6710 Midterm

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Question I.

For a white dwarf, we are only concerned with the first three equations of stellar structure: mass distribution, hydrostatic equilibrium, and equation of state. We assume an equation of state dominated by nonrelativistic electron degeneracy pressure, and solve these equations to set up the numerical analysis.

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}r} &= 4\pi\rho(r)\\ \frac{\mathrm{d}P}{\mathrm{d}r} &= -G\frac{m(r)\rho(r)}{r^2}\\ P &= K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3}\end{aligned}$$

Solving these for equations which we will iterate over numerically gives

$$\begin{aligned}dm &= 4\pi\rho(r)\,\mathrm{d}r\\ \frac{\mathrm{d}P}{\mathrm{d}r} &= -G\frac{m(r)\rho(r)}{r^2}\\ &= \frac{5K_{NR}}{3\mu_e}\left(\frac{\rho}{\mu_e}\right)^{2/3}\frac{\mathrm{d}\rho}{\mathrm{d}r}\\ \implies \frac{\mathrm{d}\rho}{\mathrm{d}r} &= \frac{-3\mu_e^{5/3}G}{5K_{NR}}\frac{m(r)\rho^{1/3}(r)}{r^2}\\ \implies \mathrm{d}\rho &= \frac{-3\mu_e^{5/3}G}{5K_{NR}}\frac{m(r)\rho^{1/3}(r)}{r^2}\,\mathrm{d}r\end{aligned}$$

We solve these numerically to find the enclosed mass distribution and the density profile (see attached python file).

Question II.

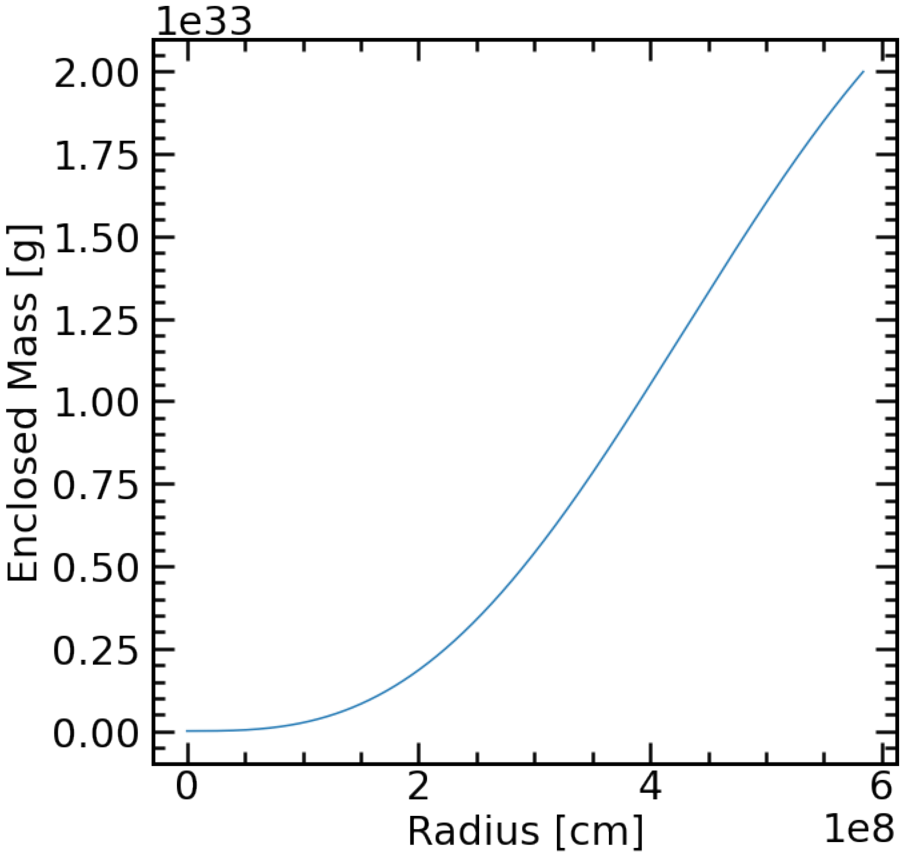

Figure 1: Mass profile for question 1

For the Sun we now must include all five equations of stellar structure. We now have

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\rho(r) \tag{1}$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G\frac{m(r)\rho(r)}{r^2} \tag{2}$$

$$P = \frac{\rho(r)}{\mu m_H}kT(r) \tag{3}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{-3\kappa}{64\pi\sigma}\frac{\rho L}{T^3r^2} \tag{4}$$

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2\rho(r)(\epsilon_{nuc} - \epsilon_\nu) \tag{5}$$

This assumes an ideal gas equation of state and energy transport dominated by radiative diffusion via electron scattering. We will take the neutrino contribution of energy generation to be negligible, and the energy generation to be dominated by the pp chain. This gives

$$\begin{aligned}\epsilon_{nuc} &= \epsilon_{pp} \\ &\simeq \epsilon_{0,pp}\rho X^2 T_6^4\end{aligned}$$

We can start similarly to question 1 to get our differential equations into forms to be solved numer-

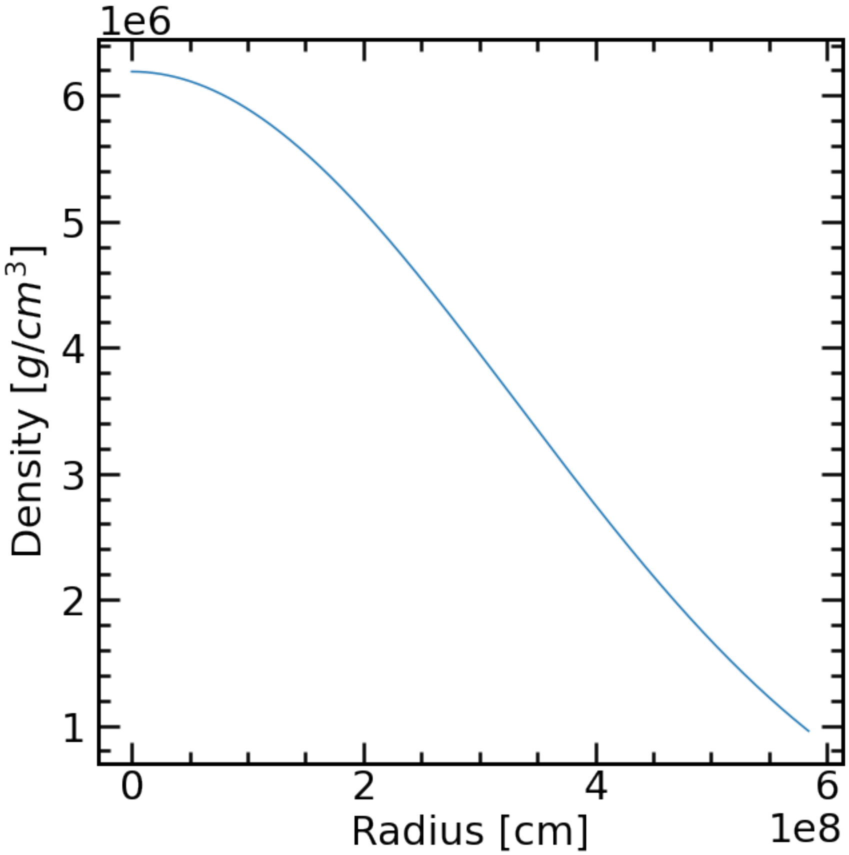

Figure 2: density profile for question 1

ically.

$$
\begin{aligned}
P &= \frac{\rho(r)}{\mu m_H} kT(r) \\
\frac{\mathrm{d}P}{\mathrm{d}r} &= -G\frac{m(r)\rho(r)}{r^2} \\
&= \frac{k}{\mu m_H}\left[\frac{\mathrm{d}\rho}{\mathrm{d}r}T + \frac{\mathrm{d}T}{\mathrm{d}r}\rho\right] \\
&= \frac{k}{\mu m_H}\left[\frac{\mathrm{d}\rho}{\mathrm{d}r}T - \frac{3\kappa}{64\pi\sigma}\frac{\rho^2 L}{T^3 r^2}\right] \\
\frac{\mathrm{d}\rho}{\mathrm{d}r} &= \frac{-G\mu m_H}{k}\frac{m(r)\rho(r)}{r^2 T(r)} + \frac{3\kappa}{64\pi\sigma}\frac{\rho^2(r)L(r)}{T^4 r^2} \\
\mathrm{d}\rho &= \left[\frac{-G\mu m_H}{k}\frac{m(r)\rho(r)}{r^2 T(r)} + \frac{3\kappa}{64\pi\sigma}\frac{\rho^2(r)L(r)}{T^4(r) r^2}\right]\mathrm{d}r
\end{aligned}
$$

For luminosity we have

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}r} &= 4\pi r^2 \rho(r) \epsilon_{pp} \\
&= 4\pi r^2 \rho^2(r) \epsilon_{0,pp} X^2 T_6^4 \\
\mathrm{d}L &= 4\pi r^2 \rho^2(r) \epsilon_{0,pp} X^2 T_6^4 \,\mathrm{d}r
\end{aligned}$$

Third we have temperature:

$$\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}r} &= \frac{-3\kappa}{64\pi\sigma} \frac{\rho L}{T^3 r^2} \\
\mathrm{d}T &= \frac{-3\kappa}{64\pi\sigma} \frac{\rho L}{T^3 r^2} \,\mathrm{d}r
\end{aligned}$$

And finally we have mass (the simplest one):

$$\mathrm{d}m = 4\pi\rho(r)\,\mathrm{d}r$$

We can numerically integrate these to get the mass, density, and temperature profiles (see attached python file). We address issues with convection and energy transport in question 3 and other issues in question 4. We see that the luminosity profile is not accurate to the Sun, indicating that there are issues with energy transport and energy generation which will resurface in our convection profile in question 3.

Question III.

Now we must consider convection. The condition for convection we will use is

$$\frac{\mathrm{d}\log T}{\mathrm{d}\log P} \leq \frac{3\kappa}{64\pi\sigma G} \frac{PL}{T^4 M}$$

We can start by taking the ideal gas equation of state (We might not use this but I will leave in this digression).

$$\begin{aligned}
P &= \frac{\rho(r)}{\mu m_H} k T(r) \\
\log P(r) &= \log\left(\frac{k}{\mu m_H}\right) + \log\rho(r) + \log T(r) \\
\log T(r) &= \log P(r) - \log\left(\frac{k}{\mu m_H}\right) - \log\rho(r) \\
\frac{\mathrm{d}\log T}{\mathrm{d}\log P} &= 1 - \frac{\mathrm{d}}{\mathrm{d}\log P(r)}\left[\log\left(\frac{k}{\mu m_H}\right)\right] - \frac{\mathrm{d}\log\rho(r)}{\mathrm{d}\log P(r)} \\
&= 1 - \frac{\mathrm{d}\log\rho(r)}{\mathrm{d}\log P(r)} \\
&= 1 - \frac{P(r)}{\rho(r)} \frac{\mathrm{d}\rho(r)}{\mathrm{d}P(r)}
\end{aligned}$$

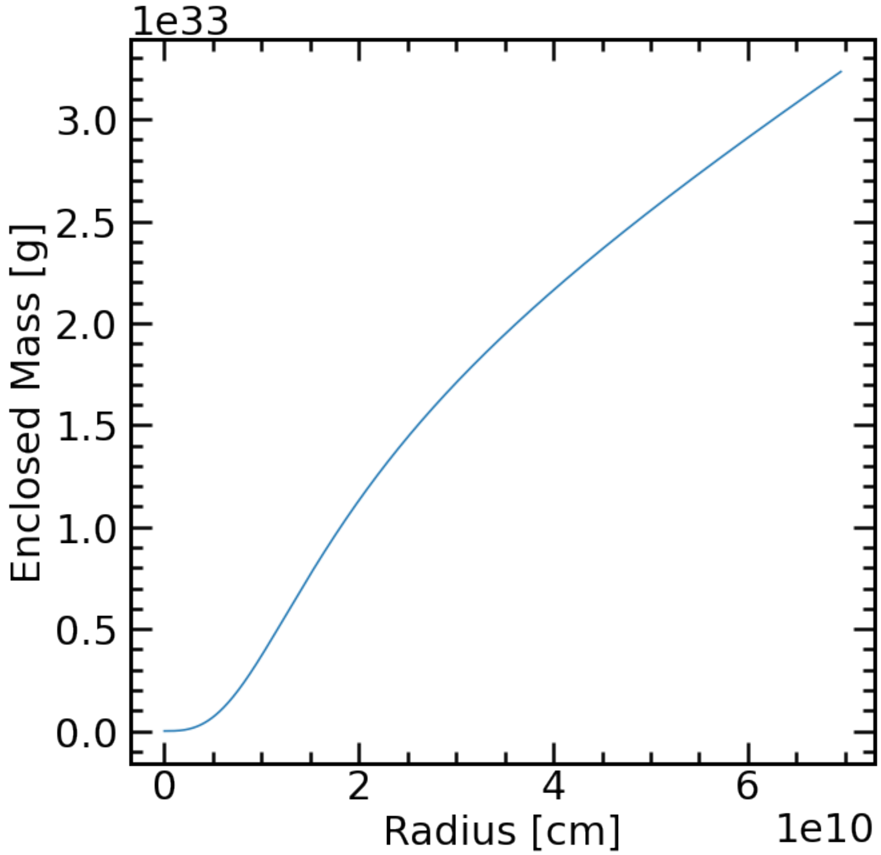

Figure 3: Mass profile of the Sun for question 2

We can also take directly the derivative of the left hand side of the convection condition to say

$$\begin{aligned}\frac{\mathrm{d}\log T}{\mathrm{d}\log P} &\leq \frac{3\kappa}{64\pi\sigma G}\frac{PL}{T^4M} \\ &= \frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}P}\end{aligned}$$

We can solve numerically for these conditions to show that the Sun is at the edge of stability when

$$\frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}P} = \frac{3\kappa}{64\pi\sigma G}\frac{PL}{T^4M}$$

We will iterate over this instead of the $\frac{\mathrm{d}\log T}{\mathrm{d}\log P}$ condition to hopefully mitigate some numerical effects of the modeling.

We expect to find that under the conditions of the Sun represented in question 2 we will not see a convection profile accurate to the (real) Sun. This is due to the incorrect luminosity profile given these assumptions, which leads us to have our convection condition satisfied at all radii. We discuss further issues with these models in question 4. (There is also a possible issue in the code where we may be drawing some values from the ith element instead of the $(1-i)$th element, but the step size is sufficiently small that this should not matter.)

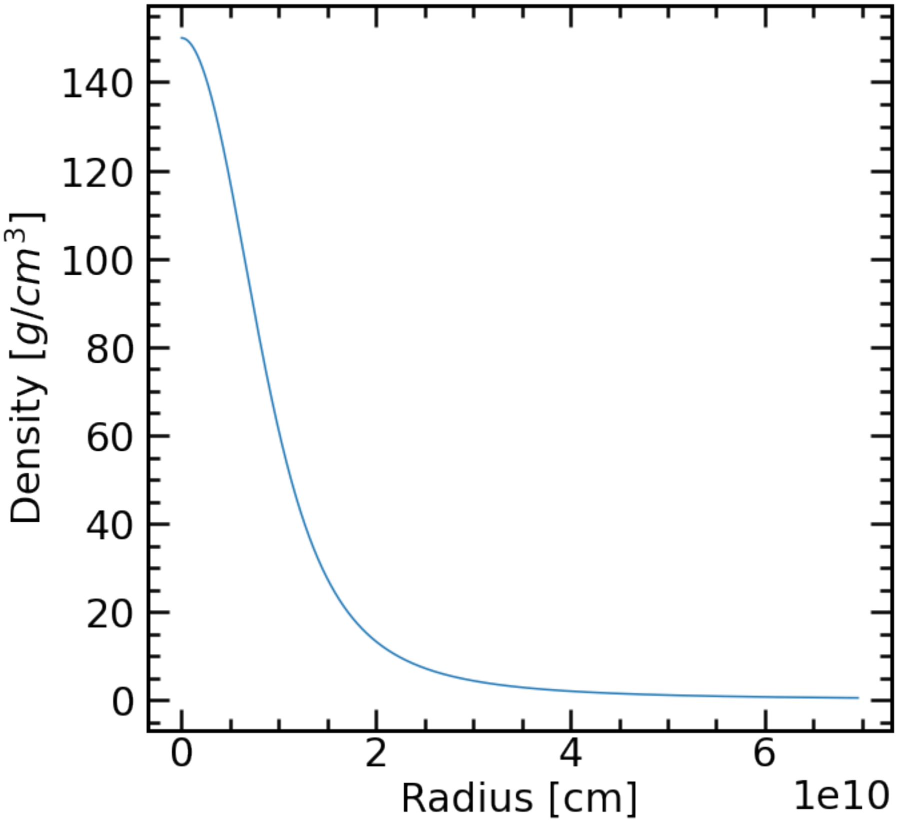

Figure 4: Temperature profile of the Sun for question 2

Question IV.

For each of our equations we made some assumptions. In mass distribution there are not many other points to account for, other than corrections to density which will be discussed later.

We in part address some of the issues of energy transport in question 3 with the addition of convection outside of the centers of stars. The convection that we see is not correct to the known convection zones in the Sun due to an incorrect luminosity profile in question 2. We do not actually work to correct the energy transport with convection. For regions with energy transport via convection we would have

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}\frac{T}{P}\left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{ad}$$

We also assume constant composition and opacity, where more accurately we would have composition (and opacity by proxy) be some functions of radius. If this was the case, we can show the

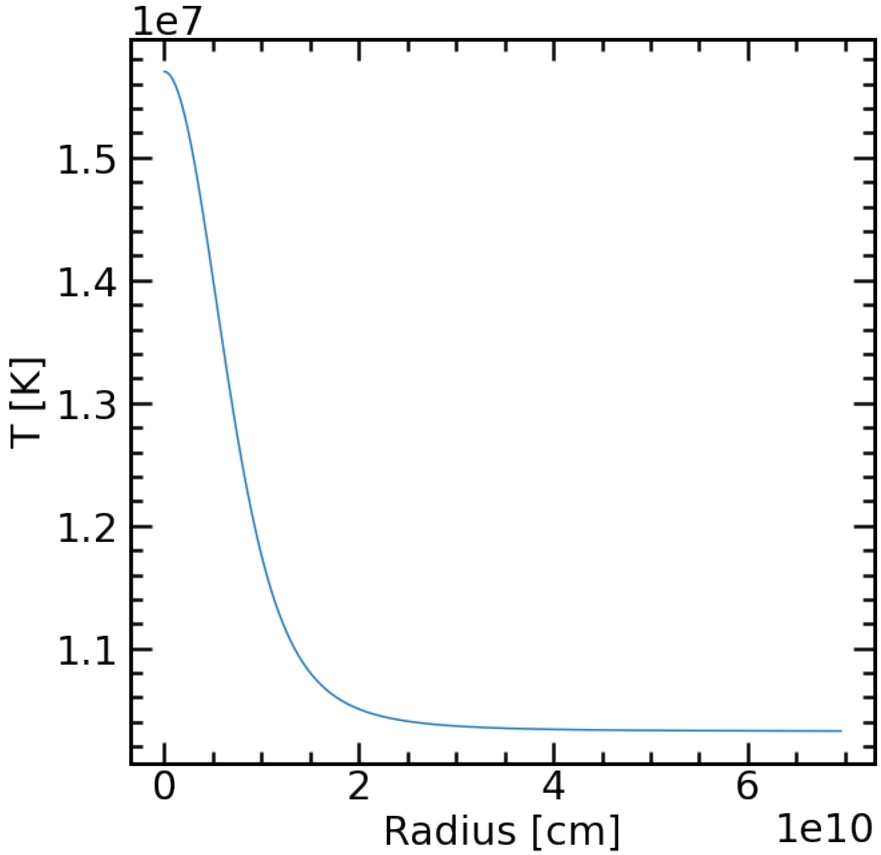

Figure 5: Temperature profile of the Sun for question 2

contribution to the ideal gas law of a radius dependent composition by

$$\begin{aligned} P &= \frac{\rho}{\mu m_H} kT \\ \frac{\mathrm{d}P}{\mathrm{d}r} &= \frac{k}{m_h} \frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{\rho(r) T(r)}{\mu(r)} \right] \\ &= \frac{k}{m_h} \left[\frac{1}{\mu(r)} \left(\rho(r) \frac{\mathrm{d}T(r)}{\mathrm{d}r} + T(r) \frac{\mathrm{d}\rho(r)}{\mathrm{d}r} \right) + \rho(r) T(r) \ln \mu(r) \right] \end{aligned}$$

We will also see an effect on opacity where

$$\begin{aligned} \kappa_{es} &= 0.20(1+X) \\ \implies \kappa_{es}(r) &= 0.20(1+X(r)) \end{aligned}$$

This would further change energy transport to have opacity as a function of radius.

We also assume energy generation entirely due to the pp chain, where we know that there is a contribution due to the CNO cycle in the Sun. The assumption that the neutrino contribution to the energy generation is negligible is okay. The contribution from the CNO cycle would be given by

$$\epsilon_{CNO} \simeq \epsilon'_{0,CNO} \rho X X_{CNO} T_6^{19.9}$$

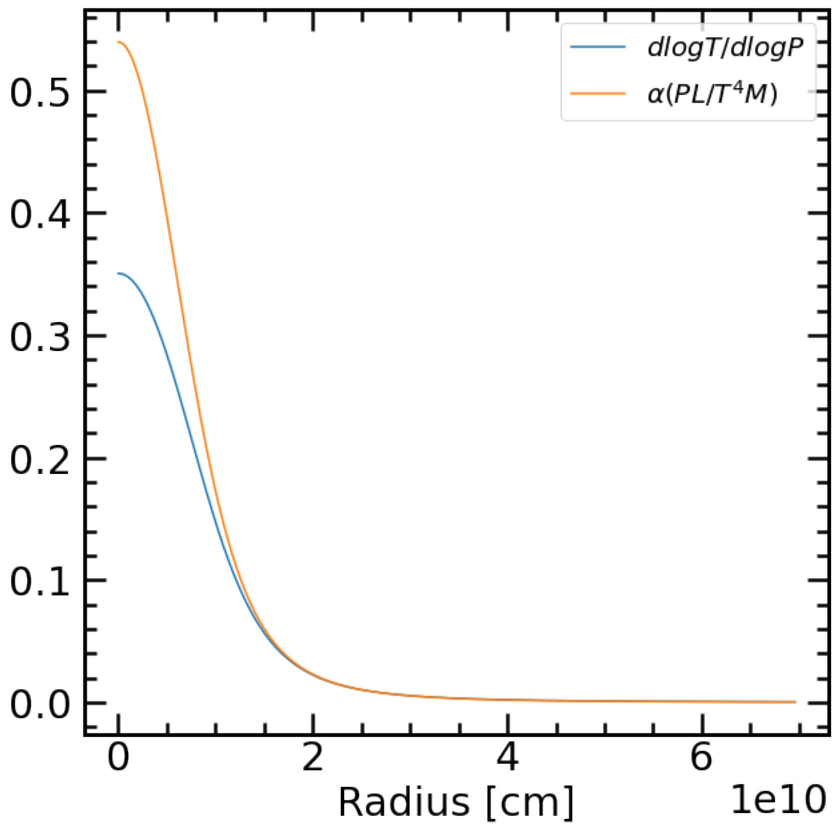

Figure 6: Convection profile of the Sun for question 3. We see that for all radii the convection condition is satisfied.

which is highly temperature dependent. The total energy generation would then be

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}r} &= 4\pi r^2 \rho(r)\epsilon_{nuc} \\ &= 4\pi r^2 \rho(r)(\epsilon_{pp} + \epsilon_{CNO})\end{aligned}$$

We also do not consider any other equations of state, where we would expect to have failure of ideal gas in different regions of the Sun. Our equation of state would more fully read

$$P = \frac{\rho}{\mu m_H} kT + \frac{4\sigma}{3c} T^4 + K_{NR} \left(\frac{\rho}{\mu_e}\right)^{5/3}$$

where electron degeneracy in the core accounts for only a small fraction of the central pressure (<1%), and we expect a contribution from radiation pressure at the surface of the Sun.

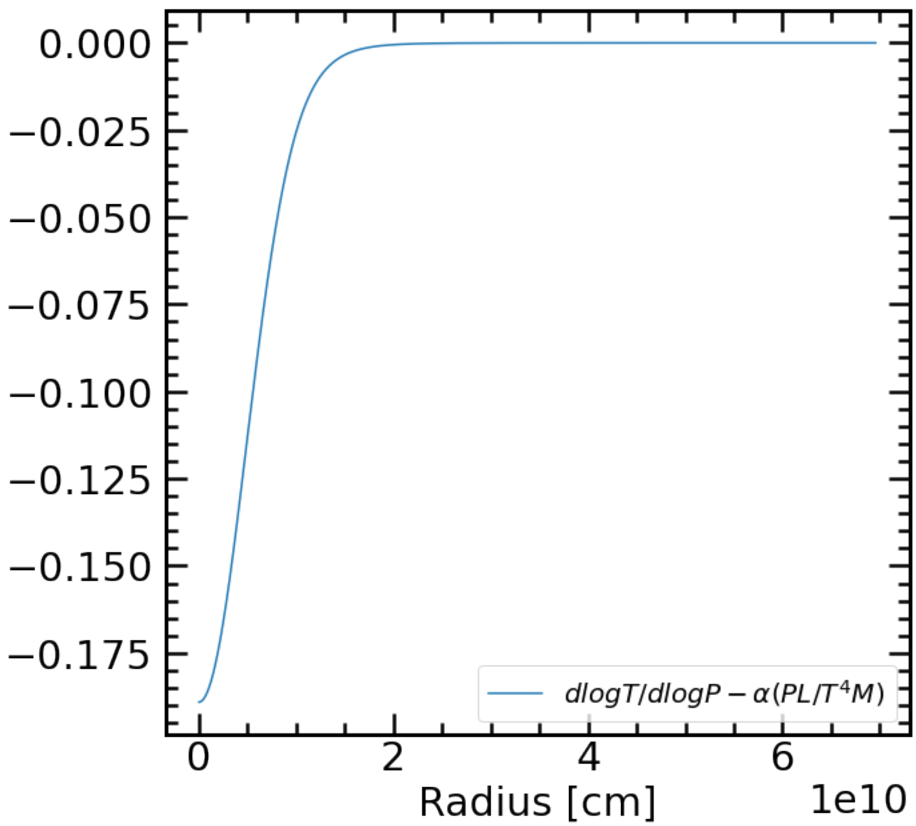

Figure 7: Convection profile of the Sun for question 3. We see that for all radii the convection condition is satisfied.