Macroeconomics I

Homework 2 - Suggested solutions

Niels-Jakob Harbo Hansen

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Introduction

If you find any errors or unclear sections in those answers, please help me and your classmates by emailing me at Niels-Jakob Harbo Hansen at `nielsjakobharbo.hansen@iies.su.se` or Jonna Olsson `jonna.olsson@ne.su.se`. Then we will update the document. Please also send any of us an email if you have other questions!

Your homeworks will be handed back in the next seminar. If you want to get feed-back earlier, just send me an email.

Problem 1

Part (a)

- A **sequential competitive equilibrium** is a set of sequences for allocations $\left\{a^*_{1,t+1}, c^*_{1t}\right\}_{t=0}^{\infty}$ and $\left\{a^*_{2,t+1}, c^*_{2t}\right\}_{t=0}^{\infty}$ and prices $\{q^*_t\}_{t=0}^{\infty}$ such that

  1. $\{a^*_{it}, c^*_{it}\}_{t=0}^{\infty}$ solves the problem of the household:

$$\max_{\{c_{i,t},a_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ for i=1,2} \tag{1}$$
$$\text{s.t.}$$
$$c_{it} + q^*_t a_{i,t+1} = e_{it} + a_{it} \forall t$$
$$\lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t} \geq 0$$

  2. Markets clear

$$\sum_{i=1}^{2} c^*_{it} = \sum_{i=1}^{2} e_{it} \qquad \forall t \tag{2}$$
$$\sum_{i=1}^{2} a^*_{i,t+1} = 0 \qquad \forall t \tag{3}$$

Part (b)

- A **date-zero competitive equilibrium** is a set of sequences for allocations $\{c^*_{1t}\}_{t=0}^{\infty}$ and $\{c^*_{2t}\}_{t=0}^{\infty}$ and prices $\{p^*_t\}_{t=0}^{\infty}$ such that

1. $\{c^*_{it}\}_{t=0}^{\infty}$ solves the problem of the household:

$$\max_{\{c_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ for i=1,2} \tag{4}$$
$$\text{s.t.} \sum_{t=0}^{\infty} p^*_t c_{it} = \sum_{t=0}^{\infty} p^*_t e_{it}$$

2. Markets clear

$$\sum_{i=1}^{2} c^*_{it} = \sum_{i=1}^{2} e_{it} \ \forall t \tag{5}$$

Part (c)

Sequential equilibrium

- The Lagrangean for the household problem reads

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) + \sum_{t=0}^{\infty} \lambda_{it} \left(c_{it} + q_t a_{i,t+1} - e_{it} - a_{it}\right) \tag{6}$$

- The first order conditions become

$$c_{it} : \beta^t u'_i(c_i t) + \lambda_{it} = 0 \tag{7}$$
$$a_{i,t+1} : \lambda_{it} q_t - \lambda_{it+1} = 0 \tag{8}$$

- Combine to get

$$\frac{q_t}{\beta} = \frac{u'(c_{i,t+1})}{u'(c_{i,t})} \ \forall i, t \tag{9}$$

- This implies[1]

$$\frac{u'(c_{1,t+1})}{u'(c_{1,t})} = \frac{u'(c_{2,t+1})}{u'(c_{2,t})} \tag{10}$$
$$\Rightarrow u'(c_{i,t+1}) = u'(c_{i,t+1}) \tag{11}$$
$$\Rightarrow c_{i,t+1} = c_{i,t} = c_i \ \forall t \tag{12}$$

- Inserting into (9) yields

$$q^*_t = \beta \tag{13}$$

[1] To see the first step notice that if $u'(c_{1,t+1}) > u'(c_{1,t})$ would imply $u'(c_{2,t+1}) > u'(c_{2,t})$ why $c_{1,t+1} + c_{2,t+1} < c_{1,t+1} + c_{2,t+1}$. This would be inconsistent with market clearing. Reversed argument can be applied if $u'(c_{1,t+1}) < u'(c_{1,t})$

- Now note that the consolidated budget constraint, $\sum_t (\beta)^t c_{it} = \sum_t (\beta)^t e_{it}$ for agent 1 reads[2]

$$\sum_t (\beta)^t c_1 = e_l + (\beta) e_h + (\beta)^2 e_l + \dots \tag{14}$$

$$\Rightarrow \sum_t (\beta)^t c_1 = e_l \sum_{t=0}^{\infty} (\beta^2)^t + e_h \beta \sum_{t=0}^{\infty} (\beta^2)^t \tag{15}$$

$$\Rightarrow c_1 \frac{1}{1-\beta} = e_l \frac{1}{1-\beta^2} + e_h \beta \frac{1}{1-\beta^2} \tag{16}$$

$$\Rightarrow c_1 = e_l \frac{1}{1+\beta} + e_h \frac{\beta}{1+\beta} \tag{17}$$

- Similar computations for agent 2 yields

$$\sum_t (\beta)^t c_1 = e_h + (\beta) e_l + (\beta)^2 e_h + \dots \tag{18}$$

$$\Rightarrow \sum_t (\beta)^t c_1 = e_h \sum_{t=0}^{\infty} (\beta^2)^t + e_l \beta \sum_{t=0}^{\infty} (\beta^2)^t \tag{19}$$

$$\Rightarrow c_1 \frac{1}{1-\beta} = e_h \frac{1}{1-\beta^2} + e_l \beta \frac{1}{1-\beta^2} \tag{20}$$

$$\Rightarrow c_2 = e_h \frac{1}{1+\beta} + e_l \frac{\beta}{1+\beta} \tag{21}$$

- Verify that this indeed is an equilibrium by checking whether c_1 and c_2 sum to $e_h + e_l$.

$$c_1 + c_2 = \frac{e_l}{1+\beta} + \frac{\beta e_h}{1+\beta} + \frac{e_h}{1+\beta} + \frac{\beta e_l}{1+\beta} = e_h + e_l \tag{22}$$

Date-zero equilibrium

- The Lagrangean reads

$$L = \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) + \lambda_i \sum_{t=0}^{\infty} (p_t c_{it} - p_t e_{it}) \tag{23}$$

- Yields the first order condtions

$$c_{1t} : \beta^t u'(c_{1t}) - \lambda_1 p_t = 0 \ \forall t \tag{24}$$

$$c_{2t} : \beta^t u'(c_{2t}) - \lambda_2 p_t = 0 \ \forall t \tag{25}$$

- From this we get

$$\frac{u'(c_{1,t+1})}{u'(c_{1,t})} = \frac{u'(c_{2,t+1})}{u'(c_{2,t})} = \frac{p_{t+1}}{\beta p_t} \tag{26}$$

[2] To see first line remember that zero is an *even* number :). To see fourth implication sign notice $(1+\beta)(1-\beta) = 1-\beta^2$

- By similar argument from above we then get

$$c_{i,t} = c_{i,t+1} \tag{27}$$

- And then

$$\frac{p_{t+1}}{p_t} = \beta \tag{28}$$

- Now use this in agent 1's budget constraint:

$$c_1 \sum_{t=0}^{\infty} p_t = \sum_{t=0}^{\infty} p_t e_1 \tag{29}$$

$$c_1 \left(p_0 + p_1 + p_2 + ...\right) = \left(e_l p_0 + e_h p_1 + e_l p_2 + ...\right) \tag{30}$$

- And notice

$$p_1 = \frac{p_1}{p_0} p_0 = \beta p_0 \tag{31}$$

$$p_2 = \frac{p_2}{p_1} p_1 = \beta\beta p_0 \tag{32}$$

$$... \tag{33}$$

$$p_t = \beta^t p_0 \tag{34}$$

- Let $p_0 = 1$ (numeraire) and use this in the budget constraint of the consumer to get

$$c_1 \sum_t \beta^t = e_l(1 + \beta^2 + \beta^4 + ...) + \beta e_h(1 + \beta^2 + \beta^4 + ...) \tag{35}$$

$$\Rightarrow c_1 = e_l \frac{1}{1+\beta} + e_h \frac{\beta}{1+\beta} \tag{36}$$

- And likewise

$$c_2 = e_h \frac{1}{1+\beta} + e_l \frac{\beta}{1+\beta} \tag{37}$$

Intuition

- Notice:

$$c_1 - c_2 = \left(e_l - e_h\right) \frac{1-\beta}{1+\beta} \leq 0 \tag{38}$$

- Thus, consumption of agent 1 < consumption of agent 2, and this difference is decreasing in β. Reason is that agent 2 gets high endowment before agent 1, which (owing to discounting) makes agent 2 richer in present value terms. When β increases agents discount future less, why the difference in consumption becomes smaller.

Part (d)

- We know from above that

$$\frac{u'(c_{1,t+1})}{u'(c_{1,t})} = \frac{u'(c_{2,t+1})}{u'(c_{2,t})} \tag{39}$$

- Under the given preferences this implies

$$\frac{c_{1,t}}{c_{1,t+1}} = 1 \Rightarrow c_{1,t} = c_{1,t+1} \tag{40}$$

- Now the consolidated budget constraint of each agent reads

$$\sum_t \beta^t c_{i,t} = e_l + \beta e_h + \beta^2 e_l + ... \tag{41}$$

$$\Rightarrow \sum_t \beta^t c_{i,t} = \sum_{t=0}^{\infty} \beta^{2t} e_l + \sum_{t=0}^{\infty} \beta^{2t} \beta e_h \tag{42}$$

$$\Rightarrow \sum_t \beta^t c_{i,t} = \frac{e_l}{1-\beta^2} + \frac{e_h \beta}{1-\beta^2} \tag{43}$$

- For agent 1 this implies

$$\frac{c_1}{1-\beta} = \frac{e_l}{1-\beta^2} + \frac{e_h \beta}{1-\beta^2} \tag{44}$$

$$\Rightarrow c_1 = e_l \frac{1-\beta}{1-\beta^2} + e_h \frac{\beta(1-\beta)}{1-\beta^2} \tag{45}$$

$$\Rightarrow c_1 = e_l \frac{1}{1+\beta} + e_h \frac{\beta}{1+\beta} \tag{46}$$

- Via market clearing this then implies

$$c_{2,t=odd} = 2e_h - c_1 = e_h \frac{2+\beta}{1+\beta} - e_l \frac{1}{1+\beta} \tag{47}$$

$$c_{2,t=even} = 2e_l - c_1 = e_l \frac{2(1+\beta)-1}{1+\beta} - e_h \frac{1}{1+\beta} \tag{48}$$

Intuition

- Notice $c_{1,t=even} = c_{1,t=odd}$ while $c_{2,t=even} < c_{1,t=odd}$, while the present value of consumption of agent 1 equals that of agent 2.
- This owes to the fact that agent 1 preferences are such that he/she dislikes consumption variation more than agent 2. Hence, agent 2 will insure agent 1 by taking all the variation in aggregate income.

Problem 2

Part (a)

- A **sequential equilibrium** is a set of sequences for allocations $\{k_{it}^*, k_{ct}^*, k_t^*, n_{it}^*, n_{ct}^*, c_t^*, i_t^*\}_{t=0}^\infty$ and prices $\{p_t^*, r_t^*, w_t^*\}_{t=0}^\infty$ such that:
  1. $\{c_t^*, i_t^*, k_{t+1}^*\}_{t=0}^\infty$ solves the household's problem:

$$\max_{\{c_t, i_t, k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_t) \tag{49}$$
$$\text{s.t.}$$
$$c_t + p_t^* i_t = r_t^* k_t + w_t^* n_t$$
$$k_{t+1} = (1-\delta)k_t + i_t$$

  2. $\{k_{ct}^*, n_{ct}^*\}_{t=0}^\infty$ solves problem of consumption good firm:

$$\max_{k_{ct}, n_{ct}} A_t^{1-\alpha} k_{ct}^\alpha n_{ct}^{1-\alpha} - r_t^* k_{ct} - w_t^* n_{ct} \ \forall t \tag{50}$$

  3. $\{k_{it}^*, n_{it}^*\}_{t=0}^\infty$ solves problem of the investment good firm:

$$\max_{k_{it}, n_{it}} p_t q_t A_t^{1-\alpha} k_{it}^\alpha n_{it}^{1-\alpha} - r_t^* k_{it} - w_t^* n_{it} \ \forall t \tag{51}$$

  4. And market clearing (feasibility) holds

$$n_{it}^* + n_{ct}^* = n_t^* \tag{52}$$
$$k_{it}^* + k_{ct}^* = k_t^* \tag{53}$$
$$c_t^* = A_t^{1-\alpha} k_{ct}^{*\alpha} n_{ct}^{*1-\alpha} \tag{54}$$
$$i_t^* = q_t A_t^{*1-\alpha} k_{it}^{*\alpha} n_{it}^{*1-\alpha} \tag{55}$$

Part (b)

- Take first order conditions to firms problem to get

$$\alpha A_t^{1-\alpha} k_{ct}^{\alpha-1} n_{ct}^{1-\alpha} = r_t^* \tag{56}$$
$$(1-\alpha) A_t^{1-\alpha} k_{ct}^{\alpha} n_{ct}^{-\alpha} = w_t^* \tag{57}$$
$$p_t q_t \alpha A_t^{1-\alpha} k_{it}^{\alpha-1} n_{it}^{1-\alpha} = r_t^* \tag{58}$$
$$p_t q_t (1-\alpha) A_t^{1-\alpha} k_{it}^{\alpha} n_{it}^{-\alpha} = w_t^* \tag{59}$$

- Divide (56) by (57) and (58) by (59)

$$\frac{\alpha}{1-\alpha}\frac{n_{ct}}{k_{ct}} = \frac{r_t^*}{w_t^*} \tag{60}$$
$$\frac{\alpha}{1-\alpha}\frac{n_{it}}{k_{it}} = \frac{r_t^*}{w_t^*} \tag{61}$$

- Equate to get

$$\frac{n_{ct}}{k_{ct}} = \frac{n_{it}}{k_{it}} \tag{62}$$

- Then divide (58) by (56) and use (62) to get

$$\begin{aligned} p_t q_t \left(\frac{k_{it}}{k_{ct}}\right)^{\alpha-1} \left(\frac{n_{it}}{n_{ct}}\right)^{1-\alpha} &= 1 \\ \Rightarrow p_t q_t \left(\frac{k_{it}}{n_{it}}\right)^{\alpha-1} \left(\frac{k_{ct}}{n_{ct}}\right)^{1-\alpha} &= 1 \\ \Rightarrow p_t &= 1/q_t \qquad (63) \end{aligned}$$

Part (c)

- Start by noting from (62) that $k_{it}/k_{ct} = n_{it}/n_{ct}$ why we can write capital and labor in each sector as the same fraction of total labor and capital:[3]

$$k_{it} = s k_t \tag{64}$$
$$n_{it} = s n_t \tag{65}$$
$$k_{ct} = (1-s) k_t \tag{66}$$
$$n_{ct} = (1-s) n_t \tag{67}$$

- Use this in (56) to get an expression for r_t

$$\begin{aligned} r_t^* &= \alpha A_t^{1-\alpha} (s k_t)^{\alpha-1} (s n_t)^{1-\alpha} \\ \Rightarrow r_t^* &= s^{\alpha-1+1-\alpha} \alpha A_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha} \\ \Rightarrow r_t^* &= \underbrace{s^0}_{=1} \alpha A_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha} \qquad (68) \end{aligned}$$

- Use this in (57) to get an expression for w_t

$$\begin{aligned} w_t^* &= (1-\alpha) A_t^{1-\alpha} (s k_t)^{\alpha} (s n_t)^{-\alpha} \\ \Rightarrow w_t^* &= \underbrace{s^0}_{=1} (1-\alpha) A_t^{1-\alpha} k_t^{\alpha} n_t^{-\alpha} \qquad (69) \end{aligned}$$

- Then plug (68) and (69) along (63) with into the budget constraint of the household to get

$$\begin{aligned} c_t + \frac{1}{q_t} i_t &= \alpha A_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha} k_t + (1-\alpha) A_t^{1-\alpha} k_t^{\alpha} n_t^{-\alpha} n_t \\ \Rightarrow c_t + \frac{1}{q_t} i_t &= A_t^{1-\alpha} k_t^{\alpha} n_t^{1-\alpha} \qquad (70) \end{aligned}$$

- Hence, the problem of the social planner can be written

[3] To see this note, that $k_{it}/k_{ct} = \frac{s_k k_t}{(1-s_k) k_t} = \frac{s_k}{(1-s_k)} = n_{it}/n_{ct} = \frac{s_n n_t}{(1-s_n) n_t} = \frac{s_n}{(1-s_n)}$ why $s_k = s_n$

$$\max_{\{c_t^*, i_t^*, k_{t+1}^*\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{71}$$

s.t.

$$c_t + \frac{1}{q_t} i_t = A_t^{1-\alpha} k_t^{\alpha} n_t^{1-\alpha}$$

$$k_{t+1} = (1-\delta)k_t + i_t$$

Intuition

- Notice that we have shown that the economy *aggregates*. That is, we started with a disaggreated economy with two sectors, each having a separate production function. But we have shown that this economy can be represented by one production function (as long as we also correct the price of the investment good by q_t - which can be thought of as the relative productivity of the investment good sector). Also notice that this result was independent of the use of preferences : the result was simply derived from the firms first order conditions.

- This result is important because it hints at something more general: Under some conditions we can represent an economy with different sectors by *one* aggregate production function.