

Hello Srijanam,

I have tried to show the main steps in the expressions derived or checked to answer your questions. Please go through them and let me know if the answers are clear.

We should also Skype each other to get a more direct interaction on this topic. My Skype ID is:

Vallurisy

Sorry for the long delay in my response due to a lot of teaching and other duties

Regards

Rami

Answer to Q1, Q2 You are correct in saying  $m = \frac{d}{\beta}$ !  
 There was a type in the published paper.

$$1) X = q + x^m$$

Euler transformation:  $X \rightarrow X^{-\beta}$

$$X^{-\beta} = q + x^{-m\beta}$$

$$\text{Set } \lambda = \beta m \Rightarrow m = \lambda/\beta \quad (\text{Note Typo } m = \lambda\beta. \text{ Should be } \lambda = m\beta)$$

$$q = (\lambda - \beta)v$$

$$X^{-\beta} = (\lambda - \beta)v + x^{-\lambda} \Rightarrow \frac{1}{X^\beta} = (\lambda - \beta)v + \frac{1}{x^\lambda}$$

Rearrangement gives:

$$\therefore \frac{1}{X^\beta} - \frac{1}{x^\lambda} = (\lambda - \beta)v \Rightarrow \frac{x^\lambda - X^\beta}{x^{\lambda+\beta}} = (\lambda - \beta)v$$

$$\therefore x^\lambda - X^\beta = (\lambda - \beta)v x^{\lambda+\beta} \quad (1.1)$$

For equation 2, one sets  $\lambda - \beta = n$

$$x^\lambda - X^\beta = (\lambda - \beta)v \Rightarrow \frac{x^\lambda}{x^{\lambda+\beta}} - 1 = (\lambda - \beta)v X^\lambda, \text{ i.e. } \frac{x^{\lambda-\beta}-1}{\lambda-\beta} = vX^\lambda$$

Set  $n = \lambda - \beta$ . This leads to

$$\frac{x^n - 1}{n} = vX^\lambda \Rightarrow \frac{(e^{\log x})^n - 1}{n} = vX^\lambda$$

$$\lim_{n \rightarrow 0} : \frac{e^{n \log x} - 1}{n} = \lim_{n \rightarrow 0} \frac{n \log x + \frac{(n \log x)^2}{2!} + \dots}{n} \Rightarrow \boxed{\log x = vX^\lambda} \quad (1.3)$$

Corless

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We rearrange the equation (1.1) and expand in powers of  $(x-1)$

$$v = \frac{x^\beta - x^{-\lambda}}{\lambda - \beta} = \underbrace{\left\{ 1 + (x-1) \right\}^{-\beta} - \left\{ 1 + (x-1) \right\}^{-\lambda}}_{\lambda - \beta}$$

$$= \frac{1}{\lambda - \beta} \left[ \left\{ x - \beta(x-1) + \frac{(-\beta)(-\beta-1)}{2}(x-1)^2 + \frac{(-\beta)(-\beta-1)(-\beta-2)}{6}(x-1)^3 \right. \right.$$

$$\left. \left. + \dots \right\} - \left\{ x - \lambda(x-1) + \frac{(-\lambda)(-\lambda-1)}{1 \cdot 2}(x-1)^2 + \frac{(-\lambda)(-\lambda-1)(-\lambda-2)}{1 \cdot 2 \cdot 3}(x-1)^3 \right. \right.$$

$$\left. \left. + \dots \right\} \right]$$

We observe, on simplification, that we obtain

$$v = (x-1) + \left( -\frac{1}{2}\lambda - \frac{1}{2} - \frac{1}{2}\beta \right) (x-1)^2 + \left( \frac{1}{6}\lambda^2 + \frac{1}{6}\beta^2 + \frac{1}{2}\lambda\beta + \frac{1}{3} \right) (x-1)^3$$

$$- \frac{1}{24} (\lambda+3+\beta) (\lambda^2 + 3\lambda + \beta^2 + 3\beta + 2) (x-1)^4 + \dots$$

Now we reverse the series to get a series for  $x$  in terms of  $v$ .

(look up Lagrange interpolation  $\xrightarrow{\text{Theorem}}$  inversion)

$$x = 1 + v + \left( \frac{1}{2}\lambda + \frac{1}{2} + \frac{1}{2}\beta \right) v^2 + \left( \frac{5}{6}\lambda\beta + \frac{\beta}{2} + \frac{\lambda}{2} + \frac{\beta^2}{3} + \frac{\lambda^2}{3} + \frac{1}{6} \right) v^3$$

$$+ \left( \frac{\lambda}{4} + \frac{\beta}{4} + \frac{11}{24}\lambda^2 + \frac{11}{24}\beta^2 + \frac{1}{24} + \frac{\lambda^3}{4} + \frac{\beta^3}{4} + \frac{13}{12}\lambda\beta + \frac{11}{12}\lambda^2\beta + \frac{13}{12}\lambda\beta^2 \right) v^4$$

$$+ \dots$$

This can be factored as.

$$X = 1 + \nu + \left( \frac{1}{2} \alpha + \frac{1}{2} + \frac{1}{2} \beta \right) \nu^2 + \frac{1}{6} (\alpha + 1 + 2\beta)(2\alpha + 1 + \beta) \nu^3 \\ + \frac{1}{24} (2\alpha + 1 + 2\beta)(\alpha + 1 + 3\beta)(\beta + 1 + 3\alpha) \nu^4 + \dots$$

Further calculation shows that.

$$X^n = 1 + n\nu + \frac{n}{2} (\alpha + \beta + n) \nu^2 + \frac{1}{6} n(\alpha + n + 2\beta)(\beta + n + 2\alpha) \nu^3 \\ + \frac{1}{24} n(\alpha + n + 2\beta)(3\beta + n + \alpha)(3\alpha + n + \beta) \nu^4 + O(\nu^5) + \dots$$

etc.

( 1, 2 )  
Corless Paper

Feb 22 4

## Brief Explanation of the Lagrange Inversion Theorem used.

Let  $f(z)$  analytic at  $z=z_0$ ,  $f'(z_0) \neq 0$ , and  $f(z_0) = w_0$  (here  $f(x_0=1) = v_0$ )

These conditions are satisfied in our series

$$w_0 = f(z_0=1) = 0 = v_0, \quad f'(1) = 1 \neq 0$$

Then the equation  $f(z) = w$  has a unique solution:

$$z = F(w) \Rightarrow x = f(v)$$

analytic at  $w = w_0 = 0$  ( $v = v_0 = 0$  at  $x = 1$ )

and

$$F(w) = z_0 + \sum_{n=1}^{\infty} F_n (w - w_0)^n$$

$$= 1 + \sum_{n=1}^{\infty} F_n (w^n) \quad (w_0 = 0)$$

$$= 1 + \sum_{n=1}^{\infty} F_n w^n = 1 + \sum_{n=1}^{\infty} F_n v^n$$

in a neighbourhood of  $w_0 = 0$ , and

$n F_n$  is the residue of  $\frac{1}{(f(z)-0)^n}$  at  $z = z_0 = 1$  ( $v = v_0 = 0$ )

i.e.  $n F_n$  is the coefficient of  $\frac{1}{z-1}$  in the Laurent expansion of  $\frac{1}{(f(z)-0)^n}$  in powers of  $(z-1)$

$$F(w) = 1 + \sum_{n=1}^{\infty} F_1 v + F_2 v^2 + F_3 v^3 + F_4 v^4 + \dots$$

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Q3.  $y = e^{st}$  is a solution of  $\dot{y}(t) = a y(t-1)$   
for some value of  $s$ .

$$\frac{dy}{dt} = s e^{st} \Rightarrow \dot{y}(t) = s e^{st} = a y(t-1)$$

$$\text{But: } a y(t-1) = a e^{s(t-1)} = a e^{-s} e^{st}.$$

$$\therefore \dot{y}(t) = a y(t-1) \Rightarrow s e^{st} = a e^{-s} e^{st}.$$

$$\therefore s = a e^{-s} \Rightarrow s e^s = a.$$

It follows from the definition of the Lambert W that

$$s = W_k(a)$$

where  $k$  could be any branch such as  $k=0, \pm 1, \pm 2, \dots$

Since  $W(z)$  is a complex multivalued inverse function:

Q4

b

Equation (3.14) clearly shows that it is possible to

obtain an antiderivative of  $W(x)$

$$w = W(x) \Rightarrow x = w e^w \Rightarrow dx = (1 + w e^w) dw.$$

$$\int w(x) dx = \int w(H+w) e^w dw =$$

$$= (w+w^2) e^w - \int (2w+1) e^w dw = w(H+w)e^w - (2w+1)e^w + 2 \int e^w dw.$$

$$= e^w (w + w^2 - 2w - 1 + 2) + C.$$

$$= (w^2 - w + 1) e^w + C = w \left( w - 1 + \frac{1}{w} \right) + C.$$

This is valid for all branches of  $W$ , in accord with

the definition  $\frac{d}{dw} w e^w \neq 0$  at any interior point of the branch

The same technique is essentially given in  
the work of F.D. Parker, Amer. Math. Monthly 62 (1955)  
439-440

on Integrals of Inverse Functions

Ref 54 — Corkiss et al. Paper

Q4

(Q4 contd)

March 3 2018 7

Example of eqn (3.16) Corkess et al.

Integrals containing inverse functions

$$\int f^{-1}(x) dx = y f(y) - \int f(y) dy \quad (3.16)$$

Corkess et al.

Consider the case  $f^{-1}(x) = W(x) = w$ .

$$\int W(x) dx \quad \text{Inverse Function}$$

$x = w e^w$

(\*)

$$\Rightarrow dx = (w e^w + 1 e^w) dw = (w+1) e^w dw \quad (*)$$

(\*)

$$\therefore \int w e^w (w+1) dw = e^w w (w+1) - \int e^w (2w+1) dw$$

$$= e^w w(w+1) - 2 e^w w - e^w - 2 \int e^w dw \quad (*)$$

$$= e^w (w^2 + w - 2w - 1 - 2) = e^w (w^2 - w - 3)$$

This will be valid for all branches of  $W$ , as

$\frac{d}{dw} w e^w \neq 0$  at any interior point of any branch

( Corkess et al ).

Note the difference from (3.15) where Corkess evaluated

$$\int x W(x) dx$$

in contrast to  $\int W(x) dx$  of (\*) above.

Q5 Pg 17

The answer to your question is from a careful reading  
of the earlier pages<sup>(14-16)</sup> and also pgs 18 and 19.

Maybe we should Skype each other to discuss  
this and other questions

Q6.

Expressions of eqns (4.16) and (4.17)

$$\frac{1}{e^{-\beta} - 1 - \sigma\beta + \gamma} = \frac{1}{(e^{-\beta} - 1)} \left\{ 1 - \frac{\sigma\beta - \gamma}{e^{-\beta} - 1} \right\} \left( \frac{-\beta}{e^{-\beta} - 1} \rightarrow (\beta - T) \right)$$

$$= \frac{1}{(e^{-\beta} - 1)} \left( 1 - \frac{\sigma\beta - \gamma}{e^{-\beta} - 1} \right)^{-1} = \frac{1}{e^{-\beta} - 1} \left\{ 1 + \frac{\sigma\beta - \gamma + (\beta - T)}{e^{-\beta} - 1} + \dots \right. \\ \left. + \left( \frac{\sigma\beta - \gamma}{e^{-\beta} - 1} \right)^k + \dots \right\}$$

$$= \frac{1}{(e^{-\beta} - 1)} \sum_{k=0}^{\infty} \left( \frac{\sigma\beta - \gamma}{e^{-\beta} - 1} \right)^k = \frac{1}{(e^{-\beta} - 1)} \sum_{k=0}^{\infty} (e^{-\beta} - 1)^{-k} (\beta - T)^k \\ (e^{-\beta} - 1) = \sum_{k=0}^{\infty} (e^{-\beta} - 1)^{-k-1} \sigma^k \beta^k \left( 1 - \frac{\gamma}{\sigma\beta} \right)^k$$

$$= \sum_{k=0}^{\infty} (e^{-\beta} - 1)^{-k-1} \sigma^k \beta^k \left[ \left( 1 - \frac{\gamma}{\sigma\beta} \right)^k \right]$$

$$= \sum_{k=0}^{\infty} (e^{-\beta} - 1)^{-k-1} \sigma^k \beta^k \sum_{m=0}^{\infty} (-1)^m \frac{\tau^m}{\sigma^m \beta^m} \binom{k}{m}$$

Now Corless et al merged the expressions as

$$\sum_{k=0}^{\infty} (e^{-\beta} - 1)^{-k-m-1} \beta^k \sigma^{k-m} (-1)^m \tau^m \binom{m+k}{m} \quad (4.16)$$

by using  $\frac{1}{\beta^m}$  in terms of  $(e^{-\beta} - 1)^{-m}$  and $\binom{k-m}{m}$  as  $\binom{k}{K}$  since both  $K$  and  $m$  are summed over  $K, m$  from 0 to  $\infty$ .

Q6 contd.

Substitution of (4.16) into (4.15) gives:

$$V = \frac{1}{2\pi i} \int_{|f|=R} \left( -e^{-f} - \sigma \right) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left( \bar{e}^{-f} - 1 \right)^{-k-m-1} f^k \sigma^k T^m (-1)^m C_{m+k}^m$$

$|f| = R$  X

$f \neq 0$  X

Note that eqn(4.15) is a consequence of eqn(4.13) having exactly one root  $V$  inside the circle (Rouche's Theorem either Ref 66-Titchmarsh or Ref 12 Carathéodory explain this well.)

Integration term by term of eqn X gives an absolutely convergent series in  $\sigma$  and  $T$  (you may have noticed that  $T=0$  gives zero contribution since the integrands for this case are regular at  $f=0$ .  $m$  starts from 1 to  $\infty$ .)

Hence

$$V = \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} c_{km} \sigma^k T^m \quad (c_{km} = C_m^{m+k})$$

(4.17)

Answer to Question 7 (Contd) After Question 6

Apr 6 2018 11

Series Expansion about the Branch Point (follows from branches of  $w_0, w_1, w_{-1}$ )

$$p^2 = 2ez + 2 \Rightarrow \frac{p^2}{2} = ez + 1 \Rightarrow \frac{p^2 - 1}{2} = ez$$

$$\Rightarrow we^w \rightarrow w = -1 \Rightarrow z = -\frac{1}{e}$$

$$\frac{p^2}{2} - 1 = e^z \cdot z = e^z \cdot we^w = we^{1+w}$$

$$= w \left( 1 + (1+w) + \frac{(1+w)^2}{2!} + \dots + \frac{(1+w)^k}{k!} \right)$$

$$= (1+w-1) \left\{ 1 + (1+w) + \frac{(1+w)^2}{2!} + \dots + \frac{(1+w)^k}{k!} \right\}$$

$$= - \left( 1 + (1+w) + \frac{(1+w)^2}{2!} + \dots + \frac{(1+w)^k}{k!} \right)$$

$$+ (1+w) \left\{ 1 + (1+w) + \frac{(1+w)^2}{2!} + \dots + \frac{(1+w)^k}{k!} \right\}$$

$$= -1 + \sum_{k \geq 1} \left\{ \frac{(1+w)^k}{k!} + \frac{(1+w)^k \cdot k}{(k-1)!} \right\} (4.2)$$

Now this Series can be inverted. Firstly we observe that the series for  $W(x)$  is of the form:

$$W(x) = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \frac{54}{5}x^6 + \frac{16807}{720}x^7 + \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$$

Answer to Question 7 contd.

The general coefficient is  $\frac{(-n)^{n-1}}{n!}$  (exponential generating function for unrooted, labelled trees)

The branch point series is.

after series inversion. (use Lagrange Inversion Theorem)

$$W(z) = -1 + \sqrt{2/e^1} \sqrt{z + \frac{1}{e}} - \frac{2}{3} \left( \frac{z + \frac{1}{e}}{e^{-1}} \right) + \frac{11\sqrt{2}}{36e} \left( \frac{z + \frac{1}{e}}{e^{-1}} \right)^{3/2} - \frac{43}{135} \frac{\left( z + \frac{1}{e} \right)^2}{e^{-2}} + \dots$$

Now the series can be more concisely written as :

$$W(z) = -1 + p - \frac{1}{3!} p^2 + \frac{11}{72!} p^3 - \frac{43}{540!} p^4 + \dots$$

$$= \sum_{l=0}^{\infty} u_l p^l \quad (9.22)$$

Series converges for  $|p| < \sqrt{2}$