5401 Problem Set 1

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Question I.

Shankar, Ex. 1.1.3

Do functions that vanish at the end points of $x = 0$ and $x = L$ form a vector space? How about *periodic functions* obeying $f(0) = f(L)$? How about functions that obey $f(0) = 4$? If the functions do not qualify, list the things that can go wrong.

The conditions necessary to show that functions constitute a vector space are (from Shankar, and I will use his notation):

-A definite rule for the vector sum, $|V\rangle + |W\rangle$

-Scalar multiplication with the features:

(a) closure: $|V\rangle + |W\rangle \in \mathbb{V}$

(b) scalar multiplication is distributive in the vectors: $a(|V\rangle + |W\rangle) = a\,|V\rangle + a\,|W\rangle$

(c) scalar multiplication is distributive in the scalars: $(a + b)\,|V\rangle = a\,|V\rangle + b\,|V\rangle$

(d) scalar multiplication is associative: $a(b\,|V\rangle) = ab\,|V\rangle$

(e) addition is commutative: $|V\rangle + |W\rangle = |W\rangle + |V\rangle$

(f) addition is associative: $|V\rangle + (|W\rangle + |Z\rangle) = (|V\rangle + |W\rangle) + |Z\rangle$

(g) $\exists$ a null vector $|0\rangle$ obeying $|V\rangle + |0\rangle = |V\rangle$

(h) for every $|V\rangle$, there exists an inverse under addition, $|-V\rangle$ obeying $|V\rangle + |-V\rangle = |0\rangle$

For the first potential vector space $\mathbb{V}_1 = \{f(x)|0 \leq x \leq L, f(0) = f(L) = 0\}$, consider the functions $g(x)$ and $h(x)$ in this set.

Vector summation: $F(x) = g(x) + h(x)$ so $F(0) = F(L) = g(0) + h(0) = 0$ is in the set, thus also satisfies closure.

Parts b-f of the scalar multiplication conditions are trivial. It follows easily that if $f(x) = 0$ for all x, then the boundary conditions are immediately satisfied, thus the null vector is in the space. We also have the additive inverse $g(x) = -f(x)$ also immediately satisfies the boundary conditions. All of the conditions are met, so this $\mathbb{V}_1 = \{f(x)|0 \leq x \leq L, f(0) = f(L) = 0\}$ is a vector space.

The second set, $\mathbb{V}_2 = \{f(x)|0 \leq x \leq L, f(0) = f(L)\}$, functions periodic in x, we have $g(x)$ and $h(x)$ where $g(0) = g(L)$ and $h(0) = h(L)$. Immediately we have $F(x) = g(x) + h(x)$ so $F(0) = g(0) + h(0) = g(L) + h(L) = F(L)$ satisfies closure and vector addition. Again, parts b-f of the scalar multiplication conditions are trivial, and it is obvious that the null vector $f(x) = 0$ is periodic. We also have the additive inverse $g(x) = -f(x)$ satisfies the periodicity of the boundary conditions since $g(0) = -f(0) = -f(L) = g(L)$. We have again satisfied all conditions, so $\mathbb{V}_2 = \{f(x)|0 \leq x \leq L, f(0) = f(L)\}$ is a vector space.

The third set, $\mathbb{V}_3 = \{f(x)|0 \leq x \leq L, f(0) = 4\}$, we immediately see cannot be a vector space. This set fails closure since $g(0) + h(0) = 8 \neq 4$, obviously fails the null vector condition since $0 \neq 4$, and by immediate extension fails the additive inverse condition.

Question II.

Shankar, Ex 1.1.4

Consider three elements from the vector space of real 2x2 matrices:

$$|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad |2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad |3\rangle = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}$$

Are they linearly independent? Support your answer with details. (Notice we are calling these matrices vectors and using kets to represent them to emphasize their role as elements of a vector space.)

It is obvious by simple inspection that:

$$|1\rangle - 2\,|2\rangle = |3\rangle$$

Since the $|3\rangle$ is a linear combination of $|1\rangle$ and $|2\rangle$, these vectors are not linearly independent.

Question III.

Shankar, Ex. 1.4.1

In a space $\mathbb{V}^n$, prove that the set of all vectors $\{|V_\perp^1\rangle, |V_\perp^2\rangle, ...\}$, orthogonal to any $|V\rangle \neq |0\rangle$, form a subspace $\mathbb{V}^{n-1}$.

By Gram-Schmidt orthonormalization it is obvious that we can choose any $|V\rangle \neq |0\rangle$ and construct a set of $n-1$ vectors perpendicular to the first. By definition, these are linearly independent, and as such they span the $\mathbb{V}^{n-1}$ subspace. Now to prove this is the set of all vectors perpendicular to $|V\rangle$.

By construction, every vector in $\mathbb{V}^{n-1}$ is orthogonal to $|V\rangle$, and every vector in the original $\mathbb{V}^n$ but outside of $\mathbb{V}^{n-1}$ is not orthogonal to $|V\rangle$. A vector of this kind $|u\rangle$ can be written as $a\,|V\rangle + \sum_{i=1}^{n-1} a_i\,|v_i\rangle$ where $a \neq 0$, so $\langle v|u\rangle = a\,\langle v|v\rangle \neq 0$ which proves any $|u\rangle$ of this type is not perpendicular to $|V\rangle$, and from this $\mathbb{V}^{n-1}$ is the set of all vectors perpendicular to $|V\rangle$.

Question IV.

Shankar, Ex. 1.4.2

Suppose $\mathbb{V}_1^{n_1}$ and $\mathbb{V}_2^{n_2}$ are two subspaces such that an element of $\mathbb{V}_1$ is orthogonal to any element of $\mathbb{V}_2$. Show that the dimensionality of $\mathbb{V}_1 \oplus \mathbb{V}_2$ is $n_1 + n_2$. (Hint: Theorem 4.)

Immediately, since all members of both sets are orthogonal, the n_1 and n_2 basis vectors form a basis of a $\mathbb{V}^{n_1+n_2}$ subspace. Left to prove is that this is in fact $\mathbb{V}^{n_1} \oplus \mathbb{V}^{n_2}$.

Since a vector can be expressed by its decomposition into the basis vectors, a vector $|u\rangle \in \mathbb{V}^{n_1+n_2}$ can be represented by:

$$|u\rangle = \sum_{i=1}^{n_1} a_i |x_i\rangle + \sum_{j=1}^{n_2} b_j |y_j\rangle$$

where $|x_i\rangle$ and $|y_j\rangle$ are the basis vectors for their respective bases in $\mathbb{V}_1^{n_1}$ and $\mathbb{V}_2^{n_2}$. Noting that the two summations are individually vectors in their respective bases, this satisfies the closure and vector summation conditions for a vector space, since this $|u\rangle$ can be any vector in $\mathbb{V}^{n_1+n_2}$. Thus, the dimensionality of the vector space $\mathbb{V}_1 \oplus \mathbb{V}_2$ is $n_1 + n_2$.

Question V.

Shankar, Ex. 1.6.2

Given Ω and Λ are Hermitian, what can you say about (1) $\Omega\Lambda$; (2)$\Omega\Lambda + \Lambda\Omega$; (3) $[\Omega, \Lambda]$; and (4) $i[\Omega, \Lambda]$?

(1) For the product of two hermitian operators, we have the relation: $(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger = \Lambda\Omega$, which means the product of two hermitian operators is hermitian itself only if the two operators commute.

(2) For the sum of the products of two hermitian operators, we have $(\Omega\Lambda + \Lambda\Omega)^\dagger = \Lambda^\dagger\Omega^\dagger + \Omega^\dagger\Lambda^\dagger = \Lambda\Omega + \Omega\Lambda$ is hermitian since addition is commutative.

(3) The commutator of two hermitian operators:

$$\begin{aligned}
[\Omega, \Lambda] &= \Omega\Lambda - \Lambda\Omega \\
[\Omega, \Lambda]^\dagger &= (\Omega\Lambda - \Lambda\Omega)^\dagger \\
&= \Lambda\Omega - \Omega\Lambda \\
&= [\Lambda, \Omega] = -[\Omega, \Lambda]
\end{aligned}$$

so the commutator is anti-hermitian.

(4) If we multiply the commutator by i:

$$\begin{aligned}
i[\Omega, \Lambda] &= i(\Omega\Lambda - \Lambda\Omega) \\
(i[\Omega, \Lambda])^\dagger &= (i(\Omega\Lambda - \Lambda\Omega))^\dagger \\
&= -i(\Lambda\Omega - \Omega\Lambda) \\
&= -i[\Lambda, \Omega] = i[\Omega, \Lambda]
\end{aligned}$$

so $i[\Omega, \Lambda]$ is a hermitian operator.