STAT 630 Problem Set 3

Nikko Cleri

September 24, 2021

Question I.

2.3.18

Consider a situation involving a server, e.g., a cashier at a fast-food restaurant, an automatic bank teller machine, a telephone exchange, etc. Units typically arrive for service in a random fashion and form a queue when the server is busy. It is often the case that the number of arrivals at the server, for some specific unit of time t, can be modeled by a Poisson(λt) distribution and is such that the number of arrivals in nonoverlapping periods are independent. In Chapter 3, we will show that λt is the average number of arrivals during a time period of length t, and so λ is the rate of arrivals per unit of time. Suppose telephone calls arrive at a help line at the rate of two per minute. A Poisson process provides a good model.

(a) What is the probability that five calls arrive in the next 2 minutes?

We have the Poisson pdf for Poisson(λt) for a random variable Y counting the number of calls received in a time t.

$$p_Y(y) = \frac{e^{-\lambda t}(\lambda t)^y}{y!} \quad y = 0, 1, 2... \quad \lambda > 0$$

where $\lambda = 2$ calls per minute and t is the time period in minutes. We have $y = 5$ and $t = 2$, so

$$P(Y = 5) = 0.16$$

(b) What is the probability that five calls arrive in the next 2 minutes and then five more calls arrive in the following two minutes?

This is simply the probability that the event in part (a) happens twice in a row, which is $(P(y = 5))^2 = 0.024$

(c) What is the probability that no calls will arrive during a ten minute period?

We have, now with $y = 0$ and $t = 10$,

$$\begin{aligned} P(Y = 0) &= e^{-20} \\ &\approx 2 \times 10^{-9} \end{aligned}$$

Question II.

2.4.4 (a,b,c) For each part, once you have the value for c, write an expression for $f(x)$ that is valid for all real x, using an indicator function as needed. Also, find the cdf for each.

Establish for which constants c the following functions are densities.

(a) $f(x) = cx$ on (0,1) and 0 otherwise

f is a density function if

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x = 1 \\ &= \int_0^1 cx\,\mathrm{d}x \\ &= \frac{c}{2}x^2\Big|_0^1 \\ &= \frac{c}{2}\end{aligned}$$

So f is a valid density if $c = 2$. The cdf is the integral of f, which is just

$$F = x^2 I_{x\in[0,1]}$$

Where

$$I_{x\in[0,1]} = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$$

(b) $f(x) = cx^n$ on (0,1) and 0 otherwise, for n a nonnegative integer

Similarly, we have f to be a valid density function if

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x = 1 \\ &= \int_0^1 cx^n\,\mathrm{d}x \\ &= \frac{c}{n+1}x^{n+1}\Big|_0^1 \\ &= \frac{c}{n+1}\end{aligned}$$

So f is a valid density if $c = n+1$. The cdf is the integral of f, which is just

$$F = x^{n+1} I_{x\in[0,1]}$$

(c) $f(x) = cx^{1/2}$ on (0,2) and 0 otherwise

Again, we integrate:

$$\begin{aligned}\int_0^2 f(x)\,\mathrm{d}x = 1 \\ &= \int_0^2 cx^{1/2}\,\mathrm{d}x \\ &= \frac{2c}{3}x^{3/2}\Big|_0^2 \\ &= \frac{c}{3}2^{5/2}\end{aligned}$$

So f is a valid density if $c = \frac{3}{2^{5/2}}$. The cdf is the integral of f, which is

$$F = x^{3/2} I_{x \in [0,2]}$$

Question III.

2.4.19 Use an indicator function to give an expression for $f(x)$ that is valid for all real x.

Consider, for $a > 0$ fixed, the function given by $f(x) = ax^{a-1}e^{-x^a}$ for $0 < x < \infty$ and 0 otherwise. Prove that f is a density function.

Again, f is a density function if the following is true:

$$\int_0^\infty f(x)\,\mathrm{d}x = 1$$
$$\int_0^\infty ax^{a-1}e^{-x^a}\,\mathrm{d}x = 1$$

We use the substitution $u = x^a \implies \mathrm{d}u = ax^{a-1}\,\mathrm{d}x$, so we have

$$\begin{aligned}\int_0^\infty ax^{a-1}e^{-x^a}\,\mathrm{d}x &= \int_0^\infty e^{-u}\,\mathrm{d}u \\ &= 1\end{aligned}$$

so f is a valid density function.

Question IV.

2.4.22 Hint: split the integral for cases $x \leq 0$ and $x > 0$

Consider the function given by $f(x) = e^{-|x|}/2$ for $-\infty < x < \infty$ and 0 otherwise. Prove that f is a density function.

We once again integrate over $\mathbb{R}$:

$$\begin{aligned}\int_{-\infty}^\infty f(x)\,\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^\infty e^{-|x|}\,\mathrm{d}x \\ &= \frac{1}{2}\left(\int_{-\infty}^0 e^{x}\,\mathrm{d}x + \int_0^\infty e^{-x}\,\mathrm{d}x\right) \\ &= \frac{1}{2} \times 2\int_0^\infty e^{-x}\,\mathrm{d}x \\ &= 1\end{aligned}$$

Since f integrates to unity it is a density function.

Question V.

2.5.3 (a,c,d,f,g). Give reasons if you say "no".

For each of the following functions F, determine whether or not F is a valid cumulative distribution function, i.e., whether or not F satisfies properties (a) through (d) of Theorem 2.5.2.

Theorem 2.5.2 states:

Let F_X be the cumulative distribution function of a random variable X. Then:

1. $0 \leq F_X(x) \leq 1 \ \forall\ x$
2. $F_X(x) \leq F_Y(y)$ whenever $x \leq y$ (F_X is nondecreasing)
3. $\lim_{x\to+\infty} F_X(x) = 1$
4. $\lim_{x\to-\infty} F_X(x) = 0$

(a) $F(x) = x$ for all $x \in R^1$

$F(x)$ is not a valid cdf. This immediately fails conditions 1, 3 and 4.

(c)

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 0 \end{cases}$$

This F is a valid cdf. The limit behavior satisfies 3 and 4, and x^2 is an increasing function (satisfies 2) and is bounded by 0 and 1 for all $x \in \mathbb{R}$ (satisfies 1).

(d)

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$$

This F is not a valid cdf, as it does not satisfy conditions 1 or 2. Example: $F(2) = 4$ and $F(10) = 1$.

(f)

$$F(x) = \begin{cases} 0 & x < 1 \\ x^2/9 & 1 \leq x \leq 3 \\ 1 & x > 1 \end{cases}$$

This is a valid cdf. F is nondecreasing and bounded by 0 and 1, with the proper limit behavior.

(g)

$$F(x) = \begin{cases} 0 & x < -1 \\ x^2/9 & -1 \leq x \leq 3 \\ 1 & x > 1 \end{cases}$$

This F is not a valid cdf as it fails condition 2. Example: $F(-1) = 1/9$ and $F(0) = 0$.

Question VI.

2.5.5. Use the `pnorm` function in R.

Let $Y \sim N(-8, 4)$. Compute each of the following, in terms of the function Φ of Definition 2.5.2 and use Table D.2 (or software) to evaluate these probabilities numerically.

Definition 2.5.2 defines the cdf of the standard normal distribution function as

$$\begin{aligned} \Phi(x) &= \int_{-\infty}^{x} \phi(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,\mathrm{d}t \end{aligned}$$

Which when transformed for an arbitrary $Y \sim N(\mu, \sigma^2)$ gives

$$P(a \leq Y \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

so for our $Y \sim N(-8, 4)$ we have

$$P(a \leq Y \leq b) = \Phi\left(\frac{b-(-8)}{2}\right) - \Phi\left(\frac{a-(-8)}{2}\right)$$

(a) $P(Y \leq -5) \approx 0.933$

(b) $P(-2 \leq Y \leq 5) \approx .0013$

(c) $P(Y \geq 3) \approx 2 \times 10^{-8}$

(d) Find the 40^{th} and 70^{th} percentiles. Use the `qnorm` function in R. The 40^{th} percentile is -8.5, and the 70^{th} percentile is -6.95.

Question VII.

2.5.8. Note: it should say $F_Y(y) = 1 - (1-y)^3$ for $1/2 \leq y \leq 1$ (Why is the definition shown in the book not a valid cdf?)

Suppose $F_Y(y) = y^3$ for $0 \leq y \leq 1/2$ and $F_Y(y) = 1 - y^3$ for $1/2 \leq y \leq 1$. Compute each of the following

(a) $P(1/3 < Y < 3/4) = 1637/1728 \approx 0.95$

(b) $P(Y = 1/3) = 0$

(c) $P(Y = 1/2) = 0$

Question VIII.

2.5.13

Let $F(x) = 0$ for $x < 0$, with $F(x) = 1/3$ for $0 \le x < 2/5$, and $F(x) = 3/4$ for $2/5 \le x < 4/5$, and $F(x) =$ for $x \ge 4/5$.

(a) Sketch a graph of F. See Figure 1.

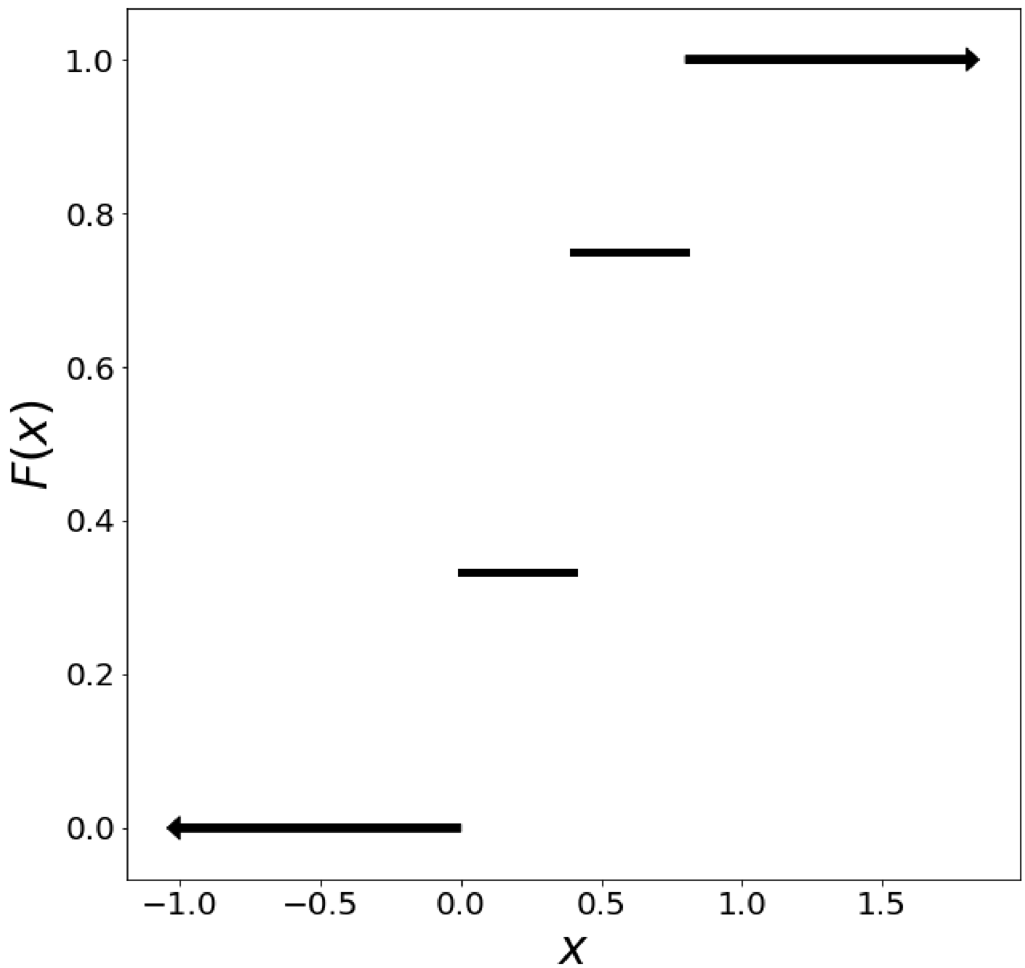

Figure 1: F vs x for question 8 part a.

(b) Prove that F is a valid cdf.

F is a valid cdf if it is (1) bounded by 0 and 1, (2) nondecreasing, (3) goes to 0 when $x \to -\infty$, and (4) goes to 1 when $x \to \infty$.

Conditions 3 and 4 are satisfied immediately given the extreme x behavior; condition 1 is true for all x (visible in Figure 1); and condition 2 is also satisfied (again see Figure 1).

(c) If X has a cumulative distribution function equal to F, then compute $P(X > 4/5)$ and $P(-1 < X < 1/2)$ and $P(X = 2/5)$ and $P(X = 4/5)$

Question IX.

2.5.19

Let Φ be as in Definition 2.5.2. Derive a formula for $\Phi(-x)$ in terms of $\Phi(x)$. (Hint: Let $s = -t$ in (2.5.2), and do not forget Theorem 2.5.2).

The normal distribution exhibits symmetry about the mean, so we can argue that the tails of the distribution are equal (above or below some $\pm x$). This gives

$$\Phi(-x) = 1 - \Phi(x)$$

We know

$$\begin{aligned}\Phi(x) &= \int_{-\infty}^{x} \phi(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,\mathrm{d}t\end{aligned}$$

Question X.

2.5.21

(a) Determine the distribution function of the Weibull(a) distribution of Problem 2.4.19.

The pdf is $f(x) = ax^{a-1}e^{-x^a}$ for $0 < x < \infty$ and 0 otherwise. To find the cdf we integrate:

$$\begin{aligned}F(x) &= \int_{-\infty}^{x} f(x')\,\mathrm{d}x' \\ &= \int_{-\infty}^{x} ax'^{a-1}e^{-x'^a} \\ &= \int_{-\infty}^{x} e^u\,\mathrm{d}u\end{aligned}$$

performing the same substitution as question 3. We then have

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^a} & x \geq 0 \end{cases}$$

(b) Find the quantile function.

To find the quantile function, we invert the cdf:

$$
\begin{aligned}
1 - e^{-x^a} &= p \\
Q &= (-\ln(1-p))^{1/a}
\end{aligned}
$$

Question XI.

2.5.24

(a) Determine the distribution function for the Laplace distribution of Problem 2.4.22.

Again, we integrate to get the cdf.

$$
\begin{aligned}
F(x) &= \frac{1}{2}\int_{-\infty}^{x} f(x')\,\mathrm{d}x' \\
&= \frac{1}{2}\int_{-\infty}^{x} e^{-|x'|}\,\mathrm{d}x' \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} e^{x'}\,\mathrm{d}x' + \int_{0}^{x} e^{-x'}\,\mathrm{d}x'\right] \\
&= \frac{1}{2}\left[1 + \int_{0}^{x} e^{-x'}\,\mathrm{d}x'\right] \\
&= \frac{1}{2}\left[1 + e^{-x} - 1\right] \\
&= \frac{1}{2}e^{-x}
\end{aligned}
$$

(b) Find the quantile function. Hint: consider the cases $p \leq 0.50$ and $p > 0.50$ separately.

Similarly to the previous problem:

$$
\begin{aligned}
\frac{1}{2}e^{-x} &= p \\
Q &= \mp\ln(2p)
\end{aligned}
$$

where we take the $-$ for $p > 0.50$ or the $+$ for $p \leq 0.50$.

Question XII.

2.6.1

Let $X \sim \text{Uniform}[L, R]$. Let $Y = cX + d$, where $c > 0$. Prove that $Y \sim \text{Uniform}[cL + d, cR + d]$.

The density function of X is

$$
f_X(x) = \begin{cases} 1 & x \in [L, R] \\ 0 & x \notin [L, R] \end{cases}
$$

so when X undergoes the transformation we have

$$f_X(x) = \begin{cases} \frac{1-d}{c} & y \in [cL+d, cR+d] \\ 0 & y \notin [cL+d, cR+d]] \end{cases}$$

So $Y \sim \text{Uniform}[cL+d, cR+d]$.

2.6.4

Let $X \sim \text{Exponential}(\lambda)$. Let $Y = cX$, where $c > 0$. Prove that $Y \sim \text{Exponential}(\lambda/c)$.

$$\begin{aligned} f_X(x) &= \lambda e^{-\lambda x} I_{(0,\infty)}(x) \\ f_Y(y) &= \frac{\lambda}{c} e^{-\lambda y/c} I_{(0,\infty)}(y) \end{aligned}$$

So $Y \sim \text{Exponential}(\lambda/c)$.

2.6.9

Let X have density function $f_X(x) = x^3/4$ for $0 < x < 2$, otherwise 0.

(a) Let $Y = X^2$. Compute the density function $f_Y(y)$ for Y

$$\begin{aligned} f_X(x) &= \frac{x^3}{4} \quad 0 < x < 2 \\ f_Y(y) &= \frac{y^{3/2}}{4} \frac{\mathrm{d}y^{1/2}}{\mathrm{d}y} \quad 0 < y < 4 \\ &= \frac{y}{8} \quad 0 < y < 4 \end{aligned}$$

(b) Let $Z = \sqrt{X}$. Compute the density function $f_Z(z)$ for Z

$$\begin{aligned} f_X(x) &= \frac{x^3}{4} \quad 0 < x < 2 \\ f_Z(z) &= \frac{z^6}{4} \frac{\mathrm{d}z^2}{\mathrm{d}z} \quad 0 < z < \sqrt{2} \\ &= \frac{z^7}{8} \quad 0 < z < \sqrt{2} \end{aligned}$$

2.6.18 Assume $\beta > 0$

Suppose that $X \sim \text{Weibull}(a)$. Determine the distribution of $Y = X^\beta$

We solved for the cdf previously:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^a} & x \geq 0 \end{cases}$$

And we transform:

$$
\begin{aligned}
F(x) &= \begin{cases} 0 & x < 0 \\ \left[1 - e^{-y^{a/\beta}}\right] \frac{\mathrm{d}y^{1/\beta}}{\mathrm{d}y} & y \geq 0 \end{cases} \\
&= \begin{cases} 0 & x < 0 \\ \left[1 - e^{-y^{a/\beta}}\right] \frac{y^{1/\beta - 1}}{\beta} & y \geq 0 \end{cases}
\end{aligned}
$$