6710 Problem Set 3

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Question I.

Most main-sequence stars are in hydrostatic equilibrium, have an equation of state dominated by the ideal gas law, and have energy transport dominated by radiation.

(a) Simplify the energy transport equation as $dT/dr \sim -T_e/R$ and with $T_{eff} \propto T_c$. using the appropriate opacity for a hot ionized star, what is the proportionality of the central temperature T_c with total mass M, luminosity L, and radius R?

$$\begin{aligned}
\frac{\partial T}{\partial r} &\rightarrow -\frac{T_c}{R} \\
&\propto -\kappa \frac{L\rho}{R^2 T_c^3} \\
\frac{T_c}{R} &\propto -\kappa \frac{ML}{R^5 T_c^3} \\
T_c &\propto -\kappa \frac{ML}{R^4 T_c^3} \\
T_c^4 &\propto -\kappa \frac{ML}{R^4} \\
T_c &\propto -\left(\kappa \frac{ML}{R^4}\right)^{1/4}
\end{aligned}$$

where $\kappa = \kappa_{es} = \frac{\sigma_{SB}}{\mu_e m_h} = 0.20(1+X)$ cm^{2}g^{-1}

(b) Now derive the proportionality of T_c with M and R using the hydrostatic equilibrium equation and the ideal gas law. Again, you can simplify by assuming that $dP/dr \sim -P_c/R$ and that the central density $\rho_c \propto \bar{\rho}$.

$$\begin{aligned}
\frac{\partial P}{\partial r} &\rightarrow -\frac{P_c}{R} \\
&\propto -\frac{GM\rho}{R^2} \\
\frac{\rho k T}{\mu m_h R} &\propto -\kappa \frac{GM\rho}{R^2} \\
T &\propto \frac{M}{R}
\end{aligned}$$

ignoring the constants.

(c) Combine the two equations above to derive the proportionality of L with M and R for a typical main sequence star.

Combining gives

$$\frac{M}{R} \propto \left(\frac{ML}{R^4}\right)^{1/4}$$
$$L \propto M^3$$

(d) The observed luminosity-mass relation is actually $L \propto M^{3.8}$ (see the notes from Lecture 1). What is the maximum mass of a star with Solar abundance ($X = 0.7$) before it exceeds the Eddington limit and blows off its outer layer?

$$\begin{aligned}
L_{edd} &= \frac{4\pi Gcm}{\kappa} \\
\kappa_{es} &= 0.20(1+X)\ \mathrm{cm^2 g^{-1}} \\
L_{edd} &= 1.47\times 10^4 \left(\frac{M}{M_\odot}\right)\ \mathrm{erg/s} \\
&= 3.8\times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot \\
L &\propto M^3 \\
\left(\frac{M}{M_\odot}\right)^3 &= 3.7\times 10^4 \left(\frac{M}{M_\odot}\right) \\
\left(\frac{M}{M_\odot}\right)^2 &= 3.7\times 10^4 \\
M_{max} &= 195\ M_\odot
\end{aligned}$$

Question II.

Energy generated by fusion in the center of a star travels slowly outward via random-walking photons. What is the average time for energy to reach the surface of the following stars? You can assume that the stars have Solar composition, are almost entirely hot and ionized, and have uniform opacity and density at all radii. (Uniform opacity and density is a bad assumption, and is why your calculations will underestimate the actual energy escape time by a factor of $\gtrsim 100$!)

From Carroll & Ostlie: $d = l\sqrt{N}$, where d is the straight line distance between two points, l is the mean free path of the particle, and N is the number of interactions. If $d = R$ the radius of the star, then we have:

$$\begin{aligned}
N &= \frac{R^2}{l^2} \\
D &= Nl \\
&= \frac{R^2}{l} \\
t &= \frac{D}{c} \\
&= \frac{R^2}{l^2} \\
&= \frac{3\kappa}{4\pi c}\frac{M}{R}
\end{aligned}$$

where D is the total distance traveled by the particle and κ is the opacity for which we assume $\kappa = \kappa_{es} = 0.20(1 + X)$ cm^{2}g^{-1}. We can use this framework to solve all of these calculations immediately:

(a) the Sun $t = 7.7 \times 10^{10}$ s

(b) Sirius A: $M = 2M_{\odot}$, $R = 1.7R_{\odot}$ $t = 9.1 \times 10^{10}$ s

(c) Betelgeuse: $M \simeq 20M_{\odot}$, $R \simeq 100R_{\odot}$ $t = 1.5 \times 10^{10}$ s

Question III.

Main sequence stars fuse hydrogen into helium by one of two pathways: p-p chain and the CNO cycle. These reactions can be approximated as:

$$\epsilon_{pp} \simeq \epsilon'_{0,pp}\rho X^2 T_6^4$$
$$\epsilon_{CNO} \simeq \epsilon'_{0,CNO}\rho X X_{CNO} T_6^{19.9}$$

These equations neglect electron screening, alternate reaction chains, and higher-order terms. The variables are density ρ, temperature $T_6 = T/(10^6 K)$, hydrogen abundance $X = 0.7$, abundance of C+N+O $X_{CNO} = 0.0081$, $\epsilon'_{0,pp} = 1.08 \times 10^{-7}$ erg s^{-1}cm^{3}g^{-2}, and $\epsilon'_{0,CNO} = 8.24 \times 10^{-26}$ erg s^{-1}cm^{3}g^{-2}. What is the fraction of energy generated by each process in the following stars?

(a) the Sun with $T_c = 15.7 \times 10^6$K, $\rho_c = 150$ g/cm^3.

We get

$$\epsilon_{pp} \simeq 0.48 \text{ erg/s/g}$$
$$\epsilon_{CNO} \simeq 0.044 \text{ erg/s/g}$$

The fraction of the total energy generation would be:

$$\frac{\epsilon_{pp}}{\epsilon_{CNO}} = 10.9$$

(b) Sirius A, with $M = 2.06M_{\odot}$ and $R = 1.71R_{\odot}$. Use this mass and radius to estimate T_c from the relation derived in problem 1b, and similarly scale the central density from the Solar value using the ratio between the mean density of Sirius and the Sun.

From 1b we know $T \propto M/R$, so we can say $T_c = 18.9 \times 10^6$K, and the density goes like M/R^3 so the central density of Sirius A is $\rho_c = 62.85$ g/cm^3. We can also assume that since $L \propto M^{3.8}$ then we have a luminosity of $L_* = 15.6L_{\odot}$, and use the same framework as before.

$$\epsilon_{pp} \simeq 0.42 \text{ erg/s/g}$$
$$\epsilon_{CNO} \simeq 0.74 \text{ erg/s/g}$$

The fraction of the total energy generation would be:

$$\frac{\epsilon_{pp}}{\epsilon_{CNO}} = 0.57$$

(c) Assuming that both stars have the same fraction of mass available for hydrogen burning (i.e., Sirius A has 2.06 times more mass available for hydrogen fusion), what is the ratio between the total H-burning fusion energy output of Sirius compared to the Sun? Why is this different from the ratio of the surface luminosities observed as $L \propto M^{3.8}$?

The ratio of the total H-burning fusion energy output would be

$$\begin{aligned} \frac{(\epsilon_{pp_{Sirius}} + \epsilon_{CNO_{Sirius}})M_{Sirius}}{(\epsilon_{pp_{\odot}} + \epsilon_{CNO_{\odot}})M_{\odot}} &\approx 4.5 \\ L_{Sirius} &\propto M_{Sirius}^{3.8} \\ &\approx 15.6\ L_{\odot} \end{aligned}$$

The ratio of the total H-burning fusion energy output of Sirius to the Sun is not equivalent to the ratio of the surface luminosities because Sirius and the Sun are dominated by different fusion processes, being in different mass regimes and different compositions, so the simple scaling assumptions that we use are not correct. This leads to an incorrect extension of the temperature of Sirius, and the CNO cycle is highly temperature dependent.

Question IV.

Massive $\gtrsim 100M_{\odot}$ stars have an equation of state dominated by radiation pressure rather than the ideal gas law. How is L proportional to M and R for massive stars?

We start off the same as in 1b with the (questionable) assumption

$$\begin{aligned} \frac{\partial P}{\partial r} &\to -\frac{P_c}{R} \\ &\propto -\frac{GM\rho}{R^2} \end{aligned}$$

We now let P_c go to the radiation pressure so we have

$$\begin{aligned} P_c &\propto T^4 \\ \frac{T^4}{R} &\propto \frac{M\rho}{R^2} \\ T^4 &\propto \frac{M\rho}{R} \\ &\propto \frac{M^2}{R^4} \\ T &\propto \frac{M^{1/2}}{R} \end{aligned}$$

Connecting back to 1a

$$\begin{aligned} L &\propto \frac{T^4R^4}{M} \\ &\propto \frac{M^2R^4}{R^4M} \\ L &\propto M \end{aligned}$$