Math 112 Exam #1

September 28, 2016

Time: 1 hour and 25 minutes
Instructions: Show all work for full credit. No outside materials or calculators allowed.
Extra Space: Use the backs of each sheet for extra space. Clearly label when doing so.

Name: Polly Nomial

ID #: ____________________

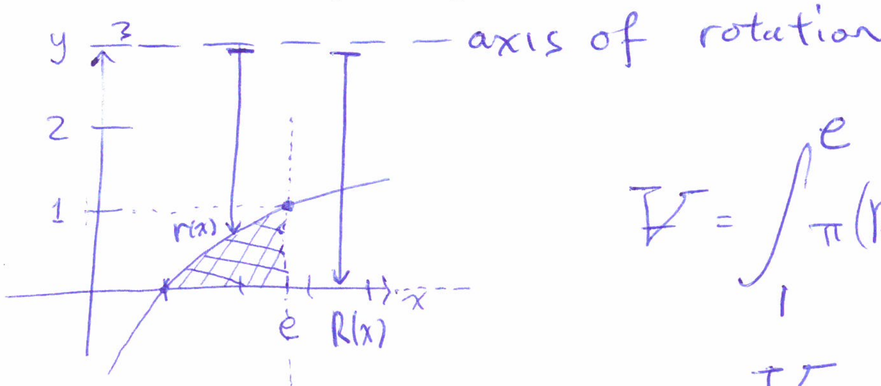
Instructor/Section: 001, 023

"I pledge by my honor that I have abided by the NJIT Academic Integrity Code."

$\sum_{n=0}^{N} a_n x^n$ (Signature)

Problem(s)	Score	Total

1. Sketch the region bound by y = ln(x), y=0, and x=e. Then set up the integral to find the volume of the figure formed by rotating this area about the line y=3. Show a sketch of the region being revolved. DO NOT SOLVE THE INTEGRAL. **(10 points)**

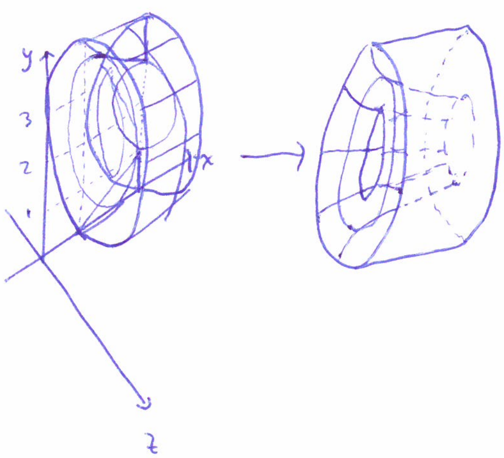


$$V = \int_1^e \pi\left(R^2(x) - r^2(x)\right)dx$$

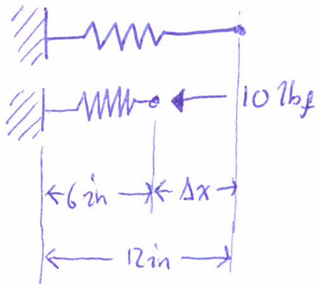
$$V = \int_1^e \pi\left[(0-3)^2 - (\cancel{\log x}\ \ln x - 3)^2\right]dx$$

$$V = \int_1^e \pi\left(9 - (\ln^2 x - 6\ln x + 9)\right)dx$$

$$V = \int_1^e \pi(6\ln x - \ln^2 x)\,dx$$



2. Suppose that a spring has a natural length of 1 foot and that a force of 10 pounds is required to hold it compressed to a length of 6 inches. How much work is done in stretching the spring from its natural length to a total length of 2 feet? **(10 points)**



work: $W = \int F\,dx$

linear spring: $F = k(x - x_0)$

$$W = \int_{x_0}^{2ft} k\,(x-x_0)\,dx$$

$$= \int_{1ft}^{2ft} k(x - x_0)\,dx = \left[k\frac{x^2}{2} - kx_0 x\right]_{x=1f}^{2ft} \quad x_0 = 1ft$$

$(x - x_0) = \Delta x = -6\,in$

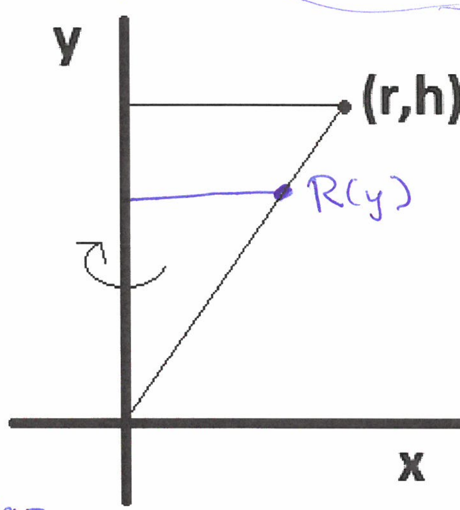
$F = -10\ lb_f$

$$\frac{F}{\Delta x} = k = \frac{(-10\ lb_f)}{(-6\ in)} = \frac{5}{3}\ lb_f/in = 20\ lb_f/ft$$

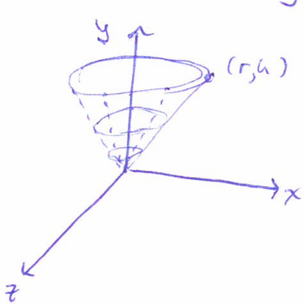
$W = (20\ lb_f/ft)(2\,ft^2)$

$W = 40\ ft\cdot lb_f$

3. Suppose a line segment from the origin to the general point (r,h) is rotated about the y-axis as shown, creating a right circular cone. Find the volume of this cone in terms of r and h by using either the method of cross sections or method of cylindrical shells **(12 points)**



use crosssections. This choice is arbitrary for this problem



- The cone has circular cross-sec.s which are perpendicular to the y-axis

$$V = \int_0^h A(y)\,dy$$

$A(y) = \pi R^2(y)$

where $R(y)$ is the cross-sec.'s radius

↳ • $R(0) = 0$

• $R(h) = r$

• this is a straight line

use $R(y) = my + b$

• $b = 0$, $m = \frac{r}{h}$

result $R(y) = \left(\frac{r}{h}\right)y$

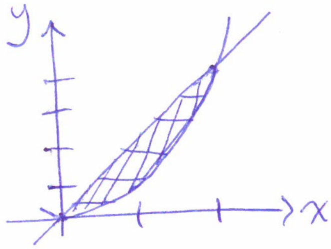
$$V = \int_0^h \pi\left(\frac{r}{h}y\right)^2 dy$$

$$V = \int_0^h \left(\frac{\pi r^2}{h^2}\right) y^2\,dy$$

$$= \left(\frac{\pi r^2}{h^2}\right)\left[\frac{1}{3}y^3\right]_{y=0}^{h}$$

$$V = \frac{\pi r^2 h}{3}$$

4. Find the volume of the figure formed by rotating the area bound between $y = x^2$ and $y = 2x$ around the y-axis. Use any method. **(10 points)**



- recommend washer method if you're not sharp with the ~~eg~~ shell method.
- using the shell method eliminates the need to change to $x = f(y)$ e.g. $y = x^2 \rightarrow x = \sqrt{y}$

Shell method:

- axis of rotation : $x = 0$
- Shell radius : $r(x) = x - 0$
- cylinder height : $h(x) = 2x - x^2$
- limits of integration : ...?

  ↳ find curve intersections

  solve $h(x) = 0$ will do.

  $2x - x^2 = 0$

  $x(2 - x) = 0$

  $a = 0$
  $b = 2$

- $$V = \int_a^b 2\pi\, h(x)\, r(x)\, dx$$

$$= \int_0^2 (2\pi)(2x - x^2)(x)\, dx$$

$$= 2\pi \int_0^2 (2x^2 - x^3)\, dx$$

$$= 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]\Bigg|_{x=0}^{2}$$

$$= 2\pi\left(\frac{2}{3}(2)^3 - \frac{1}{4}(2)^4\right) - 0$$

$$= 2\pi\left(\frac{16}{3} - 4\right) \quad 4 = \frac{12}{3}$$

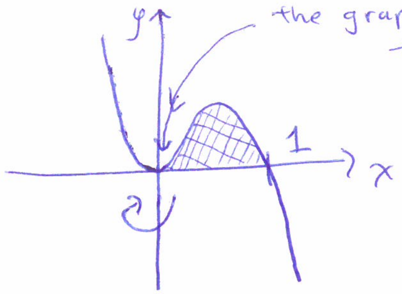
$$= 2\pi\left(\frac{4}{3}\right)$$

$$V = \frac{8}{3}\pi$$

5. Find the volume of the figure formed by rotating the area bound by the x-axis and the cubic $y = x^2 - x^3$ about the y-axis. **(12 points)**

$y = x^2(1-x)$

double zero means the graph "turns around" at this point



shell method is the only choice

axis of rotation : $x=0$

shell radius : $r(x) = x - 0$

shell height : $h(x) = (x^2 - x^3) - 0$

limits of integ. : $a=0$, $b=1$

$$V = \int_0^1 (x^2 - x^3)\, x\, dx\, (2\pi)$$

$$= 2\pi \int_0^1 (x^3 - x^4)\, dx$$

$$= 2\pi \left[\frac{x^4}{4} - \frac{x^5}{5} \right] \Bigg|_{x=0}^{1} = 2\pi \left(\frac{1}{4} - \frac{1}{5} \right) - 0$$

$$= 2\pi \left(\frac{5}{20} - \frac{4}{20} \right) = \frac{\pi}{10} = V$$

6. Evaluate $\int \frac{\cos(\sqrt{3x})}{\sqrt{x}}\, dx$ **(8 points)**:

- substitute : $u = \sqrt{3x} \rightsquigarrow du = \frac{\sqrt{3}}{2\sqrt{x}}\, dx$

  then $\frac{1}{\sqrt{x}}\, dx = \frac{2}{\sqrt{3}}\, du$

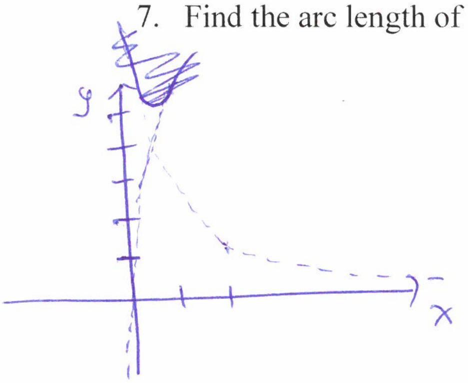
- integral becomes :

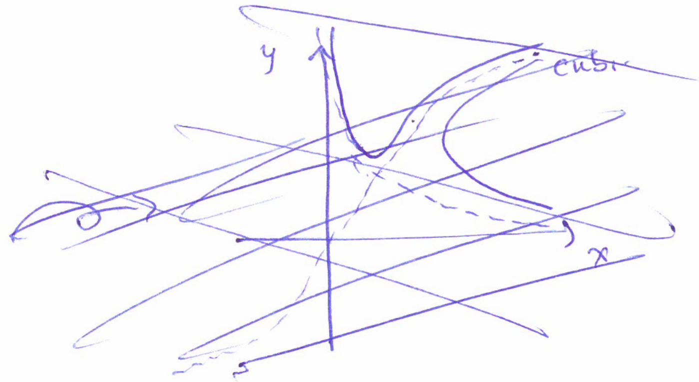
$$I = \frac{2}{\sqrt{3}} \int \cos(u)\, du$$

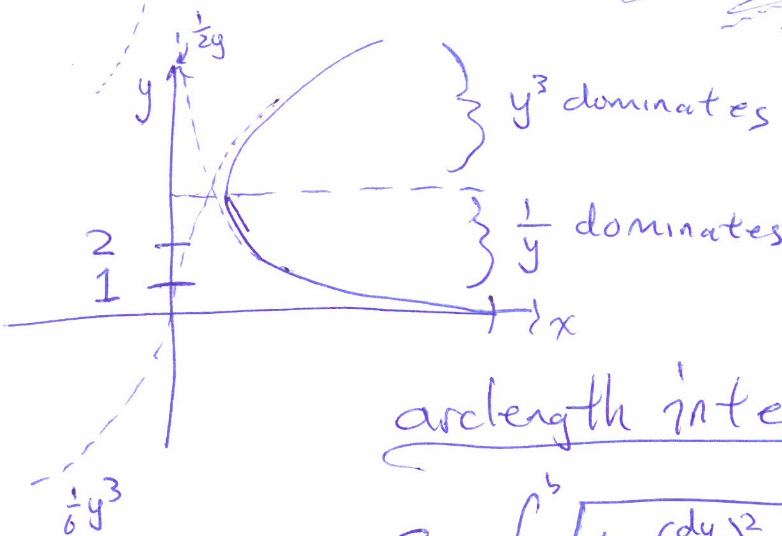
$$I = \frac{2}{\sqrt{3}} \sin(u) + C$$

$$I = \frac{2}{\sqrt{3}} \sin(\sqrt{3x}) + C$$

7. Find the arc length of the curve $x = \frac{1}{6}y^3 + \frac{1}{2y}$ between y=1 and y=2 **(12 points)**







arclength integral

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx} = \frac{3}{6}y^2 + \frac{-1}{2y^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}y^4 - \frac{1}{2}\left(\cancelto{1}{\frac{y^2}{y^2}}\right) + \frac{1}{4y^4}$$

$$= \frac{1}{4}\left(y^4+y^{-4}-2\right)$$

$$\cancel{S = \int_1^2 \sqrt{1+\frac{1}{4}(y^4+y^{-4})}\,dy}$$

$$S = \int_1^2 \sqrt{1-\frac{1}{2}+\frac{1}{4}y^4+\frac{1}{4}y^{-4}}\,dy$$

$$= \int_1^2 \sqrt{\frac{1}{4}y^4+\frac{1}{2}+\frac{1}{4}y^{-4}}\,dy$$

$$= \int_1^2 \frac{1}{2}\sqrt{y^4+2+y^{-4}}\,dy$$

$$= \int_1^2 \frac{1}{2}\sqrt{(y^2+y^{-2})^2}\,dy$$

continuation

$$S = \int_1^2 \frac{1}{2}(y^2+y^{-2})\,dy$$

$$= \frac{1}{2}\left[\frac{1}{3}y^3 - \frac{1}{y}\right]\Big|_{y=1}^{2}$$

$$= \frac{1}{2}\left[\left(\frac{8}{3}-\frac{1}{2}\right)-\left(\frac{1}{3}-\frac{1}{1}\right)\right]$$

$$= \frac{1}{2}\left(\frac{7}{3}+\frac{1}{2}\right)$$

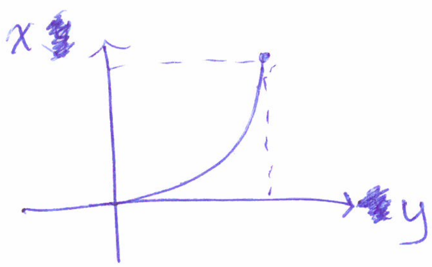
$$= \frac{1}{2}\left(\frac{14}{6}+\frac{3}{6}\right)$$

$$\cancel{\frac{1}{2}\left(\frac{15}{6}\right)}$$

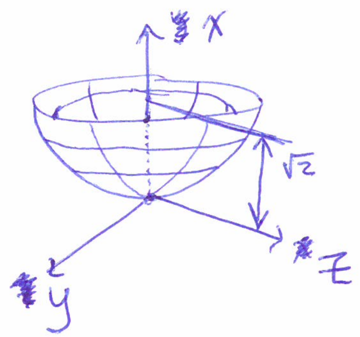
$$= \frac{1}{2}\left(\frac{17}{6}\right)$$

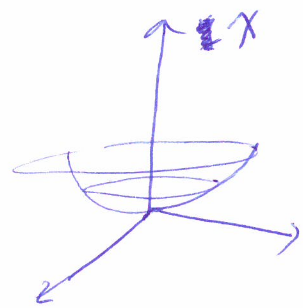
$$S = \frac{17}{12}$$

8. Find the surface area of the paraboloid formed when the curve $x = y^2$, $0 \le y \le \sqrt{2}$ is revolved around the x-axis. **(12 points)**

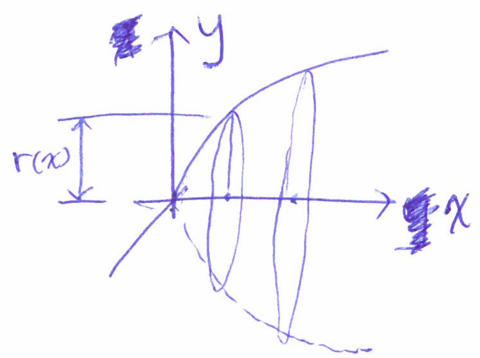


$y \in [0, \sqrt{2}]$





Simplest way is to integrate the circular profile along the x-axis to generate the surface.



~~$A = \int S(y)\,dy$~~

~~$S(y)$: circumference of the profile~~

~~$S(y) = 2\pi r$~~

$$A = \int_a^b s(x)\,dx$$

analagous to cross-section integrals $V = \int A(x)\,dx$

$s(x)$: circumference of the profile

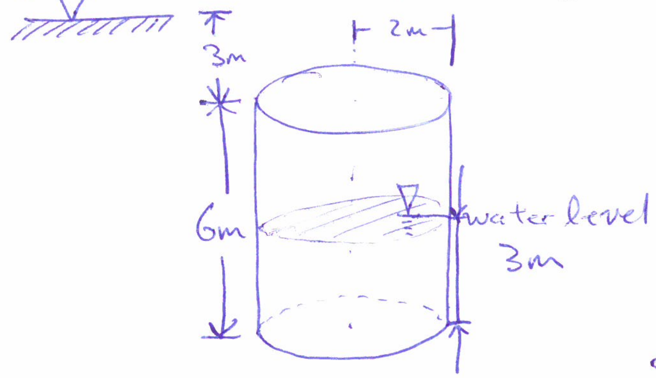
$$s(x) = 2\pi r = 2\pi\sqrt{x}$$

$$A = \int_0^2 2\pi\sqrt{x}\,dx = 2\pi\left[\tfrac{2}{3}x^{3/2}\right]\Big|_{x=0}^{2}$$

$$= \frac{4}{3}\pi\sqrt{2}^3 = \frac{4}{3}\pi\, 2\sqrt{2}$$

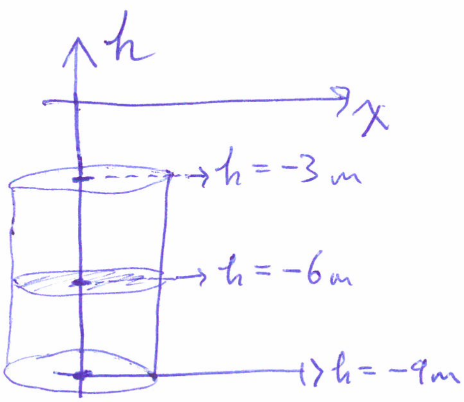
$$A = \frac{8\sqrt{2}}{3}\pi$$

9. Suppose that a cylindrical tank is buried upright underground on one of its circular bases. The tank has a height of 6 meters and a radius of 2 meters. Suppose the top of the tank is exactly 3 meters below the earth's surface. If the tank is half filled with water weighing approximately 10,000 N/m³. Find the work needed to pump all the water in this tank to the ground's surface. **(14 points)**



weight density : $\frac{mg}{V} = 10000\ N/m^3$

$\rho g = 10000\ N/m^3$



- P_e : potential energy
- W : work
- all potential energy is gravitational potential energy

$P_e = -mgh$

where h is height (altitude)

- define ground level to be $h = 0$
- then $P_e\big|_{ground} = 0$

- Conservation of energy

$W - P_{e_1} = -P_{e_2}$

- $P_{e_2} = 0$ ground level
- water has different potential energy at different heights. So we have to "sum" (integrate)

$W = P_{e_1} = -\int gh(dm)$

$= -\int_{-9m}^{-6m} \rho g h A dh$

$m = \rho V = \rho A y$

$dm = \rho dV = \rho A dy$

$$W = \int_{-9m}^{-6m} -\rho g h A \, dh$$

recap: h — vertical coordinate

A — cross-sectional area of the cylinder

$\rho g = 10000\ N/m^3$

weight density of water given

$A = \pi r^2 \qquad r = 2m$ — given

$$W = \int_{-9m}^{-6m} -(\rho g)(\pi r^2)\, h \, dh$$

$$= -\rho g \pi r^2 \left[\tfrac{1}{2} h^2\right]\Big|_{h=-9m}^{-6m}$$

$$= \frac{\rho g \pi r^2}{2}\left[81m^2 - 36m^2\right]$$

$$= \frac{\left(10{,}000 \frac{N}{m^3}\right)\pi\left(4m^2\right)}{2}\left[45m^2\right]$$

$\cancel{W = (\pi)(1{,}440{,}000\ Nm)}$

$W = (\pi)(900{,}000\ Nm)$

or $\pi 900\ kJ$