Evaluation of Pitch-Class Set Similarity Measures for Tonal Analysis (Literature Review)

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Contents

1 Introduction

The objectives of the present work are principally concerned with descriptive modelling of tonality using PC-set theory. In particular, the analytical potential of set-class similarity measures will be assessed and evaluated through comparison with similar work. The tentative goal, therefore, is to present this approach in such a way as to make it relevant to the wider community of music researchers, such as those in MIR.

Chapters 2-5 comprise a literature review containing background information and some basic theory. Chapter 2 addresses the challenges and problems involved in descriptive modelling of tonality as a means of justification for the proposed approach. Chapter 3 introduces the basic concepts of PC-set theory including set-class similarity measures. Chapter 4 gives a basic description of MDS techniques and how they can play a role in set-class analysis. Chapter 5 contains a review of the relevant tonal models that exist so as to give context work that follows.

2 Tonality

2.1 Defining Tonality

2.1.1 TODO General Definition

Tonality is a notoriously complex musical phenomenon and numerous definitions have been proposed from a variety of viewpoints. Perhaps the most general definition is that provided by Hyer /citep(Hyer2013): "... refers to the systematic arrangements of pitch phenomena and relations between them." Explanations of tonality have been provided through many different disciplines (acoustics, music theory, linguistics, cognitive psychology) and a detailed discussion of these areas is certainly beyond the scope of this work. However, it is generally agreed that tonality is an abstract cultural and cognitive construct that can have many different physical representations.

2.1.2 Babbitt's Domains

? proposed three domains to categorise different types of representation of music: acoustic (physical), auditory (perceived), graphemic (notated). Western music theory provides a lexicon for describing abstract tonal objects with terms such as note, chord and key. These objects have a hierarchical relationship and the meaning of these labels is highly dependent on musical context and the scale of observation. Musicological descriptions, which constitute the majority of reasoning about tonaility, reside mainly in the Babbitt's graphemic domain, although arguably they reflect some aspects of the other two. Each domain, whilst connected to every other, provides only a projection of the musical whole and examination of tonality from just one will most likley result in an incomplete picture. However, these three domains provide a convenient framework for the discussion that follows.

2.2 Modelling Tonality

The challenge of mathematically modelling aspects of tonality has been approached in numerous ways and from different domains. In the graphemic domain, musicologists and composers have proposed theoretical models, attempting to rethink tonal theory from a mathematical perspective. These models employ different branches of mathematics such as geometry (**?**) or group theory (**?**) to describe harmonic structure. From the auditory domain, cognitive psychologists have built models of tonal induction based on perceptual ratings of tonal stimuli (**?**).

2.2.1 Tonality as Context

Many models approach the concept of tonality as a context, within which the relations and hierarchies of tonal phenomena can be understood. A sense of tonality can be induced when musical stimuli resemble some a priori contextual category. For western music of the major-minor period, key signatures comprise a collection of categories that give context to the tonal components of music. **?** identifies three important aspects of tonality as context: dimensionality (the relatedness of categories), ambiguity (reference to two or more categories simultaneously) and timing (the dynamics of tonal context). He highlights the importance of a models capability to describe these aspects.

2.2.2 Tonality in MIR

The MIR community is primarily concerned with the extraction of tonal descriptors from audio signals such as chord and key estimates. Most systems use chroma features as a preliminary step, obtained by mapping STFT or CQ transform energies to chroma bins. Template matching is used to compare the chroma vectors to a tonal model (contextual category) using some distance measure. A commonly used tonal model for key estimation are the KK-profiles (**?**) (e.g. in **?**). Distance measures such as inner product (e.g. in **?**) and fuzzy distance (e.g. in **?**) are used to compare vectors. Statistical methods, such as HMMs, have been used for chord and key tracking (**?**). Of addition interest in the field is the concept of musical similarity (for music recommendation, structure analysis, cover detection etc.). **?** computed self-similarity matrices for visualisation of structure by correlating the MFCC feature vector time-series. **?** proposed the application of this method to tonal feature vectors.

2.2.3 Similarity

The importance of defining the similarity or closeness between musical phenomena, be it theoretical, physical or perceptual, is central to almost every model of tonality and often leads to a geometric configuration of tonal objects. The concepts of similarity and distance is discussed further in Chapter 5 where a review of spatial models of tonality is given.

2.3 The Semantic Gap

2.3.1 Acoustic Domain

? discusses, what is referred to in MIR as, the "Semantic Gap": the inability of systems to achieve success rates beyond a conspicuous boundary. He examines the fundamental methodological groundings of MIR in terms of Babbitts three domains, discussing the limits of each representation and regarding the discarnate nature of music. He concludes that the audio signal (acoustic domain) simply cannot contain all of the information that systems seek to retrieve. He points towards the the auditory domain as the chief residence of music information and urges for in not to be overlooked in MIR and wider music research.

2.3.2 Graphemic Domain

Furthermore, Wiggins criticises the purely graphemic approach and the tendency of music research to presuppose musicological axioms. **?** argues that music (tonal) theory is, rather than a theory in the scientific sense, a highly developed folk psychology (internal human theory for explaining common behaviour). Thus, the rules of music theory are not like scientific laws but rather abstract descriptions of a specific musical behaviour. This idea challenges the validity of formalising such rules in mathematics and prompts the question, "What is actually being modelled?" He concludes that to apply mathematical models to musical output alone (scales or chords) without consideration of the musical mind is a scientific failure.

2.3.3 Problems

The two assertions of Wiggins sit contrary to a number of the aspects of the tonal models discussed in 2.2. Firstly, the major-minor paradigm, upon which so many approaches are based, whilst certainly possessing cognitive significance, is still a musicological concept and therefore a misleading basis for both mathematical and cognitive approaches. A second problem is that of the numerical methods used by some MIR systems, in particular, distance measures. As will be discussed in Chapter 5, similarity (and by extension distance) is a central part of the auditory domain. MIR systems often uses distance measures from mathematics such as Mahalanobis (**?**) or Cosine (**?**) with little consideration of their perceptual or musical significance.

2.4 Systematicity

2.4.1 The Musical Surface

Having cautioned against a purely musicological approach, **?**, pp. 481 proposes a compromise: to adopt a bottom-up approach to music theory, exploring the concepts through systematic mid-level representations. He states that "methods starting at, for example, the musical surface of notes is a useful way of proceeding" The concept of musical surface is illustrated by **?**, pp. 159 with a metaphor: "…to approach a musical landscape not by drawing a map, which necessarily confines itself to a limited set of structurally important features, but by presenting a bird's-eye view of the musical surface – an aerial photograph, as it were, which details the position of every pitched component."

2.4.2 Systematic Description

? also advocates this mid-level approach, observing that surface description influences analyitcal observation and that, for an unbiased view, the researcher must be provided with the adequate raw materials with which to make more in-depth observation. Such a systematic, descriptive model would be fundamentally independent of high level concepts such as chords and key but, at the same time, capable of capturing them. **?** also discusses the importance of systematicity in terms of dimensionality, ambiguity and timing. He finds that models based on the major-minor paradigm are incapable of adequately describing tonal ambiguity even in some Western music (**?**, chap. 3).

With a systematic description of the musical surface, theories and models from different domains can be gathered and evaluated together in the same analytical arena, thus helping to bridge the gap between traditional musicology, cognitive psychology and MIR.

3 Pitch-Class Set Theory

One such method available for systematic description of the musical surface is Pitch class set theory. PC-set theory is a system for analysing the pitch content of music. It uses class equivalence relations to reduce the amount of data required to describe any sequence of pitches. This chapter will outline the basic principles. A glossary of terms is included in 12 to assist with the number of new terms that will be introduced.

3.1 Pitch Class

Pitch-class set theory uses octave equivalence. In Western equal temperament (TET), a pitch-class (PC) is an integer representing the residue class modulo 12 of a pitch /citep(Babbit1955) and indicates the position of a note within the octave. A PC-set is a collection of PCs ignoring any repetitions and the order in which they occur. PC-sets are notated as follows {0,1,2,3,4} with PCs ordered from lowest to highest as a convention (Example 1). The cardinality of a set, denoted #S, is the number of PCs it contains (Example 2). There are 4096 (2^{12}) unique PC-sets with which any segment of music can be represented.

Example 1:	PC-set	Pitch-set	S = {A4,C5,E5,A5} (A minor)
		PC-set	S = {9,0,4,9} = {0,4,9}
Example 2:	Cardinality		#S = 3

Table 1: Notes and corresponding pitch-classes

Note	C	C#	D	D#	E	F	F#	G	G#	A	A#	B
PC	0	1	2	3	4	5	6	7	8	9	10	11

3.2 Set Classification

Defining equivalence classes of PC-sets further reduces the total number of tonal objects. A set-class (SC) is a group of PC-sets related by a transformation or group of transformations. The two types of transformation commonly used are transposition and inversion. A transposition, Tn(S), transposes the set, S, by the interval, n, (by adding n to all PCs, Example 3). An inversion, I(S), inverts the set S, replacing all PCs with their inverse (12-PC, Example 4). From these two transformations it is possible to define three types of SC: Tn, TnI and I, although I-types are not commonly used.

Example 3:	Transposition	S = {0,4,9}, T3(S) = {3,7,0} = {0,3,7}
Example 4:	Inversion	S = {0,4,9}, I(S) = {11,7,2} = {2,7,11}

Transpositional (Tn):	All PC-sets that can be transformed to each by transposition belong to the same class. There are 348 distinct Tn types.
Inversional (I):	All PC-sets that can be transformed to each other by inversion belong to the same SC. There are 197 distinct I types.
Transpositional/ Inversional (TnI):	All PC-sets that can be transformed to each other by transposition, inversion or both belong to the same SC. There are 220 distinct TnI types.

The Prime Form of a PC-set is a convention for denoting the SC it belongs to. The convention was introduced by Allan Forte (**?**) for TnI types and has since been adopted by the majority of theorists. In addition, he devised a system for ordering TnI-type SCs and assigning to each one a number. For example, the Forte number 3-11 refers to the 11th SC of cardinality 3. This convention has been modified for use with Tn types by adding A and B to the names of inversionally related SCs.

One additional concept is that of cardinality-class (nC), which refers to all the SCs of cardinality n. Cardinality-class 2 is commonly referred to as interval-class (IC) and there are 6 distinct interval-classes.

Table 2: Forte's Prime form and numbering convention

PC-set	{1,4,9}
Prime Form (TnI)	{0,3,7}
Prime Form (Tn)	{0,4,7}
Forte Name (TnI)	3-11
Forte Name (Tn)	3-11B

Table 3: Numbers of objects

Object type	No. Objects
Pitch	88
Pitch set	3e26
PC	12
PC-set	4096
Tn-Type SC	348
I-Type SC	197
TnI-Type SC	220

3.3 Vector Analysis

3.3.1 Membership and Inclusion

Two concepts that are crucial in PC-set theory are membership and inclusion. Membership of a set is denoted p $\in$ S and means that PC p is a member of set S (Example 5). Inclusion in a set is denoted Q $\subset$ S and means that all members of Q are also members of set S (Example 6). Q is said to be a subset of S.

Example 5:	Membership	$4 \in \{0,4,9\}$
Example 6:	Inclusion	$\{0,4,9\} \subset \{0,1,4,5,9\}$

3.3.2 Embedding Number

? applied these concepts to SCs to develop his Embedding Number, EMB(X,Y). Given two SCs, X and Y, EMB(X,Y) is the number of instances of SC, X, which are included in (are subsets of) SC, Y (Example 7). X is ring-shifted 11 times and each unique resulting set which is included in Y adds one to the embedding number.

Example 7:	Embedding Number	X = {0,4} and Y = {0,4,8} so EMB(X,Y) = 3

Table 4: Cardinality Class

nC	Tn	I	TnI
1C	1	1	1
2C	6	6	6
3C	19		
4C			
5C			
6C			
7C			
8C			
9C	19		
10C	6		
11C	1		
12C	1		

3.3.3 Subset Vectors

An n-class subset vector of X, nCV(X), is an array of values of EMB(A,X) where A is each of the SCs in the cardinality-class, nC (Example 8). The Interval-Class Vector (ICV) is a special instance of the nCV with n equal to 2. Vector cardinality, denoted #nCV(X), is the sum of all the terms (Example 9). The length of a subset vector is given by #nC.

Subset vectors form the basis of the majority of analysis performed by PC-set theorists. In addition, many theorists have proposed modifications to the basic nCV to suit their specific purposes and some of these modifications will be discussed in context where necessary.

Example 8:	Subset Vector	S = {0,4,9}
		2CV(S) = ICV(S) = [0 0 1 1 1 0]
Example 9:	Vector Cardinality	#ICV(S) = 0+0+1+1+1+0 = 3

3.3.4 Notation

- Difference Vector

$$DV(nCV(X), nCV(Y)) = |nCV(X) - nCV(Y)|$$

- Magnitude
  - length vector

$$\|nCV(X)\| = \sqrt{\sum_{i=1}^{\#nC} (nCV(X)_i)^2}$$

- Unit Vector

$$n\hat{CV}(X) = \frac{nCV(X)}{\|nCV(X)\|}$$

- Euclidean Distance

$$d(X,Y) = \sqrt{\sum_{i=1}^{n}(X_i - Y_i)^2} = \|DV(X,Y)\|$$

- Geometric Mean

$$GM(x,y) = \sqrt{x \times y}$$
$$GMV(nCV(X), nCV(Y)) = []$$

3.4 Set-Class Similarity

3.4.1 Similarity Relations

The assessment of similarity between two SCs has been discussed in the literature for decades and a large number theoretical models have been proposed. Different models approaches the problem from different conceptual standpoints and theorists have different opinions about the contributing factors. All these models are described under the blanket term “similarity relations”. Despite the perennial fascination with the concept, little or no consensus exits as to what constitutes a good similarity relation.

? provides a comprehensive and in-depth review of a large number of similarity relations and categorises them according to some fundamental principles. Firstly, he distinguishes between methods that produce binary outcomes and those that produce a range of values. The former category, termed “plain relations”, include Forte’s R-relations (**?**) and indicate whether the two SCs are related in a specific way, which in turn may give some indication of whether they are similar. The latter category, termed “similarity measures”, indicate a degree of similarity, returning a value from a known range. This property appears to be more inline with the perceptual notion of similarity and therefore the focus of this work shall be exclusively on similarity measures.

3.4.2 Similarity Measures

The vast number and diversity of the different approaches to similarity measures can only be approached by narrowing the focus to a specific type. Here we will focus on measures that use the Tn and TnI-type SCs (3.2), and furthermore we will only consider those methods based on vector analysis (3.3). These measure usually involve the comparison of the SC’s nCVs. Of this (still sizeable) subset, **?** identifies two main categories.

Single nC:	Single nC measures compare the nCVs of the two SCs for one particular value of n. Many of the relations in this category compare the ICVs (2CVs).
Total Measures:	Total Measures consider the subsets of all cardinalities contained within in two SCs. All the relevant nCVs are compared to produce a final value.

Table 4 shows the majority of the Tn and TnI-Type, vector based similarity measures from the PC-set theoretical literature. Vector Type indicates whether the measure compares ICVs or nCVs. Card (Cardinality) indicates whether the measure is capable of comparing SCs of different cardinalities while the Measure Type indicates which of Castren's categories it belongs to. nC indicates it is a Single nC measure and TOTAL indicates it is a Total Measure. All these measure are described more thoroughly in Appendix ? (NOT INCLUDED FOR PEER REVIEW).

3.4.3 Castrens Criteria

In addition to his categorisation, **?** proposes several criteria which a good similarity relation should meet. Later, these criteria will be used in assessing the specific capabilities of various similarity measures.

Castren says that a similarity measure should:

- C1: allow comparisons between SCs of different cardinalities
- C2: provide a distinct value for every pair of SCs
- C3: provide a comprehensible scale of values such that
  - C3.1: All values are commensurable
  - C3.2: the end points are not just some extreme values but can be meaningfully associated with maximal and minimal similarity.
  - C3.3: The values are integers or other easily manageable numbers
  - C3.4: the degree of discrimination is not too coarse and not unrealistically fine
- C4: produce a uniform value for all comparable cases
- C5: observe mutually embeddable subset-classes of all meaningful cardinalities
- C6: observe also the mutual embeddable subset-classes not in common between the SCs being compared.

Table 5: Comparison table of similarity measures

THEORIST	SIMILARITY MEASURE	VECTOR TYPE	CARD	MEASURE TYPE
	K	ICV	SAME	nC
	SIM	ICV	SAME	nC
MORRIS	ASIM	ICV	ANY	nC
LORD	sf	ICV	SAME	nC
TEITELBAUM	s.i.	ICV	SAME	nC
	IcVD1	ICV	ANY	nC
	IcVD2	ICV	ANY	nC
ROGERS	COS	ICV	ANY	nC
	AMEMB2	ICV	ANY	nC
	IcVSIM	ICV	ANY	nC
	ISIM2	ICV	ANY	nC
ISAACSON	ANGLE	ICV	ANY	nC
	AK	ICV	ANY	nC
	MEMBn	nCV	ANY	nC
	TMEMB	nCV	ANY	TOTAL
RAHN	ATMEMB	nCV	ANY	TOTAL
	REL2	ICV	ANY	nC
LEWIN	REL	nCV	ANY	TOTAL
	%RELn	nC%V	ANY	nC
	T%REL	nC%V	ANY	TOTAL
CASTREN	RECREL	nC%V	ANY	TOTAL
	SATSIM	nSATV	ANY	nC
	CSATSIM	CSATV	ANY	nC
	TSATSIM	nSATV	ANY	TOTAL
BUCHLER	AvgSATSIM	nSATV	ANY	TOTAL

3.5 Perceptual Relevance

The many equivalence relations used in PC-set theory give rise to a highly abstract description of musical objects. Thus, an important question to be asked is whether these theoretical assumptions and models of similarity reflect perceptual equivalence. Here some relevant studies are discussed.

3.5.1 Octave Equivalence

Pitch is a percept that derives from a particular harmonic structure and is roughly proportional to the logarithm of the fundamental frequency. This allows pitch to be modelled as a straight line. Music psychologists have observed a strong perceptual similarity between pitches with fundamental frequencies in the ratio of 2:1. This property of octave similarity leads the straight line model of pitch to be bent into a helix. Division of the octave into a number of categories is thought to offer a more efficient cognitive representation in memory and thus confers evolutionary advantage. The resulting pitch equivalence classes are implicitly learned through exposure at an early age. TET has 12 pitch equivalence classes which, in PC-set theory, are modelled as a circular projection of the pitch helix. Thus the two most fundamental components of PC-set theory, i.e. octave equivalence and pitch-class labelling, would appear to have a solid basis in perception.

? investigated the perceived similarity of pairs of chords with varying numbers of octave related pitches. He found that in general chords with identical PC contents were perceived as more similar than chords with near identical PC contents, regardless of the octave. However, in further studies he his findings suggest that there are other factors that play a significant role (**?**).

3.5.2 Set-Class Equivalence

Some researchers have attempted to examine whether there is perceived equivalence between different manifestations of a PC-set. **?** presented subjects with sequences of tones derived by transforming two different PC-sets. They noted that subjects were able to distinguish between the different sets both in neutral and musical contexts.

? investigated the perceptual similarity of different PC-sets derived from the same set class under TnI classification. Subjects were presented with three-note melodies and asked to judge which was equivalent to a reference melody. Some melodies preserved the SC identity whilst others did not. She found transpositions to be perceived more similar than inversions and in addition she discovered that the order of the notes and melodic contour was a strong factor in perceived similarity.

Some authors have questioned the perceptual relevance of using TnI equivalence as a basis for set classification. **?** seems unconvinced by evidence for the perceptual similarity of inverted intervals. This can be illustrated by the example of major and minor triads which, while perceptually distinct, are equivalent under TnI.

3.5.3 Perceived vs Theoretical Similarity

A number of studies have been done to ascertain the connection between perceptual similarity ratings and the theoretical values obtained from some PC-set similarity measures. A large number of relevant studies are summarised by **?** and the most significant ones are mentioned here.

? used multidimensional scaling on subjects' similarity ratings between trichords and tetrachords and on the similarity values obtained from SIM. She compared the 2-dimensional solutions and found there to be little correlation.

? investigated non-traditional chords. He compared subjects' ratings with similarity assessments calculated from Forte's R-relations and Lord's sf. He also concluded there was little correspondence between the two.

? compared subjects' ratings of 4 note melodies with the theoretical values obtained from SIM. She found the ratings of subjects with more musical training to be more correlated with the SIM values.

? compared subjects' ratings of pitch sequences with corresponding values of seven ICV-based similarity measures: ASIM, MEMB2, REL2, s.i., IcVSIM and AMEMB2 and concluded there to be a strong relation.

? compared subjects' ratings of pentachords with the values obtained from 9 similarity measures. He found there to be a connection between aurally estimated ratings and the theoretical values and concluded that the abstract properties of set-classes do have some perceptual relevance. He also comments on the way in which this kind of study is conducted, suggesting that the way in which subjects are presented with the stimuli has a significant effect on the outcome.

3.6 PC-set Theory for Analysis

PC-set theory as means for descriptive modelling of tonality is not widely known outside of highly theoretical circles and the use of PC-set similarity measures seems mainly restricted to the theorists who proposed them (for example, **?**). The basic premise is simple: a musical piece is segmented and each segment described by its SC. Similarity measures can be used to assess the similarity between segments or between a segment and some reference SC.

? used a pentachordal tail segmentation policy (each successive note defines a segment that includes the preceding four notes) and compared these segments to comparison sets 7-1 (chromaticism) and 7-35 (diatonicism) using the REL distance (**??**. They claim that the visual results of their analysis "reflect pertinent aspects of our listening experience" (**?**, pp. 204).

?, chap. 5.3 uses a more systematic approach to segmentation using multiple time scales. He proposes the class-scape, a two-dimensional visualisation of a piece of music with time on the x-axis and segmentation time-scale on the y-axis. A single SC can be represented by highlighting the segment or alternatively each segment can be shaded according to its REL distance from a comparison SC. He emphasises that class-scape is an exploratory tool rather than an automated analysis system.

4 Multidimensional Scaling

Multidimensional scaling (MDS) is a numerical visualisation technique that, given a matrix of pairwise distances between objects, provides a geometric configuration of the objects in some abstract space. It provides an efficient means of observing relationships in large, complex data sets and the resulting dimensions often give valuable insight into the data as a whole.

4.1 Non-Metric MDS

Non-Metric MDS was described by **?** and it assumes that the distance matrix values are related to points in an abstract N-dimensional Euclidean space. An important consideration is that of the dimensionality of the solution. For comprehension and visualisation it is important to minimise the number of dimensions however, there is a trade-off between the number of dimensions and the accuracy of the model. For a given dimensionality, we obtain two values: Stress and r^2.

Stress	Stress is a "goodness of fit" measure which characterises the distortion that occurs in a given number of dimensions. As the number of dimensions increases the stress decreases.
r^2	r^2 is the percentage variability of the data being explained by the solution

By plotting stress against r^2 for a number of dimensionalities is possible to observe the point at which additional dimensions do not significantly improve the solution (the "elbow"). Ultimately, the choice of dimensions should be based on interpretation.

4.2 Cluster Analysis

Cluster analysis (CA) is method for dealing with dimensions that are highly separable. First, the most similar pair of objects are selected and grouped together in a cluster. The process is repeated, creating a binary tree structure. The distance between objects is then related to their separation along the branches of the tree.

4.3 MDS with Similarity Measures

Using MDS on the values produced by similarity measures is one way to approach an understanding of the constricts they are measuring. There are two potentially interesting issues to consider. Firstly, a measure may be inconsistent with itself, meaning that the geometries it produces are not "robust" (changing the set of objects changes the distances between the original set). This kind of problem cannot be observed through inspection of the values alone. The second issue is that two different measures that are both self-consistent may produce very different geometries from the same group of SCs. The question then is, what exactly do the measures measure?

5 Spatial Models of Tonality

5.1 Similarity and Distance

Judgements of similarity form the basis of many cognitive processes including the perception of tonality. Similarity between two objects is often conceived as being inversely related to distance between them in geometric space. For example, some tonal objects (chords, for example) are perceived as close to one another whereas others are further apart. In addition, the number of dimensions of the geometric space is in connection with the number of independent properties that are relevant for similarity comparisons. **?** suggests that humans are naturally predisposed to create spatial cognitive representations of perceptual stimuli due to the geometric nature of the world we have evolved to inhabit. Therefore spatial modelling of tonality, as well as helping to visualise the complex multidimensional relationships between tonal phenomena, has the potential to reflect cognitive aspects of the way they are perceived.

5.2 Spatial Representations

Throughout history theorists have proposed many spatial representations of tonality from different domains. From the graphemic domain, **?** and **?** both proposed simple 2-dimensional charts to display the proximity between keys. For representation of chords, **?** models major and minor triads as regions in a 2-dimensional space whilst **?** proposes a variety high dimensional, non-euclidean chord spaces that reflect the theoretical principles of voice leading. From the acoustic domain, **?** proposes a five-dimensional model to represent interval relations between pitches. Some theorists have attempted to incorporate relations between several levels of tonal hierarchy into one configuration. The "spiral array" of **?** is a three-dimensional mathematical model which simultaneously captures the relations between pitches, chords and keys. The "chordal-regional space" of **?** models the relations between chords within a certain key.

5.3 Cognitive Psychology

The auditory domain has been addressed through cognitive psychology by **?** who used the probe-tone methodology (**?**) to establish major and minor key profiles (12-dimensional vectors containing the perceptual stability ratings of each of the 12 pitch classes within a major or minor context). These profiles, know as Krumhansl-Kessler profiles (KK-profiles), show the hierarchy of pitches in major and minor keys. Correlating each of the 24 major and minor profiles produced a matrix of pairwise distances which was fed to a dimensional scaling algorithm. The resulting geometrical solution was found to have a double circular property (circle of fifths and relative-parallel relations) which can be modelled as the surface a 3D torus. Many spatial models of tonality have this double circular property whether it is implicit (**??**) or stated explicitly (**?**).

5.4 Set-Class Spaces

Most of these models are limited to description of music in the major-minor paradigm and are not capable of generalising beyond the "western common practice". PC-set theory, once again, provides a possible means to generalise to any kind of pitch-based music. By considering a collection of tonal objects described by SCs, a geometric space can be constructed to model their relations based on some theoretical principle. Some PC-set theorists have proposed explicit geometric spaces to model relations between SCs. The distances in these spaces are expressed by models of similarity based on voice leading (**??**) or ICVs and the Fourier transform (**??**). However, these models are only designed to represent SCs of one cardinality-class at a time and cannot model the relations between arbitrary collections of pitches.

Alternative spatial models are provided by the implicit geometries of the values produced by the SC similarity measures discussed in **??**. As mentioned in 4.3, MDS can be used on values produced by similarity measure to create a geometric space. **?** and **?** both applied MDS to the values produced from a variety of similarity measures. Samplaski used TnI-type SCs while Kuusi used Tn-type. They both found reasonably low-dimensional solutions and attempted to interpret each of the dimensions. Kuusi interpreted three dimensions as corresponding to chromaticism, wholetoneness and pentatonicism. Samplaski made similar observations but found some dimensions in the higher-dimensional spaces difficult to interpret. Nevertheless, he concluded that values from similarity measure tend to agree (with some exceptions) and that they measure constructs relating to familiar scales (diatonic, hexatonic, octatonic, etc.).

6 Similarity Measure Selection

In this section, the similarity measures will be discussed in relation to Castren's criteria in order to gauge their suitability for use in systematic surface description models. The most suitable measures will be subjected to a more thorough analysis over the course of the work.

6.1 Criteria

?? shows the various criteria met by each of the similarity measures.

Table 6: Castrens Criteria

SIMILARITY MEASURE	C1	C2	C3.1	C3.2	C3.3	C3.4	C4	C5	C6
s.i.					X	X			
sf					X	X	X		
IcVSIM	X	X				X			
ISIM2	X	X				X			
K	X	X			X	X	X		
SIM	X	X			X	X	X		
MEMBn	X	X			X	X	X		
AMEMB2	X	X	X						
ASIM	X	X	X	X		X	X		
IcVD1	X	X	X	X		X	X		
IcVD2	X	X	X	X		X			
COS	X	X	X	X		X			
ANGLE	X	X	X	X		X			
AK	X	X	X	X		X	X		
SATSIM	X	X	X						
CSATSIM	X	X	X						
REL2									
%RELn	X	X	X	X	X	X	X		
TMEMB	X	X			X		X	X	
ATMEMB	X	X	X	X		X	X	X	
TSATSIM	X	X	X	X		X		X	
AvgSATSIM	X	X	X	X		X		X	
REL	X	X	X	X		X	X	X	
T%REL	X	X	X	X	X	X	X	X	
RECREL	X	X	X	X	X	X	X	X	X

6.2 Cardinality

Measures which fail to meet criteria C1, i.e. that cannot compare SCs of different cardinalities, are clearly inadequate for systematic analysis of music, which might require the comparison of any two arbitrary segments regardless of how many PCs they contain. Both s.i. (11.3.1) and sf (11.2.1) were proposed specifically for SCs of the same cardinality and so will be excluded from further discussion. Some other measures which were intended to compare SCs of different cardinalities nonetheless have problems. Measures such as SIM (11.1.2) and K (11.1.1) give unintuitive values when the cardinalities of the SCs being compared differ greatly and, in addition, the range of values produced depends on the cardinality of the sets (failure to meet criteria C3.1). Measures of this type will also be excluded.

6.3 SC-type

An important consideration when using similarity measures is the type of SC being compared. Many of the measures are designed for comparison of TnI-type SCs, however, owing to issues riased in 3.5 regarding the perceptual relevance of invertionally related sets, here, measures will be selected for use with Tn-type SCs. This means that the measure should be able to discrimate between inversionally related sets. All the measures which exclusivly consider interval content in the comparison procedure can automatically be discounted as inversionally related sets have identical ICVs.

6.4 Measure Type

Although many theorists have suposed that dyad-class subsets are of paramount importance in similarity judgments, no thorough investigation has been carried out as to the exact perceptual significance of subset cardinality. Single nC measures presupose that subsets of one particular cardinality contribute to similarity above all others. In the interest of systematicity, we will not make this assumption instead assuming that subsets of all cardinalities are relevant and should be considered. Similarity measures that consider all subset cardinalities meet criteria C5 and are total measures. These total measures will selected for further investigation throughout the work.

6.5 Trivial Forms

- what
- why: systematicity
- what values

7 Total Measures

- in this section
- measures will be adjusted like kuusi
-

7.1 Rahn: ATMEMB

- ATMEMB 11.5.4
- Differentiates between Z-related pairs and I related
- Problems: Castren says: "divisor term is flawed... high degrees of similarity between SCs of clearly different cardinalities. General reliability and usefulness is difficulty to determine"
- trivial forms:

Max Similarity:	1
Min Similarity:	0
Average Value:	0.45
No. Values:	101
Criteria Met:	C1,C2,C3.1,C3.2,C3.4,C4,C5
I-related:	Yes
Z-related:	Yes

7.2 Lewin: REL

- REL 11.7.1
- discrimination
- problems
- two formations:
- trivial forms:

7.3 Castren: RECREL

- RECREL 11.8.5
- discrimination
- problems
- trivial forms

7.4 Buchler: AvgSATSIM and TSATSIM

- AvgSATSIM 11.9.4
- TSATSIM 11.9.5
- discrimination
- problems
- trivial froms

7.5 Trivial Forms

Trivial forms are the sets

- What are trivial forms
- typically not considered by theorist
- for systematicity
- what values do they give
- what values should they give

8 Class Scape

9 MDS

10 Structure

11 SC Similarity Measures

11.1 MORRIS

11.1.1 K

Presented in **?**, pp. 448, the K measure gives the number of intervals-classes (dyad-classes) shared by two SCs, X and Y.

$$K(X,Y) = \sum_{i=1}^{6} MIN(x_i, y_i)$$

SC Type:	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	55
Min Similarity:	0
Average Value:	10
No. Values:	35
Criteria Met:	C1,C2,C3.3,C3.4,C4
I-related:	
Z-related:	

- Problems: scale of values not the same for all value groups.

11.1.2 SIM

Presented in **?**, pp. 446, SIM compares the ICVs of two SCs (the value is the cardinality of the DV).

$$SIM\left(X,Y\right)=\sum_{i=1}^{6}\left|x_i-y_i\right|$$

or

$$SIM\left(X,Y\right)=\#DV\left(ICV\left(X\right),ICV\left(Y\right)\right)$$

SIM is a function of K:

$$SIM(X,Y)=\#ICV(X)+\#ICV(Y)-2.K(X,Y)$$

SC Type:	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	0
Min Similarity:	65
Average Value:	13
No. Values:	44
Criteria Met:	C1,C2,C3.3,C3.4,C4

- Problems: scale not the same for all value groups. course resolution when cardinalities differ greatly

11.1.3 ASIM

Presented in **?**, pp. 450, ASIM (Absolute SIM) is a scaled version of SIM to address criteria C3.1.

$$ASIM\left(X,Y\right)=\frac{SIM\left(X,Y\right)}{\#ICV\left(X\right)+\#ICV\left(Y\right)}$$

SC Type:	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	0
Min Similarity:	1
Average Value:	0.42
No. Values:	79
Criteria Met:	C1,C2,C3.1,C3.2,C3.4,C4

Problems: Fixed the scale of values, but still coarse resolution when cardinalities differ greatly. Scaling is done as the last step.

11.2 LORD

11.2.1 sf

Presented in (**?**, pp. 93), sf is similar to SIM but developed independently. sf is a subset of SIM:

$$sf\left(X,Y\right) = \frac{\#DV\left(ICV\left(X\right),ICV\left(Y\right)\right)}{2} = \frac{SIM(X,Y)}{2}$$

SC Type:	TnI
Cardinality:	Same
Vector Type:	ICV
Max Similarity:	0
Min Similarity:	9
Average Value:	3
No. Values:	10
Criteria Met:	C3.3,C3.4,C4

11.3 TEITELBAUM

11.3.1 s.i.

Presented in **?**, pp. 88, s.i. (Similarity Index) is the Euclidean distance between the carteasian coordinates defined by the ICVs of two SCs. This is equivelant to the magnitude of the distance vector.

$$s.i.(X,Y) = \sqrt{\sum_{i=1}^{6}(x_i - y_i)^2} = \|DV(ICV(X),ICV(Y))\|$$

SC Type:	TnI
Cardinality:	Same
Vector Type:	ICV
Max Similarity:	1.41
Min Similarity:	8.49
Average Value:	2.85
No. Values:	31
Criteria Met:	C3.3,C3.4
I-related:	No
Z-related:	No

- Same cardinality only
- Z-related sets not compared

11.4 ROGERS

11.4.1 IcVD$_1$

Presented in **?**, IcVD$_1$ (Distance Formula 1) is a modification of SIM (11.1.2). The ICV components are scaled before being summed. IcVD$_1$ is related to Castren's %REL$_2$ (11.8.3): %REL$_2$(X,Y) = IcVD$_1$(X,Y)$\times$ 50.

$$IcVD_1(X,Y) = \#DV\left(\frac{ICV(X)}{\#ICV(X)}, \frac{ICV(Y)}{\#ICV(Y)}\right)$$

SC Type:	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	0
Min Similarity:	2
Average Value:	0.59
No. Values:	140
Criteria Met:	C1,C2,C3.1,C3.2,C3.4,C4

11.4.2 IcVD$_2$

Presented in **?**, IcVD$_2$ (Distance Formula 2) is similar to s.i., but instead returns the Euclidean distance between the ends of the normalised ICVs.

$$IcVD_2(X,Y) = \left\| DV(IC\hat{V}(X), IC\hat{V}(Y)) \right\|$$

SC Type:	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	0
Min Similarity:	1.41
Average Value:	0.54
No. Values:	133
Criteria Met:	C1,C2,C3.1,C3.2,C3.4

- Problems: does not produce uniform values for comparable cases

11.4.3 Cos(θ)

Presented in **?**, Cosθ, again treating the ICVs as vectors in 6-dimensional space, gives the cosine of the angle between them. As the angle decreases the similarity approaches 1.

$$Cos\theta(X,Y) = \frac{ICV(X) \cdot ICV(Y)}{\|ICV(X)\| \times \|ICV(Y)\|}$$

SC Type:	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	1
Min Similarity:	0
Average Value:	0.81
No. Values:	92
Criteria Met:	C1,C2,C3.1,C3.2,C3.4

- Problems: C4

11.5 RAHN

11.5.1 AK

Presented in /citet[pp. 489]{Rahn1979}, AK is an absolute or adjusted version of Morris' K (11.1.1), addressing the the C3.1 criteria. AK is related to Morris' ASIM: AK(X,Y)=1-ASIM(X,Y).

$$AK\left(X,Y\right) = \frac{2K\left(X,Y\right)}{\#ICV\left(X\right) + \#ICV\left(Y\right)}$$

SC Type:	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	1
Min Similarity:	0
Average Value:	0.58
No. Values:	78
Criteria Met:	C1,C2,C3.1,C3.2,C3.4,C4

- Problems: single scale of values (C4), but poor discrimination for some value groups.

11.5.2 MEMB$_n$

Presented in **?**, pp. 492, MEMB$_n$ (Mutual Embedding Number) compares the nCVs of two SCs for one nC at a time. It meausres the mutual embedding of subsets such that only non-zero components of the nCVs contribute. By setting n = 2 ($MEMB_2$) it compares ICVs.

$$MEMB_n\left(X,Y\right) = \sum_{i=1}^{\#nC} nCV(X)_i + nCV(Y)_i$$

such that $nCV(X)_i > 0$ and $nCV(Y)_i > 0$.

SC Type:	TnI or Tn
Cardinality:	Any
Vector Type:	nCV
Max Similarity:	121
Min Similarity:	0
Average Value:	30
No. Values:	79
Criteria Met:	C1,C2,C3.3,C3.4,C4

- Problems: does not produce uniform scale of values for all value groups.

11.5.3 TMEMB

Presented in **?**, pp. 492, TMEMB (Total Mutual Embedding Number) counts the mutually embedded subsets of every cardinality. TMEMB is a total measure.

$$TMEMB\left(X,Y\right) = \sum_{n=2}^{12} MEMB_n\left(X,Y\right)$$

SC Type:	TnI or Tn
Cardinality:	Any
Vector Type:	nCV
Max Similarity:	6118
Min Similarity:	0
Average Value:	131
No. Values:	877
Criteria Met:	C1,C2,C3.3,C4,C5

- Problems: Different value scales for different value groups

11.5.4 ATMEMB

Presented in **?**, pp. 494, ATMEMB (Adjusted Total Mutual Embedding Number) is a scaled version of TMEMB to address criteria C3.1 (like SIM and ASIM; A and AK). ATMEMB is a total measure.

$$ATMEMB\left(X,Y\right) = \frac{TMEMB\left(X,Y\right)}{2^{\#X} + 2^{\#Y} - \left(\#X + \#Y + 2\right)}$$

SC Type:	TnI or Tn
Cardinality:	Any
Vector Type:	nCV
Max Similarity:	1
Min Similarity:	0
Average Value:	0.45
No. Values:	101
Criteria Met:	C1,C2,C3.1,C3.2,C3.4,C4,C5
I-related:	Yes
Z-related:	Yes

11.6 ISAACSON

11.6.1 AMEMB2

Proposed by **?**, pp. 8, AMEMB_2 is a scaled version MEMB_2 (11.5.2), measuring the mutual embedding of ICs.

$$AMEMB_2 = \frac{2 \times MEMB_2(X,Y)}{(\#X\,(\#X-1) + \#Y\,(\#Y-1))}$$

SC Type:	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	1
Min Similarity:	0
Average Value:	
No. Values:	
Criteria Met:	

11.6.2 IcVSIM

Presented in **?**, pp. 18, IcVSIM (Interval-Class Vector Similarity Relation) is the standard deviation of the entries in the ICV of two SCs. IcVSIM is a scaled version of s.i. (11.3.1). IdV_i is the ith term in the vector defined by ICV(X)-ICV(Y) and $\overline{DV}$ is the average (mean) of its entries.

$$IcVSIM(X,Y) = \sqrt{\frac{\sum(IdV_i - \overline{IdV})^2}{6}}$$

SC Type	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	0
Min Similarity:	3.64
Average Value:	1.2
No. Values:	121
Criteria Met:	C1,C2,C3.4

11.6.3 ISIM2

Presented in **?**, ISIM2 is a scaled version of IcVSIM (11.6.2). The squre root is taken of each term in the ICVs. Isaacson argues that each additional instance of an IC contributes less to similitude. However, **?** found ISIM2 to be inconsistent with itself when applying MDS to the values produced.

SC Type	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	
Min Similarity:	
Average Value:	
No. Values:	
Criteria Met:	C1,C2,C3.4

11.6.4 ANGLE (Isaacson & Scott)

? propose a geometric method which is identical to that of Cos/theta (11.4.3) but instead gives the size of the angle in degrees.

$$ANGLE(X, Y) = \arccos Cos\theta(X, Y)$$

SC Type	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	
Min Similarity:	
Average Value:	
No. Values:	
Criteria Met:	C1,C2,C3.1,C3.2,C3.4

11.7 LEWIN

11.7.1 REL

Presented in **?**, REL compares the nCVs of two SCs for all the nCs. Like MEMB$_n$ (11.5.2), REL only considers non-zero entries however, in this is achieved by multiplication (taking the geometric mean) of corresponding nCV terms.

$$REL(X, Y) = \frac{\sum_{i=1}^{p} \sqrt{SUB(X)_i \times SUB(Y)_i}}{\sqrt{\#SUB(X) \times \#SUB(Y)}}$$

$$REL(X, Y) = \frac{\#GMV(SUB(X), SUB(Y))}{GM(\#SUB(X), \#SUB(Y))}$$

where SUB(X) is concatenated nCVs from nCs 2-12, and has a length p.

SC Type:	TnI or Tn
Cardinality:	Any
Vector Type:	nCV
Max Similarity:	1
Min Similarity:	0
Average Value:	0.57
No. Values:	91
Criteria Met:	C1,C2,C3.1,C3.2,C3.4,C4,C5
I-related:	Yes
Z-related:	Yes

11.7.2 REL$_2$

? suggested a number of manifestations of the basic REL concept including REL$_2$ which measures only intervallic similarity.

$$REL_2(X,Y)\frac{2 \times \sum \sqrt{(x_i y_i)}}{\sqrt{(\#X(\#X-1)\#Y(\#Y-1))}}$$

SC Type:	TnI
Cardinality:	Any
Vector Type:	ICV
Max Similarity:	1
Min Similarity:	0
Average Value:	
No. Values:	
Criteria Met:	C1,C2,C3.1,C3.2,C3.4

11.8 CASTREN

11.8.1 Castren's DV

RECREL requires a different type of DV, which we shall call cDV to distinguish it from the regular DV. It consistes of two rows, $cDV_x(X,Y) = X - Y$ and $cDV_y(X,Y) = Y - X$. Any negative values in either of the rows are set to zero. In addition Castren defines the weighted difference vector (wcDV) of two vectors X and Y as:

$$wcDV = \frac{cDV(X,Y)}{\#cDV(X,Y)} \times 100$$

11.8.2 nC%V

Presented in **?** for use in %REL$_n$, nC%V(X) (n-class subset percentage vector) gives the percentage subset-class contents of an SC X. The 2C%V is the Interval percentage vector.

$$nC\%V(X)_i = \frac{nCV(X)_i}{\#nCV(X)} \times 100$$

11.8.3 %REL$_n$

Presented in **?**, %REL$_n$ (Percentage Relation) is a modification of sf (11.2.1) using the nC%Vs (11.8.2) instead of ICVs. %REL$_n$ can be used as a stand-alone measure, however it is primarily an intermediate step in RECREL (11.8.5).

$$\%REL_n(X,Y) = \frac{\#DV(nC\%V(X), nC\%V(Y))}{2}$$

SC Type	TnI or Tn
Cardinality:	Any
Measure Type:	Single nC
Vector Type:	nC%V
Max Similarity:	0
Min Similarity:	100
Average Value:	30
No. Values:	85
Criteria Met:	C1,C2,C3.1,C3.2,C3.3,C3.4,C4
I-related:	Sometimes
Z-related:	

11.8.4 T%REL

Presented in **?**, T%REL (Total Percentage Relation) is the mean average of the vlaues of %REL$_n$ for all values of n from 2 to m where, if $\#X \neq \#Y$, $m = MIN(\#X, \#Y)$ else $m = \#X - 1$.

$$T\%REL(X,Y) = \frac{\sum_{n=2}^{m} \%REL_n\,(X,Y)}{m-1}$$

SC Type:	TnI or Tn
Cardinality:	Any
Measure Type:	Total
Vector Type:	nC%V
Max Similarity:	0
Min Similarity:	100
Average Value:	63
No. Values:	79
Criteria Met:	C1,C2,C3.1,C3.2,C3.3,C3.4,C4,C5
I-related:	Yes
Z-related:	Yes

11.8.5 RECREL

Presented in **?**, RECREL (Recursive Relation) recursively compares the subsets and subsets of subsets of two SCs using %REL$_n$ (11.8.3). The comparison procedure is quite complicated and potentially involves evaluating %REL$_n$ thousands of times.

SC Type:	TnI or Tn
Cardinality:	Any
Measure Type:	Total
Vector Type:	nC%V
Max Similarity:	0
Min Similarity:	100
Average Value:	
No. Values:	89
Criteria Met:	All
I-related:	Yes
Z-related:	Yes

11.9 BUCHLER

11.9.1 nSATV

Presented in **?**, chap. 2.3 nSATV(X) (Saturation Vector) is a dual vector consisting of two rows, nSATV$_A$(X) and nSATV$_B$(X). It shows extent to which an SC is saturated with subclasses of cardinality n. The steps for computing nSATV(X) are as follows:

1. Compute the nCVs for all SCs of cardinality #X.
2. Find the minimum and maximum values for each position. These values form vectors, $Max_n(\#X)$ and $Min_n(\#X)$.
3. Compute the following two vectors: $MaxMinus = DV(nCV(X), Max_n(\#X))$ and $MinPlus = DV(nCV(X), Min_n(\#X))$
4. $nSATV_A(X)_i = MIN(MaxMinus_i, MinPlus_i)$ and $nSATV_B(X)_i = MAX(MaxMinus_i, MinPlus_i)$
5. If $MaxMinus_i = MinPlus_i$, $nSATV_A(X)_i = MaxMinus_i$ and $nSATV_B(X)_i = MinPlus_i$

11.9.2 SATSIM$_n$

Presented in **?**, chap. 2.4, SATSIM$_n$ (Saturation Similarity index) compares the nSATVs of two SCs and involves the following steps:

1. Calculate nSATV(X) and nSATV(Y)
2. Calculate the vectors nSATV$_{\text{row}}$(X) and nSATV$_{\text{row}}$(Y).
3. The function "row" maps the MaxMinus values of one nSATV to the MaxMinus values of the other. If nSATV$_A$(X)$_i$ is a MaxMinus value and nSATV$_A$(X)$_i$ is also a MaxMinus value, row = A (nSATV$_{\text{row}}$(X)$_i$ = nSATV$_A$(X)$_i$), otherwise row = B.
4. Finally SATSIM$_n$(X,Y) is given by the formula:

$$SATSIM_n(X,Y) = \frac{\#DV(nSATV_A(X), nSATV_{row}(Y)) + \#DV(nSATV_A(Y), SATV_{row}(X))}{\#DV(nSATV_A(X), SATV_B(X)) + \#DV(SATV_A(Y), SATV_B(Y))}$$

11.9.3 TODO CSATSIM

- cyclic saturation similarity
- Description
  - uses CSATV

11.9.4 AvgSATSIM

Presented in **?**, chap. 2.10, AvgSATSIM (Average Saturation Similarity index) is the mean of the values of SATSIM$_n$ where $m = MIN(\#X, \#Y)$.

$$AvgSATSIM(X,Y) = \sum_{n=2}^{m-1} SATSIM_n(X,Y)$$

11.9.5 TSATSIM

Presented in **?**, chap. 2.10, TSATSIM (Total Saturartion Vector Similarity index) is an extension of SATSIM$_n$. TSATSIM is the quotient of the sum of all SATSIM$_n$ numerators and denominators for all values of n from 2 to m-1 where $m = MIN(\#X, \#Y)$.

11.10 Comparison Table

11.11 Comparison Table 2

11.12 Details Table

12 PC-set Theory Glossary

- PC: pitch-class
- set: Unordered PC-set
- SC: Set-class
- nC: Cardinality-class
- #nC: The number of SCs in nC
- IC: Interval-class, 2C
- Tn(X): Transposition of set X by n

Table 7: Comparison table of similarity measures

THEORIST	SIMILARITY MEASURE	VECTOR TYPE	CARDINALITY
TEITELBAUM	s.i.	INTERVAL	SAME
LORD	sf	INTERVAL	SAME
	SIM	INTERVAL	SAME
	K	INTERVAL	SAME
MORRIS	ASIM	INTERVAL	ANY
	IcVD1	INTERVAL	ANY
	IcVD2	INTERVAL	ANY
ROGERS	COS	INTERVAL	ANY
	AMEMB2	INTERVAL	ANY
	IcVSIM	INTERVAL	ANY
	ISIM2	INTERVAL	ANY
ISAACSON	ANGLE	INTERVAL	ANY
	AK	INTERVAL	ANY
	MEMBn	SUBSET	ANY
	TMEMB	SUBSET	ANY
RAHN	ATMEMB	SUBSET	ANY
	REL2	INTERVAL	ANY
LEWIN	REL	SUBSET	ANY
	%RELn	SUBSET	ANY
	T%REL	SUBSET	ANY
CASTREN	RECREL	SUBSET	ANY
	SATSIM	INTERVAL	ANY
	CSATSIM	INTERVAL	ANY
	TSATSIM	SUBSET	ANY
BUCHLER	AvgSATSIM	SUBSET	ANY

- I(X): Inversion of set X
- Tn-Type: Transpositional SC-type
- I-Type: Inversional SC-type
- TnI-Type: Transpositional/inversional SC-type
- Prime Form: PC-set representing all members of an SC
- ICV: Interval-class vector, 2CV
- nCV: n-class subset vector
- nC%V: n-class subset percentage vector

Table 8: Comparison table 2 of similarity measures

SIMILARITY MEASURE	VECTOR TYPE	CARD	MEASURE TYPE	Min Min	Criteria
K	ICV	SAME	1	0-55	C1,C2,C3.3,C3.4,C4
SIM	ICV	SAME	1	65-0	C1,C2,C3.3,C3.4,C4
ASIM	ICV	ANY	1	1-0	C1,C2,C3.1,C3.2,C3.4,C4
sf	ICV	SAME	1	9-0	C3.3, C3.4, C4
s.i.	ICV	SAME	1	8.49-1.41	C3.3, C3.4
IcVD1	ICV	ANY	1	2-0	C1,C2,C3.1,C3.2,C3.4,C4
IcVD2	ICV	ANY	1	1.41-0	C1,C2,C3.1,C3.2,C3.4
COS	ICV	ANY	1	0-1	C1,C2,C3.1,C3.2,C3.4
AMEMB2	ICV	ANY	1	0-1	
IcVSIM	ICV	ANY	1	3.64-0	C1, C2, C3.4
ISIM2	ICV	ANY	1		
ANGLE	ICV	ANY	1	0-1	
AK	ICV	ANY	1	0-1	C1,C2,C3.1,C3.2,C3.4,C4
MEMBn	nCV	ANY	1	0-121	C1,C2,C3.3,C3.4,C4
TMEMB	nCV	ANY	2	0-6118	C1,C2,C3.3,C4,C5
ATMEMB	nCV	ANY	2	0-1	C1,C2,C3.1,C3.2,C3.4,C4,C5
REL2	ICV	ANY	1		
REL	nCV	ANY	2	0-1	C1,C2,C3.1,C3.2,C3.4,C4,C5
%RELn	nC%V	ANY	1	100-0	C1,C2,C3.1,C3.2,C3.3,C3.4,C4
T%REL	nC%V	ANY	2	100-0	C1,C2,C3.1,C3.2,C3.3,C3.4,C4,C5
RECREL	nC%V	ANY	2	100-0	ALL
SATSIM	SATV	ANY	1		C1,C2,C3.1
CSATSIM	ICV	ANY	1		C1,C2,C3.1
TSATSIM	SATV	ANY	2		C1,C2,C3.1
AvgSATSIM	SATV	ANY	2		C1,C2,C3.1

- SATV: Saturation vector
- CSATV: Cyclic saturation vector
- DV: Difference vector
- #X: Set cardinality
- #nCV: Vector cardinality
- Trivial forms:

Table 9: Comparison table 2 of similarity measures

SIMILARITY MEASURE	VECTOR TYPE	CARD	MEASURE TYPE	Min	Max	Avg	No. Vals
K	ICV	SAME	1	0	55	10	35
SIM	ICV	SAME	1	65	0	13	44
ASIM	ICV	ANY	1	1	0	0.42	79
sf	ICV	SAME	1	9	0	3	10
s.i.	ICV	SAME	1	8.49	1.41	2.85	31
IcVD1	ICV	ANY	1	2	0	0.59	140
IcVD2	ICV	ANY	1	1.41	0	0.54	133
COS	ICV	ANY	1	0	1	0.81	92
AMEMB2	ICV	ANY	1	0	1		
IcVSIM	ICV	ANY	1	3.64	0	1.2	121
ISIM2	ICV	ANY	1				
ANGLE	ICV	ANY	1	0	1		
AK	ICV	ANY	1	0	1	0.58	78
MEMBn	nCV	ANY	1	0	121	30	79
TMEMB	nCV	ANY	2	0	6118	131	877
ATMEMB	nCV	ANY	2	0	1	0.45	101
REL2	ICV	ANY	1				
REL	nCV	ANY	2	0	1	0.57	91
%RELn	nC%V	ANY	1	100	0	30	85
T%REL	nC%V	ANY	2	100	0	63	79
RECREL	nC%V	ANY	2	100	0	40	89
SATSIM	SATV	ANY	1				
CSATSIM	ICV	ANY	1				
TSATSIM	SATV	ANY	2				
AvgSATSIM	SATV	ANY	2				

13 Chords as PC sets

Relevant chords and their SCs

Table 10: Chord types and their SCs

Chord	PC-Set (Prime Form)	Forte Name (Tn-type)	index
maj	{0,4,7}	3-11B	25
min	{0,3,7}	3-11A	24
dim	{0,3,6}	3-10	23
aug	{0,4,8}	3-12	26
sus4	{0,2,7}	3-9	22
sus2	{0,2,7}	3-9	22
maj7	{0,1,5,8}	4-20	57
min7	{0,3,5,8}	4-26	64
hdim7	{0,2,5,8}	4-27A	65
7	{0,3,6,8}	4-27B	66
dim7	{0,3,6,9}	4-28	67
min(7)	{0,1,4,8}	4-19A	55
aug(7)	{0,3,4,8}	4-19B	56
maj(9)	{0,2,4,7}	4-22A	59
min(9)	{0,2,3,7}	4-14A	46
maj6	{0,3,5,8}	4-26	64
min6	{0,1,5,8}	4-20	57
sus4(7)	{0,2,6,7}	4-16B	51
sus4(b7)	{0,2,5,7}	4-23	61
9	{0,2,4,6,9}	5-34	129
maj9	{0,1,3,5,8}	5-27A	116
min9	{0,3,5,7,8}	5-27B	117
Pentatonic	{0,2,4,7,9}	5-35	130
Wholetone	{0,2,4,6,8,10}	6-35	192
Diatonic	{0,1,3,5,6,8,10}	7-35	276
Octatonic	{0,1,3,4,6,7,9,10}	8-28	322