Noto

Lambert W

In 1758, Lambert solved the trinomial equation $x = q + x^m$ by giving a series development for x in powers of q. Later, he extended the series to give powers of x as well [48,49]. In [28], Euler transformed Lambert's equation into the more symmetrical form

$$x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta} \tag{1.1}$$

by substituting $x^{-\beta}$ for x and setting $m = \alpha\beta$ and $q = (\alpha - \beta)v$. Euler's version of Lambert's series solution was thus

$$\begin{aligned} x^n = 1 &+ nv + \tfrac{1}{2}n(n+\alpha+\beta)v^2 \\ &+ \tfrac{1}{6}n(n+\alpha+2\beta)(n+2\alpha+\beta)v^3 \\ &+ \tfrac{1}{24}n(n+\alpha+3\beta)(n+2\alpha+2\beta)(n+3\alpha+\beta)v^4 \\ &+ \text{etc.} \end{aligned} \tag{1.2}$$

After deriving the series, Euler looked at special cases, starting with $\alpha = \beta$. To see what this means in the original trinomial equation, we divide (1.1) by $(\alpha - \beta)$ and then let $\beta \to \alpha$ to get

$$\log x = vx^\alpha\,. \tag{1.3}$$

Euler noticed that if we can solve equation (1.3) for $\alpha = 1$, then we can solve it for

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On the Lambert W function

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The Lambert W function is defined to be the multivalued inverse of the function $w \mapsto we^w$. It has many applications in pure and applied mathematics, some of which are briefly described here. We present a new discussion of the complex branches of W, an asymptotic expansion valid for all branches, an efficient numerical procedure for evaluating the function to arbitrary precision, and a method for the symbolic integration of expressions containing W.

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Function $e^w \frac{w-k}{w+k} = z$

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1 Properties

1.1 Projection on $w + k = a$

Theorem : $e^w \frac{w-k}{w+k} = z$ has 0 or 1 or 2 real roots.

$e^w \frac{w-k}{w+k} = z$ with $w+k = a$ gives $e^{w+k} \frac{w+k-2k}{a} = ze^k \Leftrightarrow e^k = -\frac{2e^a}{az}(k - \frac{a}{2}) \Leftrightarrow k = \frac{a}{2} - W(\frac{a}{2} z e^{-a/2})$. and $w = \frac{a}{2} + W(\frac{a}{2} z e^{-a/2})$

Now the equation $e^w \frac{w-k}{w+k} = z$ has exactly 2 real solutions iff $-\frac{1}{e} < \frac{a}{2} z e^{-a/2} < 0$.

- $\frac{a}{2} z e^{-a/2} = 0 \Leftrightarrow w = -k$.
- $\frac{a}{2} z e^{-a/2} = -\frac{1}{e} \Leftrightarrow a = -2W(\frac{1}{ez}) \Leftrightarrow w = -k - 2W(\frac{1}{ez})$ $-W(+\frac{1}{ez})$
- $lim_{a \to \infty} \frac{a}{2} z e^{-a/2} = sign(z) \times 0$
- $lim_{a \to -\infty} \frac{a}{2} z e^{-a/2} = -sign(z) \infty$

Thus the equation has 2 real solutions

- if $z > 0$ and $-2W(\frac{1}{ez}) < a < 0$
- if $z < -1$ and $0 < a < -2W_0(\frac{1}{ez})$ or $-2W_{-1}(\frac{1}{ez}) < a$
- if $-1 < z < 0$ and $a > 0$

2 real solutions

The equation has 1 real solution

- if $za \geq 0$
- if $a = -2W_0(\frac{1}{ez})$ or $a = -2W_{-1}(\frac{1}{ez})$

which one do you choose?

The equation has 0 real solution

- if $z < -1$ and $-2W_0(\frac{1}{ez}) < a < -2W_{-1}(\frac{1}{ez})$

Corollary : For a given z and $k = -\frac{2W^2(\frac{1}{ez})}{2W(\frac{1}{ez})+1}$ then the equation $e^w \frac{w-k}{w+k} = z$ has a double root on the point $w = \sqrt{k^2 - 2k}$. → check.

$w^2 = k^2 - 2k \Rightarrow w^2 = (a-k)^2$

$w^2 = a^2 + k^2 - 2ak$

$\frac{w-k}{w+k} = w-k$

1729 Pin Aeroplan

5h Teaching [illegible] < 10 → 3 Pm

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Hello Srijanani,

I have tried to show the main steps in the expressions derived or checked to answer your questions. Please go through them and let me know if the answers are clear.

We should also skype each other to get a more direct interaction on this topic. My Skype ID is:

Vallurisr

Sorry for the long delay in my response due to a lot of teaching and other duties.

Regards
Ram.

Answer to Q1, Q2 You are correct in saying $m = \frac{\alpha}{\beta}$!

There was a typo in the published paper.

1) $X = q + X^{m}$

Euler transformation : $X \longrightarrow X^{-\beta}$

$$X^{-\beta} = q + X^{-m\beta}$$

Set $\alpha = \beta m \Rightarrow m = \alpha/\beta$ (Note Typo $m = \alpha\beta$. Should be $\alpha = m\beta$)

yes $m = \alpha/\beta$!

$$q = (\alpha - \beta) V$$

$$X^{-\beta} = (\alpha-\beta) V + X^{-\alpha} \Rightarrow \frac{1}{X^{\beta}} = (\alpha-\beta) V + \frac{1}{X^{\alpha}}$$

Rearrangement gives:

$$\therefore \frac{1}{X^{\beta}} - \frac{1}{X^{\alpha}} = (\alpha-\beta) V \Rightarrow \frac{X^{\alpha} - X^{\beta}}{X^{\alpha+\beta}} = (\alpha-\beta) V$$

$$\therefore X^{\alpha} - X^{\beta} = (\alpha-\beta) V X^{\alpha+\beta} \qquad (1.1)$$

For equation 2, one sets $\alpha - \beta = n$

$$\frac{X^{\alpha}-X^{\beta}}{X^{\alpha+\beta}} = (\alpha-\beta) V \Rightarrow \frac{X^{\alpha}}{X^{\beta}} - 1 = (\alpha-\beta) V X^{\alpha}, \text{ i.e. } \frac{X^{\alpha-\beta}-1}{\alpha-\beta} = V X^{\alpha}$$

Set $n = \alpha - \beta$. This leads to

$$\frac{X^{n}-1}{n} = V X^{\alpha} \Rightarrow \frac{\left(e^{\log X}\right)^{n} - 1}{n} = V X^{\alpha}$$

$$\lim_{n\to 0} \therefore \frac{e^{n \log X} - 1}{n} = \lim_{n\to 0} \frac{\cancel{1} + n \log X + \frac{(n \log X)^2}{2!} + \cdots - \cancel{1}}{n} \Rightarrow \boxed{\log X = V X^{\alpha}} \quad (1.3)$$

Corless

We rearrange the equation (1.1) and expand in powers of $(X-1)$

$$V = \frac{X^{-\beta} - X^{-\alpha}}{\alpha-\beta} = \frac{\{1+(X-1)\}^{-\beta} - \{1+(X-1)\}^{-\alpha}}{\alpha-\beta}$$

$$= \frac{1}{\alpha-\beta}\Big[\Big\{1 - \beta(X-1) + \frac{(-\beta)(-\beta-1)}{2}(X-1)^2 + \frac{(-\beta)(-\beta-1)(-\beta-2)}{6}(X-1)^3 + \cdots\Big\}$$

$$- \Big\{1 - \alpha(X-1) + \frac{(-\alpha)(-\alpha-1)}{1\cdot 2}(X-1)^2 + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{1\cdot 2\cdot 3}(X-1)^3 + \cdots\Big\}\Big]$$

We observe, on simplification, that we obtain

$$V = (X-1) + \left(-\frac{1}{2}\alpha - \frac{1}{2} - \frac{1}{2}\beta\right)(X-1)^2 + \left(\frac{1}{6}\alpha^2 + \frac{1}{6}\beta^2 + \frac{1}{2}\alpha + \frac{\beta}{2} + \frac{1}{3}\right)(X-1)^3$$

$$- \frac{1}{24}(\alpha+3+\beta)(\alpha^2+3\alpha+\beta^2+3\beta+2)(X-1)^4 + \cdots$$

Now we reverse the series to get a series for X in terms of V.

(look up Lagrange ~~interpolation~~ Theorem → inversion)

$$X = 1 + V + \left(\frac{1}{2}\alpha + \frac{1}{2} + \frac{1}{2}\beta\right)V^2 + \left(\frac{5}{6}\alpha\beta + \frac{\beta}{2} + \frac{\alpha}{2} + \frac{\beta^2}{3} + \frac{\alpha^2}{3} + \frac{1}{6}\right)V^3$$

$$+ \left(\frac{\alpha}{4} + \frac{\beta}{4} + \frac{11}{24}\alpha^2 + \frac{11}{24}\beta^2 + \frac{1}{24} + \frac{\alpha^3}{4} + \frac{\beta^3}{4} + \frac{13}{12}\alpha\beta + \frac{13}{12}\alpha^2\beta + \frac{13}{12}\alpha\beta^2\right)V^4$$

$$+ \cdots$$

This can be factored as:

$$X = 1 + v + \left(\frac{1}{2}\alpha + \frac{1}{2} + \frac{1}{2}\beta\right) v^2 + \frac{1}{6}(\alpha + 1 + 2\beta)(2\alpha + 1 + \beta) v^3$$

$$+ \frac{1}{24}(2\alpha + 1 + 2\beta)(\alpha + 1 + 3\beta)(\beta + 1 + 3\alpha) v^4 + \cdots$$

Further Calculation Shows that:

$$X^n = 1 + nv + \frac{n}{2}(\alpha + \beta + n) v^2 + \frac{1}{6} n (\alpha + n + 2\beta)(\beta + n + 2\alpha) v^3$$

$$+ \frac{1}{24} n (2\alpha + n + 2\beta)(3\beta + n + \alpha)(3\alpha + n + \beta) v^4 + O(v^5) + \cdots$$

etc.

(1.2)

Corless Paper.

Brief Explanation of the
Lagrange Inversion Theorem used.

Let $f(z)$ analytic at $z = z_0$, $f'(z_0) \neq 0$, and
$f(z_0) = w_0$ (here $f(x_0 = 1) = v_0$)

These conditions are satisfied in our series
$w_0 = f\left(\begin{matrix} z_0 = 1 \\ x_0 = 1 \end{matrix}\right) = 0 = v_0$, $f'(1) = 1 \neq 0$.

Then the equation $f(z) = w$ has a unique solution: $f(x) = v$

$$z = F(w) \Rightarrow x = f(v)$$

analytic at $w = w_0 = 0$ ($v = v_0 = 0$ at $x = 1$)
and

$$F(w) = z_0 + \sum_{n=1}^{\infty} F_n (w - w_0)^n$$

$$= 1 + \sum_{n\geq 1}^{\infty} F_n (w^n) \qquad (w_0 = 0)$$

$$= 1 + \sum_{n=1}^{\infty} F_n w^n = 1 + \sum_{n=1}^{\infty} F_n v^n$$

in a neighborhood of $w_0 = 0$, ($v_0 = 0$) and

$n F_n$ is the residue of $\dfrac{1}{(f(z) - 0)^n}$ at $z = z_0 = 1$ ($x = x_0 = 1$)
($f(z_0) = 0$)
$f(1) = 0$.

i.e. $n F_n$ is the coefficient of $\dfrac{1}{z-1}$ $\left(= \dfrac{1}{x-1}\right)$ in the

Laurent expansion of $\dfrac{1}{(f(z) - 0)^n}$ in powers of $(z-1)$ $(= x-1)$

$$F(w) = 1 + \sum_{n=1}^{\infty} F_1 v + F_2 v^2 + F_3 v^3 + F_4 v^4 + \cdots$$

Q3. $y = e^{st}$ is a solution of $\dot{y}(t) = a\,y(t-1)$ for some value of s.

$\frac{dy}{dt} = s e^{st} \Rightarrow \dot{y}(t) = s e^{st} = a\,y(t-1)$

But: $a\,y(t-1) = a\,e^{s(t-1)} = a\,e^{-s}\,e^{st}$.

$\therefore \dot{y}(t) = a\,y(t-1) \Longrightarrow s\,\cancel{e^{st}} = a\,e^{-s}\,\cancel{e^{st}}$

$\therefore s = a\,e^{-s} \Rightarrow s e^{s} = a$

It follows from the definition of the Lambert W that

$s = W_k(a)$

where k could be any branch such as $k = 0, \pm 1, \pm 2, \ldots$

Since $W(z)$ is a complex multivalued inverse function

Equation (3.14) clearly shows that it is possible to obtain an antiderivative of $W(x)$.

$$w = W(x) \Rightarrow x = we^{w} \Rightarrow dx = (1 e^{w} + we^{w})\,dw.$$

$$\int W(x)\,dx = \int w(1+w)e^{w}\,dw =$$

$$= (w+w^2)e^{w} - \int (2w+1)e^{w}\,dw = w(1+w)e^{w} - (2w+1)e^{w} + 2\int e^{w}\,dw.$$

$$= e^{w}(w+w^2-2w-1+2) + C.$$

$$= (w^2-w+1)e^{w} + C \equiv x\left(w-1+\frac{1}{w}\right)+C.$$

This is valid for all branches of W, in accord with the definition $\frac{d}{dw}we^{w} \neq 0$ at any interior point of the branch

The same technique is essentially given in the work of F.D Parker, Amer. Math. Monthly 62 (1955) 439-440 on Integrals of Inverse Functions

Ref 54 — Corless et al. Paper

Q4 (Q4 contd)

March 3 2018 7

Example of eqn (3.16) Corless et al.

Integrals containing inverse functions.

$$\int f^{-1}(x)\,dx = y f(y) - \int f(y)\,dy \qquad (3.16)$$

Corless et al.

Consider the case $f^{-1}(x) = W(x) = w$.

$$\int W(x)\,dx \qquad (*1)$$

Inverse Function

$$\boxed{x = w e^{w}} \Longrightarrow dx = \left(w e^{w} + 1 e^{w}\right) dw = (w+1)e^{w}\,dw \qquad (*2)$$

$$\therefore \int w e^{w}(w+1)\,dw = e^{w} w(w+1) - \int e^{w}(2w+1)\,dw$$

$$= e^{w} w(w+1) - 2e^{w}w - e^{w} - 2\int e^{w}\,dw \qquad (*3)$$

$$= e^{w}\left(w^2 + w - 2w - 1 - 2\right) = e^{w}\left(w^2 - w - 3\right)$$

This will be valid for all branches of W, as $\frac{d}{dw} w e^{w} \neq 0$ at any interior point of any branch (Corless et al).

Note the difference from (3.15) where Corless evaluated $\int x W(x)\,dx$ in contrast to $\int W(x)\,dx$ of $(*1)$ above.

Q5 Pg 17

The answer to your question is from a careful reading of the earlier pages (14-16) and also pgs 18 and 19.

Maybe we should Skype each other to discuss this and other questions

Q6.

March 31, Apr 1 2018.

Expressions of eqns (4.16) and (4.17)

$$\frac{1}{e^{-\zeta}-1-\sigma\zeta+\tau} = \frac{1}{\left(e^{-\zeta}-1\right)\left\{1-\frac{\sigma\zeta-\tau}{e^{-\zeta}-1}\right\}} \qquad \left(e^{-\zeta}-1\right)\left(\sigma\zeta-\tau\right)$$

$$= \frac{1}{\left(e^{-\zeta}-1\right)}\left(1-\frac{\sigma\zeta-\tau}{e^{-\zeta}-1}\right)^{-1} = \frac{1}{e^{-\zeta}-1}\left\{1+\frac{\sigma\zeta-\tau}{e^{-\zeta}-1}+\left(\frac{\sigma\zeta-\tau}{e^{-\zeta}-1}\right)^{2}+\cdots + \left(\frac{\sigma\zeta-\tau}{e^{-\zeta}-1}\right)^{k}+\cdots\right\}$$

$$= \frac{1}{\left(e^{-\zeta}-1\right)}\sum_{k=0}^{\infty}\left(\frac{\sigma\zeta-\tau}{e^{-\zeta}-1}\right)^{k} = \frac{1}{\left(e^{-\zeta}-1\right)}\sum_{k=0}^{\infty}\left(e^{-\zeta}-1\right)^{-k}\left(\sigma\zeta-\tau\right)^{k}$$

$$= \sum_{k=0}^{\infty}\left(e^{-\zeta}-1\right)^{-k-1}\sigma^{k}\zeta^{k}\left(1-\frac{\tau}{\sigma\zeta}\right)^{k}$$

$$= \sum_{k=0}^{\infty}\left(e^{-\zeta}-1\right)^{-k-1}\sigma^{k}\zeta^{k}\left[\left(1-\frac{\tau}{\sigma\zeta}\right)^{k}\right]$$

$$= \sum_{k=0}^{\infty}\left(e^{-\zeta}-1\right)^{-k-1}\sigma^{k}\zeta^{k}\sum_{m=0}^{\infty}(-1)^{m}\frac{\tau^{m}}{\sigma^{m}\zeta^{m}}\binom{k}{m}$$

Now Corless et al merged the expressions as

$$\sum_{k=0}^{\infty}\left(e^{-\zeta}-1\right)^{-k-m-1}\zeta^{k}\sigma^{k-m}(-1)^{m}\tau^{m}C^{m+k}_{m} \qquad (4.16)$$

by using the $\frac{1}{\zeta^{m}}$ in terms of $\left(e^{-\zeta}-1\right)^{-m}$ and $\sigma^{k-m}\binom{k}{m}$ as σ^{k} since both k and m are summed over k, m from 0 to ∞.

Q6 contd.

Substitution of (4·16) into (4·15) gives

$$\nu = \frac{1}{2\pi i}\int_{|\zeta|=\pi} \frac{}{\left(-e^{-\zeta}-\sigma\right)} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \left(e^{-\zeta}-1\right)^{-k-m-1} \zeta^{k}\sigma^{k}\tau^{m}(-1)^{m}\, C_{m}^{m+k} \cdot \zeta \, d\zeta. \quad (*)$$

Note that eqn (4·15) is a consequence of eqn (4·13) having exactly one root ν inside the circle (Rouche's Theorem either Ref 66 – Titchmarsh or Ref 12 Carathéodory explain this well.)

Integration term by term of eqn $(*)$ gives an absolutely convergent series in σ and τ (you may have noticed that $\tau=0$ gives zero contribution since the integrands for this case are regular at $\zeta=0$. m starts from $1 \to$ to ∞.

Hence

$$\nu = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} c_{km}\,\sigma^{k}\tau^{m} \qquad \left(c_{km} = C_{m}^{m+k}\right)$$

(4·A)

Apr 6 2018

~~Answers~~ to Question ~~7~~ (After Question 6 ~~contd~~)

Series Expansion about the Branch Point (follows from branches of W_0, W_1, W_{-1})

$$p^2 = 2ez + 2 \Rightarrow \frac{p^2}{2} = ez + 1 \Rightarrow \frac{p^2}{2} - 1 = eZ$$

$$Z = we^{w} \longrightarrow w = -1 \Rightarrow z = -\frac{1}{e}$$

$$\frac{p^2}{2} - 1 = e^{1} \cdot Z = e^{1} We^{W} = We^{1+W}.$$

$$= W\left(1 + (1+W) + \frac{(1+w)^2}{2!} + \cdots + \frac{(1+W)^k}{k!} + \cdots\right)$$

$$= (1+W-1)\left\{1 + (1+W) + \frac{(1+W)^2}{2!} + \cdots + \frac{(1+W)^k}{k!} + \cdots\right\}$$

$$= -\left(1 + (1+W) + \frac{(1+W)^2}{2!} + \cdots + \frac{(1+W)^k}{k!} + \cdots\right)$$

$$+ (1+W)\left\{1 + (1+W) + \frac{(1+W)^2}{2!} + \cdots + \frac{(1+W)^k}{k!} + \cdots\right\}$$

$$= -1 + \sum_{k \geq 1}\left\{-\frac{(1+W)^k}{k!} + \frac{(1+W)^k}{(k-1)!}\right\} \qquad (4.21)$$

Now this series can be inverted. Firstly we observe that the series for $W(x)$ is of the form:

$$W(x) = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \frac{54}{5}x^6 + \frac{16807}{720}x^7 + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$$

Apr 6 2018

Answer to Question 7 cntd.

The general coefficient is $\frac{(-n)^{n-1}}{n!}$ (exponential generating function for unrooted, labelled trees)

The branch point series is:

after series inversion (use Lagrange Inversion Theorem)

$$W(z) = -1 + \sqrt{2/e^{-1}}\sqrt{z+\frac{1}{e}} - \frac{2}{3}\frac{\left(z+\frac{1}{e}\right)}{e^{-1}} + \frac{11\sqrt{2}\left(z+\frac{1}{e}\right)^{3/2}}{36e^{-3/2}}$$

$$-\frac{43}{135}\frac{\left(z+\frac{1}{e}\right)^2}{e^{-2}} + \cdots$$

Now the series can be more concisely written as:

$$W(z) = -1 + p - \frac{1}{3}p^2 + \frac{11}{72}p^3 - \frac{43}{540}p^4 + \cdots$$

$$= \sum_{l=0}^{\infty} \mu_l p^l \qquad (4.22)$$

Series converges for $|p| < \sqrt{2}$.