \frac{1}{2}mr^2 + \frac{1}{2}mh^2 + \frac{1}{2}mL^2 + m\left[r^2 + \left(\frac{1}{2}L + \frac{1}{2}h\right)^2\right] \\
&= \frac{3}{2}mr^2 + \frac{5}{12}mh^2 + \frac{3}{4}mL^2 + \frac{1}{2}MLh
\end{aligned}
$$

This is the principal axis of the inertia tensor that we need to solve the next part.

(c) If one would keep the point marked with an "x" fixed (for example, by holding on to it at this point), but would let the rest of the dumbbell fall, what is the kinetic energy of the system when the face of the cylinder is horizontal? (That is, the dumbbell bar is horizontal at the start of the movement and vertical at the end.) What is the angular momentum of around its movement axis at that time (this is the same axis as in part (b))? (Assume $L + h > 2r$.)

Conservation of energy, since there are no nonconservative forces at play:

$$
\begin{aligned}
E_i &= 0 \\
E_f &= -mg\left(\frac{h}{2} - r\right) - mg\left(L + \frac{h}{2} - r\right) + mI_{11}\omega^2 \\
I_{11}\omega^2 &= -g\left(\frac{h}{2} - r\right) - g\left(L + \frac{h}{2} - r\right) \\
\omega &\rightarrow \sqrt{\frac{g\left(L + \frac{h}{2} - r\right)}{I_{11}}} \\
L &= I_{11}\omega \\
&= \sqrt{I_{11}g(L + h - 2r)}
\end{aligned}
$$

(d) If one replaces the round cylinders with elliptical ones (where the long axes of the ellipse of both cylinders are parallel to each other), can you draw or describe the principal axes through the center of mass in this case? Around which of the axes is the moment of inertia largest, around which is it smallest? (Assume that both ellipse radii are considerably smaller than the distance between the cylinders.) Without proof or derivation, rotation around which of those axes is stable and which unstable?

The principle axes will be defined as: z is the same as before, along the massless bar axis, x and y along the semimajor and semiminor axes of the ellipses. The moment of inertia about the y axis in this case will be the largest since rotation along y will not have the semimajor axis coplanar to the rotation. Rotations about the z and y axes will be stable and rotations about x will be unstable due to the Intermediate Axis Theorem.

Reminder: The moment of inertia tensor for N particles can be described as

$$I_{\alpha\beta} = \sum_i^N m_i(|\vec{r}_i|^2\delta_{\alpha\beta} - r_{i\alpha}r_{i\beta})$$