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Research

Integrating the Lambert W Function to a Tolerance Optimization Problem

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This paper explores the integration of the Lambert W function to a tolerance optimization problem with the assessment of costs incurred by both the customer and a manufacturer. By trading off manufacturing and rejection costs, and a quality loss, this paper shows how the Lambert W function, widely used in physics, can be efficiently applied to the tolerance optimization problem, which may be the first attempt in the literature related to tolerance optimization and synthesis. Using the concept of the Lambert W function, a closed-form solution is derived, which may serve as a means for quality practitioners to make a quick decision on their optimal tolerance without resorting to rigorous optimization procedures using numerical methods. A numerical example is illustrated and a sensitivity analysis is performed. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: tolerance design; optimal specification limits; Lambert *W* function; optimization

1. INTRODUCTION

Functional performance and economic considerations are the two primary factors affecting the design of tolerances. A tight tolerance usually implies high manufacturing cost due to additional manufacturing operations, slow processing rates, additional care on the part of the operator, and a need for expensive measuring and processing equipment. The functional performance, however, can be improved by specifying a tight tolerance on a quality characteristic. On the other hand, a wide tolerance reduces the manufacturing cost but may considerably lower the product quality level. Thus, determining optimal tolerance involves a trade-off between the level of quality based on functional performance and the costs associated with the tolerance. In order to facilitate the economic trade-off, researchers typically express quality in monetary terms using a quality loss function. The quadratic loss function is widely used in the literature as a reasonable approximation of the actual loss to the customer due to the deviation of product performance from its target value. By expressing the level of quality in monetary terms, the problem of trading off quality with costs is converted into a problem of minimizing the total cost, which is the sum of quality loss and costs. The costs associated with the tolerance

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include rejection, inspection, and manufacturing costs. The problem of determining an optimal tolerance is equivalent to the problem of determining optimal specification limits, since the term tolerance refers to the distance between its lower and upper specification limits.

The analysis in this paper differs from previous studies of the tolerance optimization problem in two ways. First, a more general optimization model is proposed by simultaneously considering the quality loss incurred by the customer, and manufacturing and rejection costs incurred by the manufacturer. Second, this paper shows how the Lambert W function, widely used in physics[1], can be efficiently applied to the tolerance optimization problem, which may be the first attempt in the literature related to tolerance optimization and synthesis. There are two significant benefits from using the Lambert W function in the context of tolerance optimization. Most tolerance optimization models require rigorous optimization processes using numerical methods since closed-from solutions are rarely found. By using the Lambert W function, quality practitioners cannot only express their solutions in a closed form but they can also quickly determine their optimal tolerances without resorting to numerical methods since a number of popular mathematical softwares, including Maple and Matlab, contain the Lambert W function as an optimization component.

Following this introduction, a literature review is given in Section 2. The Lambert W function is discussed in Section 3. Generalized forms of the function relevant to the proposed tolerance optimization model are presented in Sections 3 and 4. A numerical example is conducted and a sensitivity analysis is performed.

2. *LITERATURE REVIEW*

A number of researchers have considered the problem of determining optimal tolerance. Speckhart[2], Spotts[3], Chase *et al.*[4], and Kim and Cho[5] investigated the effect of tolerances on manufacturing cost, and proposed models to determine tolerances for the minimization of manufacturing cost. Fathi[6], Phillips and Cho[7], and Kim and Cho[8] studied the issue of tolerance design from the viewpoint of functional performance, where functional performance is expressed in monetary terms by using the Taguchi quality loss concept. Tang[9] discussed an economic model for selecting the most profitable tolerance in a complete inspection plan for the case where inspection cost is a linear function of the tolerance. Fathi[6] devised a graphical approach for determining tolerances to minimize the quality loss and the rejection costs for a single quality characteristic. Tang and Tang[10] and Tang[11] further investigated screening inspection for multiple performance variables in a serial production process. Tang and Tang[12] presented a comprehensive literature review related to the design of screening procedures. Jeang[13] proposed an optimization model for the simultaneous optimization of manufacturing cost, rejection cost, and quality loss using a process capability index to establish a relationship between the tolerance and standard deviation. The model assumes a zero process bias condition; that is, the mean is equal to the target value for the quality characteristic. Further, the ratio of tolerance to the standard deviation is assumed to be constant. Jeang[14] demonstrated how response surface methodology can be employed to determine tolerances of components in an assembly. Kapur[15], Kapur and Cho[16], and Kapur and Cho[17] considered a tolerance optimization problem using truncated normal, Weibull, and multivariate normal distributions, respectively. Phillips and Cho[7] studied minimization of the quality loss and the rejection costs, where quality loss is determined empirically by applying regression analysis to historical data of losses. They presented optimization models for the first-order and second-order empirical loss functions. Moskowitz *et al.*[18] and Plante[19] conducted parametric and non-parametric studies for allocating the tolerances on design parameters affecting a response.

3. *THE LAMBERT W FUNCTION*

Lambert considered the trinomial equation $x = a + x^b$ by giving a series development for x in powers of a. This equation can be transformed into a more symmetrical form

$$x^\alpha - x^\beta = (\alpha - \beta)\nu x^{\alpha+\beta} \tag{1}$$

by substituting $x^{-\beta}$ for x and setting $x = \alpha\beta$ and $a = (\alpha - \beta)v$. Dividing Equation (1) by $(\alpha - \beta)$ and letting β converge to α, Equation (2) is obtained

$$\log x = vx^{\alpha} \tag{2}$$

Lambert's generalized series solution for x^n is given by

$$\begin{aligned} x^n = {} & 1 + nv + \tfrac{1}{2}n(n+\alpha+\beta)v^2 + \tfrac{1}{6}n(n+\alpha+2\beta)(n+2\alpha+\beta)v^3 \\ & + \tfrac{1}{24}n(n+\alpha+3\beta)(n+2\alpha+3\beta)(n+3\alpha+\beta)v^4 + \cdots \end{aligned} \tag{3}$$

Using the series given by Equation (3), Corless *et al.*[1] show that Equation (2) can be written as

$$\log x = v + \frac{2^1}{2!}v^2 + \frac{3^2}{3!}v^3 + \frac{4^3}{4!}v^4 + \frac{5^4}{5!}v^5 + \cdots \tag{4}$$

This series shown in Equation (4) converges to Lambert $W(z)$, which is defined as

$$\text{Lambert } W(z)\, \mathrm{e}^{\text{Lambert } W(z)} = z \tag{5}$$

The Lambert W function is defined to be a multivalued inverse of the function $f(z) = z\,\mathrm{e}^z$. That is, Lambert $W(z)$ can be any function that satisfies Equation (5) for all z. This function allows solving such functional equations as $g(z)\,\mathrm{e}^{g(z)} = z$ and $g(z) = \mathrm{e}^{\text{Lambert } W(\ln(z))}$, and $z\,\mathrm{e}^z = x$ and $z = \text{Lambert } W(x)$. Detailed discussions can be found in Corless *et al.*[1]. As shown in Section 5, this functional equation is encountered in the tolerance optimization models involving normally distributed random variables and use of the Lambert W function facilitates obtaining a closed-form solution for the optimal specification limits. The application of the Lambert W function to obtain a closed-form solution for the tolerance optimization problem shown in Equation (23) constitutes one of the principal contributions of this paper.

4. DETERMINING OPTIMAL TOLERANCE

Until recently, it has often been assumed that a product is judged acceptable and the customer loss is zero if the product performance falls within the specification limits. In addition, the manufacturing cost has often been ignored for the tolerance optimization problem. However, it seems more reasonable to assume that a quality loss is incurred by the customer even when product performance deviates from the target within the specification limits. A quadratic loss function is used to evaluate a quality loss within the specification limits in the proposed model. In addition to the loss incurred by the customer, costs incurred by the manufacturer, such as rejection cost and manufacturing cost, are also included.

Consider the quality characteristic Y that is normally distributed with mean μ and variance σ^2. Let $f(y)$ and $F(y)$ denote the probability density function and cumulative distribution function of Y, respectively. The lower and upper specification limits, *LSL* and *USL*, are defined as $\mu - \delta\sigma$ and $\mu + \delta\sigma$, respectively, where $\delta > 0$. Here, δ represents the number of standard deviations at which each specification limit is located from the process mean.

4.1. *Assessment of a quality loss*

One of the most important issues encountered in the area of quality engineering is the selection of a proper quality loss function (QLF) in order to relate a key quality characteristic of a product to its quality performance. The QLF is a means to quantify the quality loss of a product on a monetary scale when a product or its production process deviates from customer-identified target value(s) in terms of one or more key characteristics. This quality loss includes long-term losses related to poor reliability and the cost of warranties, excess inventory, customer dissatisfaction, and eventual loss of market share. The QLF functional relationship depends on the type of QLF used. Several forms of QLF have been discussed in the literature of statistical decision theory, where utility is viewed as the negative of quality loss. QLFs relate quality performance to three types of characteristics: 'smaller-the-better' (S-type), 'nominal-the-best' (N-type), and 'larger-the-better' (L-type). For N-type characteristics, there is an identified target value. On each side of this prespecified target, the performance of the product/process deteriorates as the value of the characteristic deviates further from the target value. Designers often set both lower specification limits (*LSL*s) and upper specification limits (*USL*s) for each N-type characteristic. For S-type characteristics, such as wear, deterioration, and noise level, the desired target value is zero. Here, the product/process designer is likely to set a *USL*. For L-type characteristics, such as strength, reliability, and life of the product, there is usually no predetermined nominal value. Zero quality loss is ideally attained when the characteristic assumes the target value of infinity.

The four desirable properties that have been identified for a QLF are unimodality, minimum value of quality loss at target, non-negative quality loss, and a continuous function. The QLF represented by a quasiconvex function can have all the four desirable characteristics. In-depth discussions of the features and characteristics of quasiconvex functions can be found in Roberts and Varberg[20], Avriel *et al.*[21] and Bazaraa and Shetty[22]. Recently, Taguchi reemphasized the applicability of this QLF and brought this loss function form to product and process design[23–25]. Separate optimization of quality characteristic values in terms of mean and variance using this quadratic QLF has become the cornerstone of Taguchi methods[23–25]. Examples for the use of the quadratic QLF are numerous[10,16,26–30]. Compared with other QLFs, such as step-loss and piecewise linear loss functions, the quadratic QLF may be a good approximation of measuring the quality of a product, particularly over the range of characteristic values in the neighborhood of the target value. If we let $L(y)$ be a measure of losses associated with the quality characteristic y whose target value is τ, then the quadratic loss function is given by

$$L(y) = k(y-\tau)^2 \tag{6}$$

where k is a positive coefficient, which can be determined from the information on losses relating to exceeding a given customer's tolerance.

4.2. Assessment of rejection cost

Product performance that falls outside the specification limits is rejected and the rejection cost is incurred by the manufacturer. Let C_{R} be the unit rejection cost incurred when a product falls below a lower specification limit or above an upper specification limit. The expected rejection cost $E[C_{\mathrm{R}}]$ is defined as

$$E[C_{\mathrm{R}}] = C_{\mathrm{R}}\left(\int_{-\infty}^{USL} f(y)\,\mathrm{d}y + \int_{LSL}^{\infty} f(y)\,\mathrm{d}y\right) \tag{7}$$

Assuming $USL - \mu = \mu - LSL$, Equation (7) is expressed as

$$E[C_R] = 2C_R \int_{USL}^{\infty} f(y)\,\mathrm{d}y \tag{8}$$

If Y is a normally distributed random variable, then using the transformation of $z = (y-\mu)/\sigma$, Equation (8) simplifies to

$$E[C_{\mathrm{R}}] = 2C_{\mathrm{R}} \int_{\delta}^{\infty} \phi(z)\,\mathrm{d}z = 2C_{\mathrm{R}}[1-\Phi(\delta)] \tag{9}$$

where $\phi(\cdot)$, $\Phi(\cdot)$, and z denote a standard normal density function, a cumulative normal distribution, and a standard normal random variable, respectively.

4.3. Assessment of manufacturing cost

Additional manufacturing operations, slow processing rates, and additional care on the part of the operator can increase the manufacturing cost incurred in order to achieve a tight tolerance. The manufacturing cost usually contributes to a significant portion of the unit cost for many products and its exclusion from the tolerance optimization model may result in a suboptimal tolerance. Enforcing the 3σ restriction[2–5,31–33] may require selecting an expensive process when tolerance selection is performed for the purposes of process selection. The manufacturing cost–tolerance relationship proposed in this paper is free of this *ad hoc* 3σ assumption. Tolerance is defined in terms of δ, μ and σ as

$$t = USL - LSL = (\mu + \delta\sigma) - (\mu - \delta\sigma) = 2\delta\sigma \tag{10}$$

where σ for the process is known beforehand and δ is the decision variable (see Section 5). The manufacturing cost is described by $C_M = a_0 + a_1 t + \varepsilon$, where ε represents the least-squares regression error. The expected manufacturing cost can then be written as $E[C_M] = a_0 + a_1 t$. This linear manufacturing cost–tolerance modeling is often applied in the literature (see Patel[34], Bjorke[35], and Chase and Parkinson[36]). Substituting $t = 2\delta\sigma$, $E[C_M]$ becomes

$$E[C_M] = a_0 + 2a_1\delta\sigma \tag{11}$$

5. PROPOSED MODEL

The expected total cost, $E[TC]$, is now given by

$$\begin{aligned} E[TC] &= E[L(y)] + E[C_R] + E[C_M] \\ &= E[L(y)] + [P(Y \le -t) + P(Y \le t)]C_R + E[C_M] \end{aligned} \tag{12}$$

Using Equations (6), (9), and (11), the proposed optimization model is formulated as

$$\text{Minimize } E[TC] = \int_{LSL}^{USL} L(y) f(y)\,\mathrm{d}y + 2C_R \int_{USL}^{\infty} f(y)\,\mathrm{d}y + a_0 + a_1 t \tag{13}$$

The expected quality loss within specification limits can be written as

$$\int_{LSL}^{USL} L(y) f_Y(y)\,\mathrm{d}y = k\left[\int_{LSL}^{USL} y^2 f(y)\,\mathrm{d}y - 2\tau \int_{LSL}^{USL} y f(y)\,\mathrm{d}y + \tau^2 \int_{LSL}^{USL} f(y)\,\mathrm{d}y\right] \tag{14}$$

where k denotes a loss coefficient. Letting $z = (y - \mu)/\sigma$, Equation (14) can be rewritten as

$$\begin{aligned} \int_{LSL}^{USL} L(y) f(y)\,\mathrm{d}y &= k\left[\int_{-\delta}^{\delta} (\mu + z\sigma)^2 \phi(z)\,\mathrm{d}z - \int_{-\delta}^{\delta} 2\tau(\mu + 2\sigma)\phi(z)\,\mathrm{d}z + \tau^2 \int_{-\delta}^{\delta} \phi(z)\,\mathrm{d}z\right] \\ &= k\left[\mu^2 \int_{-\delta}^{\delta} \phi(z)\,\mathrm{d}z + 2\mu\sigma \int_{-\delta}^{\delta} z\phi(z)\,\mathrm{d}z + \sigma^2 \int_{-\delta}^{\delta} z^2\phi(z)\,\mathrm{d}z \right. \\ &\quad \left. - 2\tau\mu \int_{-\delta}^{\delta} \phi(z)\,\mathrm{d}z - 2\tau\sigma \int_{-\delta}^{\delta} \phi(z)\,\mathrm{d}z + \tau^2 \int_{-\delta}^{\delta} \phi(z)\,\mathrm{d}z\right] \end{aligned} \tag{15}$$

where $\delta = (USL - \mu)/\sigma (> 0)$. In order to simplify Equation (15), the derivations associated with the normal probability density function are utilized as

$$\int_r^{\infty} \phi(z)\,\mathrm{d}z = 1 - \Phi(r), \quad \int_r^{\infty} z\phi(z)\,\mathrm{d}z = \phi(r) \quad \text{and} \quad \int_r^{\infty} z^2\phi(z)\,\mathrm{d}z = 1 - \Phi(r) + r\phi(r) \tag{16}$$

The detailed proofs related to Equation (16) are given in Appendix A. Using Equation (16), $E[L(y)]$ can be written as follows:

$$\int_{USL}^{LSL} L(y) f(y)\,\mathrm{d}y = k[\{1 - 2\Phi(-\delta)\}\{\mu^2 + \sigma^2 - 2\tau\mu + \tau^2\} - 2\delta\sigma^2\phi(\delta)] \tag{17}$$

Similarly, $E[C_{\mathrm{R}}]$ cost can be written as

$$E[C_{\mathrm{R}}] = 2C_{\mathrm{R}}[P(Y \le -t) + P(Y \le t)] = 2C_{\mathrm{R}} \int_{USL}^{\infty} f(y)\,\mathrm{d}y = 2C_{\mathrm{R}}[1 - \Phi(\delta)] \tag{18}$$

Further, $E[C_{\mathrm{M}}]$ is defined as $E[C_{\mathrm{M}}] = a_0 + 2a_1\delta\sigma$. The expected total cost can now be written as

$$\text{Minimize} \quad E[TC] = \Phi(\delta)[2k\{(\mu-\tau)^2 + \sigma^2\} - 2C_{\mathrm{R}}] - 2k\delta\phi(\delta) - k\{(\mu-\tau)^2 + \sigma^2\} + 2C_{\mathrm{R}} + a_0 + 2a_1\sigma\delta \tag{19}$$

To investigate the optimum, the first derivative with respect to δ is calculated as follows:

$$\frac{\partial E[TC]}{\partial \delta} = 2k\phi(\delta)\left[(\mu-\tau)^2 + \sigma^2 - 1 - \frac{C_{\mathrm{R}}}{k} + \delta^2\right] + 2a_1\sigma \tag{20}$$

Equating $\partial E[TC]/\partial\delta$ to zero and substituting

$$\phi(\delta) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-1/2\delta^2}$$

Equation (20) becomes

$$2k\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-1/2\delta^2}\left[\delta^2 + (\mu-\tau)^2 + \sigma^2 - 1 - \frac{C_{\mathrm{R}}}{k}\right] + 2a_1\sigma = 0 \tag{21}$$

The Lambert W function is used to obtain a closed-form solution for δ. The basis of the Lambert W function is established in Lemma 1, and Propositions 1 and 2, as shown in Appendix B. Proposition 2 states that if $\eta_4 = \eta_1(\chi + \eta_2)\,\mathrm{e}^{\eta_3\chi}$ where η_1, η_2, η_3, and η_4 are not functions of χ, then the solution of χ is given by

$$\chi = \pm\left(\frac{\text{Lambert } W(\eta_3\eta_4\,\mathrm{e}^{\eta_2\eta_3}/\eta_1)}{\eta_3} - \eta_2\right)^{1/2} \tag{22}$$

The left-hand side in Equation (21) can be expressed in the form of $\eta_4 = \eta_1(\chi + \eta_2)\,\mathrm{e}^{\eta_3\chi}$ using such substitutions as $\chi = \delta$, $\eta_1 = 2k/\sqrt{2\pi}$, $\eta_2 = (\mu-\tau)^2 + \sigma^2 - 1 - (C_{\mathrm{R}}/k)$, $\eta_3 = -1/2$, and $\eta_4 = 2a_1\sigma$. The closed-form solution of δ can then be obtained by substituting η_1, η_2, η_3, and η_4 into Equation (22). Thus, the optimal value of δ is

$$\delta^* = \pm\left(-2 \text{ Lambert } W\left(\frac{\sqrt{2\pi}a_1\sigma\,\mathrm{e}^{(-\frac{1}{2}((\mu-\tau)^2+\sigma^2-1-C_{\mathrm{R}}/k))}}{2k}\right) + 2\left\{(\mu-\tau)^2 + \sigma^2 - 1 - \frac{C_{\mathrm{R}}}{k}\right\}\right)^{1/2} \tag{23}$$

That is, the optimal LSL and USL are obtained as $LSL^* = \mu - \delta^*\sigma$ and $USL^* = \mu + \delta^{\sigma}$. The Lambert W function is available in a number of standard optimization softwares, such as Maple and Matlab.

5.1. *Investigation of the second derivative and the conditions for convexity*

In this section, the second derivative is computed and the conditions for obtaining the minimum value of $E[TC]$ are investigated. The second derivative of $E[TC]$ with respect to δ is

$$\frac{\partial^2 E[TC]}{\partial^2\delta} = \phi(\delta)\left[4k\delta - 2k\delta\left[(\mu-\tau)^2 + \sigma^2 - 1 - \frac{C_{\mathrm{R}}}{k} + \delta^2\right]\right] \tag{24}$$

Equation (24) can be simplified as

$$2k\delta\phi(\delta)\left[3 - \delta^2 - (\mu-\tau)^2 - \sigma^2 + \frac{C_{\mathrm{R}}}{k}\right]$$

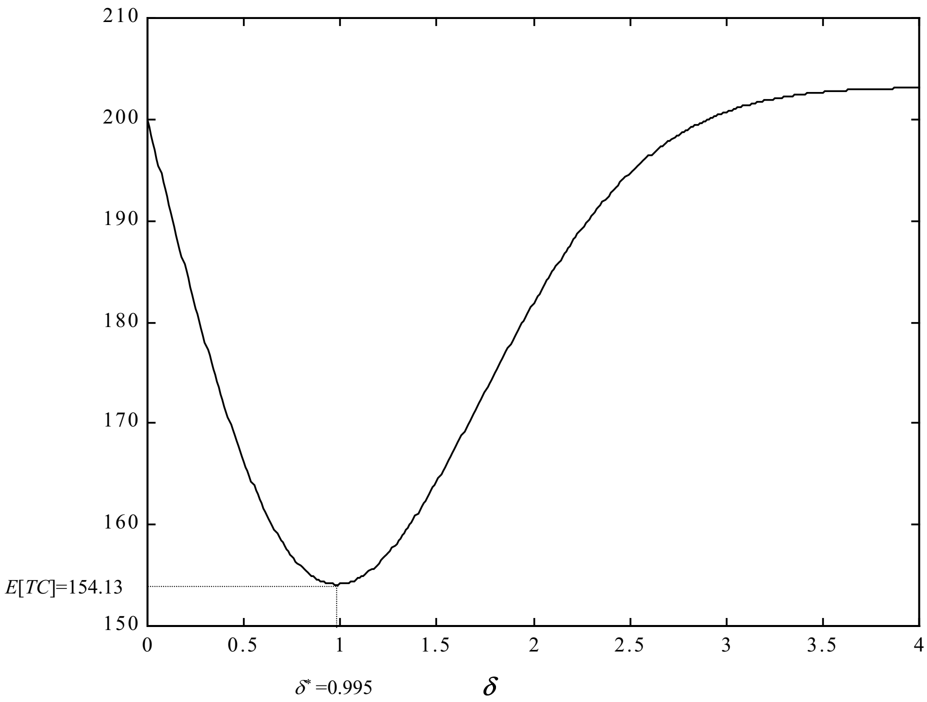

Figure 1. Relationship between $E[TC]$ and δ

After some algebra, the two conditions

$$\delta \leq \left(3 + \frac{C_R}{k} - \{(\mu - \tau)^2 + \sigma^2\}\right)^{1/2} \quad \text{and} \quad 3 + \frac{C_R}{k} - \{(\mu - \tau)^2 + \sigma^2\} \geq 0$$

need to be met in order to obtain the minimum value of $E[TC]$ at the stationary points. In many industrial situations, C_R is very large, compared with $\mu - \tau$ and σ^2 (see Phillips and Cho[37]).

6. NUMERICAL EXAMPLE

An electronic chip manufacturing company is experiencing high warranty costs and customer dissatisfaction associated with component failures in a prime product. In view of the high costs associated with the failure of the components, the company has decided to implement a 100% inspection on a key quality characteristic Y. The quality characteristic is normally distributed with a mean of 49.8 and a standard deviation of 1.0. A quality loss incurred due to a deviation from the target value of the quality characteristic is given by the quadratic function of $100(y - 50)^2$. Components with Y less than LSL and greater than USL are rejected, and $\mu = 49.8$, $\sigma = 1.0$, $\tau = 50$, $C_R = 100$, and $k = 100$.

Using regression analysis, the relationship between the manufacturing cost and the tolerance was described by the polynomial model $C_M = 100 - 0.2\delta\sigma$. Using the closed-form solution given in Equation (29), δ^* is 0.995. This optimal value can be verified from the plot of the expected total cost shown in Figure 1, where the minimum value of $E[TC]$ at $\delta^* = 0.995$ is 154.13. For $\delta^* = 0.995$, $E[L(y)]$, $E[C_R]$, $E[C_M]$, and $E[TC]$ are 22.36, 31.79, 99.98, and 154.13, respectively, and the percentages of $E[L(y)]$, $E[C_R]$, and $E[C_M]$ versus $E[TC]$ are 14, 21, and 65%, respectively. Although manufacturing cost is commonly ignored in many tolerance optimization models, it may not be a good assumption, because it is evident from this particular example that a

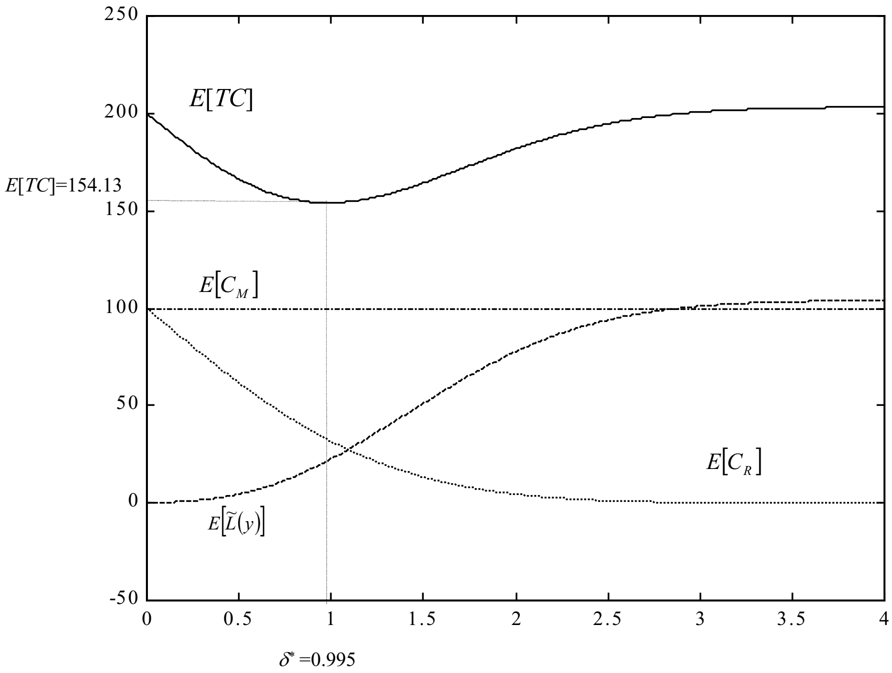

Figure 2. Plots of $E[C_{\mathrm{R}}]$, $E[L(y)]$, and $E[C_{\mathrm{M}}]$ with respect to δ

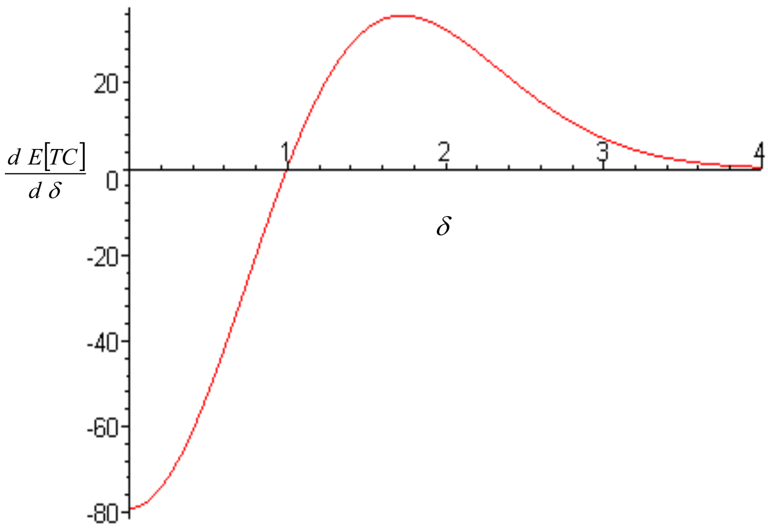

Figure 3. Plot of $\mathrm{d}E[TC]/\mathrm{d}\delta$ with respect to δ

failure to consider the manufacturing cost in tolerance optimization is likely to result in a suboptimal tolerance. Figure 2 shows how $E[TC]$, $E[C_{\mathrm{M}}]$, $E[C_{\mathrm{R}}]$ and $E[L(y)]$ vary as δ changes.

It can be seen from Figure 3 that the first derivative of the expected total cost becomes zero at the optimal value of $\delta^* = \pm 0.995$. However, the negative value is ignored since δ is always greater than zero. The conditions of convexity of the $E[TC]$ function discussed in Section 5.1 are shown in Figure 4, is which a plot of $\partial^2 E[TC]/\partial^2 \delta$

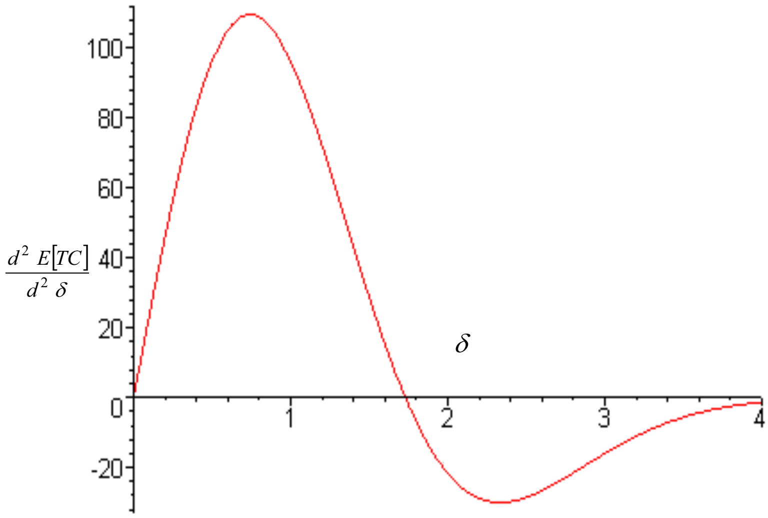

Figure 4. Plot of $\partial^2 E[TC]/\partial^2\delta$ with respect to δ

Table I. Effect of σ on δ^*

σ	μ	$\lvert\mu-\tau\rvert$	δ^*	$E[C_M]$	$E[C_R]$	$E[\tilde{L}(y)]$	$E[TC]$
1.0	49.8	0.2	2.23	99.55	12.67	86.67	198.89
1.1	49.8	0.2	2.19	99.52	14.32	105.44	219.28
1.2	49.8	0.2	2.13	99.49	16.39	125.65	241.53
1.3	49.8	0.2	2.07	99.46	19.00	147.14	265.61
1.4	49.8	0.2	2.01	99.44	22.32	169.69	291.45
1.5	49.8	0.2	1.93	99.42	26.58	193.00	319.00
1.6	49.8	0.2	1.85	99.41	32.09	216.65	348.15
1.7	49.8	0.2	1.76	99.40	39.31	240.05	378.76
1.8	49.8	0.2	1.65	99.40	48.93	262.32	410.65
1.9	49.8	0.2	1.54	99.42	61.95	282.18	443.54
2.0	49.8	0.2	1.40	99.44	79.97	297.60	477.01

Table II. Effect of δ^* on $\lvert\mu-\tau\rvert$

μ	$\lvert\mu-\tau\rvert$	σ	δ^*	$E[C_M]$	$E[C_R]$	$E[\tilde{L}(y)]$	$E[TC]$
49.0	1.0	1.0	2.00	99.60	22.42	169.59	291.61
49.1	0.9	1.0	2.05	99.59	20.01	153.82	273.42
49.2	0.8	1.0	2.09	99.58	18.08	139.40	257.06
49.3	0.7	1.0	2.13	99.57	16.54	126.47	242.58
49.4	0.6	1.0	2.16	99.57	15.31	115.12	230.00
49.5	0.5	1.0	2.19	99.56	14.34	105.42	219.32
49.6	0.4	1.0	2.21	99.56	13.60	97.42	210.57
49.7	0.3	1.0	2.22	99.56	13.05	91.16	203.76
49.8	0.2	1.0	2.23	99.55	12.67	86.67	198.89
49.9	0.1	1.0	2.24	99.55	12.45	83.97	195.96
50.0	0.0	1.0	2.24	99.55	12.37	83.06	194.99

Table III. Relationship between costs and δ

δ	σ	μ	$\lvert\mu-\tau\rvert$	$E[C_M]$	$E[C_R]$	$E[\tilde{L}(y)]$	$E[TC]$
1.0	1.0	49.8	0.2	99.80	158.61	22.62	281.02
1.2	1.1	49.8	0.2	99.76	115.01	33.47	248.25
1.4	1.2	49.8	0.2	99.72	80.70	45.29	225.71
1.6	1.3	49.8	0.2	99.68	54.73	57.12	211.54
1.8	1.4	49.8	0.2	99.64	35.86	68.12	203.62
2.0	1.5	49.8	0.2	99.60	22.68	77.69	199.97
2.2	1.6	49.8	0.2	99.56	13.83	85.51	198.91
2.4	1.7	49.8	0.2	99.52	8.13	91.56	199.21
2.6	1.8	49.8	0.2	99.48	4.59	95.98	200.05
2.8	1.9	49.8	0.2	99.44	2.48	99.05	200.97
3.0	2.0	49.8	0.2	99.40	1.28	101.08	201.75

is shown where the second derivative is greater than zero in the interval $0 \le \delta \le 1.795$. This implies that $E[TC]$ is a convex function and the local minimum becomes the global minimum in the specified interval.

The sensitivity of the optimal specification limit to the changes in the mean and standard deviation are shown in Tables I and II, respectively. It can be observed that δ^* gradually decreases as σ increases. The optimal *USL* and *LSL* are 4.46 and 5.6 when $\sigma = 1.0$ and $\sigma = 5.6$, respectively. When $\sigma = 2.0$, $E[L(y)]$ becomes larger, since an increase in tolerance implies a lower outgoing product quality. $E[C_R]$ is dependent only on δ^*, and therefore the lowest value of $E[C_R]$ is obtained when $\delta^* = 2.23$ and the highest value is obtained when $\delta^* = 1.40$. In other words, $E[C_R]$ is higher for lower values of δ^* and *vice versa*. Table I shows that $E[C_M]$ depends on δ and σ and is the lowest when σ is between 1.7 and 1.8. The effect of $|\mu - \tau|$ (i.e. process bias) on δ^* is shown in Table II. Finally, Table III shows the effect of $E[C_M]$, $E[C_R]$, $E[L(y)]$, and $E[TC]$ to the changes in δ and σ. The Maple program for the numerical example and sensitivity analysis is shown in Appendix C.

7. CONCLUSIONS AND FURTHER STUDY

Quality engineers are often faced with the problem of determining the optimal tolerance in many industrial settings. When determining the tolerance, it is important to consider a trade-off between costs incurred by a manufacturer and the customer. In this paper, we showed how these costs are quantified and integrated, and also showed how the Lambert *W* function is incorporated into this tolerance optimization problem by deriving a closed-form solution. This proposed model and solution may be appealing to quality practitioners since the Lambert function is found in a number of standard optimization softwares. Finally, we believe that the extension of the proposed model to the case of more than one performance measure within a multivariate context may be a good future study.

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APPENDIX A

First, we consider the indefinite integral of the integrand $z\phi(z)\,\mathrm{d}z$. Following the definition of a standard normal random variable, the integral can be written as

$$\int z\phi(z)\,\mathrm{d}z = \frac{1}{\sqrt{2\pi}}\int z\,\mathrm{e}^{-z^2/2}\,\mathrm{d}z$$

Substituting u into z^2, it follows that $\mathrm{d}u = 2z\,\mathrm{d}z$ and $z\,\mathrm{d}z = \mathrm{d}u/2$. Thus the integral can now be written as follows:

$$\int z\phi(z)\,\mathrm{d}z = \frac{1}{2\sqrt{2\pi}}\int \mathrm{e}^{-u/2}\,\mathrm{d}u$$

As $\int \mathrm{e}^{-u/2}\,\mathrm{d}u = -\mathrm{e}^{-u/2}$, we get

$$\int z\phi(z)\,\mathrm{d}z = \frac{1}{2\sqrt{2\pi}}\int \mathrm{e}^{-u/2}\,\mathrm{d}u = -\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-u/2} = -\phi(z)$$

Next, we consider the indefinite integral of $z^2\phi(z)\,\mathrm{d}z$. This integration can be performed by using the formula for the method of integration by parts given by

$$\int uv\,\mathrm{d}x = v\int \frac{\mathrm{d}u}{\mathrm{d}x} - \int\left[\frac{\mathrm{d}u}{\mathrm{d}x}\int v\,\mathrm{d}x\right]\mathrm{d}x$$

If $z^2\phi(z)\,\mathrm{d}z$ is separated into two parts, namely, z and $z\phi(z)$, then the integration by parts formula can be applied as follows:

$$\begin{aligned}\int z^2\phi(z)\,\mathrm{d}z &= \int z(z\phi(z))\,\mathrm{d}z\\ &= z\int z\phi(z)\,\mathrm{d}z - \int \phi(z)\,\mathrm{d}z\\ &= z\phi(z) - \Phi(z)\end{aligned}$$

APPENDIX B

Lemma 1. *Suppose $\chi \in R^1$ and the mapping $\eta : \chi \to R$ be $\eta = \chi\, e^{\chi}$, then the solution for χ is given by $\chi = \text{Lambert } W(\eta)$.*

Proof. See Corless *et al.*[1]. □

Proposition 1. *If $\eta_2 = (\chi + \eta_1)\,\mathrm{e}^{\chi}$, where η_1 and η_2 are not functions of χ, then $\chi = \text{Lambert } W(\eta_2\,\mathrm{e}^{\eta_1}) - \eta_1$.*

Proof. The proposition can be proved using Lemma 1 starting from Equation (B1) given by

$$\eta_2 = (\chi + \eta_1)\,\mathrm{e}^{\chi} \tag{B1}$$

Letting $\chi + \eta_1 = \psi$, Equation (B1) becomes

$$\eta_2 = \psi\,\mathrm{e}^{\psi - \eta_1} \tag{B2}$$

To convert Equation (B2) to the standard form $\eta = \chi\, e^{\chi}$, we first modify Equation (B2) as follows:

$$\eta_2\, e^{\eta_1} = \psi\, e^{\psi - \eta_1} \tag{B3}$$

Next, substituting $\eta_2\, e^{\eta_1} = \omega$ into the right-hand side, we obtain the standard form as $\omega = \psi\, e^{\psi - \eta_1}$. Now, recalling Lemma 1, ψ can be given by

$$\psi = \text{Lambert } W(\omega) \tag{B4}$$

Therefore, the solution for χ can be obtained as $\chi = \text{Lambert } W(\eta_2\, e^{\eta_1}) - \eta_1$. □

Proposition 2. *If $\eta_4 = \eta_1(\chi + \eta_2)\, e^{\eta_3 \chi}$, where η_1, η_2, η_3, and η_4 are not functions of χ, then*

$$\chi = \pm \left(\frac{\text{Lambert } W(\eta_3 \eta_4\, e^{\eta_2 \eta_3}/\eta_1)}{\eta_3} - \eta_2 \right)^{1/2}$$

Proof. In order to prove Proposition 2 using Lemma 1, we first consider the equation

$$\eta_4 = \eta_1(\chi + \eta_2)\, e^{\eta_3 \chi} \tag{B5}$$

Multiplying both sides with η_3/η_1, Equation (B5) becomes

$$\frac{\eta_3 \eta_4}{\eta_1} = \eta_3(\chi + \eta_2)\, e^{\eta_3 \chi} \tag{B6}$$

Letting $\psi = \eta_3(\chi + \eta_2)$, Equation (B6) can be written as

$$\frac{\eta_3 \eta_4\, e^{\eta_2 \eta_3}}{\eta_1} = \psi\, e^{\psi_1} \tag{B7}$$

Further, substituting $(\eta_3 \eta_4\, e^{\eta_2 \eta_3})/\eta_1 = \omega$, Equation (B7) is now in the standard form $\eta = \chi\, e^{\chi}$.

$$\psi = \text{Lambert } W(\omega) \tag{B8}$$

Replacing $\omega = (\eta_3 \eta_4\, e^{\eta_2 \eta_3})/\eta_1$ back into Equation (B8), the standard form in Equation (B8) can be written as

$$\psi = \text{Lambert } W \left(\frac{\eta_3 \eta_4\, e^{\eta_2 \eta_3}}{\eta_1} \right) \tag{B9}$$

Equation (B9) can be expanded to express the solution in terms of χ by substituting ψ with $\eta_3(\chi + \eta_2)$. Thus, the solution for χ is given by

$$\chi = \pm \left(\frac{\text{Lambert } W(\eta_3 \eta_4\, e^{\eta_2 \eta_3}/\eta_1)}{\eta_3} - \eta_2 \right)^{1/2}$$

APPENDIX C

Maple program for the numerical example

```
NORMALPDF:= exp( -(y-0)^2/2/1^2 ) / sqrt (2*3.141592*1^2);
NORMALCDF:= Int(NORMALPDF, y = -infinity .. x );
QUALITY LOSS:= Int(NORMALPDF, y = -infinity..x );
EXPECTED QUALITY LOSS:= Int(QUALITY LOSS * NORMALPDF, y=-x..x);
plot(EXPECTED QUALITY LOSS , x=0..4);
REJECTION COST:= 2*c*(1- NORMALCDF);
plot(REJECTION COST , x=0..4);
MANUFACTURING COST:= a0+2*a1*s*x;
plot(MANUFACTURING COST , x=0..4);
EXPECTED TOTAL COST:= EXPECTED QUALITY LOSS + REJECTION COST +
  MANUFACTURING COST;
plot((EXPECTED QUALITY LOSS, REJECTION COST, MANUFACTURING COST,
  EXPECTED TOTAL COST], x=0..4);
FIRST DERIVATIVE:= diff(EXPECTED TOTAL COST , x);
n1:= 2*k/sqrt(2*pi);
n2:= (u+t)^2+s^2-1-c/k;
n3:= -1/2;
n4:= 2*a1*s;
FIRST DERIVATIVE 1:= n1*(x^2+n2)*exp(n3*x^2)-n4=0;
SOLVE1:= solve(FIRST DERIVATIVE 1 , x);
plot(SOLVE1, x=0..4);
SECOND DERIVATIVE:= diff(FIRST DERIVATIVE 1 , x);
SOLVE2:= solve(SECOND DERIVATIVE , x);
plot(SOLVE2, x=0..4);
```

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