Suggested Solutions for Homework 5

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If you find any errors, or unclear sections, please help me and your classmates by emailing me at `karl.harmenberg@phdstudent.hhs.se`.

Problem 1

Externality 1: Firms, when deciding whether to enter, do not take in to account that their entry lowers the probability of finding a match for all other firms. Therefore, from this externality alone, there will be too many vacancy postings.[1] This is related to the parameter α. Let $M(u,v) = Au^{\alpha}v^{1-\alpha}$ be the matching function. If $\alpha = 0$, then $M(u,v) = Av$ so the probability of a firm getting a match is A, independent of v. That is, this externality does not exist for $\alpha = 0$- However, if $\alpha = 1$, then $M(u,v) = Au$ so an entering firm will not at all raise the total number of matches (total crowding out).

[1] An analogy is the decision to commute by car. Most people, when making the decision to commute, do not take in to account that their presence on the road makes several other commuters' travel time slightly longer.

Externality 2: Firms, when deciding whether to enter, do not fully take in to account social welfare, but only enter if $(1-\beta)S$ outweighs the cost of posting a vacancy. If they would maximize social welfare, they would enter if S outweighed the cost of entering. This is clearly dependent on the parameter β. If β is high, there will be too few vacancy postings, and if $\beta = 0$, this externality does not exist.

These two externalities have effects of opposite sign, externality 1 induces firms to post too many vacancies, and externality 2 induces firms to post too few vacancies. As it so happens, these effects cancel perfectly when $\alpha = \beta$.

Problem 2

For high skilled labor we have the following recursive equations,

$$rJ_h = y_h - w_h + \sigma(V_h - J_h), \tag{1}$$
$$rV_h = -c + \lambda_f(J_h - V_h), \tag{2}$$
$$rE_h = w_h + \sigma(U_h - E_h), \tag{3}$$
$$rU_h = b + \lambda_w(E_h - U_h), \tag{4}$$

and the following Nash bargaining equations,

$$S_h = J_h - V_h + E_h - U_h, \tag{5}$$
$$\beta S_h = E_h - U_h. \tag{6}$$

We also have the free entry condition,

$$V_h = 0. \tag{7}$$

Finally, we have the matching function relations,

$$\lambda_{f,h} = m(\theta_h), \tag{8}$$
$$\lambda_{w,h} = \theta m(\theta_h). \tag{9}$$

These are nine equations in the nine endogenous variables $J_h, V_h, E_h, U_h, S_h, w_h, \lambda_{f,h}, \lambda_{w,h}, \theta_h$. Therefore, solving for θ_h is purely an algebraic task. Notice that nowhere in this system does u_h or v_h show up.

Using equations (2) and (7), we get

$$J_h = \frac{c}{\lambda_{f,h}}.$$

Substituting in to (1) gives

$$w_h = y_h - (\sigma + r)\frac{c}{\lambda_{f,h}}.$$

Taking (1)-(2)+(3)-(4) gives

$$r(J_h - V_h + E_h - U_h) = y_h + c - b - (\lambda_f + \sigma)(J_h - V_h) - (\lambda_w + \sigma)(E_h - U_h).$$

Now using (5) and (6) we can rewrite this as

$$rS_h = y_h + c - b - (\lambda_f + \sigma)(1-\beta)S_h - (\lambda_w + \sigma)\beta S_h.$$

Since $(1-\beta)S_h = J_h - V_h = J_h = \frac{c}{\lambda_{f,h}}$, we can write this as

$$\frac{r}{1-\beta}\frac{c}{\lambda_{f,h}} = y_h + c - b - c - \frac{c\sigma}{\lambda_{f,h}} - \frac{\beta}{1-\beta}c\frac{\lambda_{w,h}+\sigma}{\lambda_{f,h}},$$
$$\frac{\beta}{1-\beta}(\lambda_{w,h}+\sigma) + \frac{r}{1-\beta} + \sigma = \frac{y_h - b}{c}\lambda_{f,h}.$$

Finally, using equations (8) and (9), we have an expression for θ_h,

$$\beta\theta_h m(\theta_h) + r + \sigma = (1-\beta)\frac{y_h - b}{c}m(\theta_h).$$

To close the model, we add

$$\theta_h = \frac{v_h}{u_h}, \tag{10}$$
$$\sigma(1-u_h) = \theta_h m(\theta_h)u_h. \tag{11}$$

In total, we have 11 endogenous variables and 11 equations. By identical reasoning, we get for low skilled workers analogous equations,

$$\beta\theta_l m(\theta_l) + r + \sigma = (1-\beta)\frac{y_l - b}{c}m(\theta_l),$$
$$\theta_l = \frac{v_l}{u_l},$$
$$\sigma(1-u_l) = \theta_l m(\theta_l)u_l.$$

We see that $\theta_h > \theta_l$ and thus unemployment among the highly skilled is lower than unemployment among the low skilled workers. The two labor markets are completely disjoint (they can just as easily be thought of as located on different continents).

For the second set up, let the population consist of ω workers with high productivity and $1-\omega$ workers with low productivity. We assume that firms do not know the productivity of a worker before meeting them, but know the productivity of a worker when meeting them before hiring them. Now, however, the two labor markets are not completely disjoint since a firm cannot distinguish ex ante between a low skill and a high skill worker. In total, we write down the following equations.

$$rJ_h = y_h - w_h + \sigma(V - J_h), \tag{12}$$
$$rJ_l = y_l - w_l + \sigma(V - J_l), \tag{13}$$
$$rV = -c + \lambda_f(\pi J_h + (1-\pi)J_l - V), \tag{14}$$
$$rE_h = w_h + \sigma(U_h - E_h), \tag{15}$$
$$rE_l = w_l + \sigma(U_l - E_l), \tag{16}$$
$$rU_h = b + \lambda_w(E_h - U_h), \tag{17}$$
$$rU_l = b + \lambda_w(E_l - U_l), \tag{18}$$
$$S_h = J_h - V + E_h - U_h, \tag{19}$$
$$S_l = J_l - V + E_l - U_l, \tag{20}$$
$$\beta S_h = E_h - U_h, \tag{21}$$
$$\beta S_l = E_l - U_l, \tag{22}$$
$$V = 0, \tag{23}$$
$$\lambda_f = m(\theta), \tag{24}$$
$$\lambda_w = \theta m(\theta), \tag{25}$$
$$v = \theta(\omega u_h + (1-\omega)u_l), \tag{26}$$
$$\sigma(1-u_h) = \lambda_w u_h, \tag{27}$$
$$\sigma(1-u_l) = \lambda_w u_l, \tag{28}$$
$$\pi = \frac{\omega u_h}{\omega u_h + (1-\omega)u_l}. \tag{29}$$

We have a total of eighteen equations in eighteen endogenous variables $(J_h, J_l, V, E_h, E_l, U_h, U_l, w_h, w_l, S_h, S_l \lambda_f, \lambda_w, \theta, u_h, u_l, v, \pi)$. Equations (27) and (28) give that $u_l = u_h$, so $\pi = \omega$ from (29). Write $J = \omega J_h + (1-\omega)J_l$, $y = \omega y_h + (1-\omega)y_l$, $w = \omega w_h + (1-\omega)w_l$, $E = \omega E_h + (1-\omega)E_l$, $U = \omega U_h + (1-\omega)U_l$, $S = \omega S_h + (1-\omega)S_l$.

Then the above system reduces to

$$\begin{aligned}
rJ &= y - w + \sigma(V - J), \\
rV &= -c + \lambda_f(J - V), \\
rE &= w + \sigma(U - E), \\
rU &= b + \lambda_w(E - U), \\
S &= J - V + E - U, \\
\beta S &= E - U, \\
V &= 0, \\
\lambda_f &= m(\theta), \\
\lambda_w &= \theta m(\theta), \\
v &= \theta u, \\
\sigma(1 - u) &= \lambda_w u.
\end{aligned}$$

This is the standard system we know and love, so we can solve for θ as usual from,

$$\beta\theta m(\theta) + r + \sigma = (1 - \beta)\frac{y - b}{c} m(\theta).$$

Now, using our uniquely determined θ, we can solve for w_h, w_l. The unemployment level in the two models will in general not be identical.

Since we have $y_h > y > y_l$, we have $\theta_l < \theta < \theta_h$ and $u_h < u < u_l$ where u is unemployment in the mixed labor market and u_l, u_h are the unemployment rates in the disjoint labor markets (the Beveridge curves are the same). Therefore, by the wage equation $w = \beta(y + c\theta) + (1 - \beta)b$ we have that $w_h > w > w_l$.

Problem 3

An Arrow-Debreu equilibrium is $\{c_t^*(z^t), p_t^*(z^t)\}_{t \geq 0, z^t \in Z^t}$ such that $\{c_t^*(z^t)\}$ solves the consumer's problem:

$$\max_{\{c_t(z^t)\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t)) \quad s.t. \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t(z^t)(z_t(z^t) - c_t(z^t)) = 0,$$

and the allocation is feasible,

$$z_t(z^t) = c_t^*(z^t) \qquad\qquad \forall t \geq 0, z^t \in Z^t.$$

The consumer's Lagrangian is (here I drop the $*$s)

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t)) + \lambda \left(\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t(z^t)(z_t(z^t) - c_t(z^t)) \right)$$

(note that there is only one constraint!) The first order condition is

$$\beta^t \pi(z^t) u'(c_t(z^t)) = \lambda p_t(z^t).$$

Therefore, dividing by this equation for time 0, we get

$$\frac{p_t(z^t)}{p_0(z_0)} = \beta^t \frac{\pi(z^t)}{\pi(z_0)} \frac{u'(c_t(z^t))}{u'(c_0)}$$

Finally, since in equilibrium we must have $c_t = z_t$, we get

$$p_t(z^t) = \beta^t \pi(z^t) \frac{u'(z_t(z^t))}{u'(z_0)},$$

which is a function of primitives (where we normalized $p_0(z_0) = 1$).

The value of an asset paying 2 units of consumption in period 13 for sure is

$$V_{13} = 2 \sum_{z^{13} \in Z^{13}} \beta^{13} \pi(z^{13}) \frac{u'(z_{13}(z^{13}))}{u'(z_0)}$$

The return on a riskless bond delivering 2 at maturity 13 is

$$R_{13} = \frac{2}{V_{13}} = \frac{1}{\sum_{z^{13} \in Z^{13}} \beta^{13} \pi(z^{13}) \frac{u'(z_{13}(z^{13}))}{u'(z_0)}}$$

The price of an asset paying z_t units of consumption in state z^t of period 13 is

$$\bar{V}_{13} = \sum_{z^{13} \in Z^{13}} \beta^{13} \pi(z^{13}) \frac{u'(z_{13}(z^{13}))}{u'(z_0)} z_{13}(z^{13}).$$

The expected payout of such an asset is

$$\sum_{z^{13} \in Z^{13}} \pi(z^{13}) z_{13}(z^{13})$$

so the expected return is

$$\bar{R}_{13} = \frac{\sum_{z^{13} \in Z^{13}} \pi(z^{13}) z_{13}(z^{13})}{\sum_{z^{13} \in Z^{13}} \beta^{13} \pi(z^{13}) \frac{u'(z_{13}(z^{13}))}{u'(z_0)} z_{13}(z^{13})}$$

Now we want to show that $\bar{R}_{13} > R_{13}$.

$\bar{R}_{13} > R_{13}$ is equivalent to (removing constants)

$$\frac{\sum_{z^{13} \in Z^{13}} \pi(z^{13}) z_{13}(z^{13})}{\sum_{z^{13} \in Z^{13}} \pi(z^{13}) u'(y_{13}(z^{13})) y_{13}(z^{13})} > \frac{1}{\sum_{z^{13} \in Z^{13}} \pi(z^{13}) u'(y_{13}(z^{13}))}.$$

We invert this and get

$$\frac{\sum_{z^{13} \in Z^{13}} \pi(z^{13}) u'(z_{13}(z^{13})) z_{13}(z^{13})}{\sum_{z^{13} \in Z^{13}} \pi(z^{13}) z_{13}(z^{13})} < \sum_{z^{13} \in Z^{13}} \pi(z^{13}) u'(z_{13}(z^{13})),$$

$$\sum_{z^{13} \in Z^{13}} \pi(z^{13}) u'(z_{13}(z^{13})) z_{13}(z^{13}) < \sum_{z^{13} \in Z^{13}} \pi(z^{13}) u'(z_{13}(z^{13})) \sum_{z^{13} \in Z^{13}} \pi(z^{13}) z_{13}(z^{13}).$$

Now, that u is strictly concave means that u' is strictly monotonically decreasing. Therefore, when z is large, $u'(z)$ is small. So in the product $u'(z)z$, $u'(z)$ and z counteract each other. If we rewrite the expression above in probabilistic terms, we have

$$\mathbb{E}(u'(z)z) < \mathbb{E}(u'(z))\mathbb{E}z.$$

Since $u'(z)$ and z are negatively related, this inequality holds. Alternatively (or, if you want to prove this fact), use Chebyshev's sum inequality.[2] The math is slightly involved, but the intuition should be clear: An index fund yields high return in good times and low return in bad times. The marginal utility of consumption is low in good times and high in bad times. For agents to voluntarily purchase an index fund, it must provide higher expected return to compensate for the fact that the high return will come when you least need it.

[2] http://en.wikipedia.org/wiki/Chebyshev%27s_sum_inequality

Problem 4

An equilibrium is $\{c^*_{1,t}, c^*_{2,t}\}_{t=1,2}$ and R^* such that

1. $\{c^*_{1,t}\}$ solves consumer 1's problem,

$$\max_{c_{1,1},c_{1,2}} \log c_{1,1} + \beta \log c_{1,2} \qquad s.t \qquad \begin{aligned} c_{1,2} &= R^* a_1 + \underline{y}, \\ c_{1,1} + a_1 &= \bar{y}, \\ a_1 &\geq 0. \end{aligned}$$

2. $\{c^*_{2,t}\}$ solves consumer 2's problem,

$$\max_{c_{2,1},c_{2,2}} \log c_{2,1} + \beta \log c_{2,2} \qquad s.t \qquad \begin{aligned} c_{2,2} &= R^* a_2 + \bar{y}, \\ c_{1,2} + a_2 &= \underline{y}, \\ a_2 &\geq 0. \end{aligned}$$

3. Feasibility,

$$\begin{aligned} a_1 + a_2 &= 0, \\ c_{1,1} + c_{2,1} &= \bar{y} + \underline{y}, \\ c_{1,2} + c_{2,2} &= \bar{y} + \underline{y}. \end{aligned}$$

Consumer 1's Lagrangian is

$$\mathcal{L}_1 = \log c_{1,1} + \beta \log c_{1,2} + \lambda_1(R^* a_1 + \underline{y} - c_{1,2}) + \mu_1(\bar{y} - c_{1,1} - a_1) + \xi_1 a_1.$$

The second consumer's Lagrangian is analogous. Taking first order conditions, we get

$$\begin{aligned} &\partial c_{1,1} && \frac{1}{c_{1,1}} = \mu_1, \\ &\partial c_{1,2} && \frac{\beta}{c_{1,2}} = \lambda_1, \\ &\partial a_1 && R^*\lambda_1 + \xi_1 = \mu_1. \end{aligned}$$

Eliminating the Lagrangian multipliers, we get

$$\begin{aligned} \frac{1}{c_{1,1}} &= \beta R^* \frac{1}{c_{1,2}} + \xi_1, \\ \frac{1}{c_{1,1}} &\geq \beta R^* \frac{1}{c_{1,2}} \\ c_{1,2} &\geq \beta R^* c_{1,1} \end{aligned}$$

The same reasoning holds for the second consumer, so we also get

$$c_{2,2} \geq \beta R^* c_{2,1}$$

Since the equilbrium outcome is that there is no borrowing, we have that both consumers consume their endowment. Therefore, we have

$$\begin{aligned} \bar{y} &\geq \beta R^* \underline{y}, \\ \underline{y} &\geq \beta R^* \bar{y}. \end{aligned}$$

The first inequality is always satisfied if the second holds, therefore we can summarize the constraint on the interest rate as

$$R^* \leq \frac{1}{\beta} \frac{\underline{y}}{\bar{y}}.$$

Since both consumers' Euler equation hold then, and the allocation is feasible, this describes the equilibrium interest rate.

(b) Now we have the following set up: An equilibrium is $\{c^*_{1,t}, c^*_{2,t}\}_{t=1,2}$ and R^* such that

1. $\{c^*_{1,t}\}$ solves consumer 1's problem,

$$\max_{c_{1,1}, c_{1,2}} \log c_{1,1} + \beta \log c_{1,2} \qquad s.t \qquad \begin{aligned} c_{1,2} &= R^* a_1 + \underline{y} - R^* L, \\ c_{1,1} + a_1 &= \bar{y} + L, \\ a_1 &\geq 0. \end{aligned}$$

2. $\{c^*_{2,t}\}$ solves consumer 2's problem,

$$\max_{c_{2,1}, c_{2,2}} \log c_{2,1} + \beta \log c_{2,2} \qquad s.t \qquad \begin{aligned} c_{2,2} &= R^* a_2 + \bar{y} - R^* L, \\ c_{1,1} + a_2 &= \underline{y} + L, \\ a_2 &\geq 0. \end{aligned}$$

3. Feasibility,

$$
\begin{aligned}
a_1 + a_2 &= 2L, \\
c_{1,1} + c_{2,1} &= \bar{y} + \underline{y}, \\
c_{1,2} + c_{2,2} &= \bar{y} + \underline{y}.
\end{aligned}
$$

Given an interest rate R^*, consumers' problems can be written as

$$
\begin{aligned}
&\max_{a_1 \in [0,\infty)} \log(\bar{y} + L - a_1) + \beta(R^* a_1 + \underline{y} - R^* L), \\
&\max_{a_2 \in [0,\infty)} \log(\underline{y} + L - a_2) + \beta(R^* a_2 + \bar{y} - R^* L).
\end{aligned}
$$

We can divide in to two cases: When the borrowing constraint is binding for consumer 2, and when it is not.

If the borrowing constraint is binding for consumer 2, then $a_1 = 2L$ so the constraint is not binding for consumer 1. Therefore, the Euler equation must hold for consumer 1,

$$
\begin{aligned}
\frac{1}{\bar{y} - L} &= \beta R^* \frac{1}{\underline{y} + R^* L}, \\
R^* &= \frac{1}{\beta} \frac{\underline{y}}{\bar{y} - \frac{1+\beta}{\beta} L}.
\end{aligned}
$$

Furthermore, the total utilities for consumer 1 and 2 are

$$
\begin{aligned}
&\log(\bar{y} - L) + \beta \log(\underline{y} + L) \\
&\log(\underline{y} + L) + \beta \log(\bar{y} - L)
\end{aligned}
$$

respectively.

Now, let $\beta = 1$. If $\bar{y} > \underline{y}$ then for small L, a government bond is Pareto improving (differentiate the two utility expressions with respect to L). If $\bar{y} = \underline{y}$, then a lump sum transfer has no effect.

If the borrowing constraint is not binding for consumer 2, then both Euler equations must hold with equality and by the feasibility constraints, we get $\beta R^* = 1$. Then consumption for each consumer is constant over time, and $c_2 = \frac{\underline{y} + \beta \bar{y}}{1+\beta}$. Since we must have $a_2 \geq 0$, we get that the borrowing constraint is not binding when $L \geq \frac{\beta}{1+\beta}(\bar{y} - \underline{y})$. In conclusion, we have

L	a_2	R^*
$L = 0$	0	$\frac{1}{\beta} \frac{\underline{y}}{\bar{y}}$
$0 < L \leq \frac{\beta}{1+\beta}(\bar{y} - \underline{y})$	0	$\frac{1}{\beta} \frac{\underline{y}}{\bar{y} - \frac{1+\beta}{\beta} L}$
$L > \frac{\beta}{1+\beta}(\bar{y} - \underline{y})$	$L - \frac{\beta}{1+\beta}(\bar{y} - \underline{b})$	$\frac{1}{\beta}$

Table 1: Asset holdings and interest rates under different lump sum transfers.