Macroeconomics I

Homework 2 - Suggested solutions

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Introduction

If you find any errors or unclear sections in those answers, please help me and your classmates by emailing me at Niels-Jakob Harbo Hansen at `nielsjakobharbo.hansen@iies.su.se` or Jonna Olsson `jonna.olsson@ne.su.se`,. Please also send any of us an email if you have other questions!

Your homeworks will be handed back in the next seminar. If you want to get feed-back earlier, just send me an email.

Problem 1

Part (a)

- A **sequential competitive equilibrium** is a set of sequences for allocations $\left\{a^*_{1,t+1}, c^*_{1t}\right\}_{t=0}^{\infty}$ and $\left\{a^*_{2,t+1}, c^*_{2t}\right\}_{t=0}^{\infty}$ and prices $\{q^*_t\}_{t=0}^{\infty}$ such that

  1. $\{a^*_{it}, c^*_{it}\}_{t=0}^{\infty}$ solves the problem of the household:

$$\max_{\{c_{i,t}, a_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ for i=1,2} \tag{1}$$
$$\text{s.t.}$$
$$c_{it} + q^*_t a_{i,t+1} = e_{it} + a_{it} \forall t$$
$$\lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t} \geq 0$$

  2. Markets clear

$$\sum_{i=1}^{2} c^*_{it} = \sum_{i=1}^{2} e_{it} \qquad \forall t \tag{2}$$
$$\sum_{i=1}^{2} a^*_{i,t+1} = 0 \qquad \forall t \tag{3}$$

Part (b)

- A **date-zero competitive equilibrium** is a set of sequences for allocations $\{c^*_{1t}\}_{t=0}^{\infty}$ and $\{c^*_{2t}\}_{t=0}^{\infty}$ and prices $\{p^*_t\}_{t=0}^{\infty}$ such that

1. $\{c^*_{it}\}_{t=0}^{\infty}$ solves the problem of the household:

$$\max_{\{c_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ for i=1,2} \tag{4}$$
$$\text{s.t.} \quad \sum_{t=0}^{\infty} p^*_t c_{it} = \sum_{t=0}^{\infty} p^*_t e_{it}$$

2. Markets clear

$$\sum_{i=1}^{2} c^*_{it} = \sum_{i=1}^{2} e_{it} \ \forall t \tag{5}$$

Part (c)

Sequential equilibrium

- The Lagrangean for the household problem reads

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) + \sum_{t=0}^{\infty} \lambda_{it} \left(c_{it} + q_t a_{i,t+1} - e_{it} - a_{it}\right) \tag{6}$$

- The first order conditions become

$$c_{it} : \beta^t u'_i(c_i t) + \lambda_{it} = 0 \tag{7}$$
$$a_{i,t+1} : \lambda_{it} q_t - \lambda_{it+1} = 0 \tag{8}$$

- Combine to get

$$\frac{q_t}{\beta} = \frac{u'(c_{i,t+1})}{u'(c_{i,t})} \ \forall i, t \tag{9}$$

- This implies[1]

$$\frac{u'(c_{1,t+1})}{u'(c_{1,t})} = \frac{u'(c_{2,t+1})}{u'(c_{2,t})} \tag{10}$$
$$\Rightarrow u'(c_{i,t+1}) = u'(c_{i,t+1}) \tag{11}$$
$$\Rightarrow c_{i,t+1} = c_{i,t} = c_i \tag{12}$$

- Inserting into (9) yields

$$q^*_t = \beta \tag{13}$$

- Now note that the consolidated budget constraint, $\sum_t (\beta)^t c_{it} = \sum_t (\beta)^t e_{it}$ for agent 1 reads

$$\sum_t (\beta)^t c_1 = e_h + (\beta) e_l + (\beta)^2 e_h + \dots \tag{14}$$

$$\sum_t (\beta)^t c_1 = e_h \sum_{t=0}^{\infty} (\beta^2)^t + e_l \beta \sum_{t=0}^{\infty} (\beta^2)^t \tag{15}$$

$$c_1 \frac{1}{1-\beta} = e_h \frac{1}{1-\beta^2} + e_l \beta \frac{1}{1-\beta^2} \tag{16}$$

[1] To see the first step notice that if $u'(c_{1,t+1}) > u'(c_{1,t})$ would imply $u'(c_{2,t+1}) > u'(c_{2,t})$ why $c_{1,t+1} + c_{2,t+1} < c_{1,t+1} + c_{2,t+1}$. This would be inconsistent with market clearing. Reversed argument can be applied if $u'(c_{1,t+1}) < u'(c_{1,t})$

- And for agent 2

$$\sum_t (\beta)^t c_2 = e_l + (\beta) e_h + (\beta)^2 e_l + \dots \tag{17}$$

$$\sum_t (\beta)^t c_1 = e_l \sum_{t=0}^{\infty} (\beta^2)^t + e_h \beta \sum_{t=0}^{\infty} (\beta^2)^t \tag{18}$$

$$c_2 \frac{1}{1-\beta} = e_l \frac{1}{1-\beta^2} + e_h \beta \frac{1}{1-\beta^2} \tag{19}$$

- Hence[2]

$$c_1 = e_h \frac{1-\beta}{1-\beta^2} + e_l \frac{\beta(1-\beta)}{1-\beta^2} = \frac{e_h}{1+\beta} + \frac{\beta e_l}{1+\beta} \tag{20}$$

$$c_2 = e_l \frac{1-\beta}{1-\beta^2} + e_h \frac{\beta(1-\beta)}{1-\beta^2} = \frac{e_l}{1+\beta} + \frac{\beta e_h}{1+\beta} \tag{21}$$

- Verify that this indeed is an equilibrium by checking whether c_1 and c_2 sum to $e_h + e_l$.

$$c_1 + c_2 = \frac{e_h}{1+\beta} + \frac{\beta e_l}{1+\beta} + \frac{e_l}{1+\beta} + \frac{\beta e_h}{1+\beta}) e_h + e_l \tag{22}$$

Date-zero equilibrium

- The Lagrangean reads

$$L = \sum_t \beta^t u_i(c_{it}) + \lambda_i \sum_t (p_t c_{it} - p_t e_{it}) \tag{23}$$

- Yields the first order condtions

$$c_{1t} : \beta^t u'(c_{1t}) - \lambda_1 p_t = 0 \ \forall t \tag{24}$$

$$c_{2t} : \beta^t u'(c_{2t}) - \lambda_2 p_t = 0 \ \forall t \tag{25}$$

- From this we get

$$\frac{u'(c_{1,t+1})}{u'(c_{1,t})} = \frac{u'(c_{2,t+1})}{u'(c_{2,t})} = \frac{p_{t+1}}{\beta p_t} \tag{26}$$

- By similar argument from above we then get

$$c_{i,t} = c_{i,t+1} \tag{27}$$

- And then

$$\frac{p_{t+1}}{p_t} = \beta \tag{28}$$

[2]To see second equality notice $(1+\beta)(1-\beta)$

- Now use this in agent 1's budget constraint:

$$c_1 \sum_t p_t = \sum_t p_t e_1 \tag{29}$$
$$c_1 \left(p_0 + p_1 + p_2 + ...\right) = \left(e_h p_0 + e_l p_1 + e_h p_2 + ...\right) \tag{30}$$

- And notice

$$p_1 = \frac{p_1}{p_0} p_0 = \beta p_0 \tag{31}$$
$$p_2 = \frac{p_2}{p_1} p_1 = \beta\beta p_0 \tag{32}$$
$$... \tag{33}$$
$$p_t = \beta^t p_0 \tag{34}$$

- Let $p_0 = 1$ (numeraire) and use this in the budget constraint of the consumer to get

$$c_1 \sum_t \beta^t = e_h(1 + \beta^2 + \beta^4 + ...) + \beta e_l(1 + \beta^2 + \beta^4 + ...) \tag{35}$$
$$\Rightarrow \frac{c_1}{1-\beta} = \frac{e_h}{1-\beta^2} + \frac{\beta e_l}{1-\beta^2} \tag{36}$$
$$\Rightarrow c_1 = e_h \frac{1-\beta}{1-\beta^2} + e_l \frac{\beta(1-\beta)}{1-\beta^2} \tag{37}$$
$$\Rightarrow c_1 = e_h \frac{1}{1+\beta} + e_l \frac{\beta}{1+\beta} \tag{38}$$

- And likewise

$$c_2 = e_l \frac{1}{1+\beta} + e_h \frac{\beta}{1+\beta} \tag{39}$$

Intuition

- Notice:

$$c_1 - c_2 = \left(e_h - e_l\right) \frac{1-\beta}{1+\beta} \tag{40}$$

- Thus, consumption of agent 1 > consumption of agent 2, and this difference is decreasing in β. Reason is that agent 1 gets high endowment before agent 2, which (owing to discounting) makes agent 1 richer in present value terms. When β increases agents discount future less, why the difference in consumption becomes smaller.

Part (d)

- We know from above that

$$\frac{u'(c_{1,t+1})}{u'(c_{1,t})} = \frac{u'(c_{2,t+1})}{u'(c_{2,t})} \tag{41}$$

- Under the given preferences this implies

$$\frac{c_{1,t}}{c_{1,t+1}} = 1 \Rightarrow c_{1,t} = c_{1,t+1} \tag{42}$$

- Now the consolidated budget constraint of each agent reads

$$\sum_t \beta^t c_{i,t} = e_l + \beta e_h + \beta^2 e_l + ... \tag{43}$$

$$\sum_t \beta^t c_{i,t} = \frac{e_l}{1-\beta^2} + \frac{e_h \beta}{1-\beta^2} \tag{44}$$

- For agent 1 this implies

$$\frac{c_1}{1-\beta} = \frac{e_l}{1-\beta^2} + \frac{e_h\beta}{1-\beta^2} \Rightarrow c_1 = e_l\frac{1-\beta}{1-\beta^2} + e_h\frac{\beta(1-\beta)}{1-\beta^2} \tag{45}$$

- Via market clearing this then implies

$$c_{2,t=odd} = 2e_h - c_1 = e_h\frac{2-\beta^2-\beta}{1-\beta^2} - e_l\frac{1-\beta}{1-\beta^2} \tag{46}$$

$$c_{2,t=even} = 2e_l - c_1 = e_l\frac{1-2\beta^2+\beta}{1-\beta^2} - e_h\frac{\beta(1-\beta)}{1-\beta^2} \tag{47}$$

Intuition

- Notice $c_{1,t=even} = c_{1,t=odd}$ while $c_{2,t=even} < c_{1,t=odd}$, while the present value of consumption of agent 1 equals that of agent 2.
- This ows to the fact that agent 1 preferences are such that he/she dislikes consumption variation more than agent 2. Hence, agent 2 will insure agent 1 by taking all the variation in aggregate income.

Problem 2

Part (a)

- A **sequential equilibrium** is a set of sequences for allocations $\{k^*_{it}, k^*_{ct}, k^*_t, n^*_{it}, n^*_{ct}, c^*_t, i^*_t\}_{t=0}^{\infty}$ and prices $\{p^*_t, r^*_t, w^*_t\}_{t=0}^{\infty}$ such that:

1. $\left\{c_t^*, i_t^*, k_{t+1}^*\right\}_{t=0}^{\infty}$ solves the household's problem:

$$\max_{\{c_t,i_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{48}$$
$$\text{s.t.}$$
$$c_t + p_t^* i_t = r_t^* k_t + w_t^* n_t$$
$$k_{t+1} = (1-\delta)k_t + i_t$$

2. $\{k_{ct}^*, n_{ct}^*\}_{t=0}^{\infty}$ solves problem of consumption good firm:

$$\max_{k_{ct},n_{ct}} A_t^{1-\alpha} k_{ct}^{\alpha} n_{ct}^{1-\alpha} - r_t^* k_{ct} - w_t^* n_{ct} \ \forall t \tag{49}$$

3. $\{k_{it}^*, n_{it}^*\}_{t=0}^{\infty}$ solves problem of the investment good firm:

$$\max_{k_{it},n_{it}} p_t q_t A_t^{1-\alpha} k_{it}^{\alpha} n_{it}^{1-\alpha} - r_t^* k_{it} - w_t^* n_{it} \ \forall t \tag{50}$$

4. And market clearing (feasibility) holds

$$n_{it}^* + n_{ct}^* = n_t^* \tag{51}$$
$$k_{it}^* + k_{ct}^* = k_t^* \tag{52}$$
$$c_t^* = A_t^{1-\alpha} k_{ct}^{*\alpha} n_{ct}^{*1-\alpha} \tag{53}$$
$$\tag{54}$$

Part (b)

- Take first order conditions to firms problem to get

$$\alpha A_t^{1-\alpha} k_{ct}^{\alpha-1} n_{ct}^{1-\alpha} = r_t^* \tag{55}$$
$$(1-\alpha) A_t^{1-\alpha} k_{ct}^{\alpha} n_{ct}^{-\alpha} = w_t^* \tag{56}$$
$$p_t q_t \alpha A_t^{1-\alpha} k_{it}^{\alpha-1} n_{it}^{1-\alpha} = r_t^* \tag{57}$$
$$p_t q_t (1-\alpha) A_t^{1-\alpha} k_{it}^{\alpha} n_{it}^{-\alpha} = w_t^* \tag{58}$$

- Divide (55) by (56) and (57) by (58)

$$\frac{\alpha}{1-\alpha} \frac{n_{ct}}{k_{ct}} = \frac{r_t^*}{w_t^*} \tag{59}$$
$$\frac{\alpha}{1-\alpha} \frac{n_{it}}{k_{it}} = \frac{r_t^*}{w_t^*} \tag{60}$$

- Equate to get

$$\frac{n_{ct}}{k_{ct}} = \frac{n_{it}}{k_{it}} \tag{61}$$

- Then divide (57) by (55) and use (61) to get

$$\begin{aligned} p_t q_t \left(\frac{k_{it}}{k_{ct}}\right)^{\alpha-1} \left(\frac{n_{it}}{n_{ct}}\right)^{1-\alpha} &= 1 \\ \Rightarrow p_t q_t \left(\frac{k_{it}}{n_{it}}\right)^{\alpha-1} \left(\frac{k_{ct}}{n_{ct}}\right)^{1-\alpha} &= 1 \\ \Rightarrow p_t &= 1/q_t \end{aligned} \tag{62}$$

Part (c)

- Start by noting from (61) that $k_{it}/k_{ct} = n_{it}/n_{ct}$ why we can write capital and labor in each sector as the same fraction of total labor and capital:[3]

$$k_{it} = sk_t \tag{63}$$

$$n_{it} = sn_t \tag{64}$$

$$k_{ct} = (1-s)k_t \tag{65}$$

$$n_{ct} = (1-s)n_t \tag{66}$$

- Use this in (55) to get an expression for r_t

$$\begin{aligned} r_t^* &= \alpha A_t^{1-\alpha} (sk_t)^{\alpha-1} (sn_t)^{1-\alpha} \\ \Rightarrow r_t^* &= s^{\alpha-1+1-\alpha} \alpha A_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha} \\ \Rightarrow r_t^* &= s^0 \alpha A_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha} \end{aligned} \tag{67}$$

- Use this in (56) to get an expression for w_t

$$\begin{aligned} w_t^* &= (1-\alpha) A_t^{1-\alpha} (sk_t)^{\alpha} (sn_t)^{-\alpha} \\ \Rightarrow w_t^* &= s^0 (1-\alpha) A_t^{1-\alpha} k_t^{\alpha} n_t^{-\alpha} \end{aligned} \tag{68}$$

- Then plug (67) and (68) along (62) with into the budget constraint of the household to get

$$\begin{aligned} c_t + \frac{1}{q_t} i_t &= \alpha A_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha} k_t + (1-\alpha) A_t^{1-\alpha} k_t^{\alpha} n_t^{-\alpha} n_t \\ \Rightarrow c_t + \frac{1}{q_t} i_t &= A_t^{1-\alpha} k_t^{\alpha} n_t^{1-\alpha} \end{aligned} \tag{69}$$

$$\tag{70}$$

- Hence, the problem of the social planner can be written

$$\max_{\{c_t^*, i_t^*, k_{t+1}^*\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{71}$$

s.t.

$$c_t + \frac{1}{q_t} i_t = A_t^{1-\alpha} k_t^{\alpha} n_t^{1-\alpha}$$

$$k_{t+1} = (1-\delta)k_t + i_t$$

[3] To see this note, that $k_{it}/k_{ct} = \frac{s_k k_t}{(1-s_k)k_t} = \frac{s_k}{(1-s_k)} = n_{it}/n_{ct} = \frac{s_n n_t}{(1-s_n)n_t} = \frac{s_n}{(1-s_n)}$ why $s_k = s_n$

Intuition

- Notice that we have shown that the economy *aggregates*. That is, we started with a disaggreated economy with two sectors, each having a separate production function. But we have shown that this economy can be represented by one production function (as long as we also correct the price of the investment good by q_t - which can be thought of as the relative productivity of the investment good sector). Also notice that this result was independent of the use of preferences : the result was simply derived from the firms first order conditions.
- This result is important because it hints at something more general: That we often can represent an economy with many different sectors by *one* aggregate production function.