Suggested Solutions for Homework 1

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If you find any errors, or unclear sections, please help me and your classmates by emailing me at `karl.harmenberg@phdstudent.hhs.se`.

Problem 1

The social planner solves the problem

$$\max_{\{k_{t+1},c_t\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \log c_t \qquad s.t \qquad \begin{aligned} c_t + k_{t+1} &= A_t^{1-\alpha} k_t^{\alpha}, \\ c_t, k_{t+1} &\geq 0, \end{aligned}$$

where k_0 is given.

Note: It is perhaps a bit pedantic to include $c_t, k_{t+1} \geq 0$. I prefer to include the inequalities, if nothing else to remind myself what feasible values for c_t and k_{t+1} are. If I get a solution where $c_t < 0$, I *know* that something has gone wrong.

Now we solve for the optimal consumption path. Although with some practice we (i.e., you) will be able to write down the first order conditions almost without any thought, we will do it a bit formal this first time. It never hurts to write out the full Lagrangian, and I often find it helpful in more involved problems.

The social planner's Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log c_t + \sum_{t=0}^{\infty} \lambda_t (A_t^{1-\alpha} k_t^{\alpha} - c_t - k_{t+1}).$$

We take first order conditions and get

$$\begin{aligned} \frac{\partial}{\partial c_t} \qquad & \frac{\beta^t}{c_t} - \lambda_t = 0, \\ \frac{\partial}{\partial k_{t+1}} \qquad & -\lambda_t + \alpha \lambda_{t+1} A_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} = 0. \end{aligned}$$

Substituting out λ_t, λ_{t+1}, we get

$$\frac{c_{t+1}}{c_t} = \alpha \beta A_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}.$$

We could use the resource constraint to get a second order difference equation in k_t, but it wouldn't get us much further. Instead, we use the implicit hint in the problem formulation and conjecture a constant savings rate s. That is, we guess[1] that $c_t = (1-s) A_t^{1-\alpha} k_t^{\alpha}, k_{t+1} =$

[1] The strategy here is as follows: We make a guess. We then check if the guess satisfies all conditions. Since the guess satisfies all conditions, it must be a solution. Guesses, perhaps more often in homework exercises and exams than in real life, are usually on the form "wouldn't it be nice if...". In this case, it would be nice if the savings rate was constant, so we see if that gives us a solution.

$sA_t^{1-\alpha}k_t^\alpha$ solves the problem. Substituting in, we get

$$\begin{aligned}\frac{c_{t+1}}{A_{t+1}^{1-\alpha}k_t^\alpha} &= \alpha\beta\frac{c_t}{k_{t+1}} \Leftrightarrow \\ 1-s &= \alpha\beta\frac{1-s}{s} \Leftrightarrow \\ s &= \alpha\beta.\end{aligned}$$

We conclude that the constant savings rate $s = \alpha\beta$ satisfies the first order conditions (and trivially) the resource constraint. Therefore we have found a solution to the maximization problem[2]. That is, the policy $c_t = (1-\alpha\beta)A_t^{1-\alpha}k_t^\alpha$ is optimal.

[2] Technically, we need to check that the transversality condition is satisfied as well. A reasonable exam strategy is to point out that we really ought to check the transversality condition, but to then not do it. A reasonable life strategy is perhaps to at least once really check that a transversality condition holds (for example, this one).

It is good practice to check that our answer is a reasonable one. If α is low, then we will save relatively little ($s = \alpha\beta$ will be small). This makes sense since the return to capital is quickly falling off when α is low (plot k_t^α if you don't believe me). If β is small, then the social planner cares relatively little about the future (i.e., she is impatient) so the social planner will want to consume more and save less. Since our answer seems to agree with intuition in these limiting cases, we gain a bit more confidence.

Are you surprised that the optimal savings rate is independent of technology A_t? This is due to logarithmic utility. Consider the two period problem, $\max \log c_0 + \beta \log c_1$ subject to $c_1 = A^{1-\alpha}(K-c_0)^\alpha$. Note that $\log c_1 = (1-\alpha)\log A + \alpha\log(K-c_0)$ so the first order condition is independent of A. The income and substitution effects cancel perfectly.

Problem 2

First we formulate the social planner's problem. The social planner solves

$$\max_{\{c_t,k_{t+1},n_t\}} \sum_{t=0}^{\infty} \beta^t(\log c_t - v(n_t)) \quad s.t. \quad c_t + k_{t+1} = (1-\delta)k_t + F(k_t,n_t),$$
$$c_t, k_{t+1}, n_t \geq 0.$$

The corresponding Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty}\beta^t\left(\log c_t - v(n_t) + \lambda_t((1-\delta)k_t + F(k_t,n_t) - c_t - k_{t+1})\right).$$

(note that here I put the Lagrangian multipliers inside $\beta^t(\cdot)$, which sometimes [usually] is convenient) Taking first order conditions, we get

$$\begin{aligned}
\frac{\partial}{\partial c_t} \qquad & \frac{1}{c_t} = \lambda_t, \\
\frac{\partial}{\partial n_t} \qquad & v'(n_t) = \lambda_t F_n(k_t, n_t), \\
\frac{\partial}{\partial k_{t+1}} \qquad & \lambda_t = \lambda_{t+1}\left((1-\delta) + F_k(k_{t+1}, n_{t+1})\right).
\end{aligned}$$

Eliminating the Lagrangian multipliers we get the first order conditions,

$$\begin{aligned}
v'(n_t) &= \frac{F_n(k_t, n_t)}{c_t}, \\
\frac{c_{t+1}}{c_t} &= \beta\left((1-\delta) + F_k(k_{t+1}, n_{t+1})\right).
\end{aligned}$$

Now we define a dynamic competitive equilibrium for this economy. A dynamic competitive equilibrium for this economy is $\{c_t^*, n_t^*, k_{t+1}^*, w_t^*, r_t^*\}$ such that

1. $\{c_t^*, k_{t+1}^*, n_t^*\}$ solves the consumer's problem,

$$\max_{c_t, k_{t+1}, n_t,} \sum_{t=0}^{\infty} \beta^t (\log c_t - v(n_t)) \quad s.t. \quad c_t + k_{t+1} = (1 - \delta + r_t^*)k_t + w_t^* n_t$$

and nPs.

2. $\{n_t^*, k_t^*\}$ solves the firm's problem,

$$\max_{k_t, n_t} F(k_t, n_t) - r_t^* k_t - w_t^* n_t.$$

3. Markets clear[3],

$$c_t^* + k_{t+1}^* = (1-\delta)k_t^* + F(k_t^*, n_t^*).$$

[3] The resource constraint is $c_t^* + k_{t+1}^* \leq (1-\delta)k_t^* + F(k_t^*, n_t^*)$, but we invoke e.g. local nonsatiation to get equality.

Taking first order conditions for the consumer's problem and the firm's problem gives us[4]

[4] Here I drop the stars on all the variables.

$$\begin{aligned}
v'(n_t) &= \frac{w_t}{c_t}, \\
\frac{c_{t+1}}{c_t} &= \beta(1 - \delta + r_{t+1}), \\
F_k(k_t, n_t) &= r_t, \\
F_n(k_t, n_t) &= w_t.
\end{aligned}$$

Substituting out r_{t+1} and w_t, we get the exact same equations as for the social planner solution. Therefore the competitive equilibrium and the social planner solution coincides. Since the social planner solution maximizes total welfare, so does the dynamic competitive equilibrium, and it must be Pareto optimal.

Now, we define the competitive equilibrium with taxes. A dynamic competitive equilibrium for this economy with taxes is $\{c_t^*, n_t^*, k_{t+1}^*, w_t^*, r_t^*, \tau_t^*\}$ such that

1. $\{c_t^*, k_{t+1}^*, n_t^*\}$ solves the consumer's problem,
$$\max_{c_t, k_{t+1}, n_t,} \sum_{t=0}^{\infty} \beta^t (\log c_t - v(n_t)) \quad s.t. \quad c_t + k_{t+1} = (1 - \delta + r_t^*)k_t + (1 - \tau_t^*)w_t^* n_t$$
and nPs.

2. $\{n_t^*, k_t^*\}$ solves the firm's problem,
$$\max_{k_t, n_t} F(k_t, n_t) - r_t^* k_t - w_t^* n_t.$$

3. Markets clear,
$$g + c_t^* + k_{t+1}^* = (1 - \delta)k_t^* + F(k_t^*, n_t^*).$$

4. The government's budget holds,
$$g = \tau_t^* w_t^* n_t^*.$$

Two remarks are in order. First note that only the consumer's problem is directly affected since only labor income is taxed. Second, note that the competitive equilibrium without taxes is a special case of this economy, with $g = \tau_t^* = 0$.

Taking first order conditions for the consumer's problem and the firm's problem gives us
$$\begin{aligned} v'(n_t) &= (1 - \tau_t)\frac{w_t}{c_t}, \\ \frac{c_{t+1}}{c_t} &= \beta(1 - \delta + r_t), \\ F_k(k_t, n_t) &= r_t, \\ F_n(k_t, n_t) &= w_t. \end{aligned}$$
Substituting, we get the equations
$$\begin{aligned} v'(n_t) &= (1 - \tau_t)F_n(k_t, n_t), \\ \frac{c_{t+1}}{c_t} &= \beta(1 - \delta + F_k(k_{t+1}, n_{t+1})). \end{aligned}$$
For any $g > 0$, we must have $\tau_t > 0$ (by the government budget). Therefore, a solution to the competitive equilibrium with taxes can never simultaneously be a solution to the planner's problem (since then we would have $(1 - \tau_t)F_n(k_t, n_t) = v'(n_t) = F_n(k_t, n_t) \Rightarrow F_n(k_t, n_t) = 0$ which contradicts the standard assumption that $F(k, \cdot)$ is strictly increasing in n). We conclude that the representative household will be strictly worse off in the competitive equilibrium with taxes than in the planner's solution so the equilibrium is not Pareto optimal.

Problem 3

First we carefully define a dynamic competitive equilibrium. An equilibrium is $\{(c_t^A)^*, (c_t^B)^*, (B_t^A)^*, (B_t^B)^*, q_t^*\}$ such that

1. $\{(c_t^i)^*, (B_{t+1}^i)^*\}$ solves consumer ith problem (where $i = A, B$),

$$\max_{\{c_t^i, B_{t+1}^i\}} \sum_{t=0}^{\infty} \beta^t \log c_t^i \qquad s.t. \qquad c_t^i + q_t^* B_{t+1}^i = B_t^i + \epsilon_t^i,$$

$$\text{no Ponzi scheme.}$$

2. The goods market clears,

$$(c_t^A)^* + (c_t^B)^* = \epsilon_l + \epsilon_h.$$

3. The bond market clears,

$$(B_t^A)^* + (B_t^B)^* = 0.$$

Finally, note that $B_0^A = B_0^B = 0$, there is no outstanding debt at time zero.

As previously, we drop the stars for simplicity when solving for the equilibrium. We take first order conditions for consumer i,

$$\frac{c_{t+1}^i}{c_t^i} = \frac{\beta}{q_t}.$$

Note that this is independent of i, therefore we have

$$\frac{c_{t+1}^A}{c_t^A} = \frac{c_{t+1}^B}{c_t^B}.$$

It follows that

$$\frac{c_t^A}{c_0^A} = \frac{c_1^A}{c_0^A}\frac{c_2^A}{c_1^A}\cdots\frac{c_t^A}{c_{t-1}^A} = \frac{c_1^B}{c_0^B}\frac{c_2^B}{c_1^B}\cdots\frac{c_t^A}{c_{t-1}^B} = \frac{c_t^B}{c_0^B}.$$

This, together with the goods market clearing condition, gives us that $c_t^A = c^A, c_t^B = c^B$, i.e., consumption is constant for both types of consumers.[5] Since $1 = \frac{c_{t+1}^i}{c_t^i} = \frac{\beta}{q_t}$, we can conclude that $q_t = \beta$.

Now we need to determine c^A and c^B. Here the consolidated budget constraint is useful. We can write the per period budget constraint as $\beta^t(c_t^i + \beta B_{t+1}^i) = \beta^t(B_t^i + \epsilon_t^i)$, and rearranging and summing[6] we get the consolidated budget constraint,

$$\sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t \epsilon_t^i + \lim_{t\to\infty}(B_0^i - \beta^{t+1} B_{t+1}^i).$$

[5] Mathematical argument: If $c_t^A/c_0^A > 1$, then also $c_t^B/c_0^A > 1$ and therefore $c_t^A + c_t^B > c_0^A + c_0^B$Êwhich contradicts that $c_t^A + c_t^B = \epsilon_l + \epsilon_h = c_0^A + c_0^B$. Similarly if $c_t^A/c_0^A < 1$.

[6] Note that $\sum_{t=0}^{T}(\beta^t B_t^i - \beta^{t+1} B_{t+1}^i) = B_0^i - \beta^{T+1} B_{T+1}^i$ is a telescopic sum.

By transversality and the no Ponzi condition, $\beta^{t+1}B^i_{t+1} \to 0$. By assumption, there were no outstanding debts at time zero so $B^i_0 = 0$. Therefore, the consolidated budget constraint is,

$$\sum_{t=0}^{\infty} \beta_t c^i_t = \sum_{t=0}^{\infty} \beta^t \epsilon^i_t.$$

Using that $c^i_t = c^i$, we get

$$\frac{c^i}{1-\beta} = \sum_{t=0}^{\infty} \beta^t \epsilon^i_t.$$

For A, this amounts to

$$\begin{aligned} \frac{c^A}{1-\beta} &= \epsilon_h + \beta\epsilon_l + \beta^2\epsilon_h + \ldots = \frac{\epsilon_h}{1-\beta^2} + \frac{\beta\epsilon_l}{1-\beta^2}, \\ c^A &= \frac{\epsilon_h + \beta\epsilon_l}{1+\beta}. \end{aligned}$$

Similarly, for B we get

$$c^B = \frac{\epsilon_l + \beta\epsilon_h}{1+\beta}.$$

In conclusion, the equilibrium quantities are $c^A_t = \frac{\epsilon_h+\beta\epsilon_l}{1+\beta}, c^B_t = \frac{\epsilon_l+\beta\epsilon_h}{1+\beta}$ and the equilibrium bond price is $q_t = \beta$.

Note that A is richer than B. This is because A gets the high endowment first, and they are both somewhat impatient ($\beta < 1$). As $\beta \to 1$, $c^B/c^A \to 1$.

A social planner that puts weight θ on A will solve the problem

$$\max_{\{c^A_t, c^B_t\}} \sum_{t=0}^{\infty} \beta^t (\theta \log c^A_t + (1-\theta)\log c^B_t) \quad s.t. \quad c^A_t + c^B_t = \epsilon_l + \epsilon_h.$$

This is a time separable problem (much like the firm's problem in our models) with solution (by taking first order condition)

$$\frac{\theta}{c^A_t} = \frac{1-\theta}{c^B_t}.$$

In our competitive equilibrium, we have $\frac{c^A}{c^B} = \frac{\epsilon_h+\beta\epsilon_l}{\beta\epsilon_h+\epsilon_l}$ so choosing θ such that $\frac{\theta}{1-\theta} = \frac{\epsilon_h+\beta\epsilon_l}{\beta\epsilon_h+\epsilon_l}$ would reproduce the competitive equilibrium.[7]

[7] If you want, by all means, solve for θ.

Now we consider the setup where unrestricted borrowing is not allowed. The equilibrium will be defined identically, but the consumers' problem will now be

$$\begin{aligned} \max_{\{c^i_t, B^i_t\}} \sum_{t=0}^{\infty} \beta^t \log c^i_t \qquad s.t. \qquad & c^i_t + q^*_t B^i_{t+1} = B^i_t + \epsilon^i_t, \\ & B^i_t \geq -\underline{b}. \end{aligned}$$

Note that we do not need the no Ponzi scheme condition anymore, since we have introduced a much more stringent borrowing constraint.

When $\underline{b} = 0$, the equilibrium quantities are ϵ_l and ϵ_h alternating (since no intertemporal trade is possible). We still need to solve for the bond prices.[8]

The Lagrangian for the consumer is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (\log c_t + \lambda_t (B_t + \epsilon_t - c_t - q_t B_{t+1}) + \mu_t B_{t+1})$$

with first order conditions,

$$\begin{aligned} \frac{\partial}{\partial c_t} \qquad & \frac{1}{c_t} = \lambda_t, \\ \frac{\partial}{\partial B_{t+1}} \qquad & \lambda_t q_t - \mu_t = \beta \lambda_{t+1}, \\ \frac{\partial}{\partial \lambda_t} \qquad & B_t + \epsilon_t = c_t + q_t B_{t+1}, \\ \frac{\partial}{\partial \mu_t} \qquad & B_{t+1} = 0. \end{aligned}$$

It is clear that a solution satisfies $c_t = \epsilon_t$ which gives us

$$q_t = \frac{\beta \epsilon_t}{\epsilon_{t+1}} + \frac{\mu_t}{\epsilon_t}.$$

Since it is permissible to lend (but not to borrow), consumers must abstain voluntarily from lending. That is, the shadow value of lending must be positive, $\mu_t \geq 0$. This is satisfied iff

$$q_t \geq \frac{\beta \epsilon_t}{\epsilon_{t+1}}.$$

Since this must hold for both types of consumers, we have

$$q_t \geq \frac{\beta \epsilon_h}{\epsilon_l} > \beta.$$

That is, the interest rate $1/q_t$ will be lower when consumers are borrowing constrained. This has a supply/demand intuition: If, for whatever reason, fewer people are able to borrow (or people are only able to borrow to a lesser extent), lenders have to lower their interest rates to attract borrowers.

Note that in our setting, the interest rate is not determined, we can only conclude that $q_t \geq \frac{\beta \epsilon_h}{\epsilon_l}$ (and all those interest rates together with the "consume your own basket" rule gives equilibria). By instead letting the borrowing constraint $\underline{b}$ approach 0, we get a unique interest rate $\beta \frac{\epsilon_h}{\epsilon_l}$.

[8] It might seem weird that we solve for bond prices even though no bonds are actually traded. But even though there are no trades, it doesn't mean that there is no price, it just means that the price is adjusted so that all the agents that can participate in the market voluntarily choose not to.

Interest rates have fallen globally, which perhaps suggests that more people are borrowing constrained than before, perhaps because of rising inequality.

Here is a final question: What would happen if lending constraints increased/decreased?