5500 Problem Set 8

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Question I.

Think about a quantum mechanical ideal gas of massive particles; in fact, this argument works equally well for Bose-Einstein, Fermi-Dirac, and Maxwell-Boltzmann gases. It is well known that, in a monatomic ideal gas where center-of-mass motion is the only degree of freedom, the energy of a one-particles state scales with the containing volume as $V^{-2/3}(\frac{\hbar^2}{mL^2}...)$, and in an ideal gas the same naturally applies to many-body energy eigenstates.

(a) If the container contracts or expands, the quantum states change. Suppose, however, that the probabilities of the quantum states remain unchanged, i.e., in the expression $\rho(V) = \sum_n |n(V)\rangle\, p_n\, \langle n(V)|$ the p_n remain constants. This is a model for an adiabatic process. Explain why.

Show that $\mathrm{d}S = 0$, using the von Neumann entropy in the initial state $|n(0)\rangle$:

$$\begin{aligned} S &= -k\,\mathrm{Tr}\{\rho \ln \rho\} \\ &= -k \sum_n \langle n(0)|n(0)\rangle\, p_{n_0}\, \langle n(0)|n(0)\rangle \ln p_{n_0}\, \langle n(0)|n(0)\rangle \\ &= -k \sum p_n \ln p_n \end{aligned}$$

In a final state $|n(V)\rangle$, we have

$$\begin{aligned} S &= -k\,\mathrm{Tr}\{\rho \ln \rho\} \\ &= -k \sum_n \langle n(V)|n(V)\rangle\, p_{n_V}\, \langle n(V)|n(V)\rangle \ln p_{n_V}\, \langle n(V)|n(V)\rangle \\ &= -k \sum p_n \ln p_n \end{aligned}$$

We have the same entropy in either state, therefore $\mathrm{d}S = 0$, thus this is an adiabatic process.

(b) Show that the state of the gas satisfies $pV = \frac{2}{3}U$. If we begin with the particle number

$$\begin{aligned} N &= \frac{gV}{(2\pi)^3} \int \mathrm{d}^3k \frac{ze^{-\beta\epsilon_k}}{1 - ze^{-\beta\epsilon_k}} \\ &= \frac{gV}{\lambda^3} g_{3/2}(z) \end{aligned}$$

We then have the density $n = N/V$

$$n = \frac{g}{\lambda^3} g_{3/2}(z)$$

and the corresponding internal energy

$$\begin{aligned}U &= \sum_i \langle n_i \rangle \epsilon_i \\ &= \frac{3}{2}\frac{gVkT}{\lambda^3} g_{5/2}(z)\end{aligned}$$

Attacking from the other side we have

$$\begin{aligned}\Omega &= -pV \\ &= -kT \ln \mathcal{Z} \\ &= -kT \sum_i \ln\left(1 - ze^{-\beta\epsilon_k}\right) \\ &= \frac{gV}{(2\pi)^3} \int \mathrm{d}^3k \ln 1 - ze^{-\beta\epsilon_k} \\ &= -\frac{gVkT}{\lambda^3} g_{5/2}(z)\end{aligned}$$

So we see the desired $pV = \frac{2}{3}U$.

Question II.

The familiar density of energy eigenstates $D(\epsilon)$ is defined so that it converts sums over energy eigenstates to integrals,

$$\sum_i g(\epsilon_i) \simeq \int \mathrm{d}\epsilon \, D(\epsilon) g(\epsilon)$$

The approximation is the better, the more states are involved in the sum. Suppose now that $D(\epsilon) \propto \epsilon^\alpha$ as $\epsilon \to 0$

(a) Argue that noninteracting bosons are liable to condense in a system only if the density of states is characterized by an exponent $\alpha > 0$.

From $D(\epsilon) = \gamma\epsilon^\alpha$

$$\begin{aligned}N &= \sum_i \langle n_i \rangle \\ &\to \gamma \int_0^\infty \mathrm{d}\epsilon \, \frac{\epsilon^\alpha}{e^{\beta\epsilon} - 1} \\ &= \frac{\gamma}{\beta^{1+\alpha}} \int_0^\infty \mathrm{d}x \, \frac{x^\alpha}{e^x - 1}\end{aligned}$$

which is divergent for $x \to 0$ for positive α.

(b) Argue that for a free massive particle in D dimensions, $\alpha = (D-2)/2$ holds true. We have $\epsilon_k = \frac{\hbar^2 k^2}{2m}$, so

$$
\begin{aligned}
\sum_k g(\epsilon_k) &\to \int \mathrm{d}^D k g(\epsilon_k) \\
&= \left(\frac{2m}{\hbar^2}\right)^{(D-1)/2} \sqrt{\frac{m}{\hbar^2}} \int_0^\infty \mathrm{d}\epsilon \, \epsilon^{(D-1)/2} g(\epsilon)
\end{aligned}
$$

ignoring an overall constant.

(c) The density of states is related to the number of energy eigenstates with energy less than or equal to ϵ, $N(\epsilon)$, by $D(\epsilon) = \frac{\mathrm{d}N(\epsilon)}{\mathrm{d}\epsilon}$. On the basis of this observation, argue that for massive particles in a harmonic oscillator potential in D dimensions the exponent is $\alpha = D - 1$.

The total energy is given by

$$
\epsilon = \hbar\omega \sum_i n_i + \frac{D}{2}
$$

we can make the substitution $\epsilon' = \epsilon - \hbar\omega\frac{D}{2}$. We then have

$$
\begin{aligned}
N(\epsilon) &\to \int_0^m \mathrm{d}n_1 \int_0^{m-n_1} \mathrm{d}n_2 \cdots \int_0^{m-\sum_i^{D-1}} \mathrm{d}n_D \\
&= \frac{1}{D!} m^D \\
D(\epsilon) &= \frac{\mathrm{d}N(\epsilon)}{\mathrm{d}\epsilon} \\
&= \frac{1}{\hbar\omega} \frac{m^{D-1}}{(D-1)!} \\
&\propto \epsilon^{D-1}
\end{aligned}
$$

(d) Do you have Bose-Einstein condensation in $D = 1, 2$, or 3 dimensions if the particles are free? What if they are trapped in a harmonic potential well?

For free particles we need $D - 2 > 0$, so only for three dimensions do free particles condense. In a harmonic oscillator potential we need $D - 1 > 0$ so Bose condensation can happen in 2 and 3 dimensions.