# Justifying the Classical No-Slip Boundary Condition

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## Contents

# 1 Introduction

Boundary conditions are an important part of solving any system of PDEs. In many cases, they can be just an interesting physically and philosophically than the governing equation itself. There is a typical boundary condition used when solving for the velocity of incompressible viscious fluids at solid boundaries. This is famously called the “no-slip condition” [2]. This condition states that the velocity of the fluid at this boundary should be equal to that of the boundary [2]. This has the implication that if the boundary is still, the fluid touching the boundary cannot move. Common experience contradicts this statement. A cup full of water can be emptied by pouring the fluid out. Afterwards, there is no water left in the cup. This is in contradiction to the no-slip condition since the water directly touching the cup should not be able to move, when in reality all the water has been drained. To get a handle on the issue, one has to be clear what is meant by “the fluid touching” a boundary. It is assumed that the fluid is a continuum and a fluid particle can be split further and further in two without end. This is known as the continuum hypothesis [9]. For the purposes of the project, it will be considered that the fluid layer of infinitesimal thickness at the boundary is the “fluid touching the boundary”.

## 1.1 A Paradoxical Solution

The continuum hypothesis is related to Zeno’s racecourse and raises awareness to the paradoxical implication of considering fluid in this manner[5]. The premise of Zeno’s paradox is that a sprinter attempts to run a unit distance. To cross the finish line, they must first run half the distance. Before the runner can finish that length, they must run a quarter of the total length and so on. This leaves the runner with an infinite number of lengths (with varying sizes) to travel across before they can leave the start line. Zeno’s racecourse could be applied to fluids in an attempt to justify the no-slip condition. The fluid right at the boundary, the imaginary “start line”, is unable to move because of the infinite splitting ability the fluid above it has. This is by nature a paradox since fluid can move in reality. It must turn out that something in this reasoning is wrong or incomplete, but it is a worth while exercise to consider the philosophical implications of a theory.

### 1.2 Project Goal

As entertaining as toy philosophical discussions are, to resolve this paradox and ultimately justify the use of this condition, scientific literature and physical experiments must be used. The focus of this report is to consolidate ideas presented in a handful of articles relating to the history, use, and validity of this condition. A small experiment will put some of these ideas to the test. My goal and expectation is to end up with a statement of the form "if it is assumed the fluid is something, then the no-slip condition can be assumed".

## 2 A Brief History of the Model

For a fluid with constant density and viscosity, the Navier-Stokes equation is given by the following vector PDEs. [2]

$$\begin{aligned} \rho_0 \frac{D\vec{u}}{Dt} &= -\nabla p + \mu \nabla^2 \vec{u} + \nabla \Pi \\ \nabla \cdot \vec{u} &= 0 \end{aligned} \tag{1}$$

In equations (1), we have a constant density $\rho_0$ and constant viscosity $\mu$ in addition to some known external potential $\Pi$. Note that $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$ represents the material derivative or time derivative in the reference frame of a small fluid particle [9]. It is worth highlighting the first equation in (1) is in fact three equations since we consider the system to live inside of three dimensional space. The aim of the game is to solve for all three components of the velocity, $\vec{u}$, and the pressure $p$ at all relevant points $\vec{x}$. The last piece of the puzzle necessary to close this system of equations are to assume conditions for the velocity and/or pressure at the boundary of they system.

The least controversial boundary condition that may be imposed is often referred to the impermeability condition. This assumes that the perpendicular component of the velocity of the fluid is zero at the solid boundary [2]. A straightforward *reduco ad absurdum* argument can be made to justify the use of this condition. If it is assumed that the perpendicular velocity is non-zero at the boundary at time $t$, a moment later, at time $t + \Delta t$, the fluid must have moved into the boundary or away from it. Both cases lead to absurd conclusions. A solid boundary implies that the solid has taken up some amount of space and cannot have other matter occupying the space. This would be the case if the velocity is directed towards the boundary. If the velocity is directed away from the boundary, then the velocity of the fluid partials adjacent to this point must also point away from the boundary due to the continuous nature of the velocity. We can then ask where the fluid has come from at the boundary. It cannot have come from its neighbours since their velocity is also directed away from the boundary, and it cannot have come from the boundary itself since it is solid. Even without this justification, it is fairly intuitive to conclude that fluid cannot flow in or out of a solid boundary. It is worth noting that there are instances where this assumption can break down. For example, if the boundary is porous such as a sponge, then fluid can indeed flow through this solid and the above argument fails. The remainder of our attention will be focused on the other restrictive boundary condition when the impermeability condition is assumed.

It was assumed as far back as Newton that no-slip occurs at the boundary when dealing with vortex motion [7]. Not shortly after, in 1738 Daniel Bernoulli would also assume that fluid cannot move unrestricted at the boundary [7]. An important distinction must be made between restricted motion and absolutely no motion at the boundary. The former implies there is some resistance to motion due to the solid boundary whereas the latter is stricter with what motion can occur.

Throughout the 1700's, three famous French physicists each had their school of thought on the matter. Coulomb believed there was absolutely no slippage in the tangential component of the velocity. Girard proposed there was a thin layer of fluid which stuck to the boundary and the rest of the fluid would slip over this fluid layer. Finally, Navier himself proposed there was slippage according to a proportionality relationship between the slip velocity (tangential velocity at the boundary) and the spacial derivative in the normal direction of the velocity [7]. Each idea had their own following. Some would even try and combine these ideas. For example, Poisson argued that Navier's relationship should be applied between Girard's thin layer of fluid and the slipping fluid on top of this layer. Many would devise their own experiments to try and justify their thoughts with no widespread agreement in sight.

## 2.1 Agreement on a No-Slip Model

Stokes independently derived the equations of motion (1) for fluids making him a big name in the field [7]. He was unsure about which tangential condition to impose on the velocity but ultimately agreed with Coulomb. His rational was based on the following two arguments [7].

1. "Existence of slip would imply that the friction between a solid and fluid was of a different nature from, and infinitely less than, the friction between two layers of fluid."
2. "Satisfactory agreement between the results obtained with no-slip assumption and the observations."

Prabhakara and Deshpande in their article on this condition stress the remarkably of the first argument [7]. It should be weird to model the world where fluids behave fundamentally different than solids at boundaries. This would imply something is fundamentally different about say a very viscous fluid and a solid boundary, even though reality would suggest that a lighter and less viscous fluid should flow over a denser fluid similar to flowing over a solid in the limit that the bottom fluid's viscosity becomes unbounded.

I would like to stress the importance of the second argument. Stokes compared results from experiments to theoretical data derived when assuming Coulomb's no-slip condition and found "highly satisfactory" agreement [2]. This is crucial to the development of theories. The whole goal of boundary conditions is to give an explanation and model to these physical events around us. If Stokes found that this assumption led to theoretical results which did not agree well with experimental results, we can be sure he would have rejected this assumption.

After much debate, Stokes' rational for what we will now refer to as *the* no-slip condition was finally accepted. This was only after more experiments were devised and repeated [7].

## 3 Personal Justification

Up until this point, I have read a handful of papers on the no-slip condition. I am reasonably persuaded and can trust the work of smart people who came before me. However, I would like to exercise the same scepticism other scientists had before I agree with Stokes. In an effort to convince myself, I will conduct a mini experiment of my own. My plan is to observe the displacement of the viscous fluid honey when forced to flow within the solid boundary of a kitchen pan. I will indirectly measure the velocity by drawing straight lines of food colouring and observing the displacement of these lines after the pan has been tiled to one side. Since I will be taking a before and after shot of the honey, a larger displacement would suggest the fluid had a larger velocity between the time the two photos were taken. I expect one of two things to happen. These can be summarised by the following two figures which were created in Desmos.

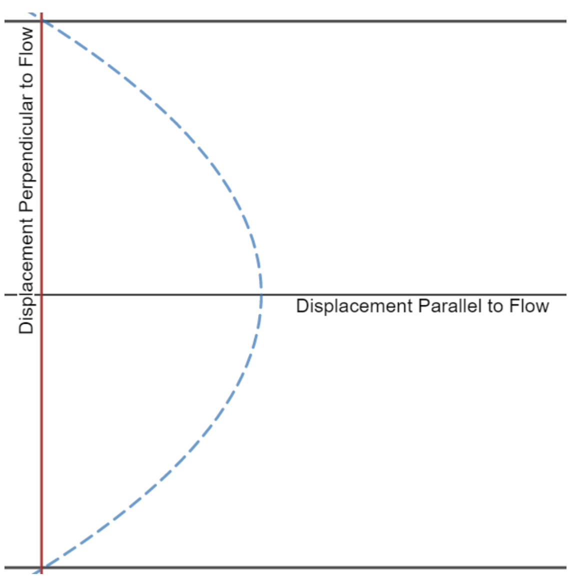


Figure 1: **No-slip:** The vertical solid red line represents the initial state of the food colouring and hence the initial state of the honey. The dashed blue line represents the final state of the food colouring. The two horizontal black lines at the top and bottom represent the boundary of the container. Notice the final displacement of the honey at the boundary is the same as the initial displacement. This would imply the honey had a zero velocity at the boundary and hence is not slipping.

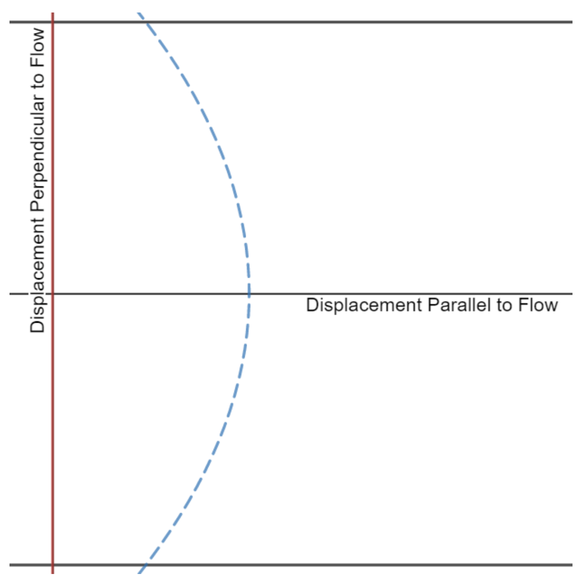


Figure 2: **Some Slip:** Similar to figure (1), the vertical solid red line is the initial displacement and the dashed blue line is the final displacement. Notice that the whole blue line has shifted rightward and the final displacement does not intersect the initial displacement. This would imply the honey had a nonzero velocity everywhere and hence must be slipping at the boundary. Note this assumes nothing about the final shape of the line, just that it was displaced everywhere including the edges.

### 3.1 Experiment Results

I started the experiment by filling the left hand side of a small bread loaf pan with honey. The honey settled to a depth of around 1cm. An image of this is shown below.

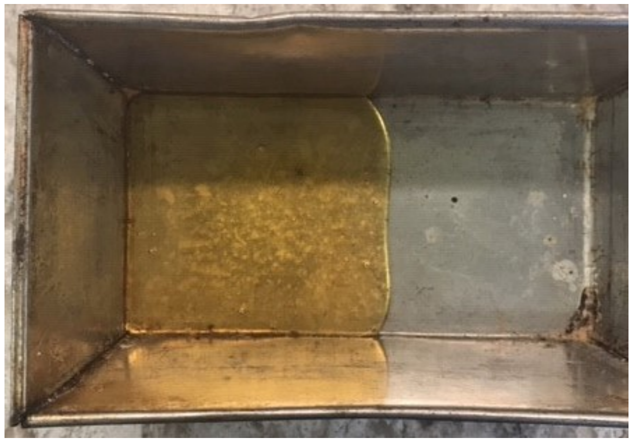

Figure 3: The yellow honey was poured in a grey loaf pan. This is its initial state after settling but before food colouring was added.

I first attempted the experiment using a thinner water based orange dye. This was mostly unsuccessful due to how quickly it defused within the honey. I used a knife dipped in the food colouring to insert a line of dye. This lead to inaccuracies in the line drawn from a wide knife, and also too much dye being deposited. When the pan was tilted, you could see the lines deform into a somewhat arc shape as expected. The large diffusivity meant it was hard to draw conclusions. The two images below showcase the initial and final state of this experiment.

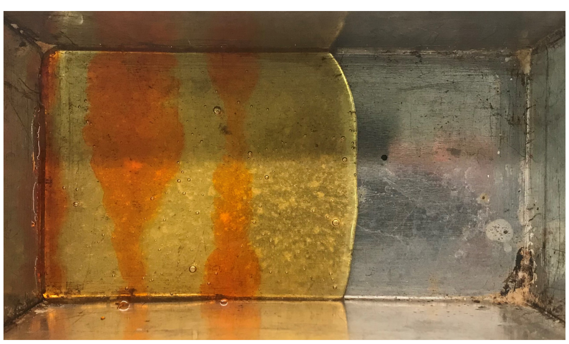

Figure 4: **Initial State:** Three vertical lines of orange dye was knifed into the honey. Within seconds the dye defused horizontally loosing much precision in the initial position of the honey.

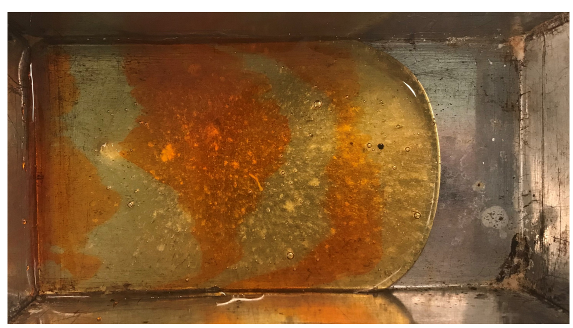

Figure 5: **Final State:** The left edge of the pan was titled up for about 5 seconds and then placed back down. The end result of was inconclusive. The already imprecise lines turned into blobs and made it hard to tell what happened along the boundary.

What can be noticed by comparing figures (4) and (5) is that the right profile of the honey went from a mostly vertical line to a parabolic looking shape. This would suggest that there is some friction between the honey and the boundary and that the fluid cannot move freely there. It would be difficult to justify no slippage occurs at the boundary using this profile. There could be many factors contributing to the leading profile from surface tension to the possible introduction of a second boundary layer between the honey and the air.

Quickly learning from the previous experiment, I found a thicker and darker blue dye that could be used. I first mixed up the honey to evenly distribute the orange dye. This introduced some suspended air bubbles but I was able to pop the largest ones that I thought might effect the experiment. I used a toothpick to carefully insert vertical lines of the thicker blue dye into the honey. In my first two attempts, I still used too much dye and there was some diffusion. I was able to get one thin clean line which can be seen as the second line from the left in figure (6). I also added a dab of dye right on the side of the pan above the start of this line. This was used to mark the initial position of the line for reference after the pan was tilted. The initial and final images of this experiment are shown below.

Figure 6: **Initial State:** Four vertical lines of blue dye was toothpicked into the honey. The first and third thicker lines are my first attempts at drawing the lines with the dye. Much less dye was used on my next two attempts which are the second and fourth line from the left. This resulted in less diffusion and more precise lines.

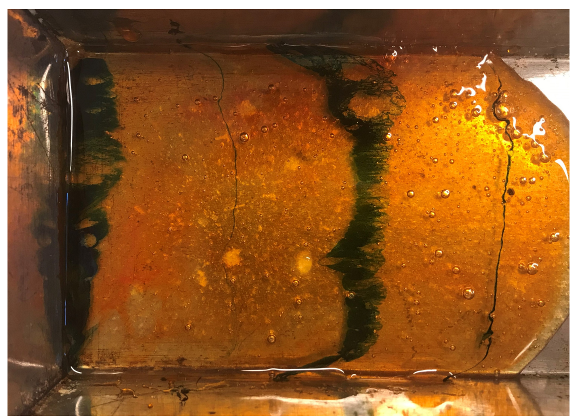

Figure 7: **Final State:** Similar to figure (5), the left edge of the pan was titled up for about 5 seconds and then placed back down. This time, the lines stayed intact. Near the middle vertically, the lines remain mostly undeformed and are simply translated to the right. Near the boundary, there is a large amount of stretching and curving of the lines.

The honey was dyed orange from the first experiment. This meant there was not large amount of contrast between the blue dye and the orange honey in the thinnest line drawn. I was able to take a close up picture of the final state to better observe what happened along the boundary. This is shown below in figure (8).

Figure 8: A close up of the second blue line drawn after tilting in the second experiment. The two blue blobs on the side of the pan can be seen which mark the initial position of the edge of the line. Notice this edge lines up perfectly below the marked blobs.

### 3.2 Discussion

As observed in figure (8), the edge of the line matches up perfectly with the initial position marked in blue dye. This result matches up with our expectation drawn in the no-slip diagram (figure 1) and we do not see the edge shifted as displayed in the some-slip diagram (figure 2). We may conclude that since there was no displacement, there was also no velocity when the pan was tilted. This gives me physical evidence of Stokes' argument that no-slip does in fact occur at the boundary. It is worth pointing out that all this analysis has been qualitative and nothing has been physically measured. This experiment was also not repeated with different fluids of different viscosity on different boundary surfaces. As such, we must be careful to conclude the statement "all viscus fluids exhibit the no-slip condition at all solid boundaries". This experiment has the primary goal of simply providing some physical evidence that no-slip may occur between many viscous fluids and many solid boundaries.

## 4 Modern Slip Theories

It is tempting to conclude this discussion after having theorised that no-slip occurs at the boundary and then physically observing no slippage. For the sake of completeness, I want to summarise some more recent findings in this field. There are a few papers arguing fluids can behave more complicated than simply freely flowing or no slipping at a boundary.

Schowalter discusses what many “real materials do” at the boundary between fluids and solids [8]. Steps for how to go about numerically modelling some everyday non-Newtonian fluids at boundaries are described in hopes of predicting how many common materials behave.

Hatzikiriakos details special classes of complex fluids that deviate from theoretical results when a no-slip boundary condition is considered [4]. These include “polymer melts, elastomers, polymer solutions, suspensions, dispersions, gels, colloidal dispersions/glasses, pastes and foams” [4]. Each class has a particular mechanism for which slip occurs and vary wildly from case to case. For example, foams create a film of bubbles between the fluid and the solid which help lubricate the boundary and allow for slip [3] [4].

Finally, Azese revisits Navier’s proportionality relationship to explain viscoelastic liquids [1]. What is personally fascinating is that Azese’s paper was published this year in 2019. 300 years after Coulomb proposed the no-slip condition, there is still active research being conducted in this field to figure out what is happening at these boundaries.

# 5 Conclusion

To reference the original goal stated at the start of the project, can a statement of the form "if it is assumed the fluid is something, then the no-slip condition can be assumed" be concluded? Using Coulomb as a starting point, his idea of no-slip can be expanded. Both Stokes' justification and the small experiment conducted provide evidence that the no-slip condition can be assumed if the fluid is Newtonian and viscous, and the flow is tangential to a solid boundary. In this case, the condition provides "highly satisfactory" results and agreement between theoretical and experimental results. The qualifiers "Newtonian" and "viscous" are necessary as described in section (4) since many everyday materials and other special classes of fluids exhibit complex slipping mechanisms rather than the straightforward no-slip condition.

I found this project fascinating. It was enlightening to research a topic which was originally troublesome for me. When first introduced to the no-slip condition, I was sceptical that real fluids can be modeled in this way. After completion of this project, I now feel more confident applying the no-slip boundary condition when solving future problems involving solid boundary conditions of viscous fluids.

## 6 References


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