5402 Problem Set 4

Nikko Cleri

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Question I.

17.2.2

$$
\begin{aligned}
H &= H_0 + H' \\
&= -\gamma B_0 S_z - \gamma B S_x
\end{aligned}
$$

For the first order we have

$$
\begin{aligned}
E_0^{(1)} &= \langle 0|H'|0\rangle \\
&= \gamma B \langle +|S_x|+\rangle \\
&= 0
\end{aligned}
$$

And to second order we have

$$
\begin{aligned}
E_+^{(2)} &= \frac{|\langle -|H'|+\rangle|^2}{E_+^{(0)} - E_-^{(0)}} \\
&= -\frac{\gamma\hbar B^2}{4B_0} \\
E_+^{(2)} &= \frac{\gamma\hbar B^2}{4B_0}
\end{aligned}
$$

The corrections to the eigenstates are given by

$$
\begin{aligned}
\left|+^{(1)}\right\rangle &= -\frac{\langle -|H'|+\rangle}{E_+^{(0)} - E_-^{(0)}} |-\rangle \\
&= \frac{B}{2B_0} |-\rangle \\
\left|-^{(1)}\right\rangle &= -\frac{B}{2B_0} |+\rangle
\end{aligned}
$$

The exact solutions can be found by diagonalizing the Hamiltonian

$$
H = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_0 & B \\ B & -B_0 \end{pmatrix}
$$

the eigenvalues are

$$
\lambda_\pm = \mp\frac{\gamma\hbar}{2}\sqrt{B_0^2 + B^2}
$$

The ground state eigenvectors (unnormalized) are

$$\begin{pmatrix} 1 \\ \frac{\sqrt{B_0^2+B^2}-B_0}{B} \end{pmatrix} \approx \begin{pmatrix} 1 \\ \frac{B}{2B_0} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{B_0^2+B^2}-B_0}{B} \\ 1 \end{pmatrix} \approx \begin{pmatrix} -\frac{B}{2B_0} \\ 1 \end{pmatrix}$$

Question II.

17.2.3

(1) From the charge density $\rho = \frac{e}{\frac{4}{3}\pi R^3}$ we get the electric field from Gauss' law as

$$\begin{aligned} E(r) &= \frac{er}{R^3} \quad r < R \\ &= \frac{e}{r^2} \quad r > R \end{aligned}$$

Since $\vec{E} = -\boldsymbol{\nabla}\phi$, we have

$$\begin{aligned} \phi_<(r) - \phi_<(0) &= -\frac{er^2}{R^3} \\ \phi_>(r) &= \frac{e}{r} \\ \phi(R) &= \frac{e}{R} \end{aligned}$$

So

$$\phi_<(0) = \frac{3e}{2R}$$

and saying $V = e\phi$ gives the desired result.

(2) We find the shift in the ground state energy as

$$\begin{aligned} E'_{100} &= \langle 100|H'|100\rangle \\ &= \langle 100|(-\frac{3e^2}{2R} + \frac{e^2r^2}{2R^3} + \frac{e^2}{r})\Theta(R-r)|100\rangle \\ &= \frac{1}{\pi a_0^3}\int_0^R 4\pi r^2 \, \mathrm{d}r \, (-\frac{3e^2}{2R} + \frac{e^2r^2}{2R^3} + \frac{e^2}{r})e^{2r/a_0} \end{aligned}$$

Using the simplification and doing the substitution $u = r/R$ gives the desired result.

Question III.

The Hamiltonian is given by

$$\begin{aligned} H &= \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + H_1 \\ H_1 &= -qxE(t) \end{aligned}$$

(a) Finding the amplitude $c_n(t)$ for finding the system in excited state n at time t

$$\begin{aligned}
c_n(t) &= -\frac{i}{\hbar}\int_{-\infty}^{t} \left\langle n^0 \middle| H_1(t') \middle| 0^0 \right\rangle e^{i\omega_{n0}t'}\, \mathrm{d}t' \\
\omega_{fi} &= \frac{E_f^0 - E_i^0}{\hbar} \\
c_n(t) &= \frac{iq}{\hbar}\int_{-\infty}^{t} E(t') \left\langle n^0 \middle| x \middle| 0^0 \right\rangle e^{i\omega_{n0}t'}\, \mathrm{d}t' \\
&= \frac{iq}{\hbar} \left\langle n^0 \middle| x \middle| 0^0 \right\rangle \int_{-\infty}^{t} E(t') e^{i\omega_{n0}t'}\, \mathrm{d}t'
\end{aligned}$$

the matrix element gives

$$\begin{aligned}
\left\langle n^0 \middle| x \middle| 0^0 \right\rangle &= \langle n| \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a + a^\dagger)\, |0\rangle \\
&= \left(\frac{\hbar}{2m\omega}\right)^{1/2} \langle n|(a + a^\dagger)|0\rangle \\
&= \left(\frac{\hbar}{2m\omega}\right)^{1/2} \delta_{n1}
\end{aligned}$$

gives

$$c_n(t) = \delta_{n1}\frac{iq}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{1/2} \int_{-\infty}^{t} E(t') e^{i\omega_{n0}t'}\, \mathrm{d}t' \tag{1}$$

(b) Find the expectation value of the momentum of the oscillator as a function of time.

$$\begin{aligned}
\langle \psi(t)|p|\psi(t)\rangle &= \sum_{n,n'} \langle \psi|\, c_n^*(t) e^{iE_n^0 t/\hbar} p c_{n'}(t) e^{-iE_{n'}^0 t/\hbar}\, |\psi\rangle \\
&= \ \langle 1||c_1(t)|^2 e^{iE_1^0 t/\hbar} p e^{-iE_1^0 t/\hbar}|1\rangle \\
&= |c_1(t)|^2 \langle 1|p|1\rangle \\
p &= i\left(\frac{2m\omega}{\hbar}\right)^{1/2} (a^\dagger - a)
\end{aligned}$$

gives

$$\begin{aligned}
\langle \psi(t)|p|\psi(t)\rangle &= |c_1(t)|^2 \langle 1|p|1\rangle \\
&= i\left(\frac{2m\omega}{\hbar}\right)^{1/2} |c_1(t)|^2 \langle 1|(a^\dagger - a)|1\rangle \\
&= i\left(\frac{2m\omega}{\hbar}\right)^{1/2} |c_1(t)|^2 (\langle 1|2\rangle - \langle 1|0\rangle) \\
&= 0
\end{aligned}$$

(c) The classical solution is

$$p(t) = \mathrm{Re}\left[\int_{-\infty}^{t} \mathrm{d}t'\, e^{-i\omega(t-t')} qE(t')\right]$$

This integral is similar to the transition amplitude $c_n(t)$, which we found earlier. This integral is dependent on unknown $E(t)$ and t, and whether or not the transition is pure.

Question IV.

The Hamiltonian is given by

$$\begin{aligned}\frac{H}{\hbar} &= \Delta\, |2\rangle\langle 2| + f(t)(|2\rangle\langle 1| + |1\rangle\langle 2|)\\ f(t) &= \frac{\lambda}{2\sqrt{\pi}\tau}\left(\exp\left\{-\left(\frac{t+T/2}{\tau}\right)^2\right\} + \exp\left\{-\left(\frac{t-T/2}{\tau}\right)\right\}\right)^2\end{aligned}$$

(a) We find the coefficients $c_1(t)$ and $c_2(t)$.

$$\begin{aligned}c_1(t) &= \delta_{11} - \frac{i}{\hbar}\int \left\langle 1^0\middle|H_1(t')\middle|1^0\right\rangle e^{i\omega_{11}t'}\,\mathrm{d}t'\\ \left\langle 1^0\middle|H_1(t')\middle|1^0\right\rangle &= \left\langle 1^0\middle|f(t')(|2\rangle\langle 1| + |1\rangle\langle 2|)\middle|1^0\right\rangle\\ &= 0\\ c_1(t) &= 1\end{aligned}$$

For $c_2(t)$ we have

$$\begin{aligned}c_2(t) &= \delta_{21} - \frac{i}{\hbar}\int \left\langle 2^0\middle|H_1(t')\middle|1^0\right\rangle e^{i\omega_{21}t'}\,\mathrm{d}t'\\ &= -\frac{i}{\hbar}\int \left\langle 2^0\middle|f(t')(|2\rangle\langle 1| + |1\rangle\langle 2|)\middle|1^0\right\rangle e^{i\omega_{21}t'}\,\mathrm{d}t'\\ &= -\frac{i}{\hbar}\int f(t')e^{i\omega_{21}t'}\,\mathrm{d}t'\end{aligned}$$

if we assume bounds of some time $[\alpha/2, \alpha/2]$, we get

$$c_2(t) = -\frac{i}{\hbar}\int_{\alpha/2}^{\alpha/2} f(t')e^{i\omega_{21}t'}\,\mathrm{d}t'$$

Mathematica gives the result in Figure 1, with a factor of $\frac{\lambda}{2\sqrt{\pi}\tau}$. We then multiply by the propagator $e^{-iE_nt/\hbar}$ for each of the coefficients.

(b) This probability is given by $|c_2(t)|^2$ (declining to do the litany of algebra here).

Question V.

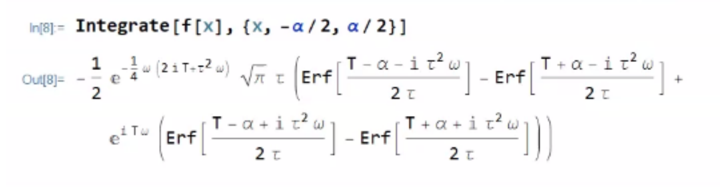

Figure 1: Integral for $c_2(t)$ in question 4

Rigid rotator in electric field has the Hamiltonian

$$H = \frac{\mathbf{L}^2}{2I} - \mathbf{d}\cdot\mathbf{E}(t)$$
$$\mathbf{E}(t) = \hat{\mathbf{z}}E_0 e^{-t/\tau}$$

(a) Find the allowed transitions and probabilities.

$$c_n(t) = -\frac{i}{\hbar}\int_0^t \left\langle f^0\middle|H_1(t')\middle|i^0\right\rangle e^{i\omega_{fi}t'}\,\mathrm{d}t'$$
$$\omega_{fi} = \frac{E_f^0 - E_i^0}{\hbar}$$
$$= \frac{i}{\hbar}\int_0^t \left\langle f^0\middle|\mathbf{d}\cdot\mathbf{E}(t)\middle|i^0\right\rangle e^{i\omega_{fi}t'}\,\mathrm{d}t'$$
$$= \frac{iq}{\hbar}\int_0^t \left\langle f^0\middle|\mathbf{z}E_z(t)\middle|i^0\right\rangle e^{i\omega_{fi}t'}\,\mathrm{d}t'$$

The states of the unperturbed Hamiltonian are the spherical harmonics

$$\left|i^0\right\rangle = Y_l^m(\theta,\phi)$$
$$\left\langle f^0\right| = Y_{l'}^{m'*}(\theta,\phi)$$

From Wigner-Eckart we have $\left\langle Y_{l'}^{m'}\middle|T_1^0\middle|Y_l^m\right\rangle \neq 0$ for $m' = m = 0$ and $l' = 1$ so the only nonzero final state is Y_1^0. Our coefficient becomes

$$c_n(t) = \frac{iq}{\hbar}\left\langle Y_1^0\middle|\mathbf{z}\middle|Y_0^0\right\rangle\int_0^t E_z(t)e^{i\omega_{fi}t'}\,\mathrm{d}t'$$
$$\left\langle Y_1^0\middle|\mathbf{z}\middle|Y_0^0\right\rangle = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{1}{2}\sqrt{\frac{1}{\pi}}\int \cos\theta r\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= \frac{\sqrt{3}r}{2}\int\cos^2\theta\sin\theta\,\mathrm{d}\theta$$
$$= -\frac{\sqrt{3}r}{6}\int u^2\,\mathrm{d}u$$
$$= -\frac{\sqrt{3}r}{6}\cos^3\theta\Big|_0^\pi$$
$$= \frac{\sqrt{3}r}{6}$$

So the transmission coefficient becomes

$$c_1(t) = \frac{iq\sqrt{3}r}{6\hbar}\int_0^t E_z(t)e^{i\omega_{10}t'}\,\mathrm{d}t'$$

Where the transition probability is $|c_1(t)|^2$.

(b) If we started at $L = 3$ then we have from Wigner-Eckart that $\left\langle Y_{l'}^{m'}\middle|T_1^0\middle|Y_3^m\right\rangle \neq 0$ when $m' = m$ and $|l-1| \leq l' \leq l+1$.

Question VI.

(i) We can find the selection rules come from the inner product with the perturbation as

$$\begin{aligned}
\left\langle n'l'm'\middle|eL_xB(t)/2Mc\middle|nlm\right\rangle &= \frac{eB(t)}{2Mc}\left\langle n'l'm'\middle|L_x\middle|nlm\right\rangle \\
&= \frac{eB(t)}{4Mc}\left\langle n'l'm'\middle|L_+ + L_-\middle|nlm\right\rangle \\
&= \frac{eB(t)}{4Mc}\left[\left\langle n'l'm'\middle|L_+\middle|nlm\right\rangle + \left\langle n'l'm'\middle|L_-\middle|nlm\right\rangle\right]
\end{aligned}$$

gives us the selection rules that these are nonzero for $m' = m \pm 1$.

(ii) The energy differences for the nonperturbed state are given by the Rydberg energy $E = \frac{R}{n^2}$ which will have the addition of the perturbation energy E' associated with the magnetic field.

(iii) The transition probabilities are given by the square of the transition amplitudes which have the form

$$\begin{aligned}
c_n(t) = \delta_{nn'} - \frac{i}{\hbar}&\int \left\langle n'l'm'\middle|H_1(t')\middle|nlm\right\rangle e^{i\omega_{nn'}t'}\,\mathrm{d}t' \\
= &\int B(t')e^{i\omega_{nn'}t'}\,\mathrm{d}t' \\
= &\int B_0e^{-\lambda t'}e^{i\omega_{nn'}t'}\,\mathrm{d}t'
\end{aligned}$$

Since the inner products will behave as in part (1) of the question. This integral gives

$$\int e^{-\lambda t'}e^{i\omega_{nn'}t'}\,\mathrm{d}t' = \frac{1}{\lambda - i\omega_{nn'}}e^{-\lambda t'}e^{i\omega_{nn'}t'}$$

The transition probabilities are $|c_n(t)|$ for each of the cases allowed in part (i).

Question VII.

18.2.5

We have the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - fx$$

which is a harmonic oscillator Hamiltonian with a shifted constant energy and translated equilibrium position. The ground state has a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(x - \frac{f}{m\omega^2}\right)^2 - \frac{1}{2m\omega^2}f^2$$

with energy $E_0 = \frac{\hbar^2\omega}{2} - \frac{f^2}{2m\omega^2}$ with wavefunction

$$\begin{aligned}
\psi_0 &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}}\left(x - \frac{f}{m\omega^2}\right)^2 \\
|0\rangle &= T\left(\frac{f}{m\omega^2}\right)\left|0^0\right\rangle \\
&= e^{\frac{i}{\hbar}\frac{f}{m\omega^2}P}\left|0^0\right\rangle
\end{aligned}$$

The probability is then

$$\begin{aligned}
P(n) &= \left|\left\langle n^0|0\right\rangle\right|^2 \\
&= \left|\left\langle n^0\right|e^{\frac{i}{\hbar}\frac{f}{m\omega^2}P}\left|0^0\right\rangle\right|^2 \\
&= \left|\left\langle n^0\right|e^{\frac{f}{\sqrt{2\hbar m\omega^2}}(a^\dagger - a)}\left|0^0\right\rangle\right|^2 \\
&= \left|\left\langle n^0\right|e^{\sqrt{\lambda}(a^\dagger - a)}\left|0^0\right\rangle\right|^2
\end{aligned}$$

Given the hint from the problem we have

$$e^{\sqrt{\lambda}(a^\dagger - a)} = e^{-\frac{\lambda}{2}}e^{\sqrt{\lambda}a^\dagger}e^{-\sqrt{\lambda}a}$$

So we have

$$\begin{aligned}
P(n) &= e^{-\lambda}\left|\left\langle n^0\right|e^{\sqrt{\lambda}a^\dagger}\left|0^0\right\rangle\right|^2 \\
&= e^{-\lambda}\left|\left\langle n^0\right|\sum_m \frac{\lambda^{m/2}}{\sqrt{m!}}\left|m^0\right\rangle\right|^2 \\
&= e^{-\lambda}\left|\frac{\lambda^{m/2}}{\sqrt{m!}}\right|^2
\end{aligned}$$

So the probability is the desired $P(n) = \frac{e^{-\lambda}\lambda^n}{n!}$.

Question VIII.

18.2.6

We have a Hamiltonian subject to the perturbation $H^1(t) = H^1\delta(t)$ with initial state $\left|i^0\right\rangle$ and final state $\left|f^0\right\rangle$.

$$\begin{aligned}
\left|\psi(0^+)\right\rangle - \left|\psi(0^-)\right\rangle &= \frac{i}{\hbar}H^1\left|\psi(0)\right\rangle \\
\left|\psi(0^+)\right\rangle &= \sum_n c_n\left|n^0\right\rangle \\
c_f &= \left\langle f^0\middle|\psi(0^+)\right\rangle \\
&= \left\langle f^0\middle|i^0\right\rangle - \frac{i}{\hbar}\left\langle f^0\middle|H^1\middle|i^0\right\rangle
\end{aligned}$$

The first inner product is zero giving the desired result.