5401 Problem Set 3

Nikko Cleri

March 4, 2020

Question I.

Shankar Ex. 4.2.2

Show that for a real wave function $\psi(x)$, the expectation value of momentum $\langle P \rangle = 0$. (Hint: Show that the probabilities for the momenta $\pm p$ are equal.) Generalize this result to the case $\psi = c\psi_r$, where ψ_r is real and c an arbitrary (real or complex) constant. (Recall that $|\psi\rangle$ and $c\,|\psi\rangle$ are equivalent.)

$$
\begin{aligned}
\langle P \rangle &= \langle \psi | P | \psi \rangle \\
&= \int_{-\infty}^{\infty} \psi^* \left[-i\hbar \frac{\partial}{\partial x} \right] \psi \,\mathrm{d}x \\
&= -i\hbar \int_{-\infty}^{\infty} \psi \frac{\partial}{\partial x} \psi \,\mathrm{d}x \\
&= \frac{-i\hbar}{2} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \psi^2 \,\mathrm{d}x \\
&= \frac{-i\hbar}{2} \, \psi^2 \bigg|_{-\infty}^{\infty} \\
&= 0
\end{aligned}
$$

This is forced since we assume ψ is normalized so it must vanish as $x \to \pm\infty$. In general, you have:

$$
\begin{aligned}
\langle P \rangle &= \langle c\psi | P | c\psi \rangle \\
&= \int_{-\infty}^{\infty} (c\psi)^* \left[-i\hbar \frac{\partial}{\partial x} \right] c\psi \,\mathrm{d}x \\
&= -|c|^2 i\hbar \int_{-\infty}^{\infty} \psi \frac{\partial}{\partial x} \psi \,\mathrm{d}x \\
&= 0
\end{aligned}
$$

Question II.

Shankar Ex. 4.2.3

Show that if $\psi(x)$ has mean momentum $\langle P \rangle$, $e^{ip_0x/\hbar}\psi(x)$ has mean momentum $\langle P \rangle + p_0$.

$$\begin{aligned}
\langle P \rangle &= \langle \psi | P | \psi \rangle \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \psi^* \left[-i\hbar \frac{\partial}{\partial x} \right] \psi \, \mathrm{d}x \\
\langle P' \rangle &= -i\hbar \int_{-\infty}^{\infty} \left(e^{ip_0x/\hbar} \psi \right)^* \frac{\partial}{\partial x} \left(e^{ip_0x/\hbar} \psi \right) \mathrm{d}x \\
&= -i\hbar \int_{-\infty}^{\infty} \left(e^{ip_0x/\hbar} \psi \right)^* \frac{\partial}{\partial x} (\psi) e^{ip_0x/\hbar} \, \mathrm{d}x - i\hbar \int_{-\infty}^{\infty} \left(e^{ip_0x/\hbar} \psi \right)^* \frac{\partial}{\partial x} \left(e^{ip_0x/\hbar} \right) \psi \, \mathrm{d}x \\
&= -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial}{\partial x} (\psi) \, \mathrm{d}x - (i\hbar)(\frac{ip_0}{\hbar}) \int_{-\infty}^{\infty} \psi^* \psi \, \mathrm{d}x \\
&= \langle P \rangle + p_0 \int_{-\infty}^{\infty} \psi^* \psi \, \mathrm{d}x \\
&= \langle P \rangle + p_0
\end{aligned}$$

Here we assume that ψ is a normalizable state, so the integration of the final step goes to unity.

Question III.

Shankar Ex. 5.2.1

A particle is in the ground state of a box of length L. Suddenly the box expands (symmetrically) to twice its size, leaving the wave function undisturbed. Show that the probability of finding the particle in the ground state of the new box is $(8/3\pi)^2$.

The ground state wave function for a particle in a box of side length L centered at $x = 0$ is:

$$|\psi_1\rangle (x) = \begin{cases} -\sqrt{\frac{2}{L}} \cos \frac{\pi x}{L} & |x| \leq \frac{L}{2} \\ 0 & |x| > \frac{L}{2} \end{cases}$$

If the box is now extended such that the side length goes to $2L$, the new ground state wavefunction becomes:

$$|\phi_1\rangle (x) = \begin{cases} -\sqrt{\frac{1}{L}} \cos \frac{\pi x}{2L} & |x| \leq \frac{L}{2} \\ 0 & |x| > \frac{L}{2} \end{cases}$$

The probability of finding the particle described by $|\psi_1\rangle$ in state $|\phi_1\rangle$ is given by the square of the inner products in the $|x| \leq \frac{L}{2}$ region:

$$|\langle\phi_1|\psi_1\rangle|^2 = \left[\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{2}{L^2} \cos\frac{\pi x}{L} \cos\frac{\pi x}{2L}\, \mathrm{d}x\right]^2$$

Solve this integral by changing variables $\frac{\pi x}{L} = y$:

$$\begin{aligned}
&= \frac{\sqrt{2}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos y \cos\frac{y}{2}\, \mathrm{d}y \\
&= \frac{8}{3\pi} \\
P &= \left(\frac{8}{3\pi}\right)^2
\end{aligned}$$

Question IV.

Shankar Ex. 5.2.2

(a) Show that for any normalized $|\psi\rangle$, $\langle\psi|H|\psi\rangle \geq E_0$, where E_0 is the lowest-energy eigenvalue (Hint: Expand $|\psi\rangle$ in the eigenbasis of H.)

First we expand $|\psi\rangle$ as:

$$\begin{aligned}
|\psi\rangle &= \int_E \langle E|\psi\rangle\, |E\rangle \\
&= \int_E a_E\, |E\rangle \\
\langle\psi|H|\psi\rangle &= \int_E \int_{E'} a_E a_{E'} \left\langle E'\middle|H\middle|E\right\rangle \\
&= \int_E |a_E|^2 E \\
&\geq \int_E |a_E|^2 E_0
\end{aligned}$$

But since $|\psi\rangle$ is a normalized wavefunction:

$$
\begin{aligned}
\langle\psi|H|\psi\rangle &\geq E_0 \int_E |a_E|^2 \\
&\geq E_0
\end{aligned}
$$

(b) Prove the following theorem: every attractive potential in one dimension has at least one bound state. Hint: since V is attractive, if we define $V(\infty) = 0$, it follows that $V(x) = -|V(x)|$ for all x. To show that there exists a bound state with $E < 0$, consider

$$
\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}
$$

and calculate

$$
E(\alpha) = \langle\psi_\alpha|H|\psi_\alpha\rangle\,, \quad H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - |V(x)|
$$

Show that $E(\alpha)$ can be made negative by a suitable choice of α. The desired result follows from the application of the theorem proved above.

If we make our potential such that $V(\infty) = 0$ and shift the definition such that V is at its minimum at $x = 0$. Considering the given wavefunction we have:

$$
\begin{aligned}
E(\alpha) &= \langle\psi_\alpha|H|\psi_\alpha\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, \psi_\alpha^*(x) \left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - |V(x)|\right] \psi_\alpha(x) \\
&= -\sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\alpha x^2} |V(x)| + \frac{\alpha\hbar^2}{4m}
\end{aligned}
$$

This tells us that the total energy of the state is a sum of a negative potential energy and a positive kinetic energy. Since we know that $E_0 \leq E(\alpha)$ for any value of the parameter α, and we know that $E(\alpha) < 0$ by construction of the potential, we know that the lowest energy eigenstate is bound, so the potential has at least one bound state.

Question V.

January 2020 Quantum Mechanics Prelim Problem

Assume you have two observables Ω, Λ that are compatible (i.e., they have a common set of eigenstates). Further assume that each eigenvalue (out of several) of either observable is degenerate, meaning that for each eigenvalue of Ω there are several eigenstates, and similarly for Λ. However, also assume that each pair of eigenvalues (ω, λ) uniquely defines one and only one state of the system (except for multiplication by a complex number), so that the set of eigenvectors $|\omega, \lambda\rangle$ is a complete and orthonormal basis of the Hilbert space. Initially, the system is in some state $|\psi\rangle$ that is completely unknown to me. I first measure observable Ω, with result ω_1, immediately followed by a measurement of Λ, with result λ_1, and then by a second measurement of Ω.

(a) Describe the extent of my knowledge about which state the system is in after each of the three measurements. Be as precise as possible. What do I know and what do I not know about the state at each point?

After the first measurement: the system is in some state $|\omega_1, \lambda_i\rangle$, where the degeneracy of Ω means that the strongest statement that can be made about its eigenvalue is that it corresponds to a family of states. Nothing here is known about Λ, so the state is not known exactly.

After the second measurement, the system is in state $|\omega_1, \lambda_1\rangle$, so we now know the system exactly, since we defined the system such that each pair of eigenvalues uniquely defines exactly one state of the system.

After the third measurement we should get the same result since the state has already collapsed to $|\omega_1, \lambda_1\rangle$.

(b) At which point along this chain do I know everything there is to know about the present state of the system?

After the second measurement the state of the system is known, and future measurements of the observables have no consequence.

(c) For which of the three measurements can I predict the exact outcome?

The exact outcome can be predicted for the third measurement since the state has already collapsed to $|\omega_1, \lambda_1\rangle$.

(d) For which of the same three measurements can I predict the probability of a given possible outcome?

Probabilities cannot be predicted for the first measurement since the state is completely unknown, but once the state collapses once (via the first measurement), the second measurement can have the probability calculated since at least something about the state. The third measurement is irrelevant since the state has already collapsed completely.

Now assume that I do completely know the state $|\psi\rangle$ of the system initially (before the first measurement), but it is not an eigenstate of either Ω or Λ. Answer the same four questions again for this case.

In this case, the first measurement will collapse the state to an eigenvalue in Ω, and the second will collapse the spectrum of Λ to one eigenvalue, leaving one known final state. The third measurement will again change nothing. As before, everything will be known about the state after the second measurement, and the exact outcome can be predicted for the third measurement since the state has completely collapse to one unique $|\omega_i, \lambda_j\rangle$. The probability of a given outcome can now be predicted for all measurements however, since the initial state of the system is known.

Question VI.

Shankar Ex. 5.2.3

Consider $v(x) = -aV_0\delta(x)$. Show that it admits a bound state of energy $E = -ma^2V_0^2/2\hbar^2$. Are there any other bound states? Hint: Solve Schrodinger's equation outside the potential for $E < 0$, and keep only the solution that has the right behavior at infinity and is continuous at $x = 0$. Draw the wavefunction and see how there is a cusp, or a discontinuous change of slope at $x = 0$. Calculate the change in slope and equate it to

$$\int_{-\varepsilon}^{\varepsilon} \left(\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right) \mathrm{d}x$$

(where ε is infinitesimal) determined from Schrodinger's equation.

We have the eigenvalue equation for $E < 0$:

$$H\,|E\rangle = E\,|E\rangle$$
$$\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\psi_E(x) = E\psi_E(x)$$

At $x \neq 0$:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_E(x) = E\psi_E(x)$$

We can propose two solutions ψ_+ and ψ_- for $x > 0$ and $x < 0$:

$$\psi_-(x) = Ae^{-kx} + Be^{kx}$$
$$\psi_-(x) = Ce^{-kx} + De^{kx}$$

And immediately the boundary conditions at infinity give:

$$\psi_- = Be^{kx}$$
$$\psi_+ = Ce^{-kx}$$

And the boundary at $x = 0$ tells us:

$$\psi_- = Be^{kx}$$
$$\psi_+ = Be^{-kx}$$

Integrating the Schrodinger equation across the boundary we get:

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \alpha\int_{-\varepsilon}^{\varepsilon}\delta(x)\psi\,\mathrm{d}x = E\int_{-\varepsilon}^{\varepsilon}\psi\,\mathrm{d}x$$
$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}\psi}{\mathrm{d}x}\bigg|_{-\varepsilon}^{\varepsilon} - \alpha\psi(0) = E\int_{-\varepsilon}^{\varepsilon}\psi\,\mathrm{d}x$$

Taking the $\varepsilon \to \infty$ limit:

$$\frac{\hbar^2}{m}Bk = \alpha B$$
$$k = \frac{m\alpha}{\hbar^2}$$
$$E = \frac{m\alpha^2}{2\hbar^2}$$

This gives us a condition on the energy E, and finding B gives us:

$$\frac{1}{k}|B|^2 = 1$$
$$B = \frac{\sqrt{m\alpha}}{\hbar}$$

This tells us that the delta function well has one bound state, dependent on the parameter α.