5500 Problem Set 11

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Question I.

The self consistency condition is given by

$$\begin{aligned}\langle s\rangle &= \tanh\left(\beta(H+\gamma\epsilon\langle s\rangle)\right)\\ &= \tanh\left(\frac{H}{kT}+\langle s\rangle\frac{T_c}{T}\right)\end{aligned}$$

Taking the derivative

$$\begin{aligned}\frac{\partial\langle s\rangle}{\partial H} &= \chi\\ &= (1-\langle s\rangle^2)\left(\chi\frac{T_c}{T}=\frac{1}{kT}\right)\\ \chi &= \frac{1-\langle s\rangle^2}{kT(1-(1-\langle s\rangle^2))\frac{T_c}{T}}\end{aligned}$$

For $T > T_c$ we have $\langle s\rangle = 0$ which gives

$$\chi = \frac{1}{kT(T-Tc)}$$

For $T \leq T_c$ we have $\langle s\rangle^2 = 3(T_c-T)/T \equiv 3t$ which gives

$$\begin{aligned}\chi &= \frac{1-3t}{kT(1-(1-3t)(1+t))}\\ &\approx \frac{1}{2kTt}\\ &= \frac{1}{2kT(T-T_c)}\end{aligned}$$

Question II.

(a) We start with Gibbs as a function of the partition function

$$\begin{aligned}G &= -kT\ln Z\\ Z &= \sum_{s=\pm1} e^{-\beta h(s)}\\ h(s) &= -(\gamma\epsilon L+H)s+\frac{1}{2}\gamma\epsilon L^2\\ Z &= e^{-\frac{\beta}{2}N\gamma\epsilon L^2}\left(e^{-\beta(\gamma\epsilon L+H)}+e^{\beta(\gamma\epsilon L+H)}\right)\end{aligned}$$

This gives the desired Gibbs

$$
\begin{aligned}
G &= -kT \ln Z \\
&= -kTN \ln\left(2\cosh[\beta(\gamma\epsilon L + H)]\right) + \frac{1}{2}\gamma\epsilon N L^2
\end{aligned}
$$

(b) Finding the extremum we take the derivative of the Gibbs

$$
\begin{aligned}
\frac{\partial G}{\partial L} &= -kTN \frac{1}{2\cosh[\beta(\gamma\epsilon L + H)]} 2\sinh\left[2\cosh[\beta(\gamma\epsilon L + H)]\right]\beta\gamma\epsilon + \gamma\epsilon N L \\
&= -\gamma\epsilon N(\tanh[2\cosh[\beta(\gamma\epsilon L + H)]] - L) \\
&= 0
\end{aligned}
$$

which is the condition we expected.

(c) With $L = 0$ we have

$$
\begin{aligned}
\frac{\partial G}{\partial L} &= -\gamma\epsilon N(\tanh[2\cosh[\beta(\gamma\epsilon L + H)]] - L) \\
\frac{\partial^2 G}{\partial L^2} &= \gamma\epsilon N\left(1 - \frac{T_c}{T}(1 - \tanh^2[\beta(\gamma\epsilon L + H)])\right)
\end{aligned}
$$

For $L = 0$ we have

$$
\frac{\partial^2 G}{\partial L^2} = \gamma\epsilon N\left(1 - \frac{T_c}{T}\right)
$$

is less than zero, so the Gibbs free energy has a maximum at $L = 0$. Since G is infinite in the infinite L limits, the other extrema must be minima.

(d) If we take $L = \pm S + \delta S$ where $S > 0$ we have

$$
\begin{aligned}
\pm S + \delta S &= \tanh\left(\beta H + \frac{T_c}{T}(\pm S + \delta S)\right) \\
&\approx \pm\tanh\frac{T_c}{T}S + \left(\beta H + \frac{T_c}{T}\delta S\right)\left(1 - \tanh^2\frac{T_c}{T}\delta S\right) \\
\delta S &= \beta H \frac{1 - S^2}{1 - \frac{T_c}{T}(1 - S^2)}
\end{aligned}
$$

δS has the same sign as the magnetic field.

(e)

Question III.

(a) Taking the derivative of g with respect to ϕ gives

$$\begin{aligned}\frac{\partial g}{\partial \phi} &= 0 \\ &= 2\alpha_2(T - T_c)\phi + 4\alpha_4\phi^3 \\ 0 &= \alpha_2(T - T_c) + 2\alpha_4\phi^2\end{aligned}$$

and the desired result follows immediately. $\phi = 0$ is also a solution but is a maximum.

(b) We can ignore the ϕ^2 term, so the equation we need to solve is

$$\left[-\gamma\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\alpha_2(T - T_c)\right]\phi(x) = h_0\delta(x)$$

If we solve this using a Fourier transform we have

$$\begin{aligned}\phi(k) &= \frac{h_0}{\gamma k^2 + 2\alpha_2(T - T_c)} \\ \phi(x) &= \frac{h_0}{2\sqrt{2\gamma\alpha_2(T - T_c)}}e^{\sqrt{2\frac{\alpha_2}{\gamma}(T-T_c)}|x|}\end{aligned}$$

where the correlation length ξ is given by

$$\xi = \frac{1}{\sqrt{2\frac{\alpha_2}{\gamma}(T - T_c)}}$$