5500 Problem Set 9

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Question I.

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\alpha x^2+\beta x} &= \int_{-\infty}^{\infty} \mathrm{d}x\, e^{\frac{\beta^2}{4\alpha}\left(-\sqrt{\alpha}x+\frac{\beta}{2\sqrt{\alpha}}\right)^2} \\
u &= \frac{2\alpha x-\beta}{2\sqrt{\alpha}} \\
\mathrm{d}x &= \frac{\mathrm{d}u}{\sqrt{\alpha}} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\alpha}} e^{\frac{\beta^2}{4\alpha}-u^2}\,\mathrm{d}u \\
&= \frac{\sqrt{\pi}e^{\beta^2/4\alpha}}{2\sqrt{\alpha}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} 2e^{-u^2}\,\mathrm{d}u \\
&= \frac{\sqrt{\pi}e^{\beta^2/4\alpha}}{2\sqrt{\alpha}}
\end{aligned}$$

Which is the desired result.

Question II.

We can start with the partition function given by

$$\begin{aligned}
Z &= \int \mathrm{d}\tau\, e^{-\beta H} \\
&= \frac{1}{N!}\left(\frac{k^2T^2}{m\omega\hbar}\right)^{\frac{3N}{2}} \\
U &= \frac{\mathrm{d}\ln Z}{\mathrm{d}\beta}
\end{aligned}$$

gives the familiar $U = 3NkT$. Then we say

$$C_v = \frac{\mathrm{d}U}{\mathrm{d}T}$$

gives us the heat capacity $C_v = 3Nk$.

Question III.

To find the density of the gas we can start again from the partition function and the internal energy

$$\begin{aligned} Z &= \int \mathrm{d}\tau\, e^{-\beta H} \\ &= \frac{1}{N!}\left(\frac{k^2T^2}{m\omega\hbar}\right)^{\frac{3N}{2}} \\ U &= \frac{\mathrm{d}\ln Z}{\mathrm{d}\beta} \end{aligned}$$

again gives $U = 3NkT$. If we then account for the density by saying $n = N/V$, this becomes

$$u = 3nkT$$

Question IV.

Starting with the energy in a given state

$$\begin{aligned} E &= \sum_{i=1}^{3N} \frac{p_i^2}{2m} + V_i \\ &= \sum_{i=1}^{3N} \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 \end{aligned}$$

The number of states with energy less than E is given by

$$\sigma(E) = \frac{\int\cdots\int_{|p|<\sqrt{2m(E-V)}} d^{3N}p \int\cdots\int_{V^N} d^{3N}x}{\Omega}$$

This gives the desired result given all of these integrals of

$$\Sigma(E) = \frac{1}{N!(3N)!}\left(\frac{E}{\hbar\omega}\right)^{3N}$$

Question V.

If we find the expectation value of the position at a given temperature

$$\langle x \rangle = \frac{\int \mathrm{d}x\, xe^{-\beta V}}{\int \mathrm{d}x\, e^{-\beta V}}$$

where V is the given

$$V(\xi) = K\xi^2 - a\xi^3$$

putting this all together we will find that length scales linearly with temperature.