5500 Problem Set 6

Nikko Cleri

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Question I.

Starting form HW5 P3: Calculate the grand partition function, and use it to derive Boyle's law $pV = NkT$.

From HW5 we can start by saying:

$$\begin{aligned}
\mu &= kT \ln n\lambda^3 \\
&= kT \ln \frac{N\lambda^3}{V} \\
Z_N &= \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N \\
\mathcal{Z} &= \sum_N z^N Z_N \\
&= \sum_N \frac{z^N}{N!}\left(\frac{V}{\lambda^3}\right)^N \\
&= e^{\frac{zV}{\lambda^3}}
\end{aligned}$$

We will now do the following to find Boyle's Law:

$$\begin{aligned}
\Omega - &-kT \ln \mathcal{Z} \\
&= -kT\left(\frac{zV}{\lambda^3}\right) \\
&= -pV \\
pV &= kT\left(\frac{zV}{\lambda^3}\right) \\
kT\frac{\partial}{\partial \mu}\log \mathcal{Z} &= kT\beta z \frac{\partial}{\partial z}\ln e^{zV/\lambda^3} \\
N &= \frac{zV}{\lambda^3} \\
\implies pV &= NkT
\end{aligned}$$

gives us Boyle's Law.

Question II.

Consider an ultrarelativistic Fermi gas, where the dispersion relation for the particles is $\epsilon = cp$ p being the momentum. Find the Fermi energy.

In the ultrarelativistic limit the relatioin $\epsilon_F = \frac{\hbar^2 k^2}{2m}$ no longer holds, and we have

$$\begin{aligned}
\epsilon_F &= p_F c \\
p_F &= \hbar k \\
&= \hbar \left(\frac{6\pi^2 n}{g} \right)^{1/3} \\
\epsilon_F &= \hbar \left(\frac{6\pi^2 n}{g} \right)^{1/3} c
\end{aligned}$$

Question III.

In a magnetic field the spin degeneracy of the electron gas is lifted: the magnetic energy for the z component of the spin m_S is $E = -g\mu_B B m_S$. Hence, in equilibrium the number of electrons in the spin-up and spin-down states is no longer the same, and the electron gas acquires a macroscopic magnetization M (magnetic moment/unit volume). Show that the resulting "Pauli paramagnetic susceptibility" $\chi = \frac{\partial M}{\partial B}$ at zero temperature and zero magnetic field is

$$\chi = \frac{g^2 \mu_B^2}{4V} \mathcal{D}(\epsilon_F) = \frac{3 g^2 \mu_B^2 n}{8 \epsilon_F}$$

Here $g \approx 2$ and $\mathcal{D}$ is the density of one particle energy eigenstates

$$\mathcal{D}(\epsilon) = \frac{\partial \mathcal{N}}{\partial \epsilon}$$

$\mathcal{N}(\epsilon)$, in turn, is the number of one-particle energy eigenstates with kinetic energy $\leq \epsilon$, as if there were no magnetic field at all.

Things to consider: This is obviously a case in which the different magnetic states are *not* degenerate. Moreover, since you are finding the susceptibility at zero magnetic field, an analysis up to the lowest nontrivial order in B suffices.

The energy of a particle in a magnetic field is given by

$$\epsilon = \frac{p^2}{2m} - \mu^* \cdot \mathbf{B}$$

The numbers of occupied energy levels in the two groups will be

$$\begin{aligned}
N^+ &= \frac{4\pi V}{3h^3} (2m(\epsilon_F + \mu^* B))^{3/2} \\
N^- &= \frac{4\pi V}{3h^3} (2m(\epsilon_F - \mu^* B))^{3/2}
\end{aligned}$$

and the magnetic oment is given by:

$$
\begin{aligned}
M &= \mu_B(N^+ - N^) \\
&= \frac{4\pi\mu_B V(2m)^{3/2}}{3h^3}((\epsilon_F + \mu_B B)^{3/2} - (\epsilon_F - \mu_B B)^{3/2}) \\
\chi &= \lim_{B\to 0}\left(\frac{M}{VB}\right) \\
&= \frac{4\pi\mu_B^2(2m)^{3/2}\epsilon_F^{1/2}}{h^3}
\end{aligned}
$$

Applying the known expression for the Fermi energy gives the desired answer.

$$\chi = \frac{3g^2\mu_B^2 n}{8\epsilon_F}$$