1 Background on Continuous Time Markov Chain

In the main text, we noted that labor market flows were regulated by a Continuos Time Markov Chain (CTMC) with flow matrix $\Lambda(t)$. We noted that this means that shares in different labor market states evolved according to

$$x(t) = x(0)P(0,t)$$

where the transition matrix $P(0,t)$ is given by

$$P(0,t) = \exp\left(\int_0^t Q(z)dz\right).$$

In this section, we will show how one can start from individuals whose labor market flows are governed by a CTMC, and build up to the aggregate behavior of shares as a function of the underlying CTMC.

1.1 Definition of a CTMC

We want to analyze a CTMC X on states $\mathcal{S} = \{\infty, \ldots, \mathcal{S}\}$ with the generating flow matrix $\Lambda(t)$. The flow matrix $\Lambda(t)$ has non-negative off-diagonal element, and the diagonal elements ensure that the row-sums are zero:

$$\Lambda(t)_{s,s} = -\sum_{s' \neq s} Q(t)_{s,s'}.$$

In this case, X is defined as a stochastic process where

$$Pr((t+h) = s'|X(t) = s) = \delta_{s,s'} + \lambda(t)_{s,s'}h + o(h), \tag{1}$$

where $\delta_{s,s'}$ is the Kronecker delta which satisfies

$$\delta_{s,s'} = \begin{cases} 1 & \text{if } s = s' \\ 0 & \text{if } s \neq s', \end{cases}$$

and where $o(h)$ means a term of smaller magnitude than h.

Equation (1) states that when the CTMC is in state s, there is a probability $h\Lambda(t)_{s,s'}$ that the process will jump to state $s' \neq s$. This means that $\Lambda(t)_{s,s'}$ gives the rate at which the process jumps from state s to state s'.

Given the formulation of the diagonal ensuring that each row sum to 0, we get that the probability that the process stays in s is approximately

$$Pr(X(t+h) = s|X(t) = s) = 1 - \sum_{s' \neq s} h\lambda_{s,s'}(t).$$

1.2 Alternative interpretation of defintion

Another way of interpreting the definition of a CTMC is that the process being in state s flows to state $s' \neq s$ with a Poisson rate $Q(t)_{s,s'}$. This relates to an equivalent definition of the CTMC Xdefined by $Q(t)$. This definition states that X is a stochastic process that waits in state s for a time $\tau_s(t)$ which is distributed as

$$P(\tau_s(t) \geq w) = \exp\left(-\int_t^{t+w} \sum_{s' \neq s} \lambda_{s,s'}(t) dz\right),$$

and that conditioned on jumping, the process jumps to s' with probability $\frac{\lambda_{s,s'}(t)}{\sum_{s'' \neq s} \lambda_{s,s''}(t)}$ conditioned on jumping at time t.

The expression above looks complicated, but is easier if $\Lambda(t)$ is constant Λ. In that case, the waiting time in state s is exponentially distributed with rate $\sum_{s' \neq s} \lambda_{s,s'}$.

The definition can also be shown to be equivalent to the definition in Section 1.1.

1.3 Transition probabilities in Markov chains

The definition in Section 1.1 is quite abstract, but it is possible to use it to formulate useful results. The most important is that the transition matrix defined by the equation

$$P_{s,s'}(t, t+u) = Pr(X(t+u) = s' | X(t) = s)$$

satisfies the equation

$$P(t, t+u) = \exp\left(\int_t^{t+u} Q(z) dz\right)$$

for any $t, u \geq 0$. This equation is useful as it allows us to use the flow matrix $\Lambda(t)$ to derive all transition matrix for any pair of time periods.

1.4 Evolution of shares and a Law of Large Numbers

In labor market research, we are interested in the evolution of the share of agents in different states when agents are governed by a CTMC. It will turn out that this follows an intuitive pattern: the vector with expected share of workers in a particular share evolve according to the transition matrix $P(0, t)$ of the underlying CTMC.

To show this more formally, we consider a model with N workers $\{1, \ldots, N\}$ which all have their movement between labor market states governed by a CTMC with flow matrix $\Lambda(t)$. We write $X_i(t)$ for the CTMC associated with individual i, where $X_i(0) = s_{i,0}$ is given.

We are interested in the expected share of workers in a particular state s at time t:

$$x_s(t) = \mathbb{E}\left(\frac{\sum_{i=1}^{n} \mathbb{I}(X_i(t) = s)}{N}\right),$$

where $\mathbb{I}$ is an indicator function. Using that the expected value of an indicator function of an event is the probability of the event, we obtain

$$x_s(t) = \frac{\sum_{i=1}^{n} Pr(X_i(t) = s)}{N}.$$

In the next step, we decompose this expression depending on the initial value of the process $X_i(t)$. Here, we use that $P(X_i(t) = s) = P(X(t) = s|X(0) = s_{i,0})$ where $X(t)$ is an arbitrary CTMC with flow matrix $\Lambda(t)$. Writing N_{s_0} for the number of individuals which start in s_0, we obtain:

$$\begin{aligned}
x_s(t) &= \frac{\sum_{i=1}^{n} \mathbb{P}(X_i(t) = s)}{N} \\
&= \frac{\sum_{i=1}^{n} \mathbb{P}(X(t) = s|X(0) = s_{i,0})}{N} \\
&= \frac{\sum_{i=1}^{n} P(0,t)_{s_{0,i},s}}{N} \\
&= \sum_{s_0 \in \mathcal{S}} \left(\frac{N_{s_0}}{N}\right) P(0,t)_{s_0,s} \\
&= \sum_{s_0 \in \mathcal{S}} x_{s_0}(0) P(0,t)_{s_{0,i},s} \\
&= \left(x(0)P(0,t)\right)_s.
\end{aligned}$$

In the second to last step, we use that $\frac{N_{s_0}}{N} = x_{s_0}(0)$. This expression shows that the vector of expected shares evolve according to the equation

$$x(t) = x(0)P(0,t)$$

as expected.

1.5 The law of large numbers and aggregate behavior

If we let $N(t)$ be vector giving the number of workers in state s at time t, we approximately have that

$$N(t) \sim Multinom(N, x(t)).$$

It is well known (Elements of Distribution Theory, Severini) that this can be approximated by

$$\frac{N(t)}{N} \sim Normal(x(t), \Sigma)$$

with

$$\Sigma_{i,j} = \begin{cases} \frac{x_i(t)}{N} & \text{if } i = j \\ -\frac{x_i(t)x_j(t)}{N} & \text{if } i \neq j \end{cases}$$

This means that the share vector has very small variance if N is sufficiently large. In our dataset, $N \approx 20,000$ and we will assume that the observed share equals the expected share $x(t)$.

1.6 Steady state in a CTMC Markov chain

If we have a CTMC X with a finite number of states and constant flow matrix Λ that is *irreducible* (which means that every state can be reached from every other state), there exists a steady state distribution π such that

$$\lim_{t\to\infty} \mathbb{P}(X(t) = s'|X(0) = s) = \pi_{s'}$$

for all s, s'. This means that regardless of the initial state, the Markov chain converges to the distribution π. This π is the unique solution to the equation

$$\pi'\Lambda = 0$$

which satisfies $\sum_s \pi_s = 1$.

When we have an inhomogenous CTMC with varying flow matrix $\Lambda(t)$, we can analogously define the steady state vector $\pi(t)$ associated with $\Lambda(t)$ as the solution to

$$\pi(t)'\Lambda(t) = 0$$

with $\sum_s \pi_s(t) = 1$. Again, this solution will exist and be unique when $\tilde{\Lambda} = \Lambda(t)$ is the flow matrix of a finite irreducible CTMC.