5402 Problem Set 2

Nikko Cleri

September 21, 2020

Question I.

Find the Clebsch-Gordan coefficients for combining two spins $s_1 = 1$ and $s_2 = 1$.

$$\begin{aligned}
|2,2\rangle &= |1,1\rangle\,|1,1\rangle \\
|2,1\rangle &= \alpha J_-\,|2,2\rangle \\
&= \alpha(|1,0\rangle\,|1,1\rangle + |1,1\rangle\,|1,0\rangle) \\
&= \frac{1}{\sqrt{2}}\,|1,0\rangle\,|1,1\rangle + \frac{1}{\sqrt{2}}\,|1,1\rangle\,|1,0\rangle \\
|2,0\rangle &= \alpha J_-\,|2,1\rangle \\
&= \frac{1}{\sqrt{6}}(|1,-1\rangle\,|1,1\rangle + |1,1\rangle\,|1,-1\rangle) + \frac{\sqrt{2}}{3}\,|1,0\rangle\,|0,1\rangle
\end{aligned}$$

now we can start taking inner products:

$$\begin{aligned}
\langle 2,1|1,1\rangle &= 0 \\
|1,1\rangle &= \frac{1}{\sqrt{2}}(|1,1\rangle\,|1,0\rangle - |1,0\rangle\,|1,1\rangle) \\
|1,0\rangle &= \alpha J_1\,|1,1\rangle \\
&= \frac{1}{\sqrt{2}}(|1,1\rangle\,|1,-1\rangle - |1,-1\rangle\,|1,1\rangle) \\
|0,0\rangle &= x\,|1,1\rangle\,|1,-1\rangle + y\,|1,0\rangle\,|1,0\rangle + z\,|1,-1\rangle\,|1,1\rangle \\
\langle 2,0|0,0\rangle &= 0 \\
\langle 1,0|0,0\rangle &= 0
\end{aligned}$$

With the constraints

$$\begin{aligned}
x - z &= 0 \\
x + z + 2y &= 0
\end{aligned}$$

So

$$|0,0\rangle = \frac{1}{\sqrt{3}}\,|1,1\rangle\,|1,-1\rangle - \frac{1}{\sqrt{3}}\,|1,0\rangle\,|1,0\rangle + \frac{1}{\sqrt{3}}\,|1,-1\rangle\,|1,1\rangle$$

We can then use the following to find the rest:

$$\langle j_1 m_1, j_2 m_2 | j m\rangle = \langle j_1(-m_1), j_2(m_2) | j(-m)\rangle$$

Question II.

Four massive spin-1/2 particles are fixed on the vertices of a regular tetrahedron. The Hamiltonian of this system consists of a sum of spin-spin interactions over each of the six pairs as follows

$$H = \alpha \sum_{i \neq j} \mathbf{S}_i \mathbf{S}_j$$

(a) Show that all three components of the total spin $\mathbf{J} = \sum_i \mathbf{S}_i$ of the system commutes with H.

We begin a litany of summation and commutator identities:

$$\begin{aligned}
[J, H] &= \sum_i \mathbf{S}_i \alpha \sum_{i \neq j} \mathbf{S}_i \mathbf{S}_j - \alpha \sum_{i \neq j} \mathbf{S}_i \mathbf{S}_j \sum_i \mathbf{S}_i \\
H &= \sum_{j \neq k} \sum_q S_{jq} S_{kq} \\
[J, H] &= \left[\sum_p S_{pi}, \sum_{j \neq k} S_{jq} S_{kq} \right] \\
&= \sum_{j \neq k} \sum_p \sum_q [S_{pi}, S_{jq} S_{kq}] \\
&= \sum_{j \neq k} \sum_p \sum_q ([S_{pi}, S_{jq}] S_{kq} + S_{jq} [S_{pi}, S_{kq}])
\end{aligned}$$

Using the identity $[S_{pi}, S_{jq}] = \sum_b i\hbar \epsilon_{iqk} S_{jb}$, where the $p \neq j$ terms are immediately zero, we have (with summation convention)

$$\begin{aligned}
[J, H] &= \epsilon_{iqb} S_{jb} S_{kq} + \epsilon_{iqc} S_{kc} S_{jq} \\
i &\to x \\
[J_x, H_x] &= \epsilon_{xyz} S_{jz} S_{ky} + \epsilon_{xyz} S_{kz} S_{jy} + \epsilon_{xzy} S_{jz} S_{ky} + \epsilon_{xzy} S_{kz} S_{jy} \\
&= 0
\end{aligned}$$

The other components follow identically.

(b) List all of the allowed energy levels for the system and the degeneracy factors of each level. Please note, you do not have to construct any eigenstates of energy explicitly.

We can find the allowable J states by taking the tensor product:

$$\begin{gathered}
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \\
(1 \oplus 0) \otimes \frac{1}{2} \otimes \\
\left(\frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} \right) \otimes \frac{1}{2}
\end{gathered}$$

The allowable J states are 0,1,2 and to get the energy levels are proportional to J^2.

Question III.

A particle of mass M is constrained to move on the surface of a sphere of radius r. Its dynamics can be described by a free Hamiltonian H_0 and a set of free eigenstates $Y_l^m(\theta, \phi)$. The sphere is then embedded into a uniform gravitational field with acceleration g directed along the $-z$ axis, so that the particle experiences the potential

$$V(\theta, \phi) = mgr\cos\theta$$

Compute the values of all non-zero matrix elements of the potential operator $V(\theta, \phi)$ in a basis consisting of those $Y_l^m(\theta, \phi)$ that have $l = 0, 1, 2$. Do not attempt to work out every single case, there are 81 of them. use symmetry arguments to save work.

$$\begin{aligned} V(\theta, \phi) &= mgr\cos\theta \\ &\propto Y_1^0 \\ &\propto T_1^0 \end{aligned}$$

The matrix elements go like

$$\begin{aligned} \langle j'm' | V | jm \rangle &\propto \langle j'm' | T_1^0 | jm \rangle \\ &\propto \langle j'm' | kqjm \rangle \end{aligned}$$

We have nonzero elements when the inequality $j + k \geq j \geq |j - k|$ is true, and $m' = m + q$. We can generically calculate the matrix elements of the potential by calculating

$$\langle l'm' | V | lm \rangle = \int_0^{2\pi} \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, Y_{l'}^{m'*} V Y_l^m$$

We will do the following example:

$$\begin{aligned} &\langle j'm' | T_1^0 | 1m \rangle \\ &\qquad k = 0 \quad q = 0 \\ &\qquad 1 + 1 \geq j' \geq 0 \\ &\qquad m' = m \end{aligned}$$

The allowable m values are $m = -1, 0, 1 = m'$ and allowable j' are $2 \geq j' \geq 0$. Computing the matrix element for $j' = 2$, $m' = -1$ with $j = 1$ and $m = -1$ gives

$$\langle 2 - 1 | V | 1 - 1 \rangle = \int_0^{2\pi} \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, Y_1^{-1*} V Y_1^{-1}$$

We can compute this for all elements of the potential operator.

Question IV.

A particle of mass m is bound in a central potential $V(r)$. The eigenstates of the system and the energy eigenvalues are $\psi_n(\vec{r})$ and E_n respectively. Now a **constant** magnetic field is turned on. Let us only consider the interaction between the magnetic field and the angular momentum of the particle. This interaction is described by the following additional term in the Hamiltonian

$$\delta H = -g\vec{B}\cdot\vec{L}$$

where $\vec{L}$ is the angular momentum of the particle and g is a constant.

(a) Show that the magnetic perturbation commutes with the original Hamiltonian, $[H, \delta H] = 0$.

$$\begin{aligned}
[H,\delta H] &= \left[\frac{P^2}{2m} + V, -gB\cdot L\right] \\
&= -g\sum_i \left[\frac{P^2}{2m} + V, B_i L_i\right] \\
&= -g\sum_i \left(\left[\frac{P^2}{2m}, B_i L_i\right] + [V, B_i L_i]\right) \\
&= -g\sum_i \left(\left[\frac{P^2}{2m}, B_i\right] L_i + B_i\left[\frac{P^2}{2m}, L_i\right] + [V, B_i]L_i + B_i[V, L_i]\right)
\end{aligned}$$

Each of these four commutators is zero since the magnetic field is constant and a rotationally symmetric Hamiltonian commutes with the components of the angular momentum operator.

(b) Explain how the eigenfunctions and energy eigenvalues of the new Hamiltonian $H + \delta H$ are related to those of the original Hamiltonian H.

Assume that for eigenvectors of the unperturbed Hamiltonian, there are corresponding eigenvectors of the perturbed Hamiltonian $|n\rangle$ with energy E_n, where these can be expanded as $|n\rangle = |n^0\rangle + |n^1\rangle + ...$, and $E_n = E_n^0 + E_n^1 + ...$ We now have the eigenvalue equation $H|n\rangle = E_n|n\rangle = (E_n^0 + ...)(|n^0\rangle + ...)$. If we take the first order terms we get

$$H^0|n^1\rangle + H^1|n^0\rangle = E_n^0|n^1\rangle + E_n^1|n^0\rangle$$

Dotting both sides with $\langle n^0|$ yields $E_n^1 = \langle n^0|H^1|n^0\rangle$ says the first order change in energy is equal to the expectation value of the perturbed Hamiltonian in the unperturbed state. If you instead dot with $\langle m^0|$ for $m \neq n$ we come to

$$\langle m^0|n^1\rangle = \frac{\langle m^0|H^1|n^0\rangle}{E_n^0 - E_m^0}$$

For $m \neq n$, this determines all components of $|n^1\rangle$ in the eigenbasis of the unperturbed Hamiltonian.

(c) Now let us assume that the magnetic field slowly varies in space, so that we have $B^i(\vec{r}) \approx B_0^i + b^{ij}r_j$. Consider the set of matrix elements

$$M_n \equiv \langle 1|\delta H|n\rangle$$

where $|1\rangle$ is the first excited state of the nonperturbed system, which is also the eigenstate of L^2 with the eigenvalue 2 (in the units where $\hbar = 1$); i.e. $L^2\,|1\rangle = 2\,|1\rangle$. Using the Wigner-Eckart theorem explain what are the conditions on the state $|n\rangle$ for the matrix element M_n to be nonzero. Can you further restrict the set of states $|n\rangle$ by using the parity symmetry of H?

Question V.

Shankar exercise 15.3.5

We have

$$\langle j|T_k^q|jm\rangle = \langle jm|kq, jm\rangle\ \langle j||T_k||j\rangle$$

This vanishes if the inequality $j + k \geq j \geq |j - k|$ is not true, which fails immediately if $k > 2j$ since $|j - k| = k - j > j$.