EC516 HW6 Solutions

Problem 6.1

(a)

$$
\begin{aligned}
H_a(j\tan(\pi/4)) &= \frac{1}{1+(\tan(\pi/4)/\omega_c)^{2N}} = (1-\delta_p)^2 \\
\frac{1}{(1-\delta_p)^2} &= 1 + (\tan(\pi/4)/\omega_c)^{2N} \\
(\tan(\pi/4)/\omega_c)^{2N} &= \frac{1}{(1-\delta_p)^2} - 1
\end{aligned}
$$

(b)

$$
\begin{aligned}
H_a(j\tan(3\pi/8)) &= \frac{1}{1+(\tan(3\pi/8)/\omega_c)^{2N}} = \delta_s^2 \\
\frac{1}{\delta_s^2} &= 1 + (\tan(3\pi/8)/\omega_c)^{2N} \\
(\tan(3\pi/8)/\omega_c)^{2N} &= \frac{1}{\delta_s^2} - 1
\end{aligned}
$$

(c)

$$
\begin{aligned}
(\tan(\pi/4)/\omega_c)^{2N}(\tan(3\pi/8)/\omega_c)^{-2N} &= \left(\frac{1}{(1-\delta_p)^2} - 1\right)\left(\frac{1}{\delta_s^2} - 1\right)^{-1} \\
(\tan(\pi/4)/\tan(3\pi/8))^{2N} &= \left(\frac{1}{(1-\delta_p)^2} - 1\right)\left(\frac{1}{\delta_s^2} - 1\right)^{-1} \\
N &= \frac{1}{2}\frac{\log\left(\left(\frac{1}{(1-\delta_p)^2} - 1\right)/\left(\frac{1}{\delta_s^2} - 1\right)\right)}{\log(\tan(\pi/4)/\tan(3\pi/8))}
\end{aligned}
$$

Plugging in the values we get $N \approx 1.679$ Rounding it up we get $N = 2$.

(d)

$$
\omega_c = \tan(\pi/4)\left(\frac{1}{(1-\delta_p)^2} - 1\right)^{-\frac{1}{2N}}
$$

Plugging $N = 2$ into the equation we get $\omega_c \approx 1.065$.

(e) We can write -1 as $e^{j\pi+2\pi k}$ for any integer k. Using this we can solve for the poles.

$$
\begin{aligned}
\left(\frac{s}{j\omega_c}\right)^{2N} &= e^{j\pi+j2\pi k} \\
\frac{s}{j\omega_c} &= e^{\frac{j(2k+1)\pi}{2N}} \\
s &= j\omega_c e^{\frac{j(2k+1)\pi}{2N}} \\
s &= \omega_c e^{\frac{j(2k+1)\pi}{2N}+j\frac{\pi}{2}} \\
s &= \omega_c e^{\frac{j(N+2k+1)\pi}{2N}}
\end{aligned}
$$

After plugging in $\omega_c = 1.065$ and $N = 2$, we find that the poles exist at $s = 1.065e^{j3\pi/4}$ and $1.065e^{-j3\pi/4}$.

(f) Before we write the difference equation, we need to get $H_d(z)$. We can get $H_d(z)$ by finding its poles and zeros.

For finding poles,we can use the equation $s = \frac{1-z^{-1}}{1+z^{-1}}$ to show that

$$
z = \frac{1+s}{1-s}
$$

The poles that we found in part (e) is in s-domain. In z-domain, the poles are in

$$
\frac{1+1.065e^{j3\pi/4}}{1-1.065e^{j3\pi/4}}, \qquad \frac{1+1.065e^{-j3\pi/4}}{1-1.065e^{-j3\pi/4}}
$$

which are approximately,

$$
0.415e^{j1.660}, \qquad 0.415e^{-j1.660}
$$

For zeros, we first find the zeros in s-domain. $|H_a(s)| = 0$ occurs for $|s| = \infty$. This maps to $z = -1$.

Now that we found the poles and zeros, we can write the frequency response. $H_d(z)$ equals to the following expression for some constant C,

$$
\begin{aligned}
H_d(z) &\approx C \cdot \frac{(z+1)^2}{\left(z-0.415e^{j1.660}\right)\left(z-0.415e^{-j1.660}\right)} \\
&\approx C \cdot \frac{z^2+2z+1}{z^2-0.830\cos(1.660)z+0.172} \\
&\approx C \cdot \frac{z^2+2z+1}{z^2+0.074z+0.373} \\
&= C \cdot \frac{1+2z^{-1}+z^{-2}}{1+0.074z^{-1}+0.373z^{-2}}
\end{aligned}
$$

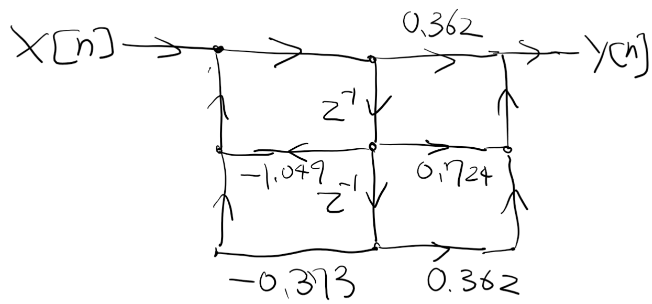
C is a constant that satisfies $H_d(1) = 1$. Thus,

$$
H_d(z) \approx= \frac{0.362+0.724z^{-1}+0.362z^{-2}}{1+0.074z^{-1}+0.373z^{-2}}
$$

Hence the difference equation is

$$y[n] = 0.362x[n] + 0.724x[n-1] + 0.362x[n-2] - 1.049y[n-1] - 0.373y[n-2]$$

Problem 6.2 Direct Form II implementation is shown below.



Flowgraph for Digital Butterworth filter from Probelm 6.1

Problem 6.3 Note that the specification is on the *digital* filter. Recall that

$$\frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \tan(\omega/2)$$

The passband frequency $\omega_p = \pi/2$ and the stopband frequency $\omega_s = 3\pi/4$ will be $\tan(\pi/4)$ and $\tan(3\pi/8)$ in the s-domain, respectively.

(c) First let's use the equation we found from 6.1(c) to solve for N.

$$N = \frac{1}{2}\frac{\log\left(\left(\frac{1}{(1-0.1)^2} - 1\right) / \left(\frac{1}{0.1^2} - 1\right)\right)}{\log(\tan(\pi/4)/\tan(3\pi/8))}$$

Plugging in $\delta_p = \delta_s = 0.1$, we get

$$N \approx 3.429$$

Round it up and we get $N = 4$.

(d) Now we find ω_c.

$$\begin{aligned}\omega_c &= \tan(\pi/4)\left(\frac{1}{(1-0.1)^2} - 1\right)^{-\frac{1}{2\cdot 4}} \\ &\approx 1.199\end{aligned}$$

(e) The poles (in s-domain) are in

$$1.199e^{j5\pi/8}, 1.199e^{j7\pi/8}, 1.199e^{-j7\pi/8}, 1.199e^{-j5\pi/8}$$

(f) The poles in z-domain are approximately

$$0.673e^{j1.766}, 0.218e^{j2.016}, 0.218e^{-j2.016}, 0.673e^{-j1.766}$$

The frequency response of the digital filter is

$$\begin{aligned}
H_d(z) &\approx C \cdot \frac{(z+1)^4}{(z-0.673e^{j1.766})(z-0.218e^{j2.016})(z-0.218e^{-j2.016})(z-0.673e^{-j1.766})} \\
&\approx C \cdot \frac{(z+1)^4}{(z^2-1.346\cos(1.766)z+0.453)(z^2-0.436\cos(2.016)z+0.048)} \\
&\approx C \cdot \frac{(z+1)^4}{(z^2+0.261z+0.453)(z^2+0.188z+0.048)} \\
&= C \cdot \frac{z^4+4z^3+6z^2+4z+1}{z^4+0.449z^3+0.550z^2+0.098z+0.022} \\
&= C \cdot \frac{1+4z^{-1}+6z^{-2}+4z^{-3}+z^{-4}}{1+0.449z^{-1}+0.550z^{-2}+0.098z^{-3}+0.022z^{-4}}
\end{aligned}$$

C that satisfies $H_d(1) = 1$ is 0.132. Thus,

$$H_d(z) \approx \frac{0.132+0.528z^{-1}+0.792z^{-2}+0.528z^{-3}+0.132z^{-4}}{1+0.449z^{-1}+0.550z^{-2}+0.098z^{-3}+0.022z^{-4}}$$

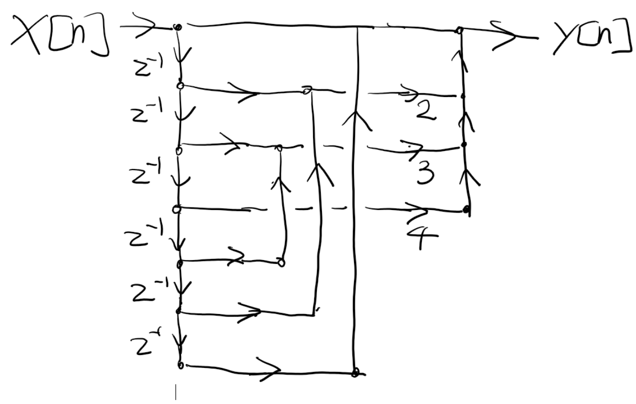
Hence the difference equation is

$$\begin{aligned}
y[n] = {} & 0.132x[n]+0.528x[n-1]+0.792x[n-2]+0.132x[n-3]+0.132x[n-4] \\
& -0.449y[n-1]+0.550y[n-2]+0.098y[n-3]+0.022y[n-4]
\end{aligned}$$

Problem 6.4

(a) Notice that some of the terms share the same coefficients. We can reduce the number of multiplication by adding some of the terms together and multiplying, instead of multiplying individually and adding later.

$$\begin{gathered}
x[n]+2x[n-1]+3x[n-2]+4x[n-3]+3x[n-4]+2x[n-5]+x[n-6] \\
\downarrow \\
(x[n]+x[n-6])+2(x[n-1]+x[n-5])+3(x[n-2]+x[n-4])+4x[n-3]
\end{gathered}$$



Flowgraph with only 3 multiplications per output sample

(b) Fourier transform of the 4-point box function $u[n] - u[n-4]$ is

$$\mathcal{F}\{u[n] - u[n-4]\} = e^{-j3\omega/2}\frac{\sin(2\omega)}{\sin(\omega/2)}$$

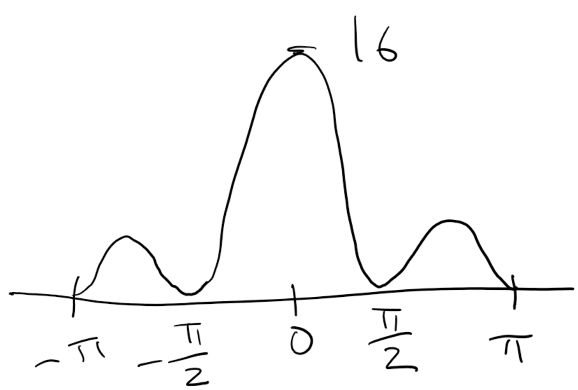
(You can check the derivation in HW2 Problem2 Part B (c)).

Since convolution is multiplication in Fourier domain, the frequency response of the filter is

$$H(e^{j\omega}) = e^{-j3\omega}\frac{\sin^2(2\omega)}{\sin^2(\omega/2)}$$

Hence the magnitude of the frequency response is

$$|H(e^{j\omega})|^2 = \frac{\sin^2(2\omega)}{\sin^2(\omega/2)}$$



Magnitude of the Frequency Response

(c) As seen previously, the phase of the frequency response is -3ω. This means that for frequency ω, there is -3ω phase is added. Because phase ϕ is equivalent to $\phi + 2\pi k$ for any integer, we are draw the phase plot for range $-\pi$ to π.

Phase of the Frequency Response