5401 Problem Set 2

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Question I.

Shankar, Ex. 1.8.2

Consider the matrix:

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

1.1) Is it Hermitian?

Ω is hermitian by inspection:

$$\Omega^\dagger = \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T \right)^* = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \Omega$$

1.2) Find its eigenvalues and eigenvectors.

Finding the characteristic for Ω and solving for the eigenvalues:

$$\begin{aligned} \Omega - \lambda I &= \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} \\ |\Omega - \lambda I| &= \lambda(-\lambda^2 + 1) \\ \lambda(-\lambda^2 + 1) &= 0 \\ \lambda_1 &= 0 \\ \lambda_2 &= 1 \\ \lambda_3 &= -1 \end{aligned}$$

Solving $(\Omega - \lambda_i I) |\lambda_i\rangle = |0\rangle$ to find the eigenvectors:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} |\lambda_1\rangle = |0\rangle$$

$$|\lambda_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} |\lambda_2\rangle = |0\rangle$$

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} |\lambda_3\rangle = |0\rangle$$

$$|\lambda_3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

1.3) Verify that $U^\dagger \Omega U$ is diagonal, U being the matrix of eigenvectors of Ω.

We construct U as:

$$U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U^\dagger = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Carrying out the multiplication:

$$
\begin{aligned}
U^\dagger \Omega U &= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\end{aligned}
$$

So this unitary transformation diagonalizes Ω.

Question II.

Shankar, Ex. 1.9.2

If H is a Hermitian operator, show that $U = e^{iH}$ is unitary. (Notice the analogy with c numbers: if θ is real, $u = e^{i\theta}$ is a number of unit modulus.)

We can express U in the following way:

$$
\begin{aligned}
U &= e^{iH} \\
&= \sum_{m=0}^{\infty} \frac{(iH)^m}{m!} \\
U^\dagger &= (e^{iH})^\dagger \\
&= \sum_{m=0}^{\infty} \frac{[(iH)^m]^\dagger}{m!} \\
&= \sum_{m=0}^{\infty} \frac{(-iH^\dagger)^m}{m!} \\
&= \sum_{m=0}^{\infty} \frac{(-iH)^m}{m!} \\
&= e^{-iH} \\
U^\dagger U &= e^{-iH} e^{iH} \\
&= 1
\end{aligned}
$$

Thus U is unitary.

Question III.

Express $e^{\lambda A} B e^{-\lambda A}$ in terms of the scalar λ and the commutators $[A, B]$, $[A,[A,B]]$ etc. (Here A and B are operators.)

If we expand the exponentials into their respective power series:

$$\begin{aligned}
e^{\lambda A} B e^{-\lambda A} &= (1 + \lambda A + \frac{1}{2}(\lambda A)^2 + ...)B(1 + (-\lambda A) + \frac{1}{2}(-\lambda A)^2 + ...) \\
&= B + \lambda AB + \frac{1}{2}(\lambda A)^2 B + ... + (-\lambda BA) + \lambda A(-\lambda)BA + \frac{1}{2}(\lambda A)^2(-\lambda)BA + ... \\
&= B + \lambda AB - \lambda BA + \frac{\lambda^2}{2} AAB - \lambda^2 ABA + ... \\
&= B + \lambda[A, B] + \frac{\lambda^2}{2}[A,[A,B]] + ...
\end{aligned}$$

Question IV.

If A is a Hermitian operator show that

$$\det\left(e^A\right) = e^{\mathrm{Tr}(A)}$$

If A is hermitian, we know that in its eigenbasis we can write the following, where a_i are the eigenvalues of A:

$$A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$$
$$e^A = \begin{bmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n} \end{bmatrix}$$

Taking the determinant it follows:

$$\begin{aligned}\det\left(e^{A}\right) &= \prod_{j=1}^{n} e^{a_j} \\ &= e^{\sum\limits_{j=1}^{n} a_j} \\ &= e^{\mathrm{Tr}(A)}\end{aligned}$$

Question V.

Simultaneous diagonalization of two Hermitian operators: Discuss the degenerate case carefully (see chapter 1, Shankar).

If we have two hermitian operators A and B such that $[A, B] = 0$, we know that we can diagonalize both simultaneously. In the case that both operators are degenerate, we can order the eigenbasis for one of the operators such that:

$$A = \begin{bmatrix} a_1 & & & & & & & \\ & \ddots & & & & & & \\ & & a_1 & & & & & \\ & & & \ddots & & & & \\ & & & & a_2 & & & \\ & & & & & \ddots & & \\ & & & & & & a_m & \\ & & & & & & & a_m \end{bmatrix}$$

Since this basis is not unique, we choose a set $\{|a_i, \alpha\rangle\}$ where α runs from 1 to m_i. Carrying on similarly to the nondegenerate case, from here we can say:

$$\begin{aligned} AB\,|a_i, \alpha\rangle &= BA\,|a_i, \alpha\rangle \\ &= a_i B\,|a_i, \alpha\rangle \end{aligned}$$

From this we can only conclude that the vector $B\,|a_i, \alpha\rangle$ is in the same eigenspace, but cannot from here conclude that this basis diagonalizes B. Since vectors from different eigenspaces are orthogonal, we can say:

$$\langle a_j, \beta|B|a_i, \alpha\rangle = 0$$

where $|a_i, \alpha\rangle$ and $|a_j, \beta\rangle$, $a_i \neq a_j$, are basis vectors. In this basis the operator B is block diagonal.

$$B = \begin{bmatrix} \boxed{\text{B}}_1 & & \\ & \ddots & \\ & & \boxed{\text{B}}_k \end{bmatrix}$$

Within each subspace i, B is given by the matrix in the block diagonal matrix B_i. If we now change bases to the eigenbasis of B_i, the operator A remains diagonal since it is unaffected by the choice of orthonormal basis in each degenerate eigenspace. If each B_i has the eigenvalues $b_i^{(1)}, b_i^{(2)} ... b_i^{(m_i)}$, the operators A and B are:

$$A = \begin{bmatrix} a_1 & & & & & & & \\ & \ddots & & & & & & \\ & & a_1 & & & & & \\ & & & \ddots & & & & \\ & & & & a_2 & & & \\ & & & & & \ddots & & \\ & & & & & & a_m & \\ & & & & & & & a_m \end{bmatrix}$$

$$B = \begin{bmatrix} b_1^{(1)} & & & & & & \\ & \ddots & & & & & \\ & & b_1^{(m_1)} & & & & \\ & & & \ddots & & & \\ & & & & b_2^{(1)} & & \\ & & & & & \ddots & \\ & & & & & & b_k^{(m_k)} \end{bmatrix}$$

Thus we have simultaneous diagonalization of the two degenerate hermitian operators A and B.

Question VI.

Shankar, Ex. 4.2.1

Consider the following operators on a Hilbert space $\mathbb{V}(C)$:

$$L_x = \frac{1}{2^{1/2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_y = \frac{1}{2^{1/2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad L_z = \frac{1}{2^{1/2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

1. What are the possible values one can obtain if L_z is measured?

   By immediate inspection, the eigenvalues of L_z are 1,0,-1.

2. Take the state in which $L_z = 1$. In this state what are $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x?

$|L_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so:

$$\begin{aligned}
\langle L_x \rangle &= \langle L_z = 1 | L_x | L_z = 1 \rangle \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= 0 \\
\left\langle L_x^2 \right\rangle &= \langle L_z = 1 | L_x^2 | L_z = 1 \rangle \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{2} \\
\Delta L_x &= \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

3. Find the normalized eigenstates and eigenvalues of L_x in the L_z basis.

   Solving the characteristic for L_x:

$$\begin{aligned}
\det(L_x - \lambda_x I) &= 0 \\
-\lambda_x(\lambda_x^2 - 1) &= 0 \\
\lambda_x &= 0, \pm 1 \\
|\lambda_x = 0\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\
|\lambda_x = 1\rangle &= \frac{1}{2} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \\
|\lambda_x = -1\rangle &= \frac{1}{2} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}
\end{aligned}$$

4. if the particle is in the state with $L_z = -1$ and L_x is measured, what are the possible outcomes and their probabilities?

   $|L_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

From the previous result:

$$
\begin{aligned}
P(L_x = 0) &= |\langle L_x = 0 | L_x = 0\rangle|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \\
P(L_x = 1) &= |\langle L_x = 1 | L_x = 1\rangle|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \\
P(L_x = -1) &= |\langle L_x = -1 | L_x = -1\rangle|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}
\end{aligned}
$$

5. Consider the state:

$$
|\psi\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2^{1/2} \end{bmatrix}
$$

in the L_z basis. if L_z^2 is measured in this state and a result of $+1$ is obtained, what is the state right after the measurement? How probable was this result? If L_z is measured immediately afterwards, what are the outcomes and respective probabilities?

$$
\begin{aligned}
|\psi\rangle &= c_{-1} |L_z = -1\rangle + c_0 |L_z = 0\rangle + c_1 |L_z = 1\rangle \\
c_{-1} &= \langle L_z = -1 | \psi\rangle = \frac{1}{\sqrt{2}} \\
c_0 &= \langle L_z = 0 | \psi\rangle = \frac{1}{2} \\
c_1 &= \langle L_z = 1 | \psi\rangle = \frac{1}{2}
\end{aligned}
$$

Since L_z^2 is measured to be $+1$, we can have either of the $|L_z = \pm 1\rangle$ states. This occurs with the probability $|c_1|^2 + |c_{-1}|^2 = \frac{3}{4}$. The resulting state is:

$$
|\psi'\rangle = c'_{-1} |L_z = -1\rangle + c'_1 |L_z = 1\rangle
$$

Normalizing:

$$
\begin{aligned}
\frac{c'_1}{c'_{-1}} &= \frac{c_1}{c_{-1}} \\
|c_{-1}|^2 + |c_1|^2 &= 1 \\
c_1 &= \sqrt{\frac{1}{3}} \\
c_{-1} &= \sqrt{\frac{2}{3}} \\
|\psi'\rangle &= c'_{-1} |L_z = -1\rangle + c'_1 |L_z = 1\rangle
\end{aligned}
$$

If we measure the state L_z immediately after, we get:

$$P(L_z = 1) = |c_1|^2 = \frac{1}{3}$$
$$P(L_z = -1) = |c_{-1}|^2 = \frac{2}{3}$$

6. A particle is in a state for which the probabilities are $P(L_z = 1) = 1/4$, $P(L_z = 0) = 1/2$, and $P(L_z = -1) = 1/4$ Convince yourself that the most general, normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z = 1\rangle + \frac{e^{i\delta_2}}{2}|L_z = 0\rangle + \frac{e^{i\delta_3}}{2}|L_z = -1\rangle$$

It was stated earlier on that if $|\psi\rangle$ is a normalized state then the state $e^{i\theta}|\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factors $e^{i\delta_i}$ multiplying the L_z eigenstates are irrelevant? [Calculate for example $P(L_x = 0)$]

The probabilities of each state are equivalent to the square of the magnitudes of their coefficients:

$$|c_1|^2 = \frac{1}{4}$$
$$c_1 = \frac{e^{i\delta_1}}{2}$$
$$|c_0|^2 = \frac{1}{2}$$
$$c_0 = \frac{e^{i\delta_2}}{\sqrt{2}}$$
$$|c_{-1}|^2 = \frac{1}{4}$$
$$c_{-1} = \frac{e^{i\delta_3}}{2}$$

From this each δ_i cannot be individually determined. While the rotation does not change a physical observable, it may have other physical significance if the state is a linear combination of eigenstates, as above:

$$\begin{aligned}
P(L_x = 0) &= |\langle L_x = 0|\psi\rangle|^2 \\
&= |\frac{1}{2}e^{i\delta_2}\langle L_x = 0|L_z = 1\rangle + \frac{1}{\sqrt{2}}e^{i\delta_2}\langle L_x = 0|L_z = 0\rangle + \frac{1}{2}e^{i\delta_3}\langle L_x = 0|L_z = -1\rangle|^2 \\
&= \frac{1}{8}(e^{-i\delta_1} - e^{-i\delta_3})(e^{-i\delta_1} - e^{-i\delta_3}) \\
&= \frac{1}{4}\cos(\delta_1 - \delta_3)
\end{aligned}$$

Therefore, the differences between the δ_i can be measured and have significance.