# Evaluation of Pitch-Class Set Similarity Measures for Tonal Analysis (Draft)

Nicholas Harley
SMC Masters Thesis - UPF
Supervisor: A. Martorell

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**Abstract**

This work explores the limitations of exiting approaches to computational modelling and description of tonality. PC-set theory is presented as alternative or complement to existing approaches. The methodology emphasised systematicity and perceptual relevance. SC similarity is presented as tool for aiding in the representation of SC information. A survey of SC similarity measures is conducted. Systematic SC descriptions of specific examples are represented through techniques rovolving around SC similarity. The analytical potential of the techniques is evidenced in the retrieval of musicological information from the model.

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## PC-Set Theory Glossary

**#X** Set cardinality.
**#nC** Size of nC.
**#nCV** Vector cardinality.

**DV** Difference Vector.

**I(X)** Inversion.
**I-Type** Inversional SC-Type.
**IC** Interval Class.
**ICV** Interval-class Vector.

**nC** Cardinality Class.
**nC%V** n-class subset percentage vector.
**nCV** n-class subset vector.
**nSATV** n-class subset saturation vector.

**PC** Pitch Class.
**PC-Set** Pitch Class Set.
**Prime Form** PC-set representing all members of an SC.

**SC** Set Class.

**Tn(X)** Transposition.
**Tn-Type** Transpositional SC-type.
**TnI-Type** Transpositional/Inversional SC-Type.
**Trivial Form** SCs 1-1, 11-1 and 12-1.
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# 1 Introduction

The objectives of the present work are principally concerned with descriptive modelling of tonality using PC-set theory. In particular, the analytical potential of set-class similarity measures will be assessed and evaluated through comparison with similar work. The tentative goal, therefore, is to present this approach in such a way as to make it relevant to the wider community of music researchers, such as those in MIR.

## 1.1 Problem

1. Existing computational description of tonality is often limited in its scope.
2. The predominant use of template matching for chord and key estimations limits the knowledge of music-capable systems to repertoire the major-minor paradigm. This narrow view of tonality is insufficient for even some western music.
3. The predominant use of non-musical similarity measures in MIR such as Euclidean distance and correlation seems counter-intuitive in many cases.
4. The above may be contributing to the “semantic gap”.

## 1.2 Objectives

1. To adopt a systematic approach to description of tonality.
2. To justify set class analysis as a systematic descriptive tool, capable of relating useful musical information as well as having a degree of perceptual relevance.
3. To position set class analysis among existing methods of tonal description such as chroma vector time-series.
4. To create a comprehensive and practical survey of set class similarity measures.
5. To examine the measures in terms of systematicity and perceptual relevance.
6. To develop techniques for the representation of set class information so as to expose meaningful musicological information.
7. To utilise set class similarity in these representation.
8. To evaluate the utility of these similarity measures through exploration of SC information to extract a priori knowledge about specific pieces. The analytical potential of the model will be evidenced through specific analysis examples.

## 1.3 Outline

Chapters 2-5 comprise a literature review containing background information and some basic theory. Chapter 2 addresses the challenges and problems involved in descriptive modelling of tonality as a means of justification for the proposed approach. Chapter 3 introduces the basic concepts of PC-set theory including set-class similarity measures. Chapter 4 gives a basic description of MDS techniques and how they can play a role in set-class analysis. Chapter 5 contains a review of the relevant tonal models that exist so as to give context work that follows. Chapter 6 evaluates the similarity measures from the literature in terms of some specific criteria and in doing so a subset of measures are selected for further investigation. Chapter 7 examines this subset of measures more closely and comparisons are made. Chapter 8 outlines the segmentation techniques required for a systematic description of a piece while Chaprer 9 presents three techniques for representing this data for the extraction of meaningful musical information. Chapters 9.4 and 12 present the conclusions and discussion of future work respectively.

# 2 Tonality

## 2.1 Defining Tonality

### 2.1.1 General Definition

Tonality is a notoriously complex musical phenomenon and numerous definitions have been proposed from a variety of viewpoints. Perhaps the most general definition is that provided by Hyer (2013): “... refers to the systematic arrangements of pitch phenomena and relations between them." Explanations of tonality have been provided through many different disciplines (acoustics, music theory, linguistics, cognitive psychology) and a detailed discussion of these areas is certainly beyond the scope of this work. However, it is generally agreed that tonality is an abstract cultural and cognitive construct that can have many different physical representations.

### 2.1.2 Babbitt's Domains

Babbitt (1965) proposed three domains to categorise different types of representation of music: acoustic (physical), auditory (perceived), graphemic (notated). Western music theory provides a lexicon for describing abstract tonal objects with terms such as note, chord and key. These objects have a hierarchical relationship and the meaning of these labels is highly dependent on musical context and the scale of observation. Musicological descriptions, which constitute the majority of reasoning about tonaility, reside mainly in the Babbitt's graphemic domain, although arguably they reflect some aspects of the other two. Each domain, whilst connected to every other, provides only a projection of the musical whole and examination of tonality from just one will most likely result in an incomplete picture. However, these three domains provide a convenient framework for the discussion that follows.

## 2.2 Modelling Tonality

The challenge of mathematically modelling aspects of tonality has been approached in numerous ways and from different domains. In the graphemic domain, musicologists and composers have proposed theoretical models, attempting to rethink tonal theory from a mathematical perspective. These models employ different branches of mathematics such as geometry (Tymoczko, 2012) or group theory (Ring, 2011) to describe harmonic structure. From the auditory domain, cognitive psychologists have built models of tonal induction based on perceptual ratings of tonal stimuli (Krumhansl, 1990).

### 2.2.1 Tonality as Context

Many models approach the concept of tonality as a context, within which the relations and hierarchies of tonal phenomena can be understood. A sense of tonality can be induced when musical stimuli resemble some a priori contextual category. For western music of the major-minor period, key signatures comprise a collection of categories that give context to the tonal components of music. Martorell (2013) identifies three important aspects of tonality as context: dimensionality (the relatedness or "closeness" of categories), ambiguity (reference to two or more categories simultaneously) and timing (the dynamics of tonal context). He highlights the importance of a models capability to describe these aspects.

### 2.2.2 Tonality in MIR

The MIR community is primarily concerned with the extraction of tonal descriptors from audio signals such as chord and key estimates. Most systems use chroma features as a preliminary step, obtained by mapping STFT or CQ transform energies to chroma bins. Template matching is used to compare the chroma vectors to a tonal model (contextual category) using some distance measure. A commonly used tonal model for key estimation are the KK-profiles (Krumhansl, 1990) (5.3) (e.g. in Gómez 2006). Distance measures such as inner product (e.g. in Gómez 2006) and fuzzy distance (e.g. in Purwins et al. 2000) are used to compare vectors. Statistical methods, such as HMMs, have been used for chord and key tracking (Chai, 2005). Of addition interest in the field is the concept of musical similarity (for music recommendation, structure analysis, cover detection etc.). Foote (2000) computed self-similarity matrices for visualisation of structure by correlating the MFCC feature vector time-series. Gómez (2006) proposed the application of this method to tonal feature vectors.

### 2.2.3 Similarity

The importance of defining the similarity or closeness between musical phenomena, be it theoretical, physical or perceptual, is central to almost every model of tonality and often leads to a geometric configuration of tonal objects. The concepts of similarity and distance is discussed further in Chapter 5 where a review of spatial models of tonality is given.

## 2.3 The Semantic Gap

### 2.3.1 Acoustic Domain

Wiggins (2009) discusses, what is referred to in MIR as, the "Semantic Gap": the inability of systems to achieve success rates beyond a conspicuous boundary. He examines the fundamental methodological groundings of MIR in terms of Babbitts three domains, discussing the limits of each representation and regarding the discarnate nature of music. He concludes that the audio signal (acoustic domain) simply cannot contain all of the information that systems seek to retrieve. He points towards the the auditory domain as the chief residence of music information and urges for in not to be overlooked in MIR and wider music research.

### 2.3.2 Graphemic Domain

Furthermore, Wiggins criticises the purely graphemic approach and the tendency of music research to presuppose musicological axioms. Wiggins (2012) argues that music (tonal) theory is, rather than a theory in the scientific sense, a highly developed folk psychology (internal human theory for explaining common behaviour). Thus, the rules of music theory are not like scientific laws but rather abstract descriptions of a specific musical behaviour. This idea challenges the validity of formalising such rules in mathematics and prompts the question, "What is actually being modelled?" He concludes that to apply mathematical models to musical output alone (scales or chords) without consideration of the musical mind is a scientific failure.

### 2.3.3 Problems

The two assertions of Wiggins sit contrary to a number of the aspects of the tonal models discussed in 2.2. Firstly, the major-minor paradigm, upon which so many approaches are based, whilst certainly possessing cognitive significance, is still a musicological concept and therefore a misleading basis for both mathematical and cognitive approaches. A second problem is that of the numerical methods used by some MIR systems, in particular, distance measures. As will be discussed in Chapter 5, similarity (and by extension distance) is a central part of the auditory domain. MIR systems often uses distance measures from mathematics such as Mahalanobis (Tzanetakis and Cook, 1999) or Cosine (Foote, 2000) with little consideration of their perceptual or musical significance.

## 2.4 Systematicity

### 2.4.1 The Musical Surface

Having cautioned against a purely musicological approach, Wiggins (2009, pp. 481) proposes a compromise: to adopt a bottom-up approach to music theory, exploring the concepts through systematic mid-level representations. He states that "methods starting at, for example, the musical surface of notes is a useful way of proceeding" The concept of musical surface is illustrated by Huovinen and Tenkanen (2007, pp. 159) with a metaphor: "...to approach a musical landscape not by drawing a map, which necessarily confines itself to a limited set of structurally important features, but by presenting a bird's-eye view of the musical surface – an aerial photograph, as it were, which details the position of every pitched component."

### 2.4.2 Systematic Description

Martorell (2013) also advocates this mid-level approach, observing that surface description influences analyitcal observation and that, for an unbiased view, the researcher must be provided with the adequate raw materials with which to make more in-depth observation. Such a systematic, descriptive model would be fundamentally independent of high level concepts such as chords and key but, at the same time, capable of capturing them. Martorell (2013) also discusses the importance of systematicity in terms of dimensionality, ambiguity and timing. He finds that models based on the major-minor paradigm are incapable of adequately describing tonal ambiguity even in some Western music (Martorell, 2013, chap. 3).

With a systematic description of the musical surface, theories and models from different domains can be gathered and evaluated together in the same analytical arena, thus helping to bridge the gap between traditional musicology, cognitive psychology and MIR.

# 3 Pitch-Class Set Theory

One such method available for systematic description of the musical surface is Pitch class set theory. PC-set theory is a system for analysing the pitch content of music. It uses class equivalence relations to reduce the amount of data required to describe any collection of pitches. This chapter will outline the basic principles.

## 3.1 Pitch Class

Pitch-class set theory uses octave equivalence. In Western equal temperament (TET), a pitch-class (PC) is an integer representing the residue class modulo 12 of a pitch (**?**)Babbit1955) and indicates the position of a note within the octave. A PC-set is a collection of PCs ignoring any repetitions and the order in which they occur. PC-sets are notated as follows {0,1,2,3,4} with PCs ordered from lowest to highest as a convention (Example 1). The cardinality of a set, denoted #S, is the number of PCs it contains (Example 2). There are 4096 ($2^{12}$) unique PC-sets with which any segment of music can be represented.

Table 1: Notes and corresponding pitch-classes

| Note | C | C# | D | D# | E | F | F# | G | G# | A | A# | B |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| PC | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

| | | | |
|---|---|---|---|
| Example 1: | PC-set | Pitch-set | S = {A4,C5,E5,A5} (A minor) |
| | | PC-set | S = {9,0,4,9} = {0,4,9} |
| Example 2: | Cardinality | | #S = 3 |

## 3.2 Set Classification

Defining equivalence classes of PC-sets further reduces the total number of tonal objects. A set-class (SC) is a group of PC-sets related by a transformation or group of transformations. The two types of transformation commonly used are transposition and inversion. A transposition, Tn(S), transposes the set, S, by the interval, n, (by adding n to all PCs, Example 3). An inversion, I(S), inverts the set S, replacing all PCs with their inverse (12-PC, Example 4). From these two transformations it is possible to define three types of SC: Tn, TnI and I.

| | | |
|---|---|---|
| Example 3: | Transposition | S = {0,4,9}, T3(S) = {3,7,0} = {0,3,7} |
| Example 4: | Inversion | S = {0,4,9}, I(S) = {11,7,2} = {2,7,11} |

| | |
|---|---|
| Transpositional (Tn): | All PC-sets that can be transformed to each by transposition belong to the same class. There are 351 distinct Tn types. |
| Inversional (I): | All PC-sets that can be transformed to each other by inversion belong to the same SC. There are 200 distinct I types. |
| Transpositional/ Inversional (TnI): | All PC-sets that can be transformed to each other by transposition, inversion or both belong to the same SC. There are 223 distinct TnI types. |

The Prime Form of a PC-set is a convention for denoting the SC it belongs to. The convention was introduced by Allan Forte (Forte, 1973) for TnI types and has since been adopted by the majority of theorists. In addition, he devised a system for ordering TnI-type SCs and assigning to each one a cardinality-ordinal number. For example, the Forte number 3-11 refers to the 11th SC of cardinality 3. This convention has been modified for use with Tn types by adding A and B to the names of inversionally related SCs.

One additional concept is that of cardinality-class (nC), which refers to all the SCs of cardinality n. Cardinality-class 2 is commonly referred to as interval-class (IC) and there are 6 distinct interval-classes.

Table 2: Forte's Prime form and numbering convention

| | |
|---|---|
| PC-set | {1,4,9} |
| Prime Form (TnI) | {0,3,7} |
| Prime Form (Tn) | {0,4,7} |
| Forte Name (TnI) | 3-11 |
| Forte Name (Tn) | 3-11B |

Table 3: Numbers of objects

| Object type | No. Objects |
|---|---:|
| Pitch | 88 |
| Pitch set | 3e26 |
| PC | 12 |
| PC-set | 4096 |
| Tn-Type SC | 348 |
| I-Type SC | 197 |
| TnI-Type SC | 220 |

Table 4: Cardinality Class

| | | $\#nC$ | |
|---|---:|---:|---:|
| n | Tn | I | TnI |
| 1C | 1 | 1 | 1 |
| 2C | 6 | 6 | 6 |
| 3C | 19 | 12 | 12 |
| 4C | 43 | 28 | 29 |
| 5C | 66 | 35 | 38 |
| 6C | 80 | 35 | 50 |
| 7C | 66 | 35 | 38 |
| 8C | 43 | 28 | 29 |
| 9C | 19 | 12 | 12 |
| 10C | 6 | 6 | 6 |
| 11C | 1 | 1 | 1 |
| 12C | 1 | 1 | 1 |

## 3.3 Vector Analysis

### 3.3.1 Membership and Inclusion

Two concepts that are crucial in PC-set theory are membership and inclusion. Membership of a set is denoted p $\in$ S and means that PC p is a member of set S (Example 5). Inclusion in a set is denoted Q $\subset$ S and means that all members of set Q are also members of set S (Example 6). Q is said to be a subset of S.

| | | |
|---|---|---|
| Example 5: | Membership | 4 $\in$ {0,4,9} |
| Example 6: | Inclusion | {0,4,9} $\subset$ {0,1,4,5,9} |

### 3.3.2 Embedding Number

Lewin (1979) applied these concepts to SCs to develop his Embedding Number, EMB(X,Y). Given two SCs, X and Y, EMB(X,Y) is the number of instances of SC, X, which are included in (are subsets of) SC, Y (Example 7). X is ring-shifted 11 times and each unique resulting set which is included in Y adds one to the embedding number.

| | | |
|---|---|---|
| Example 7: | Embedding Number | X = {0,4} and Y = {0,4,8} |
| | | so EMB(X,Y) = 3 |

### 3.3.3 Subset Vectors

An n-class subset vector of X, nCV(X), is an array of values of EMB(A,X) where A is each of the SCs in the cardinality-class, nC (Example 8). The Interval-Class Vector (ICV) is a special instance of the nCV with n equal to 2. Vector cardinality, denoted #nCV(X), is the sum of all the terms in the vector (Example 9). The length of a subset vector is given by the number of SCs in the cardinality class, #nC.

Subset vectors form the basis of the majority of analysis performed by PC-set theorists. In addition, many theorists have proposed modifications to the basic nCV to suit their specific purposes and some of these modifications will be discussed in context where necessary.

| | | |
|---|---|---|
| Example 8: | Subset Vector | S = {0,4,9} |
| | | 2CV(S) = ICV(S) = [0 0 1 1 1 0] |
| Example 9: | Vector Cardinality | #ICV(S) = 0+0+1+1+1+0 = 3 |

### 3.3.4 Notation

Some additional vector notation is required for the comparison procedures of the similarity measures described in 3.4.

**Difference Vector** is the absolute difference between corresponding terms in the nCVs of two SCs, X and Y:

$$DV(nCV(X), nCV(Y)) = |nCV(X) - nCV(Y)|$$

**Vector Magnitude** is the length of the nCV in euclidean space:

$$\|nCV(X)\| = \sqrt{\sum_{i=1}^{\#nC} (nCV(X)_i)^2}$$

**Unit Vector** is the normalised nCV (unit length):

$$nC\hat{V}(X) = \frac{nCV(X)}{\|nCV(X)\|}$$

**Euclidean Distance** is the distance between the points defined by two nCVs in n-dimensional Euclidean space:

$$d(X,Y) = \sqrt{\sum_{i=1}^{n} (X_i - Y_i)^2} = \|DV(X,Y)\|$$

## 3.4 Set-Class Similarity

### 3.4.1 Similarity Relations

The assessment of similarity between two SCs has been discussed in the literature for decades and a large number theoretical models have been proposed. Different models approache the problem from different conceptual standpoints and theorists have different opinions about the contributing factors. All these models are described under the blanket term "similarity relations". Despite the perennial fascination with the concept, little or no consensus exits as to what constitutes a good similarity relation.

Castrén (1994) provides a comprehensive and in-depth review of a large number of similarity relations and categorises them according to some fundamental principles. Firstly, he distinguishes between methods that produce binary outcomes and those that produce a range of values. The former category, termed "plain relations", include Forte's R-relations (Forte, 1973) and indicate whether the two SCs are related in a specific way, which in turn may give some indication of whether they are similar. The latter category, termed "similarity measures", indicate a degree of similarity, returning a value from a known range. This property appears to be more inline with the perceptual notion of similarity and therefore the focus of this work shall be exclusively on similarity measures.

### 3.4.2 Similarity Measures

The vast number and diversity of the different approaches to similarity measures can only be approached by narrowing the focus to a specific type. Here we will focus on measures that use the Tn and TnI-type SCs (3.2), and furthermore we will only consider those methods based on vector analysis (3.3). These measure usually involve the comparison of the SCs' nCVs. Of this (still sizeable) subset, Castrén (1994) identifies two main categories.

| | |
|---|---|
| Single nC: | Single nC measures compare the nCVs of the two SCs for one particular value of n. Many of the relations in this category compare ICVs (2CVs). |
| Total Measures: | Total Measures consider the subsets of all cardinalities contained within in two SCs. All the relevant nCVs are compared to produce a final value. |

Table 4 shows the majority of the Tn and TnI-Type, vector based similarity measures from the PC-set theoretical literature organised by theorist. Vector Type indicates whether the measure compares ICVs or nCVs. Card (Cardinality) indicates whether the measure is capable of comparing SCs of different cardinalities while the Measure Type indicates which of Castren's categories it belongs to. nC indicates it is a Single nC measure and TOTAL indicates it is a Total Measure. All these measure are described more thoroughly in A.

### 3.4.3 Castren's Criteria

In addition to his categorisation, Castrén (1994) proposes several criteria which a good similarity relation should meet. Later, these criteria will be used in assessing the specific capabilities of various similarity measures.

Castren says that a similarity measure should:

- C1: allow comparisons between SCs of different cardinalities
- C2: provide a distinct value for every pair of SCs
- C3: provide a comprehensible scale of values such that
  - C3.1: All values are commensurable
  - C3.2: the end points are not just some extreme values but can be meaningfully associated with maximal and minimal similarity.
  - C3.3: The values are integers or other easily manageable numbers
  - C3.4: the degree of discrimination is not too coarse and not unrealistically fine
- C4: produce a uniform value for all comparable cases
- C5: observe mutually embeddable subset-classes of all meaningful cardinalities

Table 5: Comparison table of similarity measures

| THEORIST | SIMILARITY MEASURE | VECTOR TYPE | CARD | MEASURE TYPE |
|---|---|---|---|---|
| | K | ICV | SAME | nC |
| | SIM | ICV | SAME | nC |
| MORRIS | ASIM | ICV | ANY | nC |
| LORD | sf | ICV | SAME | nC |
| TEITELBAUM | s.i. | ICV | SAME | nC |
| | IcVD1 | ICV | ANY | nC |
| | IcVD2 | ICV | ANY | nC |
| ROGERS | COS | ICV | ANY | nC |
| | AMEMB2 | ICV | ANY | nC |
| | IcVSIM | ICV | ANY | nC |
| | ISIM2 | ICV | ANY | nC |
| ISAACSON | ANGLE | ICV | ANY | nC |
| | AK | ICV | ANY | nC |
| | MEMBn | nCV | ANY | nC |
| | TMEMB | nCV | ANY | TOTAL |
| RAHN | ATMEMB | nCV | ANY | TOTAL |
| | REL2 | ICV | ANY | nC |
| LEWIN | REL | nCV | ANY | TOTAL |
| | %RELn | nC%V | ANY | nC |
| | T%REL | nC%V | ANY | TOTAL |
| CASTREN | RECREL | nC%V | ANY | TOTAL |
| | SATSIM | nSATV | ANY | nC |
| | CSATSIM | CSATV | ANY | nC |
| | TSATSIM | nSATV | ANY | TOTAL |
| BUCHLER | AvgSATSIM | nSATV | ANY | TOTAL |

- C6: observe also the mutual embeddable subset-classes not in common between the SCs being compared.

## 3.5 Perceptual Relevance

The many equivalence relations used in PC-set theory give rise to a highly abstract description of musical objects. Thus, an important question to be asked is whether these theoretical assumptions and models of similarity reflect perceptual equivalence. This chapter contains a summary and discussion of some relevant studies.

### 3.5.1 Octave Equivalence

Pitch is a percept that derives from a particular harmonic structure and is roughly proportional to the logarithm of the fundamental frequency. This allows pitch to be perceptually modelled as a straight line. Music psychologists have observed a strong perceptual similarity between pitches with fundamental frequencies in the ratio of 2:1. This property of octave similarity leads the straight line model of pitch to be bent into a helix. Division of the octave into a number of categories is thought to offer a more efficient cognitive representation in memory and thus confers evolutionary advantage. The resulting pitch equivalence classes are implicitly learned through exposure at an early age. TET has 12 pitch equivalence classes which, in PC-set theory, are modelled as a circular projection of the pitch helix. Thus the two most fundamental components of PC-set theory, i.e. octave equivalence and pitch-class labelling, would appear to have a solid basis in perception.

Gibson (1988) investigated the perceived similarity of pairs of chords with varying numbers of octave related pitches. He found that in general chords with identical PC contents were perceived as more similar than chords with near identical PC contents, regardless of the octave of the pitch components. However, in further studies he his findings suggest that there are other factors that play a significant role (Gibson, 1993).

### 3.5.2 Set-Class Equivalence

Some researchers have attempted to examine whether there is perceived equivalence between different manifestations of a PC-set. Krumhansl et al. (1987) presented subjects with sequences of tones derived by transforming two different PC-sets. They noted that subjects were able to distinguish between the different sets both in neutral and musical contexts.

Millar (1984) investigated the perceptual similarity of different PC-sets derived from the same set class under TnI classification. Subjects were presented with three-note melodies and asked to judge which was equivalent to a reference melody. Some melodies preserved the SC identity whilst others did not. She found transpositions to be perceived more similar than inversions and in addition she discovered that the order of the notes and melodic contour was a strong factor in perceived similarity.

Some authors have questioned the perceptual relevance of using TnI and I equivalence as a basis for set classification. Deutsch (1982) seems unconvinced by evidence for the perceptual similarity of inverted intervals. This can be illustrated by the example of major and minor triads which, while perceptually distinct, are equivalent under TnI and I equivalence.

### 3.5.3 Perceived vs Theoretical Similarity

A number of studies have been done to ascertain the connection between perceptual similarity ratings and the theoretical values obtained from some PC-set similarity measures. A large number of relevant studies are summarised by Kuusi (2001) and the most significant ones are mentioned here.

Bruner (1984) used multidimensional scaling on subjects' similarity ratings between trichords and tetrachords and on the similarity values obtained from SIM. She compared the 2-dimensional solutions and found there to be little correlation.

Gibson (1986) investigated non-traditional chords. He compared subjects' ratings with similarity assessments calculated from Forte's R-relations and Lord's similarity function. He also concluded there was little correspondence between the two.

Stammers (1994) compared subjects' ratings of 4 note melodies with the theoretical values obtained from SIM. She found the ratings of subjects with more musical training to be more correlated with the SIM values.

Lane (1997) compared subjects' ratings of pitch sequences with corresponding values of seven ICV-based similarity measures: ASIM, MEMB2, REL2, s.i., IcVSIM and AMEMB2 and concluded there to be a strong relation.

Kuusi (2001) compared subjects' ratings of pentachords with the values obtained from 9 similarity measures. He found there to be a connection between aurally estimated ratings and the theoretical values and concluded that the abstract properties of set-classes do have some perceptual relevance. He also comments on the way in which this kind of study is conducted, suggesting that the way in which subjects are presented with the stimuli has a significant effect on the outcome.

## 3.6 PC-set Theory for Analysis

PC-set theory as means for descriptive modelling of tonality is not widely known outside of highly theoretical circles and the use of set-class similarity measures seems mainly restricted to the theorists who proposed them (for example, Isaacson 1996). The basic premise is simple: a musical piece is segmented and each segment described by its SC. Similarity measures can be used to assess the similarity between segments or between a segment and some reference SC.

Huovinen and Tenkanen (2007) used a pentachordal tail segmentation policy (each successive note defines a segment that includes the preceding four notes) and compared these segments to comparison sets 7-1 (chromaticism) and 7-35 (diatonicism) using the REL distance (A.7.1). They claim that the visual results of their analysis "reflect pertinent aspects of our listening experience" (**?**, pp. 204).

Martorell (2013, chap. 5.3) uses a more systematic approach to segmentation using multiple time scales. He proposes the class-scape, a two-dimensional visualisation of a piece of music with time on the x-axis and segmentation time-scale on the y-axis. A single SC can be represented by highlighting the segment or alternatively each segment can be shaded according to its REL distance from a comparison SC. He emphasises that the class-scape is an exploratory tool rather than an automated analysis system.

Perhaps the most crucial aspect of using SC descriptions for tonal analysis is the way in which a piece of music is segmented. The issue of segmentation will be discussed further in Chapter 8.

### 3.7 Chords as PC-Sets

So far, PC-set theory and the similarity measures have been discussed with only the broadest reference to familiar musical or musicologial concepts. In order to use these techniques for real music description and analysis some effort should be made to link SC theoretical concepts to musicological ones.

The table below shows a collection of familiar chord types, cadence types and scale types with their corresponding Tn-type prime-form SC, Forte Name and index number (the position in an ordered list of all 351 Tn-type SCs). The table is divided into five vertical segments: three-note chords, four-note chords, five-note chords, cadences and scales.

In PC-set theory it is only the chord type that is relevant seeing as transpositionally related (Tn-type) SCs are considered equivelant. For example all major chords are considered equivelant regardless of the root.

## 4 Multidimensional Scaling

Multidimensional scaling (MDS) is a numerical visualisation technique that, given a matrix of pairwise distances between objects, provides a geometric configuration of the objects in some abstract space. It provides an efficient means of observing relationships in large, complex data sets and the resulting dimensions often give valuable insight into the data as a whole.

### 4.1 Non-Metric MDS

Non-Metric MDS was described by Shepard (1962) and it assumes that the distance matrix values are related to points in an abstract N-dimensional Euclidean space. An important consideration is that of the dimensionality of the solution. For comprehension and visualisation it is important to minimise the number of dimensions however, there is a trade-off between the number of dimensions and the accuracy of the model. For a given dimensionality, we obtain two values: Stress and $r^2$.

| | |
|---|---|
| Stress | Stress is a "goodness of fit" measure which characterises the distortion that occurs in a given number of dimensions. As the number of dimensions increases the stress decreases. |
| $r^2$ | $r^2$ is the percentage variability of the data being explained by the solution. |

By plotting stress against $r^2$ for a number of dimensionalities is possible to observe the point at which additional dimensions do not significantly improve the solution (the "elbow"). Ultimately, the choice of dimensions should be based on interpretation.

### 4.2 Cluster Analysis

Cluster analysis (CA) is method for dealing with dimensions that are highly separable. First, the most similar pair of objects are selected and grouped together in a cluster. The process is repeated, creating a binary tree structure. The distance between objects is then related to their separation along the branches of the tree.

Table 6: Chord types and their SCs

| Chord | Tn-Type SC (Prime Form) | Forte Name (Tn-type) | index |
|---|---|---|---|
| maj | {0,4,7} | 3-11B | 25 |
| min | {0,3,7} | 3-11A | 24 |
| dim | {0,3,6} | 3-10 | 23 |
| aug | {0,4,8} | 3-12 | 26 |
| sus4 | {0,2,7} | 3-9 | 22 |
| sus2 | {0,2,7} | 3-9 | 22 |
| maj7 | {0,1,5,8} | 4-20 | 57 |
| min7 | {0,3,5,8} | 4-26 | 64 |
| hdim7 | {0,2,5,8} | 4-27A | 65 |
| 7 | {0,3,6,8} | 4-27B | 66 |
| dim7 | {0,3,6,9} | 4-28 | 67 |
| min(7) | {0,1,4,8} | 4-19A | 55 |
| aug(7) | {0,3,4,8} | 4-19B | 56 |
| maj(9) | {0,2,4,7} | 4-22A | 59 |
| min(9) | {0,2,3,7} | 4-14A | 46 |
| maj6 | {0,3,5,8} | 4-26 | 64 |
| min6 | {0,1,5,8} | 4-20 | 57 |
| sus4(7) | {0,2,6,7} | 4-16B | 51 |
| sus4(b7) | {0,2,5,7} | 4-23 | 61 |
| 9 | {0,2,4,6,9} | 5-34 | 129 |
| maj9 | {0,1,3,5,8} | 5-27A | 116 |
| min9 | {0,3,5,7,8} | 5-27B | 117 |
| V-I/IV-I | {0,1,3,5,8} | 5-27A | 116 |
| V7-I | {0,1,3,5,6,8} | 6-Z25A | 176 |
| V-IV | {0,2,4,6,7,9} | 6-33B | 189 |
| Pentatonic | {0,2,4,7,9} | 5-35 | 130 |
| Wholetone | {0,2,4,6,8,10} | 6-35 | 192 |
| Diatonic | {0,1,3,5,6,8,10} | 7-35 | 276 |
| Octatonic | {0,1,3,4,6,7,9,10} | 8-28 | 322 |

### 4.3 MDS with Similarity Measures

Using MDS on the values produced by similarity measures is one way to approach an understanding of the constructs they are measuring. There are two potentially interesting issues to consider. Firstly, a measure may be inconsistent with itself, meaning that the geometries it produces are not "robust" (changing the set of objects changes the distances between the original set). This kind of problem cannot be observed through inspection of the values alone. The second issue is that two different measures that are both self-consistent may produce very different geometries from the same group of SCs. The question then is, what exactly do the measures measure?

## 5 Spatial Models of Tonality

This chapter describes existing attempts to characterise the dimensionality of tonality as context.

### 5.1 Similarity and Distance

Judgements of similarity form the basis of many cognitive processes including the perception of tonality. Similarity between two objects is often conceived as being inversely related to distance between them in geometric space. For example, some tonal objects (chords, for example) are perceived as close to one another whereas others are further apart. In addition, the number of dimensions of the geometric space is in connection with the number of independent properties that are relevant for similarity comparisons. Gärdenfors (2000) suggests that humans are naturally predisposed to create spatial cognitive representations of perceptual stimuli due to the geometric nature of the world we have evolved to inhabit. Therefore spatial modelling of tonality, as well as helping to visualise the complex multidimensional relationships between tonal phenomena, has the potential to reflect cognitive aspects of the way they are perceived.

## 5.2 Spatial Representations

Throughout history theorists have proposed many spatial representations of tonality from different domains. From the graphemic domain, Weber (1851) and Schoenberg (1954) both proposed simple 2-dimensional charts to display the proximity between keys. For representation of chords, Riemann (1877) models major and minor triads as regions in a 2-dimensional space whilst Tymoczko (2011) proposes a variety high dimensional, non-euclidean chord spaces that reflect the theoretical principles of voice leading. From the acoustic domain, Shepard (1982) proposes a five-dimensional model to represent interval relations between pitches. Some theorists have attempted to incorporate relations between several levels of tonal hierarchy into one configuration. The "spiral array" of Chew (2000) is a three-dimensional mathematical model which simultaneously captures the relations between pitches, chords and keys. The "chordal-regional space" of Lerdahl (2001) models the relations between chords within a certain key.

## 5.3 Cognitive Psychology

The auditory domain has been addressed through cognitive psychology by Krumhansl (1990) who used the probe-tone methodology (Krumhansl and Shepard, 1979) to establish major and minor key profiles (12-dimensional vectors containing the perceptual stability ratings of each of the 12 pitch classes within a major or minor context). These profiles, know as Krumhansl-Kessler profiles (KK-profiles), show the hierarchy of pitches in major and minor keys. Correlating each of the 24 major and minor profiles produced a matrix of pairwise distances which was fed to a dimensional scaling algorithm. The resulting geometrical solution was found to have a double circular property (circle of fifths and relative-parallel relations) which can be modelled as the surface a 3D torus. Many spatial models of tonality have this double circular property whether it is implicit (Weber, 1851; Schoenberg, 1954) or stated explicitly (Lerdahl, 2001).

## 5.4 Set-Class Spaces

Most of these models are limited to description of music in the major-minor paradigm and are not capable of generalising beyond the "western common practice". PC-set theory, once again, provides a possible means to generalise to any kind of pitch-based music. By considering a collection of tonal objects described by SCs, a geometric space can be constructed to model their relations based on some theoretical principle. Some PC-set theorists have proposed explicit geometric spaces to model relations between SCs. The distances in these spaces are expressed by models of similarity based on voice leading (Cohn, 2003; Tymoczko, 2012) or ICVs and the Fourier transform (Quinn, 2006, 2007). However, these models are only designed to represent SCs of one cardinality-class at a time and cannot model the relations between arbitrary collections of pitches.

Alternative spatial models are provided by the implicit geometries of the values produced by the SC similarity measures discussed in **??**. As mentioned in 4.3, MDS can be used on values produced by similarity measure to create a geometric space. Kuusi (2001) and Samplaski (2005) both applied MDS to the values produced from a variety of similarity measures. Samplaski used TnI-type SCs while Kuusi used Tn-type. They both found reasonably low-dimensional solutions and attempted to interpret each of the dimensions. Kuusi interpreted three dimensions as corresponding to chromaticism, wholetoneness and pentatonicism. Samplaski made similar observations but found some dimensions in the higher-dimensional spaces difficult to interpret. Nevertheless, he concluded that values from similarity measure tend to agree (with some exceptions) and that they measure constructs relating to familiar scales (diatonic, hexatonic, octatonic, etc.).

# 6 Similarity Measure Selection

So far, PC-set theory has been presented as viable means for systematic decriptive modelling of tonality and brief reference has been made to the extensive existing literature on SC similarity measures (3.4). In this section, the large number of measures will be discussed in relation to Castren's criteria (3.4.3) in order to gauge their suitability for use in systematic surface description models. The most suitable models will be adopted for examination over the course of the work.

## 6.1 Criteria

Castren's criteria (see 3.4.3) for similarity measures provide a basis for assement of similarity measures for our purposes. A detailed descriptions and justification for the criteria can be found in Castrén (1994, chap. 2), however here we will focus on one or two specific aspects. The table below shows the list of similarity measures with marks indicating whether each of the criteria is met. In sections 6.2 to 6.4 specific criteria are used to exclude measures from further consideration with justification in terms of systematicity and perceptual relevance.

Table 7: Castren's Criteria

| SIMILARITY MEASURE | C1 | C2 | C3.1 | C3.2 | C3.3 | C3.4 | C4 | C5 | C6 |
|---|---|---|---|---|---|---|---|---|---|
| s.i. | | | | | X | X | | | |
| sf | | | | | X | X | X | | |
| IcVSIM | X | X | | | | X | | | |
| ISIM2 | X | X | | | | X | | | |
| K | X | X | | | X | X | X | | |
| SIM | X | X | | | X | X | X | | |
| MEMBn | X | X | | | X | X | X | | |
| AMEMB2 | X | X | X | | | | | | |
| ASIM | X | X | X | X | | X | X | | |
| IcVD1 | X | X | X | X | | X | X | | |
| IcVD2 | X | X | X | X | | X | | | |
| COS | X | X | X | X | | X | | | |
| ANGLE | X | X | X | X | | X | | | |
| AK | X | X | X | X | | X | X | | |
| SATSIM | X | X | X | | | | | | |
| CSATSIM | X | X | X | | | | | | |
| REL2 | | | | | | | | | |
| %RELn | X | X | X | X | X | X | X | | |
| TMEMB | X | X | | | X | | X | X | |
| ATMEMB | X | X | X | X | | X | X | X | |
| TSATSIM | X | X | X | X | | X | | X | |
| AvgSATSIM | X | X | X | X | | X | | X | |
| REL | X | X | X | X | | X | X | X | |
| T%REL | X | X | X | X | X | X | X | X | |
| RECREL | X | X | X | X | X | X | X | X | X |

## 6.2 Cardinality

Measures which fail to meet criteria C1, i.e. that cannot compare SCs of different cardinalities, are clearly inadequate for systematic analysis of music, which might require the comparison of any two arbitrary segments regardless of how many PCs they contain. Both s.i. (A.3.1) and sf (A.2.1) were proposed specifically for SCs of the same cardinality and so will be excluded from further discussion. Some other measures which were intended to compare SCs of different cardinalities nonetheless have problems. Measures such as SIM (A.1.2) and K (A.1.1) give unintuitive values when the cardinalities of the SCs being compared differ greatly and, in addition, the range of values produced depends on the cardinality of the sets (failure to meet criteria C3.1). Measures of this type will also be excluded.

### 6.3 SC-Type

An important consideration when using similarity measures is the type of SC being compared. Many of the measures are designed for comparison of TnI-type SCs, however, owing to issues riased in 3.5 regarding the perceptual relevance of invertionally related sets, here, measures will be selected for use with Tn-type SCs. This means that the measure should be able to discriminate between inversionally related sets. All the single-nC measures which exclusively consider interval content (ICVs) in the comparison procedure can therefore be discounted, as inversionally related sets have identical ICVs.

### 6.4 Measure Type

Although many theorists have supposed that interval-class subsets are of paramount importance in similarity judgments, no thorough investigation has been carried out as to the exact perceptual significance of subset cardinality. Single-nC measures presuppose that subsets of one particular cardinality contribute to similarity above all others. In the interest of systematicity, we will not make this assumption instead assuming that subsets of all cardinalities are equally relevant and should be considered. Similarity measures that exhaustively consider all subset cardinalities meet criteria C5 and are total measures (see 3.4.2). The six total measures from 3.4.2 shall therefore become the focus of this work.

## 7 Total Measures

In previous chapters we have examined the fundamental aspects of tonal models and proposed PC-set theory as a descriptive tool, potentially capable of capturing the three important elements of tonality as context. Through a desire for both perceptual relevance and systematicity a subset of these techniques has been identified as most pertinent, specifically Tn-type SCs and total similarity measures.

In this chapter the six total similarity measures will be examined more closely and placed in practical context through examination of the values they produce. Section **??** describes the concept of trivial forms and for each measure the method for handling these cases is described. Section 7.8 gives a brief comparison of the measures.

### 7.1 Trivial Forms

Three of the 351 Tn-type SCs are known as trivial forms: 1-1, 11-1 and 12-1. Due to their lack of musical or harmonic significance, these SCs are usually excluded from the work of SC-theorists. However, it is important that they be included in any systematic description and that their similarity to other sets be given a meaningful value. The next chapter will discuss each of the total measures and specify how each of the trivial forms is to be dealt with if it was not made explicit by the theorist.

The total measures which will be discussed make comparisons based on the subset content of a set. SC 1-1, which has no subsets, is rarely accounted for in such measures and in these cases a simple method will be used: Comparisons involving X = 1-1 and Y will be given the value $1\#Y$. Thus, the value will be the ratios of the cardinalities with 1 indicating maximum similarity.

### 7.2 Rahn: ATMEMB

Details on how to calculate ATMEMB are give in A.5.4. In his analysis of the measure, Castren concludes that "divisor term is flawed, resulting in values suggesting suspiciously high degrees of dissimilarity between SCs of clearly different cardinalities. The general reliability and usefulness of the measure is difficulty to determine" (Castrén, 1994, pp. 89). The trivial forms 11-1 and 12-1 are accommodated explicitly by the

Table 8: Trivial Forms

| | |
|---|---|
| 1-1 | {0} |
| 11-1 | {0,1,2,3,4,5,6,7,8,9,10} |
| 12-1 | {0,1,2,3,4,5,6,7,8,9,10,11} |

formulation of Rahn (1979), however SC 1-1 is not and thus values will be obtained using the method specified in 7.1.

## 7.3 Lewin: REL

Details on how to calculate REL are given in A.7.1. From the basic equation it is possible to define three different formulations depending on the exact nature of SUB(X). In each formulation the trivial forms 11-1 and 12-1 are accommodated. The three formulations are as follows:

1. SUB(X) consists of the concatenated nCVs from 2 to 12. Here comparisons involving SC 1-1 will be evaluated with the method specified in 7.1.
2. SUB(X) consists of the concatenated nCVs from 1 to 12 ($1CV(X) = #X%). This formulation accommodates SC 1-1.
3. Martorell (2013) specifies an alternative formulation where SUB(X) begins with the ICV (2CV) followed by the concatenated nCVs from 1 to 12. This formulation accommodates SC 1-1.

## 7.4 Buchler: AvgSATSIM and TSATSIM

Details on how to calculate AvgSATSIM and TSATSIM are given in A.9.3 and A.9.4 respectively. Comparisons involving SC 1-1 are not accommodated and thus the method specified in 7.1 will be used to provide values. Comparisons involving SCs 11-1 and 12-1 are accommodated except for the single comparison that involves both. This is because their $\mathrm{MAX}_n(\#\mathrm{X})$ and $\mathrm{MIN}_n(\#\mathrm{X})$ vectors are equal and thus all terms of the nSATVs are 0. The value for this comparison will be set to 0 (indicating maximal similarity). For comparisons involving ICs the value will be given by $\mathrm{SATSIM}_2(\mathrm{X,Y})$ (see A.9.2).

## 7.5 Castren: T%REL and RECREL

Details on how to calculate T%REL and RECREL are given in A.8.4 and A.8.5 respectively. Comparisons involving SCs 11-1 and 12-1 are accommodated in both by Castren's formulation. Comparisons involving SC 1-1 will be given values by the method specified in 7.1. Castren comments that some T%REL values are too high to be intuitively plausible. Finally, it should be noted that the basic algorithm provided by Castren for calculating RECREL is not feasible for large sets. Comparisons of such sets require tables of pre-computed branch values.

## 7.6 Scale of Values

The values of each measure will be adjusted to the same scale for comparability by the same method as Kuusi (2001, pp. 48)). This scale is from 0 to 100 with with 0 indicating maximum similarity. The modified values will be signalled by adding the symbol "prime" to the name.

## 7.7 Comparison of Chord Types

For a preliminary idea of the utility of the total measures it is useful to visualise the vlaues produced for comparisons involving the common tonal objects described in 3.7. This information can be visualised as 2D grids with each square corresponding to the comparison between two tonal objects and coloured according the distance between them i.e. the value of MEASURE-prime (see 7.6). Figures 1 and 2 show two such grids for ATMEMB and AvgSATSIM respectively.

Table 9: Adjustment for MEASURE-prime scale

| Measure-prime(X,Y) |
| --- |
| (1-ATMEMB(X,Y))*100 |
| (1-REL(X,Y))*100 |
| T%REL(X,Y) |
| RECREL(X,Y) |
| AvgSATSIM(X,Y)*100 |
| TSATSIM(X,Y)*100 |

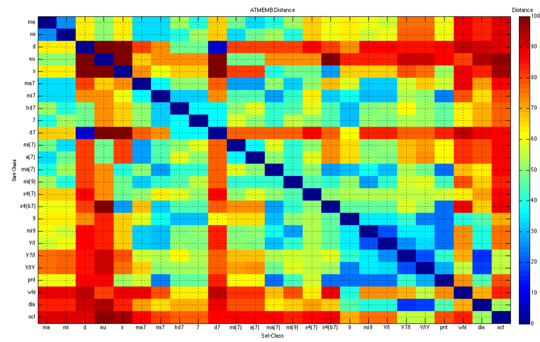

Figure 1: ATMEMB distance between tonal objects

As can be seen, the values for these chord types are quite dissimilar for the different measures. Thus, measure selection will be an important part of the analysis depending on their specific discriminatory power. Therefore, these plots will form a useful reference guide when selecting parameters and values for the visualisation techniques described in Chapter 9.

## 7.8 Total Measure Comparison

For a more quantitative comparison of the measures, the absolute difference between corresponding values can be calculated and plotted on similar grids. This gives an overall visualisation of where the measures most disagree. Figure 3 shows a matrix of plots each comparing the values of two measures. Each plot displays only half the table as the they are symmetric. The lighter blue and green areas indicate higher discrepancy between the measures' values.

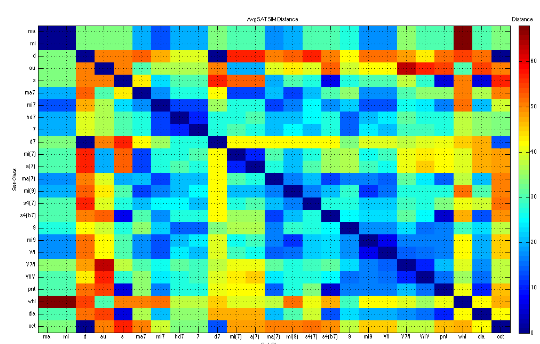

Figure 2: AvgSATSIM distance between tonal objects

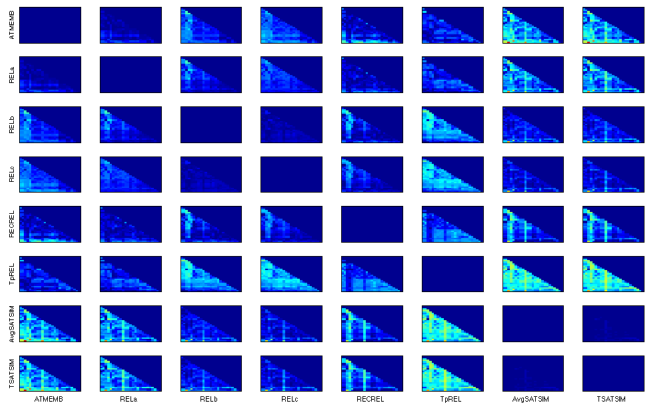


Figure 3: Absolute difference between measures' values

A more compact representation of these comparisons can be obtained by correlation of the vectors containing the values from each measure. Figure 4 shows a grid where each square corresponds to the comparison between two measures and is coloured according the correlation of the values for the common chord types. For the purposes of comparison, Figure 5 shows a similar plot to Figure 4 but rather displays the correlation involving all the 61425 pairwise Tn-type SC comparisons for each measure.

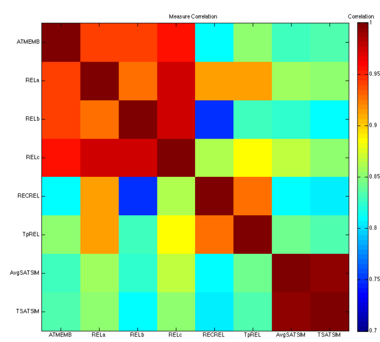


Figure 4: Correlation between measures' values for common chords

## 7.9 Chapter Conclusions

- general comments about measure discrimination?
  - high/low
  - major minor
  - cardinality
  - symmetric sets
  - inversionally related sets
- these plots will aid selection of comparison sets
- abs diff plot
  - which measures are closest
  - avgsatsim - tsatsim
  - which measures are furthest
  - closer inspection

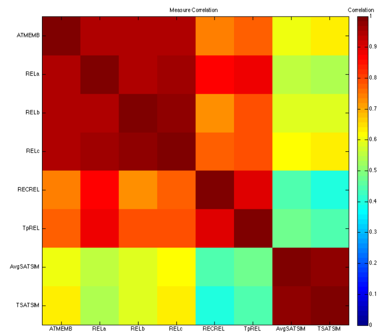


Figure 5: Correlation between measures' values for all SCs

- chord corr plot
  - measures tend to agree
  - most different measures
  - recrel - relb: why?
- SC corr plot
  - means of comparison
  - common chord types have common properties that the measures tend to treat in a similar way.
- repertiore
  - what repertoire contains what chords
  - billboard
  - beatles

# 8 Segmentation

As mentioned in 3.6, segmentation is a very important stage of the analysis. The segmentation policy used strongly influences the SC content that can be observed, with larger segments tending to contain higher cardinality sets. Using a sliding window to segment the music requires careful consideration of the specific window length and hop size depending not only on the type and cardinality of sets that the analyst desires to observe but also on the way in which the data will then be visualised.

In this chapter we outline two segmentation policies:

1. A sliding window with fixed length and hop size.
2. The systematic segmentation policy of Martorell (2013).

## 8.1 Systematic Segmentation

Martorell (2013, chap. 5.3) specifies a fully systematic segmentation policy which exhaustively records every change in the PC material of a piece. The segments are of variable length but together capture the entire SC contents of the music. With each segment indexed in time by its centre, the data structure, known as a cass-scape, is a sparse 3-dimensional binary matrix with axes representing time, time-scale (segment length) and set class.

Martorell (2013, chap. 5.3.5) specifies two compact representations of this data as a means of observing the the global SC content of a piece.

Class-matrix: The class-matrix is a 2d projection the class-scape obtained by removing the time-scale information. This, in effect, acts to expand each point to the actual duration of the segment it represents.

Class-vector: The class-vector is a further reduction of data showing the relative active duration of each class in the class-matrix and is expressed as a percentage of the total duration of the piece.

The systematic segmentation policy was applied to a MIDI representation of the C major Prelude from Book 1 of The Well Tempered Clavier by Bach. Figures 6 and 7 show the class-matrix and class-vector respectively for the Bach prelude. This information gives a global indications as to the types of sonorities contained within the piece.

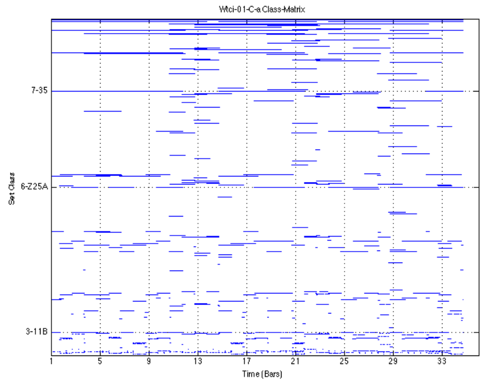


Figure 6: Class-matrix for Bach WTCi Prelude 1

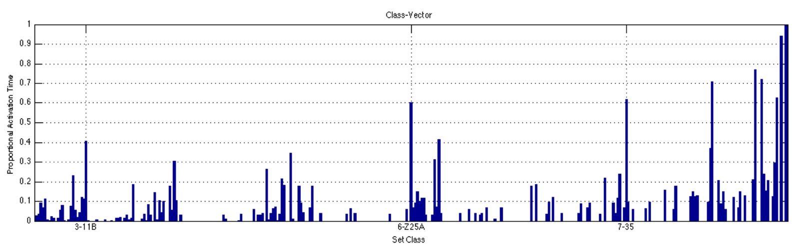


Figure 7: Class-vector for Bach WTCi Prelude 1

From this complete information it is also possible to view statistical information about which SCs or type of SCs appear in which time scale. Figure 8 plots the average segment length and standard deviation for each SC. Figure 9 shows the same information for each cardinality class.

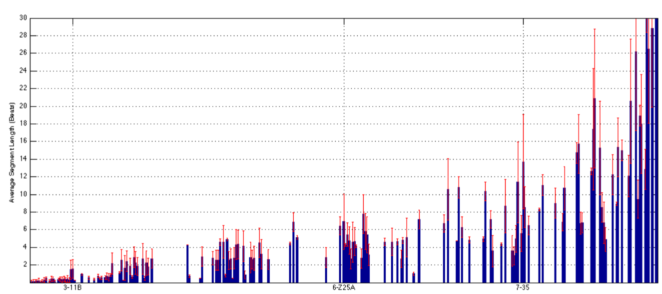


Figure 8: Average segment length and STD for each SC in Bach WTCi Prelude 1

## 8.2 Sliding Window

Segmentation using a sliding window with fixed window and hop size is used to obtain an SC time series of one specific time-scale. This time series is then used for the visualisation techniques outlined in the next chapter. The purpose of the sliding window is to allow the analyst to focus on particular sets or tune in to one particular cardinality. This tuning process is performed through selection of an appropriate window and

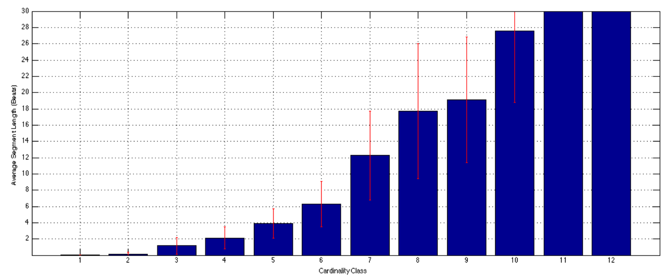


Figure 9: Average Segment length and STD for each nC in Bach WTCi Prelude 1

hop size. The selection of window and hop size can be informed by exploration of the data obtained from the systematic segmentation.

For example, from Figure 9 in the previous section it can be seen that three note chords have an average window length of between one and two beats. Specifically, from Figure 8, major chords (3-11B) have an average window length of closer to two beats. Figure 10 displays the results of a sliding window segmentation of the piece with a window size of two beats and hop size of one beat. The SC contents extracted from this segmentation are displayed in red on top of the class-vector. This representation gives an indication of what proportion of the overall class contents have been retrieved by the sliding window segmentation.

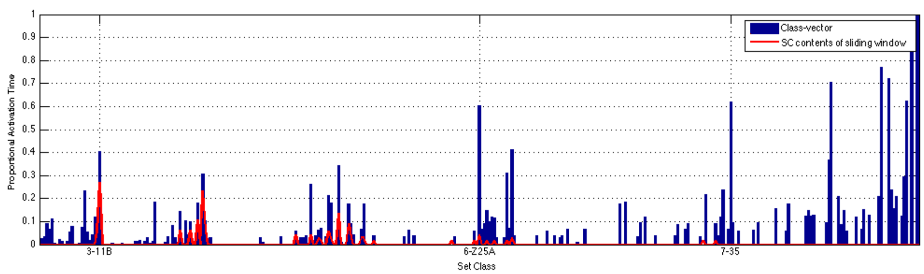


Figure 10: SC contents from sliding window segmentation (red) superimposed on class-vector (black)

The plot shows that the particular segmentation policy has captured a higher proportion of SC 3-11B than other types. This specific targeting of SCs will be exploited in the visualisation techniques described in 9.

# 9 Visualisation

This chapter outlines three visualisation techniques that may be used in conjunction to represent SC class information of a musical piece so as to retrieve meaningful analysis.

## 9.1 Distance Plot

The distance plot provides a simple means of capturing how the pitch content of a piece evolves in time with respect to a specific SC. It involves segmenting the piece using a fixed sliding window and calculating the distance between each segment and a comparison set.

There are three interdependent parameters which must be selected according to the specific intentions of the analyst: Segmentation (window and hop size), comparison set and similarity/distance measure. The segmentation determines the captured SC content which should be targeted according to its relationship to the comparison set. This relationship is determined by the measure used which must possess an adequate degree of discrimination so as to produce noticeable changes in the time series.

Figure 11 shows an example distance time series plotted in as a line for the Bach Prelude. The same sliding window segmentation as Figure 10 was used so as to capture three note chords, specifically major chords (3-11B). Each segment was compared to SC 3-11B using the ATMEMB-prime distance.


Figure 11: ATMEMB-prime distance plot (3-11B)

As Figure 10 indicates a higher proportion of SC 3-11B present in the segments, it is unsurprising to see the line frequently displaying a distance of 0 where the segment coincides with the comparison set. A plot such as this gives an indication as to the tonal progression of the piece, indicating where the sonority most resembles that of a simple major triads. Higher parts of the curve indicate a departure from these triads and possibly more complicated or less familiar harmonic passages.

- different measures (strengths and weaknesses)
- different comparison sets (5-27A,6-Z27A,7-35,8-32)

## 9.2 Auto-correlation

The distance plots described in the previous section can give a basic view of the tonal progression of a piece. In many cases they contain recurring patterns where passages with similar SC content producing similar curves. A further visualisation technique is autocorrelation. By autocorrelating the distance plot it may be possible to capture some structural elements of the piece. The autocorrelation time series will contain peaks corresponding to "similar" passages. For example, an piece with a structure of A-B-A will have an autocorrelation plot with a slight peak corresponding to beginning of the repeat of section A.

Figure 12 shows the autocorrelation function of the distance plot in Figure 11. The peaks that appear at regular intervals correspond to an approximate tonal tonal pattern that recurs throughout the piece.

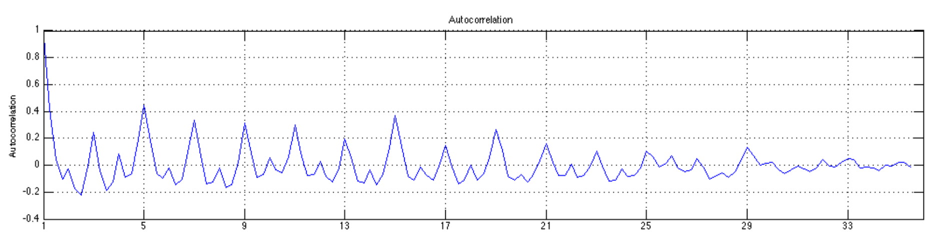


Figure 12: Autocorrelation of distance plot

## 9.3 Self-similarity matrix

As an alternative to using a specific comparison set, the distance of a segment to other segments in the piece could be observed. A systematic way of doing this is computing the self-similarity matrix which provides a simples means for discovery of repetitions in a time series. The patterns present in a visualisation of a self-similarity matrix can indicate similar passages and structural elements of a piece.

Figure 13 shows the self-similarity matrix computed from the sliding window SC time-series. The diagonal line highlighted in red represents the repetition of a four bar sequence: bars 5-11 in the tonic key are repeated in bars 15-19 with a transposition to the dominant key.

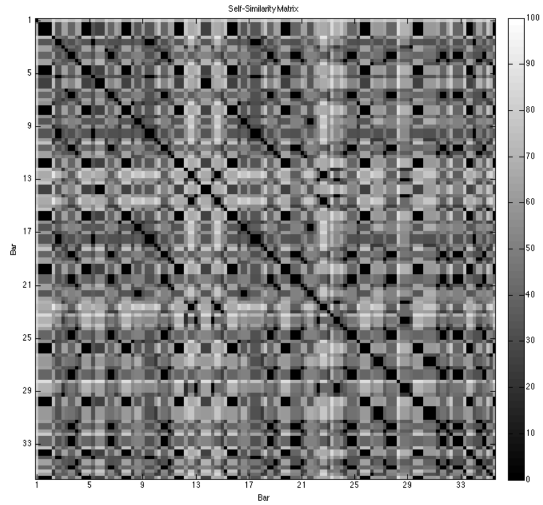


Figure 13: Self-similarity matrix of SC time series

### 9.4 Chapter Conclusions

1. The techniques outlined in this chapter have proved capable of exposing some relevant musicological understanding of the example piece.
2. The sliding window segmentation can be tuned to target specific SCs using the global statistics obtained from a fully systematic segmentation policy.
3. The distance plot can represent intuitions about the tonal progression of a piece through comparison with a reference set.
4. The autocorrelation can highlight recurring patterns in the distance plot which can correspond to structurally important elements of the piece.
5. The self-similarity matrix is capable of capturing not only exact repetitions within a piece, but transposed, inverted and reversed repetitions.
6. Working on the assumption that the similarity measures have some perceptual relevance, these representation can be used not only to find repetitions in SC content, but also passages where the some quality of perceptual distance or ratio is preserved. These relationships might have a sophisticated musicological basis and might not be as easily observable from listening or from the score.

## 10 Analysis Tool

1. Due to the complex relationship between the selection of the various parameters involved in this kind of analysis, it is desirable to work in an interactive and exploratory environment.
2. The analysis tool combines all of the presented representation techniques in a single Matlab GUI.
3. This kind of tool provides a test bench for exploring the capabilities of the analytic techniques. Preliminary results can quickly be obtained and compared.

## 11 Conclusions

1. This work has attempted to assess the analytical potential of six SC similarity measures.
2. Systematic SC descriptions of music have been discussed in terms of their perceptual and analytical relevance and have been shown to possess a lot potential as a starting point in many music research areas.

3. The importance of individual sets that correspond to chords of interest is less than than a hierarchical understanding of these sets' subsets and supersets and their evolution in time.
4. A combination of two segmentation policies have been presented to work in conjunction for extracting a SC description of a musical piece.
5. Three techniques have been presented for representing SC information. These techniques exploit theoretical similarity measures to retrieve structural information about a musical piece.
6. The distance plot tracks the change in SC contents of a piece with respect to a comparison set and is capable of exposing recurring sequences or sonically and perceptually similar sequences.
7. This recurrences can be quantified by autocorrelation of the distance plot. Peaks in the autocorrelation can point to important structural boundaries in a piece.
8. The self-similarity matrix gives a comprehensive comparison between all SCs in a time series and has been shown to expose structural repetitions.
9. The analysis tool allows all of these techniques to be used in conjunction, enabling the analyst to explore the numerous combinations and approaches. The demonstrations here are just the beginning of what could potentially be explored.
10. A systematic SC description combined with these techniques for representation of the data could be employed in MIR systems for the automatic detection of structure and musical similarity.

# 12 Future Work

1. A more concrete and quantitative analysis of the discriminatory power of the SC similarity measures will better inform the selection of appropriate comparison sets and measure.
2. The addition of peak selecting and structural marking in the analysis tool would inform tests as to the suitability of the proposed techniques for automatic structual segmentation.
3. A more thorough understanding of the SC contents of a piece would enable more exhaustive exploitation of the systematic SC description. Understanding cadential sets is a step in this direction, however understanding the supersets that result from say a typical progression of major and minor triads would bring this approach closer the requirements of MIR researchers in chord recognition.
4. Implicit SC spaces from multi-dimensional scaling of similarity measure values. Such spaces could provide deeper understanding of tonal progressions by analogy to trajectories in space. Partial SC spaces that include only a subset of all SCs could replicate some perceptual intuitions about the piece. Spaces could be specifically constructed for a piece through targeting of the relevant SC content as in the techniques specified here. The SCs incorporated may not necessarily be familiar chord types but could be supersets of progressions or represent the sonority of different structural elements.
5. There are many further ways in the which data could be extracted from a systematic SC description and represented so as to expose relevant tonal information about a piece.
   - The use of multiple measures, comparison sets and sliding windows could allow for a more finely tuned targeting of structural information.
   - Computationally combining different distance plots (multiplying, convolving, correlating etc.) could reinforce or weaken a particular analytical hypothesis.
   - Peaks present in an approximate differential of a distance plot would correspond to rapid changes in the tonal content and might signify some point of musical interest.

## References


M. Babbitt. The Use of Computers in Musicological Research. *Perspectives of New Music*, 3(2):74–83, 1965.

C. L. Bruner. The Perception of Contemporary Pitch Structures. *Music Perception: An Interdisciplinary Journal*, 2(1):25–39, 1984.

MH Buchler. Relative Saturation of Subsets and Interval Cycles as a Means for Determining Set-Class Similarity. (ic):1–204, 1997.

JA Burgoyne, J Wild, and I Fujinaga. An Expert Ground Truth Set for Audio Chord Recognition and Music Analysis. *ISMIR*, (Ismir), 2011. URL `http://ismir2011.ismir.net/papers/OS8-1.pdf`.

M. Castrén. *RECREL. A similarity measure for set-classes.* Sibelius Academy, Helsinki, 1994.

W. Chai. *Automated Analysis of Musical Structure.* PhD thesis, MIT, 2005.

E. Chew. *Towards a mathematical model of tonality.* PhD thesis, M.I.T., 2000.

R. Cohn. A Tetrahedral Graph of Tetrachordal Voice-Leading Space. *Music Theory Online*, 9(4), 2003. URL `http://www.mtosmt.org/issues/mto.03.9.4/mto.03.9.4.cohn_frames.html`.

D. Deutsch. The Procession of Pitch Combinations. In *Deutsch D. (ed.): The Psychology of Music*, chapter 9, pages 271–316. Academic Press, New York, 1982.

J. Foote. Automatic audio segmentation using a measure of audio novelty. In *Multimedia and Expo, 2000. ICME 2000. 2000 IEEE*, volume 00, pages 452–455, 2000. ISBN 0780365364.

A. Forte. *The Structure of Atonal Music*, volume 61. Yale University Press, London, 1973.

P. Gärdenfors. *Conceptual Spaces: The Geometry of Thought*, volume 4. MIT Press, Cambridge, MA, 2000.

D. Gibson. The Aural Perception of Nontraditional Chords in Selected Theoretical Relationships: A Computer-Generated Experiment. *Journal of Research in Music Education*, 34(1):5–23, 1986.

D. Gibson. The Aural Perception of Similarity in Nontraditional Chords Related by Octave Equivalence. *Journal of Research in Music Education*, 36(1):5–17, 1988.

D. Gibson. The Effects of Pitch and Pitch-class Content on their Aural Perception of Dissimilarity in Complementary Hexachords. *Psychomusicology*, 12(1):58–72, 1993.

E. Gómez. *Tonal description of music audio signals.* PhD thesis, UPF, 2006.

Christopher Harte. *Towards Automatic Extraction of Harmony Information from Music Signals.* PhD thesis, 2010.

E. Huovinen and A. Tenkanen. Bird's-Eye Views of the Musical Surface: Methods for Systematic Pitch-Class Set Analysis. *Music Analysis*, 26(1-2):159–214, March 2007. ISSN 02625245.

Brian Hyer. Tonality. *Grove Music Online*, 2013.

E. J. Isaacson. Similarity of Interval-Class Content between Pitch-Class Sets : The IcVSIM Relation. *Journal of Music Theory*, 34(1):1–28, 1990.

E. J. Isaacson. Issues in the Study of Similarity in Atonal Music. *Music Theory Online*, 2(7), 1996. URL `http://www.mtosmt.org/issues/mto.96.2.7/mto.96.2.7.isaacson.html`.

C. L. Krumhansl. *Cognitive Foundations of Musical Pitch.* Oxford University Press, New York, 1990.

C. L. Krumhansl and R. N. Shepard. Quantification of the Hierarchy of Tonal Functions Within a Diatonic Context. *Journal of Experimental Psychology*, 5(4):579–594, 1979.

C L Krumhansl, G J Sandell, and D C Sergeant. The Perception of Tone Hierarchies and Mirror Forms in Twelve-Tone Serial Music. *Music Perception: An Interdisciplinary Journal*, 5(1):31–77, 1987.

T. Kuusi. *Set-class and chord: examining connection between theoretical resemblance and perceived closeness.* PhD thesis, 2001.

R. C. Lane. *A Multidimensional Scaling Study of Seven Theoretical Indexes of Intervallic Similarity and Musicians' Perceptions Among Twenty-One Pitch-Class Sets.* Phd, University of North Carolina, 1997.

F. Lerdahl. *Tonal Pitch Space.* Oxford University Press, New York, 2001.

D. Lewin. A response to a response: on Pcset relatedness. *Perspectives of new music*, 18(1):498–502, 1979.

C. H. Lord. Intervallic Similarity Relations In Atonal Set Analysis. *Journal of Music Theory*, 25(1):91–111, 1981.

A. Martorell. *Modelling Tonal Context Dynamics by Temporal Multi-Scale Analysis.* PhD thesis, UPF, 2013.

J. K. Millar. *The Aural Perception of Pitch-Class Set Relations: A Computer-Assisted Investigation.* Phd, University of North Texas, 1984.

Robert Morris. A Similarity Index for Pitch-Class Sets. *Perspectives of New Music*, 18(1):445–460, 1979.

H. Purwins, B. Blankertz, and K. Obermayer. A new method for tracking modulations in tonal music in audio data format. In *International Joint Conference on Neural Network*, volume 6, pages 270–275, 2000. ISBN 0-7695-0619-4.

I. Quinn. General Equal-Tempered Harmony (Introduction and Part I). *Perspectives of New Music*, 44(2): 114–158, 2006.

I. Quinn. General Equal-Tempered Harmony: Parts 2 and 3. *Perspectives of New Music*, 45(1):4–63, 2007.

J. Rahn. Relating sets. *Perspectives of New Music*, 18(1):483–498, 1979. URL `http://www.jstor.org/stable/10.2307/832998`.

H. Riemann. *Musikalische Syntaxis: Grundriss einer harmonischen Satzbildungslehre.* Leipzig, 1877.

S. Ring. *Tonality and Transformation.* Oxford University Press, 2011.

D. W. Rogers. *Interval-Class Content and Similarity Functions: A Critique of a Critique.* Eastman School of Music, A seminar handout, 1992.

A. Samplaski. Mapping the Geometries of Pitch-Class Set Similarity Measures via Multidimensional Scaling. *Music Theory Online*, 11(2), 2005. URL `http://www.mtosmt.org/issues/mto.05.11.2/mto.05.11.2.samplaski.html`.

A. Schoenberg. *Structural Functions of Harmony.* Norton, New York, 1969 edition, 1954.

D. Scott and E. J. Isaacson. The interval angle: A similarity measure for pitch-class sets. *Perspectives of New Music*, 36(2):107–142, 1998.

R. N. Shepard. The Analysis of Proximities: Multidimensional Scaling With an Unknown Distance Function (I & II). *Psychometrika*, 27(2,3):125–140, 219–246, 1962.

R. N. Shepard. Geometrical approximations to the structure of musical pitch. *Psychological Review*, 89(4): 305–333, 1982. ISSN 1939-1471(Electronic);0033-295X(Print).

D. Stammers. *Set Theory in the Perception of Atonal Pitch Relations.* Phd, Cambridge University, 1994.

R. Teitelbaum. Intervallic Relations in Atonal Music. *Journal of Music Theory*, 9(1):72–127, 1965.

D. Tymoczko. *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice*. Oxford University Press, New York, 2011.

D. Tymoczko. The Generalized Tonnetz. *Journal of Music Theory*, 56(1):1–5, 2012.

G. Tzanetakis and P. Cook. A framework for audio analysis based on classification and temporal segmentation. In *EUROMICRO Conference, 1999. ...*, volume 2, pages 61–67. Ieee, 1999. ISBN 0-7695-0321-7.

G. Weber. *The Theory of Musical Composition*. Cocks, London, 1851.

G. A. Wiggins. Semantic Gap?? Schemantic Schmap!! Methodological Considerations in the Scientific Study of Music. In *2009 11th IEEE International Symposium on Multimedia*, pages 477–482, 2009. ISBN 978-1-4244-5231-6.

G. A. Wiggins. Music, mind and mathematics: theory, reality and formality. *Journal of Mathematics and Music*, 6(2):111–123, July 2012. ISSN 1745-9737.

# A SC Similarity Measures

This chapter contains a concise summary of the SC similarity measures from the literature organised by theorist. Each section specifies the publication in which the measure was proposed and brief description of the theoretical approach adopted by the theorist. A mathmatical formula is given where possible using standard notation. A reference for notation can be found in 3.3.4 and commonly used symbols are defined in the glossary. Where a mathematical formula does suffice, the comparison proceedure is described in words. In addition, each section contains a table specifying important statistics:

- SC-Type: the type of SC the measure compares (Tn or TnI)
- Cardinality: whether the measure can compare SCs of different cardinalities.
- Vector Type: the type of vector used in the comparison procedure (see 3.3).
- Max Similarity: the value that indicates maximum similarity from all possible comparisons.
- Min Similarity: the value that indicates minimum similarity from all possible comparisons.
- Average Value: the average value from all possible comparisons.
- No. Values: the number of distinct values produced from all possible comparisons.
- Criteria Met: a list of Castren's criter which the measure meets.
- I-related: whether the measure descriminates between inversionally related sets.
- Z-related: whether the measure discriminated between Z-related sets.

## A.1 MORRIS

### A.1.1 K

Presented in Morris (1979, pp. 448), the K measure gives the number of intervals-classes (dyad-classes) shared by two SCs, X and Y.

$$K(X,Y) = \sum_{i=1}^{6} MIN(x_i, y_i)$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 55 |
| Min Similarity: | 0 |
| Average Value: | 10 |
| No. Values: | 35 |
| Criteria Met: | C1,C2,C3.3,C3.4,C4 |
| I-related: | No |
| Z-related: | No |

- Problems: scale of values not the same for all value groups.

### A.1.2 SIM

Presented in Morris (1979, pp. 446), SIM compares the ICVs of two SCs (the value is the cardinality of the DV).

$$SIM\left(X,Y\right) = \sum_{i=1}^{6} |x_i - y_i|$$

or

$$SIM\left(X,Y\right) = \#DV\left(ICV\left(X\right), ICV\left(Y\right)\right)$$

SIM is a function of K:

$$SIM(X,Y) = \#ICV(X) + \#ICV(Y) - 2.K(X,Y)$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 0 |
| Min Similarity: | 65 |
| Average Value: | 13 |
| No. Values: | 44 |
| Criteria Met: | C1,C2,C3.3,C3.4,C4 |
| I-related: | No |
| Z-related: | No |

- Problems: scale not the same for all value groups. course resolution when cardinalities differ greatly

### A.1.3 ASIM

Presented in Morris (1979, pp. 450), ASIM (Absolute SIM) is a scaled version of SIM to address criteria C3.1.

$$ASIM\left(X,Y\right)=\frac{SIM\left(X,Y\right)}{\#ICV\left(X\right)+\#ICV\left(Y\right)}$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 0 |
| Min Similarity: | 1 |
| Average Value: | 0.42 |
| No. Values: | 79 |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.4,C4 |
| I-related: | No |
| Z-related: | No |

Problems: Fixed the scale of values, but still coarse resolution when cardinalities differ greatly. Scaling is done as the last step.

## A.2 LORD

### A.2.1 sf

Presented in (Lord, 1981, pp. 93), sf (Similarity Function) is similar to SIM but developed independently. sf is a subset of SIM:

$$sf\left(X,Y\right)=\frac{\#DV\left(ICV\left(X\right),ICV\left(Y\right)\right)}{2}=\frac{SIM(X,Y)}{2}$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Same |
| Vector Type: | ICV |
| Max Similarity: | 0 |
| Min Similarity: | 9 |
| Average Value: | 3 |
| No. Values: | 10 |
| Criteria Met: | C3.3,C3.4,C4 |
| I-related: | No |
| Z-related: | No |

## A.3 TEITELBAUM

### A.3.1 s.i.

Presented in Teitelbaum (1965, pp. 88), s.i. (Similarity Index) is the Euclidean distance between the carteasian coordinates defined by the ICVs of two SCs. This is equivelant to the magnitude of the difference vector.

$$s.i.(X,Y) = \sqrt{\sum_{i=1}^{6}(x_i - y_i)^2} = \|DV(ICV(X), ICV(Y))\|$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Same |
| Vector Type: | ICV |
| Max Similarity: | 1.41 |
| Min Similarity: | 8.49 |
| Average Value: | 2.85 |
| No. Values: | 31 |
| Criteria Met: | C3.3,C3.4 |
| I-related: | No |
| Z-related: | No |

- Same cardinality only
- Z-related sets not compared

## A.4 ROGERS

### A.4.1 $\text{IcVD}_1$

Presented in Rogers (1992), $\text{IcVD}_1$ (Distance Formula 1) is a modification of SIM (A.1.2). The ICV components are scaled before being summed. $\text{IcVD}_1$ is related to Castren's $\%\text{REL}_2$ (A.8.3): $\%\text{REL}_2(\text{X,Y}) = \text{IcVD}_1(\text{X,Y}) \times 50$.

$$IcVD_1(X,Y) = \#DV\left(\frac{ICV(X)}{\#ICV(X)}, \frac{ICV(Y)}{\#ICV(Y)}\right)$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 0 |
| Min Similarity: | 2 |
| Average Value: | 0.59 |
| No. Values: | 140 |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.4,C4 |
| I-related: | No |
| Z-related: | No |

### A.4.2 $\text{IcVD}_2$

Presented in Rogers (1992), $\text{IcVD}_2$ (Distance Formula 2) is similar to s.i. (A.3.1), but instead returns the Euclidean distance between the ends of the normalised ICVs.

$$IcVD_2(X,Y) = \left\| DV(IC\hat{V}(X), IC\hat{V}(Y)) \right\|$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 0 |
| Min Similarity: | 1.41 |
| Average Value: | 0.54 |
| No. Values: | 133 |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.4 |
| I-related: | No |
| Z-related: | No |

- Problems: does not produce uniform values for comparable cases

### A.4.3 Cos($\theta$)

Presented in Rogers (1992), Cos$\theta$, gives the cosine of the angle between the ICVs in six-dimensional euclidean space. As the angle decreases the similarity approaches 1.

$$Cos\theta(X,Y) = \frac{ICV(X) \cdot ICV(Y)}{\|ICV(X)\| \times \|ICV(Y)\|}$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 1 |
| Min Similarity: | 0 |
| Average Value: | 0.81 |
| No. Values: | 92 |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.4 |
| I-related: | No |
| Z-related: | No |

- Problems: C4

## A.5 RAHN

### A.5.1 AK

Presented in /citet[pp. 489]{Rahn1979}, AK is an absolute or adjusted version of Morris' K (A.1.1), addressing the C3.1 criteria. AK is related to Morris' ASIM: AK(X,Y)=1-ASIM(X,Y).

$$AK\left(X,Y\right) = \frac{2K\left(X,Y\right)}{\#ICV\left(X\right) + \#ICV\left(Y\right)}$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 1 |
| Min Similarity: | 0 |
| Average Value: | 0.58 |
| No. Values: | 78 |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.4,C4 |
| I-related: | No |
| Z-related: | No |

- Problems: single scale of values (C4), but poor discrimination for some value groups.

### A.5.2 MEMB$_n$

Presented in Rahn (1979, pp. 492), MEMB$_n$ (Mutual Embedding Number) compares the nCVs of two SCs for one nC at a time. It meausres the mutual embedding of subsets such that only non-zero components of the nCVs contribute. By setting n = 2 (MEMB$_2$) it compares ICVs.

$$MEMB_n\,(X,Y) = \sum_{i=1}^{\#nC} nCV(X)_i + nCV(Y)_i$$

such that nCV(X)$_i$>0 and nCV(Y)$_i$>0.

| | |
|---|---|
| SC Type: | TnI or Tn |
| Cardinality: | Any |
| Vector Type: | nCV |
| Max Similarity: | 121 |
| Min Similarity: | 0 |
| Average Value: | 30 |
| No. Values: | 79 |
| Criteria Met: | C1,C2,C3.3,C3.4,C4 |
| I-related: | Yes* |
| Z-related: | Yes* |

- Problems: does not produce uniform scale of values for all value groups.

### A.5.3 TMEMB

Presented in Rahn (1979, pp. 492), TMEMB (Total Mutual Embedding Number) counts the mutually embedded subsets of every cardinality. TMEMB is a total measure.

$$TMEMB\,(X,Y) = \sum_{n=2}^{12} MEMB_n\,(X,Y)$$

| | |
|---|---|
| SC Type: | TnI or Tn |
| Cardinality: | Any |
| Vector Type: | nCV |
| Max Similarity: | 6118 |
| Min Similarity: | 0 |
| Average Value: | 131 |
| No. Values: | 877 |
| Criteria Met: | C1,C2,C3.3,C4,C5 |
| I-related: | Yes |
| Z-related: | Yes |

- Problems: Different value scales for different value groups

### A.5.4 ATMEMB

Presented in Rahn (1979, pp. 494), ATMEMB (Adjusted Total Mutual Embedding Number) is a scaled version of TMEMB to address criteria C3.1 (like SIM and ASIM; A and AK). ATMEMB is a total measure.

$$ATMEMB\,(X,Y) = \frac{TMEMB\,(X,Y)}{2^{\#X} + 2^{\#Y} - (\#X + \#Y + 2)}$$

| | |
|---|---|
| SC Type: | TnI or Tn |
| Cardinality: | Any |
| Vector Type: | nCV |
| Max Similarity: | 1 |
| Min Similarity: | 0 |
| Average Value: | 0.45 |
| No. Values: | 101 |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.4,C4,C5 |
| I-related: | Yes |
| Z-related: | Yes |

## A.6 ISAACSON

### A.6.1 AMEMB2

Proposed by Isaacson (1990, pp. 8), $\mathrm{AMEMB}_2$ (Adjusted $\mathrm{MEMB}_2$) is a scaled version $\mathrm{MEMB}_2$ (A.5.2), measuring the mutual embedding of ICs.

$$AMEMB_2 = \frac{2 \times MEMB_2(X,Y)}{(\#X\,(\#X-1) + \#Y\,(\#Y-1))}$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 1 |
| Min Similarity: | 0 |
| Average Value: | |
| No. Values: | |
| Criteria Met: | |

### A.6.2 IcVSIM

Presented in Isaacson (1990, pp. 18), IcVSIM (Interval-Class Vector Similarity Relation) is the standard deviation of the entries in the ICVs of two SCs. IcVSIM is a scaled version of s.i. (A.3.1). $\mathrm{IdV}_i$ is the ith term in the vector defined by ICV(X)-ICV(Y) and $\overline{DV}$ is the average (mean) of its entries.

$$IcVSIM(X,Y) = \sqrt{\frac{\sum(IdV_i - \overline{IdV})^2}{6}}$$

| | |
|---|---|
| SC Type | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 0 |
| Min Similarity: | 3.64 |
| Average Value: | 1.2 |
| No. Values: | 121 |
| Criteria Met: | C1,C2,C3.4 |
| I-related: | No |
| Z-related: | No |

### A.6.3 ISIM2

Presented in Isaacson (1996), ISIM2 is a scaled version of IcVSIM (A.6.2). The squre root is taken of each term in the ICVs. Isaacson argues that each additional instance of an IC contributes less to similitude. However, Samplaski (2005) found ISIM2 to be inconsistent with itself when applying MDS to the values produced.

| | |
|---|---|
| SC Type | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | |
| Min Similarity: | |
| Average Value: | |
| No. Values: | |
| Criteria Met: | C1,C2,C3.4 |

### A.6.4 ANGLE (Isaacson & Scott)

Scott and Isaacson (1998) propose a geometric method which is identical to that of Cos/theta (A.4.3) but instead gives the size of the angle in degrees.

$$ANGLE(X,Y) = \arccos Cos\theta(X,Y)$$

| | |
|---|---|
| SC Type | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | |
| Min Similarity: | |
| Average Value: | |
| No. Values: | |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.4 |
| I-related: | No |
| Z-related: | No |

## A.7 LEWIN

### A.7.1 REL

Presented in Lewin (1979), REL compares the nCVs of two SCs for all the nCs. Like $\text{MEMB}_n$ (A.5.2), REL only considers non-zero entries however, this is achieved by multiplication (taking the geometric mean) of corresponding nCV terms.

$$REL(X,Y) = \frac{\sum_{i=1}^{p} \sqrt{SUB(X)_i \times SUB(Y)_i}}{\sqrt{\#SUB(X) \times \#SUB(Y)}}$$

where SUB(X) consists of concatenated nCVs and has a length p.

| | |
|---|---|
| SC Type: | TnI or Tn |
| Cardinality: | Any |
| Vector Type: | nCV |
| Max Similarity: | 1 |
| Min Similarity: | 0 |
| Average Value: | 0.57 |
| No. Values: | 91 |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.4,C4,C5 |
| I-related: | Yes |
| Z-related: | Yes |

### A.7.2 $\text{REL}_2$

Rahn (1979) suggested a number of manifestations of the basic REL concept including $\text{REL}_2$ which measures only intervallic similarity.

$$REL_2(X,Y) \frac{2 \times \sum \sqrt{(x_i y_i)}}{\sqrt{(\#X(\#X-1)\#Y(\#Y-1))}}$$

| | |
|---|---|
| SC Type: | TnI |
| Cardinality: | Any |
| Vector Type: | ICV |
| Max Similarity: | 1 |
| Min Similarity: | 0 |
| Average Value: | |
| No. Values: | |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.4 |

## A.8 CASTREN

### A.8.1 Castren's Difference Vector

Castren specifies a different type of DV, which we shall call cDV to distinguish it from the regular DV. It consistes of two rows, $cDV_x(X,Y) = X - Y$ and $cDV_y(X,Y) = Y - X$. Any negative values in either of the rows are set to zero. In addition Castren defines the weighted difference vector (wcDV) of two vectors X and Y as:

$$wcDV = \frac{cDV(X,Y)}{\#cDV(X,Y)} \times 100$$

### A.8.2 nC%V

Presented in Castrén (1994) for use in %REL$_n$, nC%V(X) (n-class subset percentage vector) gives the percentage subset-class contents of an SC, X. The 2C%V is the Interval percentage vector.

$$nC\%V(X) = \frac{nCV(X)}{\#nCV(X)} \times 100$$

### A.8.3 %REL$_n$

Presented in Castrén (1994), %REL$_n$ (Percentage Relation) is a modification of sf (A.2.1) using the nC%Vs (A.8.2) instead of ICVs. %REL$_n$ can be used as a stand-alone measure, however it is primarily intended as an intermediate step in T%REL and RECREL (A.8.4 and A.8.5).

$$\%REL_n(X,Y) = \frac{\#DV(nC\%V(X), nC\%V(Y))}{2}$$

| | |
|---|---|
| SC Type | TnI or Tn |
| Cardinality: | Any |
| Measure Type: | Single nC |
| Vector Type: | nC%V |
| Max Similarity: | 0 |
| Min Similarity: | 100 |
| Average Value: | 30 |
| No. Values: | 85 |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.3,C3.4,C4 |
| I-related: | Sometimes |
| Z-related: | Sometimes |

### A.8.4 T%REL

Presented in Castrén (1994), T%REL (Total Percentage Relation) is the mean average of the vlaues of %REL$_n$ for all values of $n$ from 2 to $m$ where, if $\#X \neq \#Y$, $m = MIN(\#X, \#Y)$ else $m = \#X - 1$.

$$T\%REL(X,Y) = \frac{\sum_{n=2}^{m} \%REL_n(X,Y)}{m-1}$$

| | |
|---|---|
| SC Type: | TnI or Tn |
| Cardinality: | Any |
| Measure Type: | Total |
| Vector Type: | nC%V |
| Max Similarity: | 0 |
| Min Similarity: | 100 |
| Average Value: | 63 |
| No. Values: | 79 |
| Criteria Met: | C1,C2,C3.1,C3.2,C3.3,C3.4,C4,C5 |
| I-related: | Yes |
| Z-related: | Yes |

### A.8.5 RECREL

Presented in Castrén (1994), RECREL (Recursive Relation) recursively compares the subsets and subsets of subsets of two SCs using %REL$_n$ (A.8.3). The comparison procedure is quite complicated and potentially involves evaluating %REL$_n$ thousands of times.

| | |
|---|---|
| SC Type: | TnI or Tn |
| Cardinality: | Any |
| Measure Type: | Total |
| Vector Type: | nC%V |
| Max Similarity: | 0 |
| Min Similarity: | 100 |
| Average Value: | |
| No. Values: | 89 |
| Criteria Met: | All |
| I-related: | Yes |
| Z-related: | Yes |

## A.9 BUCHLER

### A.9.1 nSATV

Presented in Buchler (1997, chap. 2.3) nSATV(X) (Saturation Vector) is a dual vector consisting of two rows, nSATV$_A$(X) and nSATV$_B$(X). It shows extent to which an SC is saturated with subclasses of cardinality n. The steps for computing nSATV(X) are as follows:

1. Compute the nCVs for all SCs of cardinality #X.
2. Find the minimum and maximum values for each vector position. These values form vectors $Max_n(\#X)$ and $Min_n(\#X)$.
3. Compute the following two vectors: $MaxMinus = DV(nCV(X), Max_n(\#X))$ and $MinPlus = DV(nCV(X), Min_n(\#X))$
4. $nSATV_A(X)_i = MIN(MaxMinus_i, MinPlus_i)$ and $nSATV_B(X)_i = MAX(MaxMinus_i, MinPlus_i)$
5. If $MaxMinus_i = MinPlus_i$, $nSATV_A(X)_i = MaxMinus_i$ and $nSATV_B(X)_i = MinPlus_i$

### A.9.2 SATSIM$_n$

Presented in Buchler (1997, chap. 2.4), SATSIM$_n$ (Saturation Similarity index) compares the nSATVs of two SCs and involves the following steps:

1. Calculate nSATV(X) and nSATV(Y)
2. Calculate the vectors nSATV$_{\text{row}}$(X) and nSATV$_{\text{row}}$(Y).
3. The function "row" maps the MaxMinus values of one nSATV to the MaxMinus values of the other. If nSATV$_A$(X)$_i$ is a MaxMinus value and nSATV$_A$(X)$_i$ is also a MaxMinus value, row = A (nSATV$_{\text{row}}$(X)$_i$ = nSATV$_A$(X)$_i$), otherwise row = B.
4. Finally SATSIM$_n$(X,Y) is given by the formula:

$$SATSIM_n(X,Y) = \frac{\#DV(nSATV_A(X), nSATV_{row}(Y)) + \#DV(nSATV_A(Y), SATV_{row}(X))}{\#DV(nSATV_A(X), SATV_B(X)) + \#DV(SATV_A(Y), SATV_B(Y))}$$

### A.9.3 AvgSATSIM

Presented in Buchler (1997, chap. 2.10), AvgSATSIM (Average Saturation Similarity index) is the mean of SATSIM$_n$ values where $m = MIN(\#X, \#Y)$.

$$AvgSATSIM(X,Y) = \frac{\sum_{n=2}^{m-1} SATSIM_n(X,Y)}{m-2}$$

### A.9.4 TSATSIM

Presented in Buchler (1997, chap. 2.10), TSATSIM (Total Saturartion Vector Similarity index) is an extension of SATSIM$_n$. TSATSIM is the quotient of the sum of all SATSIM$_n$ numerators and denominators for all values of n from 2 to m-1 where $m = MIN(\#X, \#Y)$.