6720 Problem Set 2

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Resources: Carroll & Ostlie, Schneider
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Question I.

(a) We know the following:

$$V_p = \sqrt{\frac{(1+e)\mu}{(1-e)a}}$$
$$V_a = \sqrt{\frac{(1-e)\mu}{(1+e)a}}$$

Where v_p and v_a are the pericenter and apocenter velocities, respectively, $\mu = GM$ and a is the semimajor axis of the orbit. We can solve for the eccentricity e by taking the ratio of these two.

$$\begin{aligned}
\frac{V_p}{V_a} &= \frac{\sqrt{\frac{(1+e)\mu}{(1-e)a}}}{\sqrt{\frac{(1-e)\mu}{(1+e)a}}} \\
&= \frac{1+e}{1-e} \\
\frac{V_p}{V_a}(1-e) &= 1+e \\
\frac{V_p}{V_a} - \frac{V_p}{V_a}e &= 1+e \\
\frac{V_p}{V_a} - 1 &= \left(\frac{V_p}{V_a}+1\right)e \\
e &= \frac{\frac{V_p}{V_a}-1}{\frac{V_p}{V_a}+1}
\end{aligned}$$

This gives $e = 0.88$.

(b) Let's conserve energy at the pericenter and apocenter:

$$\begin{aligned}
\frac{1}{2}mv_p^2 - \frac{GMm}{r_p} &= \frac{1}{2}mv_a^2 - \frac{GMm}{r_a} \\
r_p &= (1-e)a \\
r_a &= (1+e)a \\
\frac{1}{2}mv_p^2 - \frac{GMm}{(1-e)a} &= \frac{1}{2}mv_a^2 - \frac{GMm}{(1+e)a}
\end{aligned}$$

We conveniently have m in every term, and we can rearrange to solve for a.

$$\begin{aligned}
\frac{1}{2}v_p^2 - \frac{GM}{(1-e)a} &= \frac{1}{2}v_a^2 - \frac{GM}{(1+e)a} \\
\frac{1}{2}v_p^2 - \frac{1}{2}v_a^2 &= \frac{GM}{(1-e)a} - \frac{GM}{(1+e)a} \\
\frac{1}{2}v_p^2 - \frac{1}{2}v_a^2 &= \frac{GM}{a}\left(\frac{1}{1-e} - \frac{1}{1+e}\right) \\
a &= GM\left[\frac{\left(\frac{1}{1-e} - \frac{1}{1+e}\right)}{\frac{1}{2}v_p^2 - \frac{1}{2}v_a^2}\right]
\end{aligned}$$

The term in the brackets includes only known quantities, and we can use Kepler to get M in terms of the period T and semimajor axis a.

$$\begin{aligned}
T^2 &= \frac{4\pi^2}{GM}a^3 \\
M &= \frac{4\pi^2}{GT^2}a^3
\end{aligned}$$

So

$$a = GM\left[\frac{\left(\frac{1}{1-e} - \frac{1}{1+e}\right)}{\frac{1}{2}v_p^2 - \frac{1}{2}v_a^2}\right] \tag{1}$$

$$= \frac{4\pi^2}{T^2}a^3\left[\frac{\left(\frac{1}{1-e} - \frac{1}{1+e}\right)}{\frac{1}{2}v_p^2 - \frac{1}{2}v_a^2}\right] \tag{2}$$

$$a^{-2} = \frac{4\pi^2}{T^2}\left[\frac{\left(\frac{1}{1-e} - \frac{1}{1+e}\right)}{\frac{1}{2}v_p^2 - \frac{1}{2}v_a^2}\right] \tag{3}$$

$$a = \left(\frac{4\pi^2}{T^2}\left[\frac{\left(\frac{1}{1-e} - \frac{1}{1+e}\right)}{\frac{1}{2}v_p^2 - \frac{1}{2}v_a^2}\right]\right)^{-1/2} \tag{4}$$

All of these are now known quantities, so this gives $a = 1.4 \times 10^{14}$ m, or ~ 950 AU.

(c) The mass of the black hole can be calculated using Kepler's third law as seen before

$$M = \frac{4\pi^2}{GT^2} a^3$$

gives a black hole mass of $3.7 \times 10^6 \ M_\odot$

(d)

$$S = R\theta$$

Where $S = 2a$ and $\theta = s$. This gives a distance to the Galactic center of $R = 7.5$ kpc (pretty close!).

Question II.

If we take $\alpha = 0$, we have

$$v_r = \Theta - \Theta_0 \sin l$$

at $\alpha = 0$, $v_r - \Theta_{max}$, and taking the notational liberties of changing Θs to Vs we have

$$V_{max} = V(R) - V_0 \sin l$$

Solving for $V(R)$ gives the desired

$$V(R) = V_{max} + V_0 \sin l$$

Question III.

(a) We can start with the known quantities $R_0 \approx 8$ kpc and $v_0 \approx 220$ km/s. We can find the period of the revolution of the Sun around the center of the Milky Way as

$$v = \frac{2\pi R}{T}$$
$$T = \frac{2\pi R_0}{v_0}$$

This gives a period of $T = 223$ Myr, and we know the age of the Sun is approximately 4.5 Gyr. This implies that the Sun has completed approximately 20 revolutions around the center of the galaxy since it's formation.

(b) The Cretaceous period ended ~ 65 Myr ago, which corresponds to ~ 0.3 periods of the Sun revolving around the Galactic Center. This gives and angle of $104°$.

Question IV.

(a) We can solve the following to find the amount of interstellar extinction

$$d = 10^{(m_V - M_V - A + 5)/5}$$
$$A = m_V - M_V + 5 - 5 \log d$$

This gives an extinction of $A = 17$ magnitudes (this seems pretty high to me, but maybe this makes sense for something in the MW).

(b) This is found simply by dividing my the distance given, so we have an attenuation per kpc of 2.27 magnitudes per kpc.

Question V.

(a) If we assume a solar-neighborhood stellar density of 0.14 $\mathrm{M_\odot\ pc^{-3}}$, with the same assumptions for orbital velocity and radius as question 3, we have

$$\rho(r) = \frac{V^2}{4\pi G r^2}$$
$$= 0.014\ \mathrm{M_\odot\ pc^{-3}}$$

One tenth the stellar density.

(b) The enclosed mass is the integral of the density over the volume, which gives (assuming spherical symmetry)

$$\rho(r) = \frac{\rho_0}{1 + (r/a)^2}$$
$$M_r = 4\pi \int_0^r \rho(r') r'^2 \, \mathrm{d}r'$$

This integral gives $-\rho_0 a^2 \left(a \arctan \frac{r'}{a} - r'\right)\Big|_0^r$, which when evaluated at the bounds gives the desired

$$M_r = 4\pi\rho_0 a^2 \left[r - a \arctan \frac{r}{a}\right]$$

Question VI.

(a) The color is given by the central $(H - Ks)$ (~ 1.7 from Figure 2) minus the intrinsic scatter of 0.07, which gives 1.63.

(b)

$$E(H - Ks) = -A_{Ks}\left(1 - \left(\frac{\lambda_H}{\lambda_{Ks}}\right)^{-\alpha}\right)$$
$$1.63 = -A_{Ks}\left(1 - \left(\frac{1.677}{2.168}\right)^{-2.21}\right)$$

This gives $A_{Ks} = 2.1$ magnitudes.

(c) We can use the distance modulus with attenuation like we did in question 4

$$
\begin{aligned}
d &= 10^{(m-M-A+5)/5} \\
&= 7.9\text{kpc}
\end{aligned}
$$