Z-TRANSFORM EXAMPLES for EE/TE 3302 (Summer '04)

NOTE:
All these problems are from the DSP (EE4361) course that was taught in Spring '04. This set includes the problems from the DSP text book. Only the problems which have a check mark are relevant to EE/TE 3302

ONLY THE CHECK MARKED PROBLEMS ARE OF INTEREST.

PROBLEMS

3.1 Determine the z-transform of the following signals.

(a) $x(n) = \{3, 0, 0, 0, 0, \underset{\uparrow}{6}, 1, -4\}$

(b) $x(n) = \begin{cases} (\frac{1}{2})^n, & n \geq 5 \\ 0, & n \leq 4 \end{cases}$

3.2 Determine the z-transforms of the following signals and sketch the corresponding pole-zero patterns.

(a) $x(n) = (1 + n)u(n)$

(b) $x(n) = (a^n + a^{-n})u(n)$, a real

(c) $x(n) = (-1)^n 2^{-n} u(n)$

(d) $x(n) = (na^n \sin \omega_0 n)u(n)$

(e) $x(n) = (na^n \cos \omega_0 n)u(n)$

(f) $x(n) = Ar^n \cos(\omega_0 n + \phi)u(n)$, $0 < r < 1$

(g) $x(n) = \frac{1}{2}(n^2 + n)(\frac{1}{3})^{n-1} u(n-1)$

(h) $x(n) = (\frac{1}{2})^n [u(n) - u(n - 10)]$

3.3 Determine the z-transforms and sketch the ROC of the following signals.

✓**(a)** $x_1(n) = \begin{cases} (\frac{1}{3})^n, & n \geq 0 \\ (\frac{1}{2})^{-n}, & n < 0 \end{cases}$

(b) $x_2(n) = \begin{cases} (\frac{1}{3})^n - 2^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

(c) $x_3(n) = x_1(n + 4)$

✓**(d)** $x_4(n) = x_1(-n)$

3.4 Determine the z-transform of the following signals.

(a) $x(n) = n(-1)^n u(n)$

(b) $x(n) = n^2 u(n)$

(c) $x(n) = -na^n u(-n - 1)$

(d) $x(n) = (-1)^n \left(\cos \frac{\pi}{3} n\right) u(n)$

(e) $x(n) = (-1)^n u(n)$

(f) $x(n) = \{\underset{\uparrow}{1}, 0, -1, 0, 1, -1, \ldots\}$

3.5 Determine the regions of convergence of right-sided, left-sided, and finite-duration two-sided sequences.

3.6 Express the z-transform of

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

in terms of $X(z)$. [*Hint:* Find the difference $y(n) - y(n - 1)$.]

✓**3.7** Compute the convolution of the following signals by means of the z-transform.

$$x_1(n) = \begin{cases} (\frac{1}{3})^n, & n \geq 0 \\ (\frac{1}{2})^{-n}, & n < 0 \end{cases}$$

$$x_2(n) = (\tfrac{1}{2})^n u(n)$$

3.8 Use the convolution property to:

(a) Express the z-transform of

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

in terms of $X(z)$.

(b) Determine the z-transform of $x(n) = (n + 1)u(n)$. [*Hint:* Show first that $x(n) = u(n) * u(n)$.]

3.9 The z-transform $X(z)$ of a real signal $x(n)$ includes a pair of complex-conjugate zeros and a pair of complex-conjugate poles. What happens to these pairs if we multiply $x(n)$ by $e^{j\omega_0 n}$? (*Hint:* Use the scaling theorem in the z-domain.)

3.10 Apply the final value theorem to determine $x(\infty)$ for the signal

$$x(n) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

3.11 Using long division, determine the inverse z-transform of

$$X(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}}$$

if **(a)** $x(n)$ is causal and **(b)** $x(n)$ is anticausal.

3.12 Determine the causal signal $x(n)$ having the z-transform

$$X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2}$$

3.13 Let $x(n)$ be a sequence with z-transform $X(z)$. Determine, in terms of $X(z)$, the z-transforms of the following signals.

(a) $x_1(n) = \begin{cases} x\left(\dfrac{n}{2}\right), & \text{if } n \text{ even} \\ 0, & \text{if } n \text{ odd} \end{cases}$

(b) $x_2(n) = x(2n)$

3.14 Determine the causal signal $x(n)$ if its z-transform $X(z)$ is given by:

(a) $X(z) = \dfrac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$

(b) $X(z) = \dfrac{1}{1-z^{-1}+\frac{1}{2}z^{-2}}$

(c) $X(z) = \dfrac{z^{-6}+z^{-7}}{1-z^{-1}}$

(d) $X(z) = \dfrac{1+2z^{-2}}{1+z^{-2}}$

(e) $X(z) = \dfrac{1}{4}\dfrac{1+6z^{-1}+z^{-2}}{(1-2z^{-1}+2z^{-2})(1-0.5z^{-1})}$

(f) $X(z) = \dfrac{2-1.5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}$

(g) $X(z) = \dfrac{1+2z^{-1}+z^{-2}}{1+4z^{-1}+4z^{-2}}$

(h) $X(z)$ is specified by a pole-zero pattern in Fig. P3.14. The constant $G = \frac{1}{4}$.

(i) $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}$

(j) $X(z) = \dfrac{1-az^{-1}}{z^{-1}-a}$

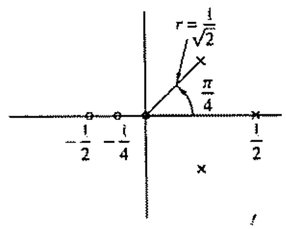


Figure P3.14

3.15 Determine all possible signals $x(n)$ associated with the z-transform

$$X(z) = \frac{5z^{-1}}{(1-2z^{-1})(3-z^{-1})}$$

3.16 Determine the convolution of the following pairs of signals by means of the z-transform.

(a) $x_1(n) = (\frac{1}{4})^n u(n-1), \quad x_2(n) = [1+(\frac{1}{2})^n]u(n)$

(b) $x_1(n) = u(n), \quad x_2(n) = \delta(n) + (\frac{1}{2})^n u(n)$

(c) $x_1(n) = (\frac{1}{2})^n u(n), \quad x_2(n) = \cos \pi n u(n)$

(d) $x_1(n) = nu(n), \quad x_2(n) = 2^n u(n-1)$

3.17 Prove the final value theorem for the one-sided z-transform.

3.18 If $X(z)$ is the z-transform of $x(n)$, show that:

(a) $Z\{x^*(n)\} = X^*(z^*)$

(b) $Z\{\text{Re}[x(n)]\} = \frac{1}{2}[X(z) + X^*(z^*)]$

(c) $Z\{\text{Im}[x(n)]\} = \frac{1}{2}[X(z) - X^*(z^*)]$

(d) If

$$x_k(n) = \begin{cases} x\left(\frac{n}{k}\right), & \text{if } n/k \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

then

$$X_k(z) = X(z^k)$$

(e) $Z\{e^{j\omega_0 n}x(n)\} = X(ze^{-j\omega_0})$

3.19 By first differentiating $X(z)$ and then using appropriate properties of the z-transform, determine $x(n)$ for the following transforms.

(a) $X(z) = \log(1-2z), \quad |z| < \frac{1}{2}$

(b) $X(z) = \log(1-z^{-1}), \quad |z| > \frac{1}{2}$

3.20 **(a)** Draw the pole-zero pattern for the signal

$$x_1(n) = (r^n \sin \omega_0 n)u(n) \qquad 0 < r < 1$$

(b) Compute the z-transform $X_2(z)$, which corresponds to the pole-zero pattern in part (a).

(c) Compare $X_1(z)$ with $X_2(z)$. Are they indentical? If not, indicate a method to derive $X_1(z)$ from the pole-zero pattern.

3.21 Show that the roots of a polynomial with real coefficients are real or form complex-conjugate pairs. The inverse is not true, in general.

3.22 Prove the convolution and correlation properties of the z-transform using only its definition.

3.23 Determine the signal $x(n)$ with z-transform

$$X(z) = e^z + e^{1/z} \qquad |z| \neq 0$$

3.24 Determine, in closed form, the causal signals $x(n)$ whose z-transforms are given by:

(a) $X(z) = \dfrac{1}{1+1.5z^{-1}-0.5z^{-2}}$

(b) $X(z) = \dfrac{1}{1-0.5z^{-1}+0.6z^{-2}}$

Partially check your results by computing $x(0)$, $x(1)$, $x(2)$, and $x(\infty)$ by an alternative method.

3.25 Determine all possible signals that can have the following z-transforms.

(a) $X(z) = \dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$

(b) $X(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$

3.26 Determine the signal $x(n)$ with z-transform

$$X(z) = \frac{3}{1 - \frac{10}{3}z^{-1} + z^{-2}}$$

if $X(z)$ converges on the unit circle.

3.27 Prove the complex convolution relation given by (3.2.22).

3.28 Prove the conjugation properties and Parseval's relation for the z-transform given in Table 3.2.

3.29 In Example 3.4.1 we solved for $x(n)$, $n < 0$, by performing contour integrations for each value of n. In general, this procedure proves to be tedious. It can be avoided by making a transformation in the contour integral from z-plane to the $w = 1/z$ plane. Thus a circle of radius R in the z-plane is mapped into a circle of radius $1/R$ in the w-plane. As a consequence, a pole inside the unit circle in the z-plane is mapped into a pole outside the unit circle in the w-plane. By making the change of variable $w = 1/z$ in the contour integral, determine the sequence $x(n)$ for $n < 0$ in Example 3.4.1.

3.30 Let $x(n)$, $0 \le n \le N - 1$ be a finite-duration sequence, which is also real-valued and even. Show that the zeros of the polynomial $X(z)$ occur in mirror-image pairs about the unit circle. That is, if $z = re^{j\theta}$ is a zero of $X(z)$, then $z = (1/r)e^{j\theta}$ is also a zero.

3.31 Compute the convolution of the following pair of signals in the time domain and by using the one-sided z-transform.

(a) $x_1(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$, $\quad x_2(n) = \{\underset{\uparrow}{1}, 1, 1\}$

(b) $x_1(n) = (\frac{1}{2})^n u(n)$, $\quad x_2(n) = (\frac{1}{3})^n u(n)$

(c) $x_1(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$, $\quad x_2(n) = \{4, 3, \underset{\uparrow}{2}, 1\}$

(d) $x_1(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$, $\quad x_2(n) = \{\underset{\uparrow}{1}, 1, 1\}$

Did you obtain the same results by both methods? Explain.

3.32 Determine the one-sided z-transform of the constant signal $x(n) = 1$, $-\infty < n < \infty$.

3.33 Prove that the Fibonacci sequence can be thought of as the impulse response of the system described by the difference equation $y(n) = y(n-1) + y(n-2) + x(n)$. Then determine $h(n)$ using z-transform techniques.

3.34 Use the one-sided z-transform to determine $y(n)$, $n \ge 0$ in the following cases.

(a) $y(n) + \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) = 0; \quad y(-1) = y(-2) = 1$

(b) $y(n) - 1.5y(n-1) + 0.5y(n-2) = 0; \quad y(-1) = 1, y(-2) = 0$

(c) $y(n) = \frac{1}{2}y(n-1) + x(n)$

$x(n) = (\frac{1}{3})^n u(n), \quad y(-1) = 1$

(d) $y(n) = \frac{1}{4}y(n-2) + x(n)$

$x(n) = u(n)$

$y(-1) = 0; \quad y(-2) = 1$

3.35 Show that the following systems are equivalent.

(a) $y(n) = 0.2y(n-1) + x(n) - 0.3x(n-1) + 0.02x(n-2)$

(b) $y(n) = x(n) - 0.1x(n-1)$

3.36 Consider the sequence $x(n) = a^n u(n)$, $-1 < a < 1$. Determine at least two sequences that are not equal to $x(n)$ but have the same autocorrelation.

3.37 Compute the unit step response of the system with impulse response

$$h(n) = \begin{cases} 3^n, & n < 0 \\ (\frac{2}{5})^n, & n \ge 0 \end{cases}$$

3.38 Compute the zero-state response for the following pairs of systems and input signals.

(a) $h(n) = (\frac{1}{3})^n u(n), x(n) = (\frac{1}{2})^n \left(\cos \frac{\pi}{3} n\right) u(n)$

(b) $h(n) = (\frac{1}{2})^n u(n), x(n) = (\frac{1}{3})^n u(n) + (\frac{1}{2})^{-n} u(-n-1)$

(c) $y(n) = -0.1y(n-1) + 0.2y(n-2) + x(n) + x(n-1)$

$x(n) = (\frac{1}{3})^n u(n)$

(d) $y(n) = \frac{1}{2}x(n) - \frac{1}{2}x(n-1)$

$x(n) = 10\left(\cos \frac{\pi}{2} n\right) u(n)$

(e) $y(n) = -y(n-2) + 10x(n)$

$x(n) = 10\left(\cos \frac{\pi}{2} n\right) u(n)$

(f) $h(n) = (\frac{2}{5})^n u(n), x(n) = u(n) - u(n-7)$

(g) $h(n) = (\frac{1}{2})^n u(n), x(n) = (-1)^n, -\infty < n < \infty$

(h) $h(n) = (\frac{1}{2})^n u(n), x(n) = (n+1)(\frac{1}{4})^n u(n)$

3.39 Consider the system

$$H(z) = \frac{1 - 2z^{-1} + 2z^{-2} - z^{-3}}{(1 - z^{-1})(1 - 0.5z^{-1})(1 - 0.2z^{-1})} \qquad \text{ROC: } 0.5 < |z| < 1$$

(a) Sketch the pole-zero pattern. Is the system stable?

(b) Determine the impulse response of the system.

3.40 Compute the response of the system

$$y(n) = 0.7y(n-1) - 0.12y(n-2) + x(n-1) + x(n-2)$$

to the input $x(n) = nu(n)$. Is the system stable?

3.41 Determine the impulse response and the step response of the following causal systems. Plot the pole-zero patterns and determine which of the systems are stable.

(a) $y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$

(b) $y(n) = y(n-1) - 0.5y(n-2) + x(n) + x(n-1)$

(c) $H(z) = \dfrac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$

(d) $y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$

(e) $y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2)$

3.42 Let $x(n)$ be a causal sequence with z-transform $X(z)$ whose pole-zero plot is shown in Fig. P3.42. Sketch the pole-zero plots and the ROC of the following sequences:

(a) $x_1(n) = x(-n+2)$

(b) $x_2(n) = e^{j(\pi/3)n} x(n)$

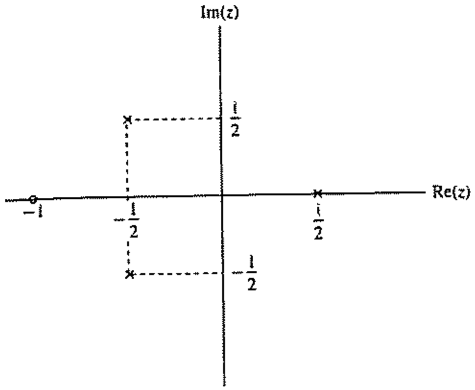


Figure P3.42

3.43 We want to design a causal discrete-time LTI system with the property that if the input is

$$x(n) = (\tfrac{1}{2})^n u(n) - \tfrac{1}{4}(\tfrac{1}{2})^{n-1} u(n-1)$$

then the output is

$$y(n) = (\tfrac{1}{3})^n u(n)$$

✓**(a)** Determine the impulse response $h(n)$ and the system function $H(z)$ of a system that satisfies the foregoing conditions.
✓**(b)** Find the difference equation that characterizes this system.
(c) Determine a realization of the system that requires the minimum possible amount of memory.
(d) Determine if the system is stable.

3.44 Determine the stability region for the causal system

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

by computing its poles and restricting them to be inside the unit circle.

3.45 Consider the system

$$H(z) = \frac{z^{-1} + \frac{1}{2}z^{-2}}{1 - \frac{3}{5}z^{-1} + \frac{2}{25}z^{-2}}$$

Determine:
(a) The impulse response
(b) The zero-state step response
(c) The step response if $y(-1) = 1$ and $y(-2) = 2$

3.46 Determine the system function, impulse response, and zero-state step response of the system shown in Fig P3.46.

3.47 Consider the causal system

$$y(n) = -a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$$

Determine:
(a) The impulse response

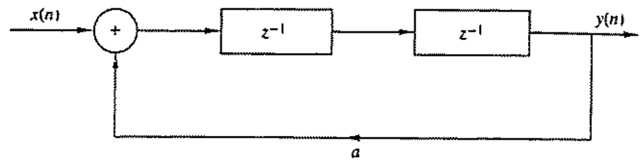


Figure P3.46

(b) The zero-state step response
(c) The step response if $y(-1) = A \neq 0$
(d) The response to the input

$$x(n) = \cos \omega_0 n \qquad 0 \le n < \infty$$

3.48 Determine the zero-state response of the system

$$y(n) = \tfrac{1}{2}y(n-1) + 4x(n) + 3x(n-1)$$

to the input

$$x(n) = e^{j\omega_0 n} u(n)$$

What is the steady-state response of the system?

3.49 Consider the causal system defined by the pole-zero pattern shown in Fig. P3.49.
(a) Determine the system function and the impulse response of the system given that $H(z)|_{z=1} = 1$.
(b) Is the system stable?
(c) Sketch a possible implementation of the system and determine the corresponding difference equations.

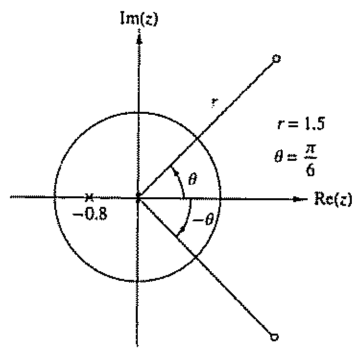


Figure P3.49

✓**3.50** An FIR LTI system has an impulse response $h(n)$, which is real valued, even, and has finite duration of $2N+1$. Show that if $z_1 = re^{j\omega_0}$ is a zero of the system, then $z_1 = (1/r)e^{j\omega_0}$ is also a zero.

3.51 Consider an LTI discrete-time system whose pole-zero pattern is shown in Fig. P3.51.
(a) Determine the ROC of the system function $H(z)$ if the system is known to be stable.

3.3. a

$$x_1(n) = \begin{cases} (1/3)^n, & n \geq 0 \\ (1/2)^{-n}, & n < 0 \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n + \sum_{n=-\infty}^{-1} \left(\frac{z}{2}\right)^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n + \sum_{k=1}^{\infty} \left(\frac{z}{2}\right)^k$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n + \left[-1 + \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k\right]$$

$$= \left[\frac{1}{1 - (1/3z)}\right] + \left[-1 + \frac{1}{1 - (z/2)}\right] \quad - (1)$$

$\downarrow$

ROC: $\left|\frac{1}{3z}\right| < 1$ or $|z| > \frac{1}{3}$

ROC $\left|\frac{z}{2}\right| < 1$

$|z| < 2$

$$= \frac{1}{1 - \frac{1}{3} z^{-1}} + \left[-1 + \frac{1}{1 - z/2}\right], \quad \text{ROC } \boxed{\frac{1}{3} < |z| < 2}$$

$$= \frac{1}{(1 - \frac{1}{3} z^{-1})} + \frac{\frac{1}{2} z}{(1 - z/2)}, \quad \boxed{\frac{1}{3} < |z| < 2}$$

$$= \frac{\frac{5}{3} z}{(z - \frac{1}{3})(2 - z)}, \quad \frac{1}{3} < |z| < 2$$

3.3. d.

$$x_4(n) = x_1(-n)$$

$$= \begin{cases} \left(\frac{1}{3}\right)^{-n} & -n \geqslant 0 \\ \left(\frac{1}{2}\right)^{n} & -n < 0 \end{cases}$$

or

$$= \begin{cases} \left(\frac{1}{3}\right)^{-n} & n \leq 0 \\ \left(\frac{1}{2}\right)^{n} & n > 0. \end{cases}$$

we can use the definition or the time reversal property,

$x(-n) \longleftrightarrow X(z^{-1})$, with appropriate mod. to ROC.

using the time reversal property, we have

$$X_4(z) = \frac{5/3\ z^{-1}}{(z^{-1} - 1/3)(2 - z^{-1})}, \quad \frac{1}{3} < |z^{-1}| < 2$$

or

$$X_4(z) = \frac{-5/2\ z}{(z-3)(z-1/2)}, \quad \boxed{\frac{1}{2} < |z| < 3}$$

3.7

$$x_1(n) = \begin{cases} (1/3)^n & n \geq 0 \\ (1/2)^{-n} & n < 0 \end{cases}$$

$$x_2(n) = \left(\tfrac{1}{2}\right)^n u(n)$$

$$y(n) = x_1(n) * x_2(n)$$

$$Y(z) = X_1(z) \cdot X_2(z)$$

$$= \underbrace{\frac{-5/3\, z}{(z-1/3)(z-2)}}_{\substack{\text{ROC} \\ \frac{1}{3} < |z| < 2}} \cdot \underbrace{\frac{z}{(z-1/2)}}_{\substack{\text{ROC} \\ |z| > \frac{1}{2}}} . \quad — (1)$$

ROC

$$\frac{1}{2} < |z| < 2.$$

[Note: we used results from # 3.3.a

$$\frac{Y(z)}{z} = \frac{-5/3\, z}{(z-1/3)(z-2)(z-1/2)}$$

$$= \frac{A}{(z-1/3)} + \frac{B}{(z-1/2)} + \frac{C}{(z-2)}$$

$$= \frac{-2}{(z-1/3)} + \frac{10/3}{(z-1/2)} + \frac{-4/3}{z-2}$$

3.7 (continued)

$$Y(z) = \underbrace{\frac{-2z}{z-1/3} + \frac{10/3\, z}{z-1/2}}_{\text{Causal}} + \underbrace{\frac{-4/3\, z}{(z-2)}}_{\text{anticausal}}$$

$1/2 < |z| < 2$

$$\boxed{y(n) = \left[-2\left(1/3\right)^n + \left(\tfrac{10}{3}\right)\left(1/2\right)^n\right] u(n) + \left[\left(4/3\right)\cdot(2)^n\, u(-1-n)\right]}$$

or

$$\boxed{y(n) = \begin{cases} -2\cdot\left(\tfrac{1}{3}\right)^n + \left(\tfrac{10}{3}\right)\left(1/2\right)^n, & n \geq 0 \\ \left(4/3\right)\cdot 2^n, & n < 0 \end{cases}}$$

3.11 $\quad X(z) = \dfrac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}}$

(a) $x(n)$ is causal:

$$
\begin{array}{r|l}
 & 1 + 4z^{-1} + 7z^{-2} + \cdots \\
\hline
1 - 2z^{-1} + z^{-2} & 1 + 2z^{-1} \\
 & 1 - 2z^{-1} + z^{-2} \\
\cline{2-2}
 & \quad 4z^{-1} - z^{-2} \\
 & \quad 4z^{-1} - 8z^{-2} + 4z^{-3} \\
\cline{2-2}
 & \qquad 7z^{-2} - 4z^{-3} \\
 & \qquad \vdots
\end{array}
$$

$$x(n) = \{\underset{\uparrow}{1}, 4, 7 \cdots\}$$

(b) $x(n)$ is anticausal.

$$
\begin{array}{r|l}
 & 2z + 5z^{2} + 8z^{3} + \cdots \\
\hline
z^{-2} - 2z^{-1} + 1 & 2z^{-1} + 1 \\
 & 2z^{-1} - 4 + 2z \\
\cline{2-2}
 & \quad 5 - 2z \\
 & \quad 5 - 10z + 5z^{2} \\
\cline{2-2}
 & \qquad 8z - 5z^{2} \\
 & \qquad 8z - 16z^{2} + 8z^{3} \\
\cline{2-2}
 & \qquad \vdots
\end{array}
$$

$$x(n) = \{\cdots\cdots 8, 5, 2, \underset{\uparrow}{0}\}.$$

3.14.a $$X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} = \frac{z^2+3z}{z^2+3z+2}$$

$$= \frac{z(z+3)}{(z+1)(z+2)}.$$ $x(n)$ is causal

$\Rightarrow$ ROC: $|z| > 2$

$$\frac{X(z)}{z} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$= \frac{2}{z+1} + \frac{-1}{(z+2)}$$

$$X(z) = \frac{2z}{z+1} - \frac{z}{z+2}$$

$$= 2 \cdot \frac{1}{1-(-1)z^{-1}} - \frac{1}{1-(-2)z^{-1}}.$$

$$\boxed{x(n) = \left[2(-1)^n - (-2)^n\right] u(n)}$$

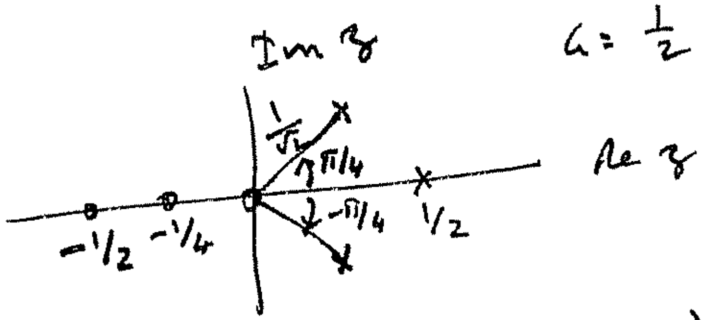
3.14.d

$$X(z) = \frac{1 + 2z^{-2}}{1 + z^{-2}} = 1 + \frac{z^{-2}}{1 + z^{-2}}.$$

$$\boxed{x(n) = \delta(n) + \left[\cos \frac{\pi}{2}(n-2)\right] u(n-2)}$$

using entry # 7, Table 3.3, pg. 174 of text and the time shift property.

3.14.h



$$X(z) = \frac{1}{2} \frac{z \cdot (z + 1/4)(z + 1/2)}{(z - 1/2)\,(z - [1/2 + j1/2])\,(z - [1/2 - j1/2])}$$

$$= \frac{1}{2} \frac{z\,(z + 1/4)\,(z + 1/2)}{(z - 1/2)\,(z^2 - z + 1/2)}.$$

$$\frac{X(z)}{z} = \frac{A}{(z - 1/2)} + \frac{(Bz + C)}{(z^2 - z + 1/2)}$$

3.14.h (continued)

$$\frac{X(z)}{z} = \frac{3/2}{(z-1/2)} + \frac{(-z+11/8)}{(z^2-z+1/2)}$$

$$X(z) = \frac{3/2\, z}{(z-1/2)} + \frac{-(z-1/2)\cdot z}{z^2-z+1/2}$$

$$+ \frac{-7/8\, z}{(z^2-z+1/2)} \quad \text{—} ①$$

Note:

$$(\alpha^n \cos w_0 n)u(n) \longleftrightarrow \frac{1-\alpha\cos w_0 \cdot z^{-1}}{1-2\alpha\cos w_0 z^{-1}+z^{-2}}$$

$$= \frac{z(z-\alpha\cos w_0)}{z^2-(2\alpha\cos w_0)z+\alpha^2}, \quad |z|>|\alpha| \quad \text{—} ②$$

and

$$(\alpha^n \sin w_0 n)u(n) \longleftrightarrow \frac{\alpha \sin w_0 \cdot z^{-1}}{1-2\alpha\cos w_0 z^{-1}+z^{-2}}$$

$$= \frac{(\alpha \sin w_0)\cdot z}{z^2-(2\alpha\cos w_0)\cdot z+\alpha^2},$$

$$|z|>|\alpha| \quad \text{—} ③$$

3.14.h (Continued)

Comparing 1, 2 and 3, we can write:

$$\boxed{x(n) = (3/2)\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n\right) u(n) + (7/4)\cdot\left(\frac{1}{\sqrt{2}}\right)^n \sin(\pi/4\, n)\, u(n)}$$

3.15
$$X(z) = \frac{5z^{-1}}{(1-2z^{-1})(3-z^{-1})}$$
$$= \frac{A}{(1-2z^{-1})} + \frac{B}{(1-\frac{1}{3}z^{-1})}$$
$$= \frac{1}{(1-2z^{-1})} + \frac{-1}{(1-\frac{1}{3}z^{-1})}$$

ROC1 : $x(n)$ is Causal, $|z| > 2$

$$x(n) = \left[2^n - \left(\tfrac{1}{3}\right)^n\right] u(n)$$ ← unstable

ROC2 : $x(n)$ is anti-causal, $|z| < \frac{1}{3}$.

$$x(n) = \left[\left(\tfrac{1}{3}\right)^n - 2^n\right] u(-n-1)$$ ← unstable.

ROC3 : $x(n)$ is two-sided.

$\frac{1}{3} < |z| < 2$ → stable.

$$x(n) = -\left(\tfrac{1}{3}\right)^n u(n) - (2)^n u(-n-1),$$

3.26

$$X(z) = \frac{3}{1 - \frac{10}{3} z^{-1} + z^{-2}}$$

$$= \frac{3z^2}{(z^2 - \frac{10}{3} z + 1)}$$

ROC includes unit circle.

$$\frac{X(z)}{z} = \frac{A}{(z - 1/3)} + \frac{B}{(z-3)}$$

$$= \frac{-3/8}{(z - 1/3)} + \frac{(27/8)}{(z-3)}$$

$$X(z) = \frac{-3/8\, z}{(z - 1/3)} + \frac{27/8\, z}{(z-3)}$$

Causal   Anticausal

$$\boxed{x(n) = -\frac{3}{8} \cdot \left(\frac{1}{3}\right)^n u(n) - \left(\frac{27}{8}\right) \cdot (3)^n u(-n-1)}$$

3.34 a

$$y(n) + \frac{1}{2} y(n-1) - \frac{1}{4} y(n-2) = 0$$

$$y(-1) = y(-2) = 1$$

$$Y^+(z) + \frac{1}{2}\left[z^{-1} Y^+(z) + y(-1)\right]$$

$$- \frac{1}{4}\left[z^{-2} Y^+(z) + z^{-1} y(-1) + y(-2)\right] = 0$$

or

$$Y^+(z)\left[1 + \frac{1}{2} z^{-1} - \frac{1}{4} z^{-2}\right] + \left[\frac{1}{2} y(-1) - \frac{1}{4} z^{-1} y(-1) - \frac{1}{4} y(-2)\right] = 0$$

$$\Longrightarrow$$

$$Y^+(z) = \frac{\left(-\frac{1}{4} z^{-1} + \frac{1}{4}\right)}{\left[1 + 1/2\, z^{-1} - 1/4\, z^{-2}\right]}$$

$$= \frac{z\left(\cdot 1/4\, z - 1/4\right)}{\left(z^2 + 1/2\, z - 1/4\right)}$$

$$\Longrightarrow$$

$$\frac{Y^+(z)}{z} = \frac{A}{(z - 0.309)} + \frac{B}{(z + 0.809)}$$

Note! Roots from MATLAB

$$= \frac{0.154}{(z - 0.309)} + \frac{-0.404}{(z + 0.809)}$$

$$y(n) = \left[(0.154)(0.309)^n - (0.404)(-0.809)^n\right] u(n) \quad //$$

3.41.a Assume Causal System, zero I.C.

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$$

Impulse Response:

$$Y(z) = 3/4\, z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - 3/4\, z^{-1} + 1/8\, z^{-2})}$$

$$= \frac{z^2}{(z^2 - 3/4\, z + 1/8)}$$

$$\frac{H(z)}{z} = \frac{A}{(z-1/2)} + \frac{B}{(z-1/4)}$$

$$= \frac{2}{(z-1/2)} + \frac{-1}{(z-1/4)}$$

$$H(z) = \frac{2z}{z-1/2} + \frac{-z}{z-1/4}$$

$$\Longrightarrow \boxed{h(n) = \left[2\cdot\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n\right] u(n)}$$

STABLE because poles are inside the unit circle.

3.41a (continued)

Step Response.

Let $x(n) = u(n) \Longrightarrow X(z) = \frac{z}{z-1}$.

$$Y(z) = H(z) \cdot X(z)$$

$$= \frac{z^3}{(z-1/2)(z-1/4)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{A}{(z-1/2)} + \frac{B}{(z-1/4)} + \frac{C}{(z-1)}.$$

$$= \frac{-2}{(z-1/2)} + \frac{1/3}{(z-1/4)} + \frac{8/3}{(z-1)}$$

$$\Longrightarrow \boxed{y(n) = \left[-2 \cdot \left(\tfrac{1}{2}\right)^n + \tfrac{1}{3}\left(\tfrac{1}{4}\right)^n + 8/3\right] u(n)}$$

Step Response

3.41.b Causal, zero I.C.

$$y(n) = y(n-1) - \frac{1}{2}y(n-2) + x(n) + x(n-1)$$

$$Y(z)[1 - z^{-1} + \frac{1}{2}z^{-2}] = X(z)[1+z^{-1}]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1+z^{-1})}{(1-z^{-1}+\frac{1}{2}z^{-2})}$$

$$= \frac{z(z+1)}{(z^2 - z + 1/2)} = \frac{z^2+z}{z^2-z+1/2} \quad \text{— (1)}$$

Note that

$$[\alpha^n \cos w_0 n]u(n) \longleftrightarrow \frac{1-\alpha z^{-1}\cos w_0}{1-2\alpha z^{-1}\cos w_0 + \alpha^2 z^{-2}} \quad |z|>|\alpha| \quad \text{— (2A)}$$

$$= \frac{z^2 - \alpha\cos w_0 \cdot z}{[z^2 - (2\alpha\cos w_0)z + \alpha^2]} \quad \text{— (2B)}$$

and

$$[\alpha^n \sin w_0 n]u(n) \longleftrightarrow \frac{(\alpha\sin w_0)z}{z^2-(2\alpha\cos w_0)z+\alpha^2}, \quad |z|>|\alpha|$$

— (3)

3.41.b (continued)

① can be written as

$$H(z) = \frac{z^2 + \frac{1}{2}z}{(z^2 - z + \frac{1}{2})} + \frac{\frac{1}{2}z}{(z^2 - z + \frac{1}{2})} \quad — ④$$

Comparing ④ to (2A) & (2B), we have:

$$\boxed{h(n) = \left[\left(\frac{1}{\sqrt{2}}\right)^n \cos\frac{\pi}{4}n + \left(\frac{1}{\sqrt{2}}\right)^n \sin\frac{\pi}{4}n\right] u(n)}$$

Step Response:

$$Y(z) = H(z) \cdot \frac{z}{(z-1)}$$

$$= \frac{(z^2 + z)z}{(z^2 - z + \frac{1}{2})(z+1)}$$

$$\frac{Y(z)}{z} = \frac{A}{(z-1)} + \frac{(Bz + C)}{(z^2 - z + \frac{1}{2})}$$

$$= \frac{4}{z-1} + \frac{(-3z + 2)}{(z^2 - z + \frac{1}{2})}$$

3.41 (b) Continued

$$Y(z) = \frac{4z}{(z-1)} + \frac{-3z^2+2z}{(z^2-z+1/2)}$$

$$= \frac{4z}{(z-1)} - \frac{3\left(z^2-\frac{1}{2}z\right)}{(z^2-z+1/2)} + \frac{1/2\, z}{(z^2-z+1/2)}$$

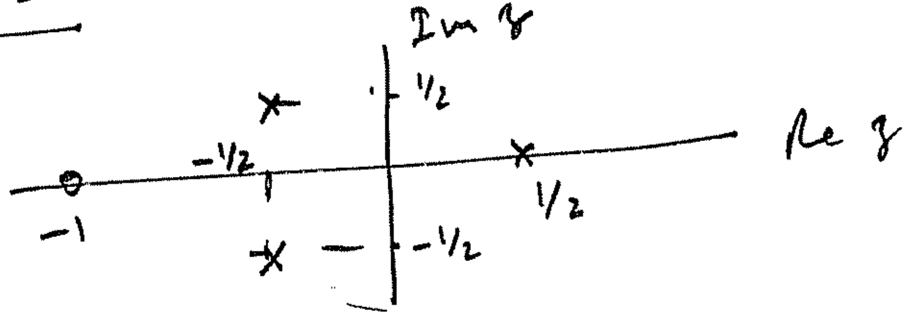
$\Longrightarrow$

$$y(n) = \left[4u(n) - 3\cdot\left(\frac{1}{\sqrt{2}}\right)^n \cos\frac{\pi}{4}n + \left(\frac{1}{\sqrt{2}}\right)^n \cos\frac{\pi}{4}n\right] u(n)$$

Step Response

$x(n) \to$ Causal

3.42 a



$$X(z) = \frac{(z+1)}{(z-1/2)(z^2+z+\frac{1}{2})}, \quad |z| > \frac{1}{\sqrt{2}}.$$

a. $x_1(n) = x(-n+2)$

$= x(-(n-2))$

$x_1(n)$ is the sequence that is flipped and right shifted by 2 samples of $x(n)$.

Let $x_2(n) = x(-n) \to x_1(n) = x_2(n-2)$.

$X_2(z) = X(z^{-1}), \quad |z^{-1}| > 1/\sqrt{2}$

$\Rightarrow |z| < \sqrt{2}.$

$X_1(z) = z^{-2} X_2(z)$

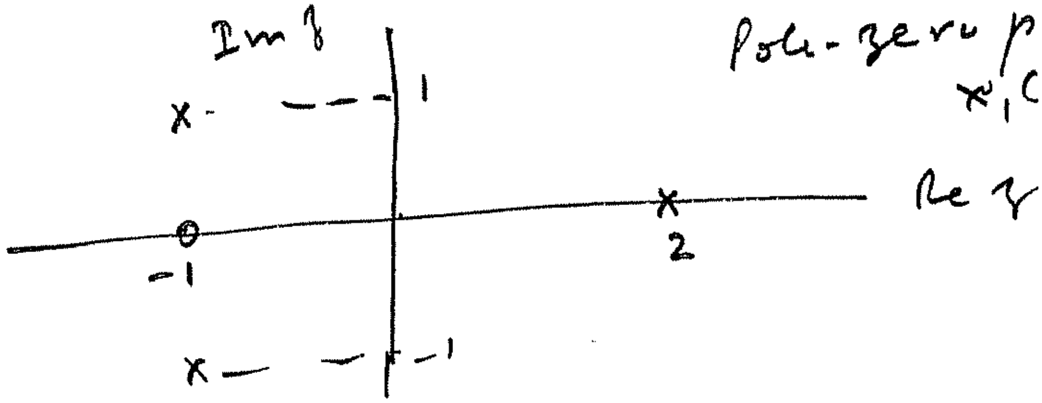
$$= z^{-2} X(z^{-1}) = z^{-2} \frac{(z^{-1}+1)}{(z^{-1}-1/2)(z^{-2}+z^{-1}+1/2)}$$

3.42 a (continued)

$$X_1(z) = \frac{(z^{-3} + z^{-2})}{(z^{-1} - 1/2)(z^{-2} + z^{-1} + 1/2)}$$

$$= \frac{(1 + z)}{(1 - \frac{1}{2}z)(1 + z + \frac{1}{2}z^2)}$$

$$= \frac{4(z+1)}{(z-2)(z^2+2z+2)}$$



Pole-zero plot of $X_1(z)$

ROC: $|z| < \sqrt{2}$.

Zeros: $z = -1$, 2 zeros at $z = \infty$

Poles: $z = 2$, $z = -1 \pm j1$

3.43

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n) - \tfrac{1}{4}\left(\tfrac{1}{2}\right)^{n-1} u(n-1)$$

$$y(n) = \left(\tfrac{1}{3}\right)^n u(n)$$

Causal system.

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{4}z^{-1}\cdot\frac{1}{1-(\frac{1}{2})z^{-1}}$$

$$= \frac{(1-\frac{1}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})}.$$

$$Y(z) = \frac{1}{(1-\frac{1}{3}z^{-1})}.$$

a: Impulse resp

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$= \frac{A}{(1-\frac{1}{4}z^{-1})} + \frac{B}{(1-\frac{1}{3}z^{-1})}$$

$$= \frac{3}{(1-\frac{1}{4}z^{-1})} + \frac{-2}{(1-\frac{1}{3}z^{-1})}$$

$$\Rightarrow h(n) = \left[3\cdot\left(\tfrac{1}{4}\right)^n - 2\left(\tfrac{1}{3}\right)^n\right]u(n)$$

3.43 b (continued)

Difference eqn.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 - \frac{1}{2} z^{-1})}{(1 - \frac{7}{12} z^{-1} + \frac{1}{12} z^{-2})}$$

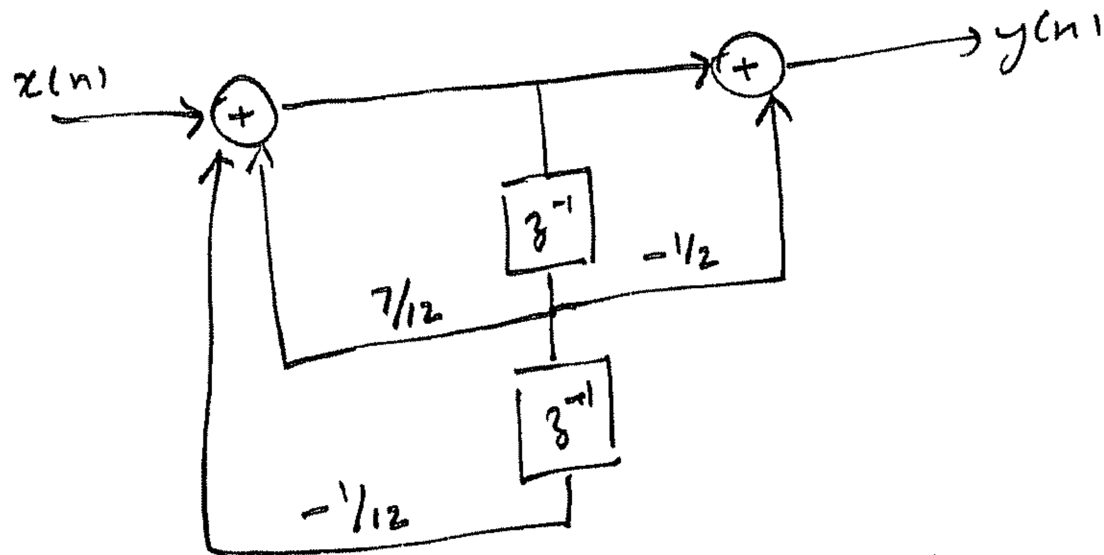
or

$$(1 - \tfrac{7}{12} z^{-1} + \tfrac{1}{12} z^{-2}) Y(z) = (1 - \tfrac{1}{2} z^{-1}) X(z)$$

$\Longrightarrow$ Diff Eqn

$$\boxed{y(n) - \tfrac{7}{12} y(n-1) + \tfrac{1}{12} y(n-2) = x(n) - \tfrac{1}{2} x(n-1)}$$

c Direct form II.



d. System is stable because the poles are inside the unit circle and the system is causal.