Having Fun with Lambert W(x) Function

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Abstract

This short note presents the Lambert W(x) function and its possible application in the framework of physics related to the Pierre Auger Observatory. The actual numerical implementation in C++ consists of Halley's and Fritsch's iteration with branch-point expansion, asymptotic series and rational fits as initial approximations.

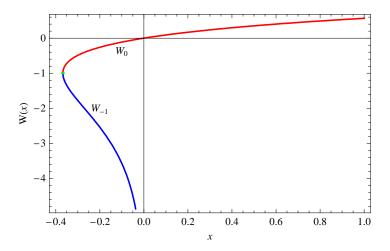


Figure 1: The two branches of the Lambert W function, $W_{-1}(x)$ in blue and $W_0(x)$ in red. The branching point at $(-e^{-1}, -1)$ is denoted with a green dash.

1 Introduction

The Lambert W(x) function is defined as the inverse function of

$$y \exp y = x,\tag{1}$$

the solution being given by

$$y = W(x), (2)$$

or shortly

$$W(x)\exp W(x) = x. (3)$$

Since the $x \mapsto x \exp x$ mapping is not injective, no unique inverse of the $x \exp x$ function exists. As can be seen in Fig. 1, the Lambert function has two real branches with a branching point located at $(-e^{-1}, -1)$. The bottom branch, $W_{-1}(x)$, is defined in the interval $x \in [-e^{-1}, 0]$ and has a negative singularity for $x \to 0^-$. The upper branch is defined for $x \in [-e^{-1}, \infty]$.

The earliest mention of problem of Eq. (1) is attributed to Euler. However, Euler himself credited Lambert for his previous work in this subject. The W(x) function started to be named after Lambert only recently, in the last 10 years or so. The letter W was chosen by the first implementation of the W(x) function in the Maple computer software.

Recently, the W(x) function amassed quite a following in the mathematical community. Its most faithful proponents are suggesting to elevate it among the present set of elementary functions, such as $\sin(x)$, $\cos(x)$, $\ln(x)$, etc. The main argument for doing so is the fact that it is the root of the simplest exponential polynomial function.

While the Lambert W function is simply called W in the mathematics software tool *Maple*, in the *Mathematica* computer algebra framework this function is implemented under the name ProductLog (in the recent versions an alias LambertW is also supported).

There are numerous, well documented applications of W(x) in mathematics, physics, and computer science [1, 3]. Here we will give two examples that arise from the physics related to the Pierre Auger Observatory.

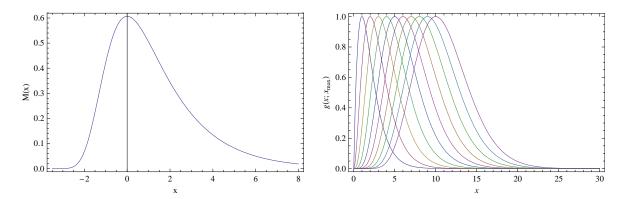


Figure 2: *Left:* The Moyal function M(x). *Right:* A family of one-parametric Gaisser-Hillas functions $g(x; x_{max})$ for x_{max} in the range from 1 to 10 with step 1.

1.1 Moyal function

Moyal function is defined as

$$M(x) = \exp\left(-\frac{1}{2}\left(x + \exp(-x)\right)\right). \tag{4}$$

Its inverse can be written in terms of the two branches of the Lambert W function,

$$\mathbf{M}_{\pm}^{-1}(x) = \mathbf{W}_{0,-1}(-x^2) - 2\ln x. \tag{5}$$

and can be seen in Fig. 2 (left).

Within the event reconstruction of the data taken by the Pierre Auger Observatory, the Moyal function is used for phenomenological recovery of the saturated signals from the photomultipliers.

1.2 Gaisser-Hillas function

In astrophysics the Gaisser-Hillas function is used to model the longitudinal particle density in a cosmic-ray air showers [4]. We can show that the inverse of the three-parametric Gaisser-Hillas function,

$$G(X; X_0, X_{\text{max}}, \lambda) = \left[\frac{X - X_0}{X_{\text{max}} - X_0}\right]^{\frac{X_{\text{max}} - X_0}{\lambda}} \exp\left(\frac{X_{\text{max}} - X}{\lambda}\right), \tag{6}$$

is intimately related to the Lambert W function. Using rescale substitutions,

$$x = \frac{X - X_0}{\lambda} \quad \text{and}$$
 (7)

$$x_{\text{max}} = \frac{X_{\text{max}} - X_0}{\lambda},\tag{8}$$

the Gaisser-Hillas function is modified into a function of one parameter only,

$$g(x; x_{\text{max}}) = \left[\frac{x}{x_{\text{max}}}\right]^{x_{\text{max}}} \exp(x_{\text{max}} - x). \tag{9}$$

The family of one-parametric Gaisser-Hillas functions is shown in Fig. 2 (right). The problem of finding an inverse,

$$g(x; x_{\text{max}}) \equiv a \tag{10}$$

for $0 < a \le 1$, can be rewritten into

$$-\frac{x}{x_{\text{max}}} \exp\left(-\frac{x}{x_{\text{max}}}\right) = -a^{1/x_{\text{max}}} e^{-1}.$$
 (11)

According to the definition (1), the two (real) solutions for x are obtained from the two branches of the Lambert W function,

$$x_{1,2} = -x_{\text{max}} W_{0,-1}(-a^{1/x_{\text{max}}} e^{-1}) = -x_{\text{max}} W_{0,-1}(-x_{\text{max}} \sqrt{a}/e).$$
(12)

Note that the branch -1 or 0 simply chooses the right or left side relative to the maximum, respectively.

2 Numerics

Before moving to the actual implementation let us review some of the possible nimerical and analytical approaches.

2.1 Recursion

For x > 0 and W(x) > 0 we can take the natural logarithm of (3) and rearrange it,

$$W(x) = \ln x - \ln W(x). \tag{13}$$

It is clear, that a possible analytical expression for W(x) exhibits a degree of self similarity. The W(x) function has multiple branches in the complex domain. Due to the x > 0 and W(x) > 0 conditions, the Eq. (13) represents the positive part of the $W_0(x)$ principal branch, but as it turns out, in this form it is suitable for evaluation when $W_0(x) > 1$, i.e. when x > e.

Unrolling the self-similarity (13) as a recursive relation, one obtains the following curious expression for $W_0(x)$,

$$W_0(x) = \ln x - \ln(\ln x - \ln(\ln x - \dots)), \tag{14}$$

or in a shorthand of a continued logarithm,

$$W_0(x) = \ln \frac{x}{\ln \frac{x}{\ln x}}.$$
 (15)

The above expression is clearly a form of successive approximation, the final result given by the limit, when it exists.

For x < 0 and W(x) < 0 we can multiply both sides of Eq. (3) with -1, take logarithm, and rewrite it to get a similar expansion for the $W_{-1}(x)$ branch,

$$W(x) = \ln(-x) - \ln(-W(x)). \tag{16}$$

Again, this leads to a similar recursive expression,

$$W_{-1}(x) = \ln(-x) - \ln(-(\ln(-x) - \ln(-(\ln(-x) - \dots)))), \tag{17}$$

or as a continued logarithm,

$$W_{-1}(x) = \ln \frac{-x}{-\ln \frac{-x}{-\ln \frac{-x}{x}}}.$$
 (18)

For this continued logarithm we will use the symbol $R_{-1}^{[n]}(x)$ where n denotes the depth of the recursion.

Starting from yet another rearrangement of Eq. (3),

$$W(x) = \frac{x}{\exp W(x)},\tag{19}$$

we can obtain a recursion relation for the $-e^{-1} < x < e$ part of the principal branch $W_0(x) < 1$,

$$W_0(x) = \frac{x}{\exp\frac{x}{\exp^{\frac{x}{2}}}}. (20)$$

2.2 Halley's iteration

We can apply Halley's root-finding method [8] to derive an iteration scheme for f(y) = W(y) - x by writing the second-order Taylor series

$$f(y) = f(y_n) + f'(y_n)(y - y_n) + \frac{1}{2}f''(y_n)(y - y_n)^2 + \cdots$$
 (21)

Since root y of f(y) satisfies f(y) = 0 we can approximate the left-hand side of Eq. (21) with 0 and replace y with y_{n+1} . Rewriting the obtained result into

$$y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n) + \frac{1}{2}f''(y_n)(y_{n+1} - y_n)}$$
(22)

and using Newton's method $y_{n+1} - y_n = -f(y_n)/f''(y_n)$ on the last bracket, we arrive at the expression for the Halley's iteration for Lambert function

$$W_{n+1} = W_n + \frac{t_n}{t_n s_n - u_n'},\tag{23}$$

where

$$t_n = W_n \exp W_n - x, \tag{24}$$

$$s_n = \frac{W_n + 2}{2(W_n + 1)},\tag{25}$$

$$u_n = (W_n + 1) \exp W_n. \tag{26}$$

This method is of the third order, i.e. having $W_n = W(x) + \mathcal{O}(\varepsilon)$ will give $W_{n+1} = W(x) + \mathcal{O}(\varepsilon^3)$. Supplying this iteration with sufficiently accurate first approximation of the order of $\mathcal{O}(10^{-4})$ will thus give a machine-size floating point precission $\mathcal{O}(10^{-16})$ in at least two iterations.

2.3 Fritsch's iteration

For both branches of Lambert function a more efficient iteration scheme exists [9],

$$W_{n+1} = W_n(1 + \varepsilon_n), \tag{27}$$

where ε_n is the relative difference of successive approximations at iteration n,

$$\varepsilon_n = \frac{W_{n+1} - W_n}{W_n}. (28)$$

The relative difference can be expressed as

$$\varepsilon_n = \left(\frac{z_n}{1 + W_n}\right) \left(\frac{q_n - z_n}{q_n - 2z_n}\right),\tag{29}$$

where

$$z_n = \ln \frac{x}{W_n} - W_n, \tag{30}$$

$$q_n = 2(1 + W_n) \left(1 + W_n + \frac{2}{3}z_n\right).$$
 (31)

The error term in this iteration is of a fourth order, i.e. with $W_n = W(x) + \mathcal{O}(\varepsilon_n)$ we get $W_{n+1} = W(x) + \mathcal{O}(\varepsilon_n^4)$.

Supplying this iteration with a sufficiently reasonable first guess, accurate to the order of $\mathcal{O}(10^{-4})$, will therefore deliver machine-size floating point precission $\mathcal{O}(10^{-16})$ in only one iteration and excessive $\mathcal{O}(10^{-64})$ in two! We have to find reliable first order approximation that can be fed into the Fritsch iteration. Due to the lively landscape of the Lambert function properties, the approximations will have to be found in all the particular ranges of the function behavior.

3 Initial approximations

The following section deals with finding the appropriate initial approximations in the whole definition ranges of the two branches of the Lambert function.

3.1 Branch-point expansion

The inverse of the Lambert function, $W^{-1}(y) = y \exp y$, has two extrema located at $W^{-1}(-1) = -e^{-1}$ and $W^{-1}(-\infty) = 0^-$. Expanding $W^{-1}(y)$ to the second order around the minimum at y = -1 we obtain

$$W^{-1}(y) \approx -\frac{1}{e} + \frac{(y+1)^2}{2e}.$$
 (32)

The inverse $W^{-1}(y)$ is thus in the lowest order approximated with a parabolic term implying that the Lambert function will have square-root behavior in the vicinity of the branch point $x = -e^{-1}$,

$$W_{-1,0}(x) \approx -1 \mp \sqrt{2(1+ex)}$$
. (33)

To obtain the additional terms in expression (33) we proceed by defining an inverse function, centered and rescaled around the minimum, i.e. $f(y) = 2(eW^{-1}(y-1) + 1)$. Due

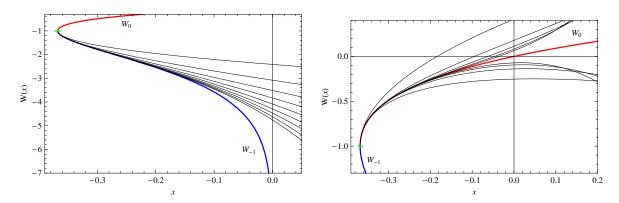


Figure 3: Successive orders of the branch-point expansion for the $W_{-1}(x)$ on the left and $W_0(x)$ on the right.

to the centering and rescaling the Taylor series of this function around y=0 becomes particularly neat,

$$f(y) = y^2 + \frac{2}{3}y^3 + \frac{1}{4}y^4 + \frac{1}{15}y^5 + \cdots$$
 (34)

Using this Taylor expansion we can derive coefficients [10] of the corresponding inverse function

$$f^{-1}(z) = 1 + W\left(\frac{z-2}{2e}\right) = \tag{35}$$

$$=z^{1/2} - \frac{1}{3}z + \frac{11}{72}z^{3/2} - \frac{43}{540}z^2 + \cdots$$
 (36)

From Eq. (35) we see that z=2(1+ex). Using $p_{\pm}(x)=\pm\sqrt{2(1+ex)}$ as independent variable we can write this series expansion as

$$W_{-1,0}(x) \approx B_{-1,0}^{[n]}(x) = \sum_{i=0}^{n} b_i p_{\pm}^i(x),$$
 (37)

where the lowest few coefficients b_i are

i	0	1	2	3	4	5	6	7	8	9
b_i	-1	1	$-\frac{1}{3}$	11 72	$-\frac{43}{540}$	769 17 280	$-\frac{221}{8505}$	680 863 43 545 600	$-\frac{1963}{204120}$	226 287 557 37 623 398 400

3.2 Asymptotic series

Another useful tool is the asymptotic expansion [2] where using

$$A(a,b) = a - b + \sum_{k} \sum_{m} C_{km} a^{-k-m-1} b^{m+1}$$
(38)

where C_{km} are related to the Stirling number of the first kind, the Lambert function can be expressed as

$$W_{-1,0}(x) = A(\ln(\mp x), \ln(\mp \ln(\mp x)))$$
(39)

with $a = \ln x$, $b = \ln \ln x$ for the W₀ branch and $a = \ln(-x)$, $b = \ln(-\ln(-x))$ for the W₋₁ branch. The first few terms are

$$A(a,b) = a - b + \frac{b}{a} + \frac{b(-2+b)}{2a^2} + \frac{b(6-9b+2b^2)}{6a^3} + \frac{b(-12+36b-22b^2+3b^3)}{12a^4} + \frac{b(60-300b+350b^2-125b^3+12b^4)}{60a^5} + \cdots$$
(40)

3.3 Rational fits

A useful quick-and-dirty approach to the functional approximation is to generate large enough sample of data points $\{w_i \exp w_i, w_i\}$. These points evidently lie on the Lambert function. Within some appropriately chosen range of w_i values the points are fitted with a rational approximation

$$Q(x) = \frac{\sum_{i} a_{i} x^{i}}{\sum_{i} b_{i} x^{i}},\tag{41}$$

varying the order of the polynomials in the nominator and denominator, and choosing the one that has the lowest maximal absolute residual in a particular interval of interest.

For the $W_0(x)$ branch, the first set of equally-spaced w_i component was chosen in a range that produced w_i exp w_i values in an interval [-0.3, 0]. The optimal rational fit turned out to be

$$Q_0(x) = x \frac{1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}$$
(42)

where the coefficients for this first approximation $Q_0^{[1]}(x)$ are

i	1	2	3	4
	5.931375839364438			
b_i	6.931373689597704	16.82349461388016	16.43072324143226	5.115235195211697

For the second fit of the $W_0(x)$ branch a w_i range was chosen so that the w_i exp w_i values were produced in the interval [0.3, 2e] giving rise to the second optimal rational fit $Q_0^{[2]}(x)$ of the same form as in Eq. (42) but with coefficients

i	1	2	3	4
a_i	2.445053070726557	1.343664225958226	0.148440055397592	0.0008047501729130
b_i	3.444708986486002	3.292489857371952	0.916460018803122	0.0530686404483322

For the $W_{-1}(x)$ branch one rational approximation of the form

$$Q_{-1}(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5}$$
(43)

with the coefficients

i	0	1	2	3
a_i	-7.81417672390744	253.88810188892484	657.9493176902304	
b_i		-60.43958713690808	99.9856708310761	682.6073999909428
i	4	5		
b_i	962.1784396969866	1477.9341280760887		

is enough.

4 Implementation

Note that the most recent and up-to-date implementation [11] can be found at: https://github.com/DarkoVeberic/LambertW

To quantify the accuracy of a particular approximation $\widetilde{W}(x)$ of the Lambert function W(x) we can introduce a quantity $\Delta(x)$ defined as

$$\Delta(x) = -\log_{10}|\widetilde{W}(x) - W(x)|,\tag{44}$$

so that it gives us a number of correct decimal places the approximation $\widetilde{W}(x)$ is producing for some parameter x.

In Fig. 4 all mentioned approximations for the $W_0(x)$ are shown in the linear interval $[-e^{-1}, 0.3]$ on the left and logarithmic interval $[0.3, 10^5]$ on the right. For each of the approximations an use interval is chosen so that the number of accurate decimal places is maximized over the whole definition range. For the $W_0(x)$ branch the resulting piecewise approximation

$$\widetilde{W}_{0}(x) = \begin{cases} B_{0}^{[9]}(x) & ; -e^{-1} \leqslant x < -0.32358170806015724 \\ Q_{0}^{[1]}(x) & ; -0.32358170806015724 \leqslant x < 0.14546954290661823 \\ Q_{0}^{[2]}(x) & ; 0.14546954290661823 \leqslant x < 8.706658967856612 \\ A_{0}(x) & ; 8.706658967856612 \leqslant x < \infty \end{cases}$$

$$(45)$$

is accurate in the definition range $[-e^{-1}, 7]$ to at least 5 decimal places and to at least 3 decimal places in the whole definition range. The $B_0^{[9]}(x)$ is from Eq. (37), $Q_0^{[1]}(x)$ and $Q_0^{[2]}(x)$ are from Eq. (42), and $A_0(x)$ is from Eq. (40).

The final piecewise approximation $\widetilde{W}_0(x)$ is shown in Fig. 5 in black line. Using this approximation a single step of the Halley iteration (in red) and the Fritsch iteration (in blue) is performed and the resulting number of accurate decimal places is shown. As can be seen both iterations produce machine-size accurate floating point numbers in the whole definition interval except for the [9,110] interval where the Halley method requires another step of the iteration. For that reason we have decided to use only (one step of) the Fritsch iteration in the C++ implementation of the Lambert function.

In Fig. 6 (left) the same procedure is shown for the $W_{-1}(x)$ branch. The final approximation

$$\widetilde{W}_{-1}(x) = \begin{cases} B_{-1}^{[9]}(x) & ; -e^{-1} \leqslant x < -0.30298541769 \\ Q_{-1}(x) & ; -0.30298541769 \leqslant x < -0.051012917658221676 \\ R_{-1}^{[9]}(x) & ; -0.051012917658221676 \leqslant x < 0 \end{cases}$$
(46)

is accurate to at least 5 decimal places in the whole definition range $[-e^{-1}, 0]$ and where $B_{-1}^{[9]}(x)$ is from Eq. (37), $Q_{-1}(x)$ is from Eq. (43), and $R_{-1}^{[9]}(x)$ is from Eq. (18).

In Fig. 6 (right) the combined approximation $\widetilde{W}_{-1}(x)$ is shown in black line. The values after one step of the Halley iteration are shown in red and after one step of the Fritsch

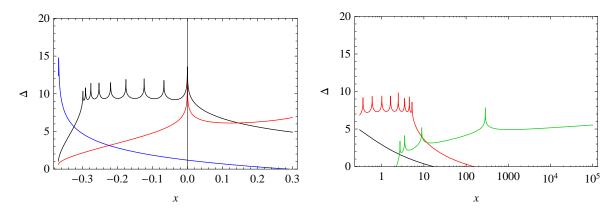


Figure 4: Combining different approximations of $W_0(x)$ into final piecewise function. The number of accurate decimal places $\Delta(x)$ is shown for two ranges, linear interval $[-e^{-1}, 0.3]$ on the left and logarithmic interval $[0.3, 10^5]$ on the right. The approximation are branch-point expansion $B_0^{[9]}(x)$ from Eq. (37) in blue, rational fits $Q_0^{[1]}(x)$ and $Q_0^{[2]}(x)$ from Eq. (42) in black and red, respectively, and asymptotic series $A_0(x)$ from Eq. (40) in green.

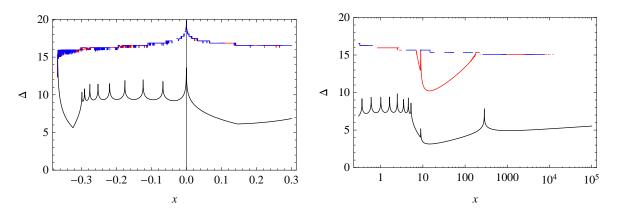


Figure 5: Final values of the combined approximation $\widetilde{W}_0(x)$ (black) from Fig. 4 after one step of the Halley iteration (red) and one step of the Fritsch iteration (blue).

iteration in blue. Similarly as for the previous branch, the Fritsch iteration is superior, yielding machine-size accurate results in the whole definition range, while the Halley is accurate to at least 13 decimal places.

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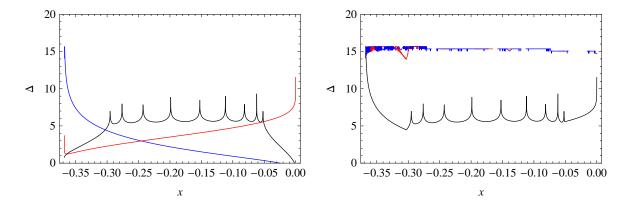


Figure 6: *Left:* Approximations of the $W_{-1}(x)$ branch. The branch point expansion $B_{-1}^{[9]}(x)$ is shown in blue, the rational approximation $Q_{-1}(x)$ in black, and the logarithmic recursion $R_{-1}^{[9]}$ in red. *Right:* Combined approximation is accurate to at least 5 decimal places in the whole definition range. The results after applying one step of the Halley iteration are shown in red and after one step of the Fritsch iteration in blue.

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A Implementation in C++

This program is distributed in the hope that it will be useful,

Note that this is the 2010 version of the implementation. The most up-to-date sources can be found at https://github.com/DarkoVeberic/LambertW

```
A.1 Lambert.h
                                                                                            but WITHOUT ANY WARRANTY; without even the implied warranty of
                                                                                            MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE. See the
                                                                                            GNU General Public License for more details.
  Implementation of Lambert W function
                                                                                           You should have received a copy of the GNU General Public License along with this program. If not, see <a href="http://www.gnu.org/licenses/">http://www.gnu.org/licenses/</a>.
  Copyright (C) 2009 Darko Veberic, darko.veberic@kit.edu
  This program is free software: you can redistribute it and/or modify it under the terms of the GNU General Public License as published by
                                                                                         #include <iostream>
                                                                                         #include <cmath>
#include <limits>
#include "LambertW.h"
   the Free Software Foundation, either version 3 of the License, or
  (at your option) any later version.
  This program is distributed in the hope that it will be useful,
  but WITHOUT ANY WARRANTY; without even the implied warranty of MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE. See the
                                                                                         using namespace std;
  GNU General Public License for more details.
                                                                                         namespace LambertWDetail {
  You should have received a copy of the GNU General Public License along with this program. If not, see <a href="http://www.gnu.org/licenses/">http://www.gnu.org/licenses/</a>.
                                                                                           const double kInvE = 1/M_E;
  25 Jun 2009
                                                                                            template<int n>
                                                                                            inline double BranchPointPolynomial(const double p);
#ifndef _utl_LambertW_h_
#define _utl_LambertW_h_
                                                                                            template<>
/** Approximate Lambert W function
  Accuracy at least 5 decimal places in all definition range.
                                                                                            double
                                                                                            BranchPointPolynomial<1>(const double p)
  See LambertW() for details.
  \param branch: valid values are 0 and -1
   \param x: real-valued argument \f$\geq-1/e\f$
                                                                                                -1 + p;
  \ingroup math
                                                                                            template<>
double LambertWApproximation(const double x);
                                                                                            double
                                                                                            BranchPointPolynomial<2>(const double p)
/** Lambert W function
                                                                                                -1 + p*(1 + p*(-1./3));
  Lambert function f={\rm W}(x)\ is defined as a solution
  to the f=x=y^f expression and is also known as "product logarithm". Since the inverse of f=y^f is not single-valued, the Lambert function has two real branches f=x^f m f=x^f and f=x^f m f=x^f.
                                                                                            template<>
                                                                                            double
  f{\rm W} O(x)\f$ has real values in the interval
                                                                                            BranchPointPolynomial<3>(const double p)
  f^{-1/e,\inf y}\f and f^{\infty W}_{-1}(x)\f has real values
  in the interval \f$[-1/e,0]\f$.

Accuracy is the nominal double type resolution
                                                                                                 -1 + p*(1 + p*(-1./3 + p*(11./72)));
  (16 decimal places).
   \param branch: valid values are 0 and -1
  \param x: real-valued argument \f$\geq-1/e\f$ (range depends on
                                                                                            template<>
  branch)
                                                                                            BranchPointPolynomial<4>(const double p)
template<int branch>
double LambertW(const double x):
                                                                                              return
                                                                                                 -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540))));
                                                                                            inline
A.2 Lambert.cc
                                                                                            BranchPointPolynomial<5>(const double p)
  Implementation of Lambert W function
                                                                                                  -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 +
                                                                                                p*(769./17280))));
  Copyright (C) 2009 Darko Veberic, darko.veberic@kit.edu
  This program is free software: you can redistribute it and/or modify it under the terms of the GNU General Public License as published by
  the Free Software Foundation, either version 3 of the License, or (at your option) any later version.
                                                                                            template<>
                                                                                            double
```

BranchPointPolynomial<6>(const double p)

```
return
                                                                                     AsymptoticExpansion<4>(const double a, const double b)
     -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 + p*(769./17280
                                                                                        const double ia = 1 / a;
     + p*(-221./8505)))));
                                                                                        return a - b + b / a * (1 + ia *
                                                                                            (0.5*(-2 + b) + ia *
(1/6.*(6 + b*(-9 + b*2)) + ia *
template<>
inline
                                                                                                 1/12.*(-12 + b*(36 + b*(-22 + b*3)))
                                                                                              )
double
BranchPointPolynomial<7>(const double p)
                                                                                         ):
                                                                                     }
    -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 + p*(769./17280 + p*(-221./8505 + p*(680863./43545600)))))));
                                                                                     template<>
                                                                                     double
                                                                                     AsymptoticExpansion<5>(const double a, const double b)
template<>
inline
                                                                                       const double ia = 1 / a;
return a - b + b / a *
   (1 + ia *
double
BranchPointPolynomial<8>(const double p)
                                                                                            (0.5*(-2 + b) + ia *
  return
                                                                                              (1/6.*(6 + b*(-9 + b*2)) + ia *
(1/12.*(-12 + b*(36 + b*(-22 + b*3))) + ia *
1/60.*(60 + b*(-300 + b*(350 + b*(-125 + b*12))))
    -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 + p*(769./17280 + p*(-221./8505 + p*(680863./43545600 + p*(-1963./204120))))))));
                                                                                              )
                                                                                            )
template<>
                                                                                         ):
inline
BranchPointPolynomial<9>(const double p)
  return
                                                                                     template<int branch>
    -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 + p*(769./17280
+ p*(-221./8505 + p*(680863./43545600 + p*(-1963./204120
+ p*(226287557./37623398400.))))))));
                                                                                     class Branch {
                                                                                       template<int order>
static double BranchPointExpansion(const double x)
}
                                                                                        { return BranchPointPolynomial<order>(eSign * sqrt(2*(M_E*x+1))); }
template<int order>
inline double AsymptoticExpansion(const double a, const double b);
                                                                                        // Asymptotic expansion
                                                                                        // Corless et al. 1996, de Bruijn (1981)
                                                                                        template<int order>
template<>
inline
                                                                                        double
double
                                                                                        AsymptoticExpansion(const double x)
AsymptoticExpansion<0>(const double a, const double b)
                                                                                          const double logsx = log(eSign * x);
const double logslogsx = log(eSign * logsx);
  return a - b;
}
                                                                                          return LambertWDetail::AsymptoticExpansion<order>(logsx, logslogsx);
template<>
                                                                                        template<int n>
inline
                                                                                        static inline double RationalApproximation(const double x);
double
AsymptoticExpansion<1>(const double a, const double b)
                                                                                        // Logarithmic recursion
                                                                                        template<int order>
  return a - b + b / a;
                                                                                        static inline double LogRecursion(const double x)
                                                                                        { return LogRecursionStep<order>(log(eSign * x)); }
                                                                                        // generic approximation valid to at least 5 decimal places
template<>
inline
                                                                                        static inline double Approximation(const double x);
double
                                                                                        // enum { eSign = 2*branch + 1 }; this doesn't work on gcc 3.3.3
AsymptoticExpansion<2>(const double a, const double b)
                                                                                        static const int eSign = 2*branch + 1;
  const double ia = 1 / a;
return a - b + b / a * (1 + ia * 0.5*(-2 + b));
                                                                                        template<int n>
}
                                                                                        static inline double LogRecursionStep(const double logsx)
                                                                                        { return logsx - log(eSign * LogRecursionStep<n-1>(logsx)); }
template<>
                                                                                     // Rational approximations
double
AsymptoticExpansion<3>(const double a, const double b)
                                                                                     template<>
  const double ia = 1 / a;
return a - b + b / a *
  (1 + ia *
                                                                                     inline
                                                                                     double
       (0.5*(-2 + b) + ia *
                                                                                     Branch<0>::RationalApproximation<0>(const double x)
           1/6.*(6 + b*(-9 + b*2))
                                                                                       return x*(60 + x*(114 + 17*x)) / (60 + x*(174 + 101*x));
    );
                                                                                     template<>
                                                                                     template<>
template<>
inline
```

```
}
Branch<0>::RationalApproximation<1>(const double x)
  // branch 0, valid for [-0.31,0.3]
                                                                                   // stopping conditions
  return
x * (1 + x *
                                                                                   template<>
       (5.931375839364438 + x *
                                                                                    template<>
         (11.392205505329132 + x ×
                                                                                   inline
            (7.338883399111118 + x*0.6534490169919599)
                                                                                    double
         )
                                                                                   Branch<0>::LogRecursionStep<0>(const double logsx)
                                                                                     return logsx:
     (1 + x *
                                                                                   }
       (6.931373689597704 + x *
         (16.82349461388016 + x *
(16.43072324143226 + x*5.115235195211697)
                                                                                   template<>
         )
                                                                                    template<>
       )
                                                                                   inline
    );
                                                                                    double
}
                                                                                    Branch<-1>::LogRecursionStep<0>(const double logsx)
                                                                                     return logsx;
template<>
                                                                                   7
template<>
inline
                                                                                    template<>
double
Branch<0>::RationalApproximation<2>(const double x)
                                                                                    inline
                                                                                    double
  // branch 0, valid for [-0.31,0.5]
                                                                                   Branch<0>::Approximation(const double x)
  return
x * (1 + x *
                                                                                      if (x < -0.32358170806015724) {
       (4.790423028527326 + x *
                                                                                        if (x < -kInvE)
         (6.695945075293267 + x * 2.4243096805908033)
                                                                                        return numeric_limits<double>::quiet_NaN();
else if (x < -kInvE+1e-5)</pre>
    ) /
                                                                                          return BranchPointExpansion<5>(x);
    (1 + x *
(5.790432723810737 + x *
                                                                                        else
                                                                                          return BranchPointExpansion<9>(x);
                                                                                      } else {
  if (x < 0.14546954290661823)
         (10.986445930034288 + x *
           (7.391303898769326 + x * 1.1414723648617864)
         )
                                                                                        return Rational Approximation <1>(x);
else if (x < 8.706658967856612)
    );
                                                                                           return RationalApproximation<3>(x);
}
                                                                                        else
                                                                                          return AsymptoticExpansion<5>(x);
                                                                                     }
template<>
                                                                                   }
template<>
inline
                                                                                   template<>
double
Branch<0>::RationalApproximation<3>(const double x)
                                                                                   double
  // branch 0, valid for [0.3,7]
                                                                                    Branch<-1>::Approximation(const double x)
  return
    x * (1 + x *
                                                                                      if (x < -0.051012917658221676) {
                                                                                        if (x < -kInvE+1e-5) {
   if (x < -kInvE)
       (2.4450530707265568 + x *
         (1.3436642259582265 + x * (0.14844005539759195 + x * 0.0008047501729129999)
                                                                                            return numeric_limits<double>::quiet_NaN();
         )
                                                                                           else
                                                                                            return BranchPointExpansion<5>(x);
                                                                                        } else {
  if (x < -0.30298541769)</pre>
    ) /
     (1 + x *
       (3.4447089864860025 + x *
(3.2924898573719523 + x *
(0.9164600188031222 + x * 0.05306864044833221)
                                                                                            return BranchPointExpansion<9>(x);
                                                                                           else
                                                                                            return RationalApproximation<4>(x);
         )
      )
                                                                                      } else {
                                                                                        if (x < 0)
    );
}
                                                                                        return LogRecursion<9>(x);
else if (x == 0)
                                                                                        return -numeric_limits<double>::infinity();
template<>
template<> inline
                                                                                          return numeric_limits<double>::quiet_NaN();
                                                                                     }
double
Branch<-1>::RationalApproximation<4>(const double x)
{
  // branch -1, valid for [-0.3,-0.05]
                                                                                   // iterations
  return
    (-7.814176723907436 + x * (253.88810188892484 + x * 657.9493176902304)
                                                                                    inline
                                                                                    double
    ) /
(1 + x *
                                                                                   HalleyStep(const double x, const double w)
       (-60.43958713690808 + x * (99.98567083107612 + x *
                                                                                      const double ew = exp(w);
                                                                                      const double wew = w * ew;
                                                                                      const double wewx = wew - x;

const double w1 = w + 1;

return w - wewx / (ew * w1 - (w + 2) * wewx/(2*w1));
           (682.6073999909428 + x * (962.1784396969866 + x * 1477.9341280760887)
           )
         )
```

```
inline
                                                                                        static double Recurse(const double x, const double w)
double
                                                                                       { return IterationStep(x, w); }
FritschStep(const double x, const double w)
   const double z = log(x/w) - w;
                                                                                     // identity
  const double w1 = w + 1;

const double q = 2 * w1 * (w1 + (2/3.)*z);

const double eps = z / w1 * (q - z) / (q - 2*z);

return w * (1 + eps);
                                                                                     template<class T>
struct Depth<0, T> {
    static double Recurse(const double x, const double w)
                                                                                       { return w; }
}
                                                                                     };
  double IterationStep(const double x, const double w)
                                                                                } // LambertWDetail
inline
double
Iterate(const double x, double w, const double eps = 1e-6)
                                                                                template<int branch>
  for (int i = 0; i < 100; ++i) {
                                                                                LambertWApproximation(const double x)
     const double ww = IterationStep(x, w);
if (fabs(ww - w) <= eps)</pre>
                                                                                   return LambertWDetail::Branch<branch>::Approximation(x);
                                                                                7
    return ww;
w = ww;
                                                                                // instantiations
cerr << "c
return w;
}</pre>
   cerr << "convergence not reached." << endl;
                                                                                 template double LambertWApproximation<0>(const double x);
                                                                                template double LambertWApproximation<-1>(const double x);
                                                                                 template<int branch>
templates
                                                                                double LambertW(const double x):
   double IterationStep(const double x, const double w)
                                                                                template<>
struct Iterator {
                                                                                LambertW<0>(const double x)
                                                                                   if (fabs(x) > 1e-6 && x > -LambertWDetail::kInvE + 1e-5)
   double
   Do(const int n, const double x, const double w)
                                                                                       LambertWDetail::
     for (int i = 0; i < n; ++i)
                                                                                          Iterator<LambertWDetail::FritschStep>::
       w = IterationStep(x, w);
                                                                                            Depth<1>::
     return w;
                                                                                              Recurse(x, LambertWApproximation<0>(x));
                                                                                   else
                                                                                     return LambertWApproximation<0>(x);
   template<int n>
                                                                                template<>
   double
   Do(const double x, const double w)
                                                                                LambertW<-1>(const double x)
    for (int i = 0; i < n; ++i)
w = IterationStep(x, w);</pre>
                                                                                   if (x > -LambertWDetail::kInvE + 1e-5)
     return w;
                                                                                       LambertWDetail::
                                                                                          Iterator < Lambert \\ WDetail:: Fritsch \\ Step > ::
   template<int n, class = void>
                                                                                            Depth<1>::
   struct Depth {
                                                                                              Recurse(x, LambertWApproximation<-1>(x));
     static double Recurse(const double x, double w)
     { return Depth<n-1>::Recurse(x, IterationStep(x, w)); }
                                                                                     return LambertWApproximation<-1>(x);
   // stop condition
                                                                                // instantiations
  template < class T > struct Depth < 1, T > {
                                                                                template double LambertW<0>(const double x);
template double LambertW<-1>(const double x);
```