Suggested Solutions for Homework 2

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If you find any errors, or unclear sections, please help me and your classmates by emailing me at `karl.harmenberg@phdstudent.hhs.se`.

Problem 1

I happen to be born in Sweden, so I used Swedish data (found at www.scb.se). For this exercise, you are of course free to use whatever program you want. I did this in R, and if you are interested, I can post the code on the course page.

Some potential pitfalls:

1. Your data is not inflation adjusted (i.e., not real).

2. Your data is not seasonally adjusted. Since we are interested in business cycles (and not e.g. harvest seasonality), we have to make sure that the data is seasonally adjusted, otherwise the results will be driven by seasonal variation.

3. Your calculation of percent standard deviations is not sound. In particular, before applying the filter you should take the logarithm of all variables. Therefore, for the log variable, the percentage standard deviation is just the standard deviation of the cyclical term whereas the standard deviation for the non-log variables is $sd(X_c/X_{tr})$ (where X is the variable).

I used Swedish data from 1993 to 2012. In figure 1 we can see log GDP, and the trend. Note that the data is seasonally adjusted, otherwise it would look much more jagged.

Comparing the cyclical components of output, investment and consumption in Figure 2, we see already by eyeballing that investment and consumption follow output closely, but are more and less volatile.

	Percent standard deviation	Correlation with output
Y	1.74	1.00
C	0.71	0.64
I	6.95	0.91
Z	1.27	0.74
N	1.18	0.69
L	1.13	0.64

Table 1: Statistics for output (Y), consumption (C), investment (I), hours worked (N), employed (L) and productivity (Z).

We summarize our findings in a Table 1. It is seen that all variables are strongly procyclical. I roughly has a standard deviation three

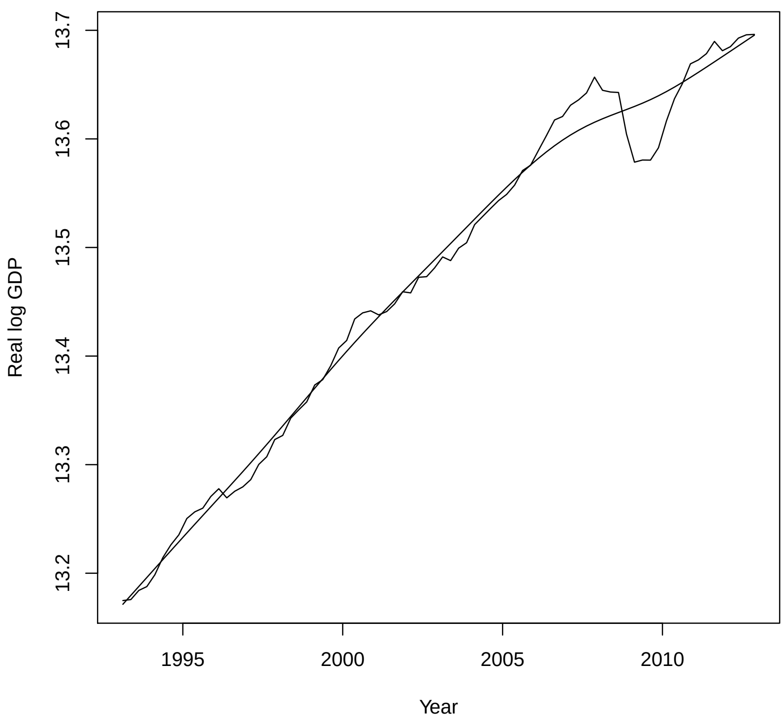


Figure 1: Log real GDP and log real GDP trend.

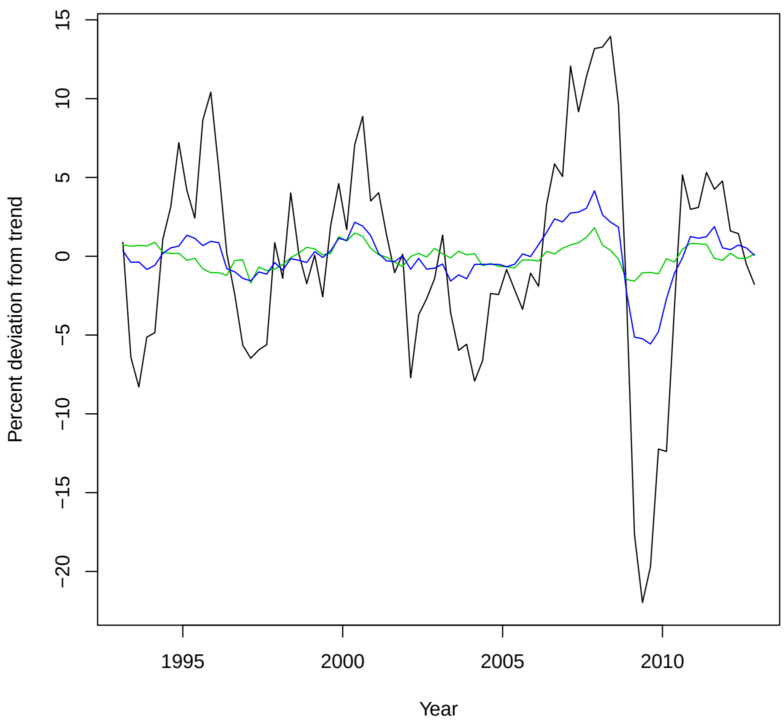
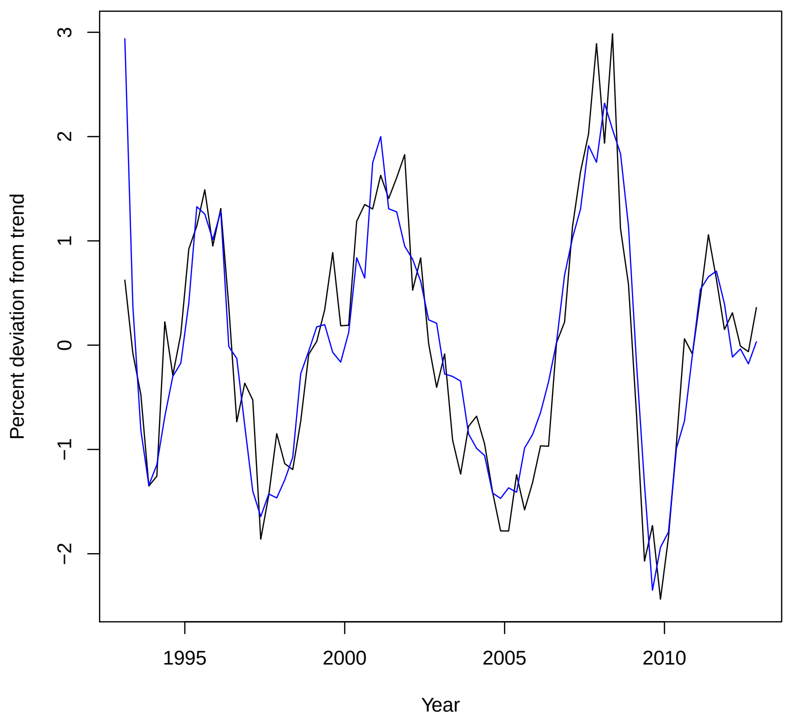


Figure 2: Cyclical component of real GDP (blue), investment (black) and consumption (yellow).

times the standard deviation of Y, but consumption has a smaller standard deviation, in line with the stylized facts.

Furthermore, hours worked and employment follow each others closely, which is in line with the stylized fact that hours worked per employed does not change much. (see Figure 3) Their standard deviations are a bit smaller than the standard deviation of output, but not by a huge amount.

Figure 3: Cyclical component of hours worked (black) and employed (blue).



Problem 2

The consumer's maximization problem is

$$\max_{\{c_t, a_{t+1}, l_t\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t (\log c_t + \psi \log l_t) \quad s.t \quad c_t + \beta a_{t+1} = a_t + w_t(1 - l_t),$$

$$nPs.$$

The corresponding Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (\log c_t + \psi \log l_t + \lambda_t (a_t + w_t(1 - l_t) - c_t - \beta a_{t+1}))$$

with first order conditions

$$\frac{1}{c_t} = \lambda_t$$
$$\frac{\psi}{l_t} = \lambda_t w_t$$
$$\lambda_t = \lambda_{t+1}$$

which gives us

$$c_t = c_{t+1}$$
$$l_t = \frac{\psi c_t}{w_t}$$

That is, consumption is constant and leisure is inversely related with the wage. If the wage is constant, we have

$$c = \frac{w}{\psi} l.$$

Now we use the consolidated budget constraint to determine the level of consumption (and leisure). We have

$$\sum_{t=0}^{\infty} \beta^t c = \sum_{t=0}^{\infty} \beta^t w(1-l)$$

so

$$c = w(1-l) = w\left(1 - \frac{\psi}{w} c\right) \Leftrightarrow$$
$$\frac{w+\psi}{w} c = w \Leftrightarrow$$
$$c = \frac{w}{1+\psi}$$
$$l = \frac{\psi}{1+\psi}$$

Now, when we have an answer, we should check limiting cases. If the wage is really low, then consumption will be really low. If ψ is really small, then $c = w$, which reflects that the consumer will work and earn w in each period. So it seems like our answer makes sense.

For the second part, we use that

$$\frac{d\log(1-l_t)}{d\log w_t} = \frac{d(1-l_t)}{dw_t}\frac{w_t}{1-l_t} = -\frac{dl_t}{dw_t}\frac{w_t}{1-l_t}.$$

Now, we use that $l_t = \frac{\psi c_t}{w_t}$ and differentiate with respect to w_t. Note that we can do this naively since we are keeping $\lambda_t = \frac{1}{c_t}$ constant, otherwise we could not treat c_t as a constant when differentiating. We get

$$\frac{d\log(1-l_t)}{d\log w_t}|_{\lambda_t} = \frac{\psi c_t}{w_t(1-l_t)} = \frac{l_t}{1-l_t} = 2.$$

Therefore, the Frisch λ-constant elasticity is 2 if $1 - l_t = 1/3$.

Problem 3

By the general equivalence of competitive equilibria and social planner's equilibria, we can solve this either by defining and solving a competitive equilibrium or by defining and solving a social planner's problem. I will solve the social planner's problem, since we are ultimately not interested in wages and interest rates.

The social planner's maximization problem is

$$\max_{\{c_t, k_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \mathbb{E}\left(\left(\log c_t + \frac{\chi_t}{\nu} l_t^{\nu}\right) \xi_t\right) \quad s.t. \quad c_t + k_{t+1}\eta_t = z_t k_t^{\alpha}(1-l_t)^{1-\alpha}.$$

Note that in this stochastic setting, the solution is no longer a sequence of consumption and capital, but a contingency plan: Given everything that has happened until period t, we will do this. That is, c_t and k_{t+1} are functions of all the previous shocks $\chi_0, \ldots, \chi_t, \xi_0, \ldots, \xi_t, \nu_0, \ldots, \nu_t, z_0, \ldots, z_t$.

The (stochastic) Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\mathbb{E}\left(\left(\log c_t + \frac{\chi_t}{\nu} l_t^{\nu}\right) \xi_t\right) + \lambda_t(z_t k_t^{\alpha}(1-l_t)^{1-\alpha} - c_t - k_{t+1}\eta_t)\right)$$

where λ_t is now a random variable. The first order conditions are

$$\begin{aligned}
\partial/\partial c_t \qquad & \frac{\xi_t}{c_t} = \lambda_t, \\
\partial/\partial k_{t+1} \qquad & \lambda_t \eta_t = \alpha\beta\mathbb{E}_t\left(\lambda_{t+1}\frac{F(k_{t+1}, 1-l_{t+1}, z_{t+1})}{k_{t+1}}\right), \\
\partial/\partial l_t \qquad & \chi_t \xi_t l_t^{\nu-1} = (1-\alpha)\lambda_t \frac{F(k_t, 1-l_t, z_t)}{1-l_t}.
\end{aligned}$$

Eliminating λ_t, λ_{t+1}, we get the Euler equation,

$$\frac{\xi_t \eta_t}{c_t} = \beta\mathbb{E}_t\left(\frac{\xi_{t+1}\alpha F(k_{t+1}, 1-l_{t+1}, z_{t+1})}{c_{t+1}k_{t+1}}\right), \tag{1}$$

and the intratemporal first order condition

$$\chi_t l_t^{\nu-1} = \frac{1-\alpha}{1-l_t}\frac{F(k_t, 1-l_t, z_t)}{c_t}. \tag{2}$$

Setting $s_t = \frac{k_{t+1}\eta_t}{F(k_t, 1-l_t, z_t)}$, the Euler equation reduces to

$$\frac{s_t}{1-s_t} = \alpha\beta\mathbb{E}_t\left(\frac{\xi_{t+1}}{\xi_t}\frac{1}{1-s_{t+1}}\right).$$

Write $S_t = \frac{s_t}{1-s_t}$ to get

$$S_t = \alpha\beta\mathbb{E}_t\left(\frac{\xi_{t+1}}{\xi_t}(1+S_{t+1})\right).$$

Now, we can solve forward. $S_{t+1} = \alpha\beta\mathbb{E}_{t+1}\left(\frac{\xi_{t+2}}{\xi_{t+1}}(1+S_{t+2})\right)$ so

$$S_t = \alpha\beta\frac{\mathbb{E}_t\xi_{t+1}}{\xi_t} + \alpha^2\beta^2\mathbb{E}_t\left(\frac{\xi_{t+2}}{\xi_t}\right) + \alpha^2\beta^2\mathbb{E}_t\left(\frac{\xi_{t+2}}{\xi_t}S_{t+2}\right).$$

If we continue to solve forward, we get

$$S_t = \sum_{r=1}^{\infty}\alpha^r\beta^r\frac{\mathbb{E}_t\xi_{t+r}}{\xi_t} + \lim_{r\to\infty}\alpha^r\beta^r\mathbb{E}_t\left(\frac{\xi_{t+r}}{\xi_t}S_{t+r}\right).$$

We therefore conclude that $S_t = \sum_{r=1}^{\infty}\alpha^r\beta^r\frac{\mathbb{E}_t\xi_{t+r}}{\xi_t}$. To do this formally, we invoke the transversality condition.[1] Therefore, we get the formula,

$$S_t = \frac{\alpha\beta}{1-\alpha\beta}\frac{\mathbb{E}(\xi)}{\xi_t},$$

since the ξ_ts are i.i.d.

Now, we substitute back s_t and get

$$\frac{s_t}{1-s_t} = \frac{\alpha\beta\mathbb{E}(\xi)}{1-\alpha\beta}\frac{1}{\xi_t},$$

$$s_t = \frac{\alpha\beta\frac{\mathbb{E}(\xi)}{1-\alpha\beta}\frac{1}{\xi_t}}{1+\frac{\alpha\beta\mathbb{E}(\xi)}{1-\alpha\beta}\frac{1}{\xi_t}} = \frac{\alpha\beta\mathbb{E}(\xi)}{(1-\alpha\beta)\xi_t+\alpha\beta\mathbb{E}(\xi)}.$$

Reality checks: s_t is increasing in both α and β, which is good. And note that if $\xi_t = \mathbb{E}\xi_t$, then we get the savings rate from problem set 1, $s_t = \alpha\beta$. It is independent of η_t, which is due to the substitution and wealth effects canceling for log utility. It is decreasing in ξ_t which makes sense, if the consumer cares extra much about a particular period, the consumer will save less during that period. χ_t only affects the work-leisure tradeoff, not the savings rate. z_t shifts the level but not share of savings.

Substituting back in to the first order condition for leisure, we get

$$\frac{1-l_t}{l_t^{1-\nu}} = \frac{1-\alpha}{\chi_t}\left(1+\frac{\alpha\beta}{1-\alpha\beta}\frac{\mathbb{E}\xi}{\xi_t}\right) = \frac{1-\alpha}{\chi_t}\left(1-\frac{\mathbb{E}\xi}{\xi}+\frac{1}{1-\alpha\beta}\frac{\mathbb{E}\xi}{\xi_t}\right),$$

and therefore l_t is increasing in χ_t and ξ_t, which matches at least my a priori intuition. Furthermore, leisure is decreasing in β and increasing in α which also matches our intuition.

Note: These exercises are quite neat, and some (I) might argue kind of fun. But we can really only hope to use the constant savings rate trick when we have 1) log utility and 2) full depreciation. Try for yourself to do the first exercise from the previous homework with CRRA utility or $\delta \neq 1$.

[1] Assume not, then there exists a sequence $S_{t+r_k} \to \infty$. Since ξ_{t+r} is bounded, this amounts to $\lim\alpha^s\beta^s\mathbb{E}_t(S_{t+s}) \geq \epsilon$. On the other hand, the transversality condition says that $\lim_{r\to\infty}\beta^t\mathbb{E}u'(c_r)k_r \leq 0$. Or, equivalently, $\lim_{r\to\infty}\beta^r\mathbb{E}\frac{k_r}{c_r} = \lim_{r\to\infty}\beta^r\mathbb{E}\frac{1}{(1-s_t)F_k(k_t,1-l_t,z_t)} \leq 0$. But since $\mathbb{E}(S_r) \geq \epsilon/(\alpha^r\beta^r)$, and F_k is bounded from below (there is a maximum possible capital, since all the stochastic elements are finite valued) we get a contradiction.