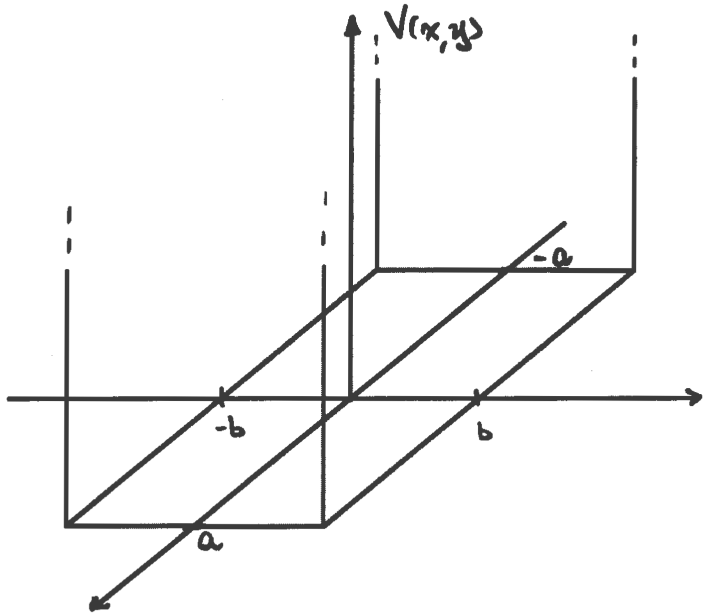
- Problem 1

Consider a infinitely deep 2D quantum well of size $-a < x < a$ and $-b < y < b$, where $b \neq a$.

$-1-$Determine the eigenvalues and eigenfunctions of the confined states. Determine the degeneracy.

$-2-$Write a state function for the system that has equal probability in the first three states.



1)

$$\left[-\frac{\hbar^2}{2m}\nabla + V(x,y)\right]\psi_n(x,y) = E_n\psi_n(x,y)$$

$$-\frac{\hbar^2}{2m}\left[\frac{\delta}{\delta x}\psi_n(x,y) + \frac{\delta}{\delta y}\psi_n(x,y)\right] = E_n\psi_n(x,y)$$

$$\psi_n(x,y) = X(x)\,Y(y) \qquad\qquad E = \varepsilon_x + \varepsilon_y$$

$$\begin{cases} -\frac{\hbar^2}{2m}\left[\frac{d}{dx}X(x)\right] = \varepsilon_x X(x) \\ -\frac{\hbar^2}{2m}\left[\frac{d}{dy}Y(y)\right] = \varepsilon_y Y(y) \end{cases}$$

$$X(x) = A_x \sin(K_x x) + B_x \cos(K_x x)$$

$$Y(y) = A_y \sin(K_y y) + B_y \cos(K_y y)$$

Degeneracy

$$E_{n_{x_1}, n_{y_1}} = E_{n_{x_2}, n_{y_2}} \qquad n_{x_1} \neq n_{x_2}, \; n_{y_1} \neq n_{y_2}$$

$$\frac{\cancel{\hbar^2} \cancel{\pi^2}}{\cancel{2m}} \left(\frac{n_{x_1}^2}{a^2} + \frac{n_{y_1}^2}{b^2} \right) = \frac{\cancel{\hbar^2} \cancel{\pi^2}}{2m} \left(\frac{n_{x_2}^2}{a^2} + \frac{n_{y_2}^2}{b^2} \right)$$

$$\frac{1}{a^2} (n_{x_1}^2 - n_{x_2}^2) = \frac{1}{b^2} (n_{y_2}^2 - n_{y_1}^2)$$

$$\left(\frac{b}{a} \right)^2 = \frac{n_{y_2}^2 - n_{y_1}^2}{n_{x_1}^2 - n_{x_2}^2}$$

$$n_{y_2}^2 - n_{y_1}^2 , \; n_{x_1}^2 - n_{x_2}^2 \in \mathbb{N}$$

$$\Rightarrow \frac{n_{y_2}^2 - n_{y_1}^2}{n_{x_1}^2 - n_{x_2}^2} \in \mathbb{Q}$$

$a, b \in \mathbb{R} : \left(\frac{b}{a} \right)^2 \in \mathbb{Q} \Rightarrow$ degeneracy

$a, b \in \mathbb{R} : \left(\frac{b}{a} \right)^2 \in \mathbb{R} / \mathbb{Q} \Rightarrow$ no degeneracy

2)

$$\Psi(x,y) = \frac{1}{\sqrt{3}} \left(\varphi_{00}(x,y) + \varphi_{10}(x,y) + \varphi_{01}(x,y) \right)$$

$$\psi(\pm a, y) = 0$$

$$\psi(x, \pm b) = 0$$

$$\begin{cases} X(x) = \dfrac{1}{\sqrt{a}} \cos\left(\dfrac{n_x \pi}{2a} x\right) \\ Y(y) = \dfrac{1}{\sqrt{b}} \cos\left(\dfrac{n_y \pi}{2b} y\right) \end{cases} \quad n \text{ odd}$$

$$\begin{cases} X(x) = \dfrac{1}{\sqrt{a}} \sin\left(\dfrac{n_x \pi}{2a} x\right) \\ Y(y) = \dfrac{1}{\sqrt{b}} \sin\left(\dfrac{n_y \pi}{2b} y\right) \end{cases} \quad n \text{ even}$$

$$E = \varepsilon_x + \varepsilon_y = \frac{\hbar^2 \pi^2 n_x^2}{8 m a^2} + \frac{\hbar^2 \pi^2 n_y^2}{8 m b^2} = \frac{\hbar^2 \pi^2}{8m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right)$$

$$\psi_n(x,y) = X(x)\,Y(y) = \begin{cases} \dfrac{1}{\sqrt{ab}} \cos\left(\dfrac{n_x \pi}{2a} x\right) \cos\left(\dfrac{n_y \pi}{2b} y\right), & n_x, n_y \text{ odd} \\ \dfrac{1}{\sqrt{ab}} \sin\left(\dfrac{n_x \pi}{2a} x\right) \sin\left(\dfrac{n_y \pi}{2b} y\right), & n_x, n_y \text{ even} \\ \dfrac{1}{\sqrt{ab}} \cos\left(\dfrac{n_x \pi}{2a} x\right) \sin\left(\dfrac{n_y \pi}{2b} y\right), & n_x \text{ odd} \;\; n_y \text{ even} \\ \dfrac{1}{\sqrt{ab}} \sin\left(\dfrac{n_x \pi}{2a} x\right) \cos\left(\dfrac{n_y \pi}{2b} y\right), & n_x \text{ even} \;\; n_y \text{ odd} \end{cases}$$

$$\nexists\, n_{x_1}, n_{x_2} \in \mathbb{N} : n_{x_1} - n_{x_2} = \frac{1}{\sqrt{2}} (n_{y_2} - n_{y_1}) , \quad n_{y_2}, n_{y_1} \in \mathbb{N}$$

No degeneracy

- Problem 2

You have a harmonic oscillator with $\alpha = (mk/\hbar^2)^{1/4}$, where k is the oscillator spring constant and $\omega = (k/m)^{1/2}$ the corresponding frequency. The eigenfunctions solution of the Schroedinger equation are given by:

$$\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi}\,2^n\,n!}\right)^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_n(\alpha x)$$

and the corresponding generating function of the Hermite polynomia is:

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n e^{-\xi^2}}{d\xi^n}$$

Consider a 2D harmonic oscillator where k_x is twice k_y.

– Determine the solution of the Schroedinger equation by writing an explicit expression of the eigenfuntion and eigenvalues. Are there any degeneracy in the system? What kind?

– Using the lowering and raising operators technique compute the expectation values of $\langle x\, p_y\rangle$ and $\langle p_x\, y\rangle$ for the harmonic oscillator states. NOTE - watch for the coordinates on which a^+ and a^- operate on.

1)

$$K_x = 2K_y \Rightarrow \alpha_y = \left(\frac{mK}{2\hbar^2}\right)^{1/4} = 2^{-\frac{1}{4}}\left(\frac{mK}{\hbar^2}\right)^{1/4} = \frac{\alpha_x}{2^{1/4}}$$

$$\varphi_n(x,y) = \left(\frac{\alpha_x}{\sqrt{\pi}\,2^{n_x} n_x!}\right)^{1/2}\left(\frac{\alpha_y}{\sqrt{\pi}\,2^{n_y} n_y!}\right)^{1/2} e^{-\frac{\alpha_x^2 x^2}{2}} e^{-\frac{\alpha_y^2 y^2}{2}} H_{n_x}(\alpha_x x) H_{n_y}(\alpha_y y)$$

$$\alpha_x \cdot \alpha_y = 2^{-\frac{1}{4}}\alpha_x^2 = 2^{-\frac{1}{4}}\alpha^2$$

$$\varphi_n(x,y) = \left(\frac{\alpha^2}{2^{1/4}\sqrt{\pi}\,2^{n_x}2^{n_y}\,n_x!\,n_y!}\right) e^{-\frac{1}{2}\left(\alpha^2 x^2 + \frac{1}{\sqrt{2}}\alpha^2 y^2\right)} H_{n_x}(\alpha x)\, H_{n_y}(2^{-\frac{1}{4}}\alpha y)$$

$$\omega_y = \left(\frac{K}{2m}\right)^{1/2} = 2^{-\frac{1}{2}}\omega_0$$

$$E_{n_x,n_y} = \left(n_x+\frac{1}{2}\right)\hbar\omega_0 + \frac{1}{\sqrt{2}}\left(n_y+\frac{1}{2}\right)\hbar\omega_0$$

$$= \left(\frac{1}{2}+\frac{1}{2\sqrt{2}} + n_x + \frac{1}{\sqrt{2}}n_y\right)\hbar\omega_0$$

Degeneracy

$$E_{n_{x_1},n_{y_1}} = E_{n_{x_2},n_{y_2}} \qquad n_{x_1} \neq n_{x_2},\ n_{y_1} \neq n_{y_2}$$

$$\left(\frac{1}{2}+\frac{1}{2\sqrt{2}} + n_{x_1} + \frac{1}{\sqrt{2}}n_{y_1}\right)\cancel{\hbar\omega_0} = \left(\frac{1}{2}+\frac{1}{2\sqrt{2}} + n_{x_2} + \frac{1}{\sqrt{2}}n_{y_2}\right)\cancel{\hbar\omega_0}$$

$$\cancel{\frac{1}{2}} + \cancel{\frac{1}{2\sqrt{2}}} + n_{x_1} + \frac{n_{y_1}}{\sqrt{2}} = \cancel{\frac{1}{2}} + \cancel{\frac{1}{2\sqrt{2}}} + n_{x_2} + \frac{n_{y_2}}{\sqrt{2}}$$

$$n_{x_1} - n_{x_2} = \frac{1}{\sqrt{2}}\left(n_{y_2} - n_{y_1}\right)$$

2)

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(\pm\hbar\frac{d}{dx} + m\omega x\right) \qquad p_x = -i\hbar\frac{d}{dx} \qquad a_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i p_x + m\omega x\right)$$

$$p_x = i\sqrt{\frac{\hbar\omega m}{2}}\,(a_+ - a_-) \qquad a_\pm^x = \frac{1}{\sqrt{2\hbar m\omega}}\left(\pm\hbar\frac{d}{dx} + m\omega x\right)$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-) \qquad a_\pm^y = \frac{1}{\sqrt{2\hbar m\omega}}\left(\pm\hbar\frac{d}{dy} + m\omega y\right)$$

$$\langle x p_y\rangle = \langle \psi_n | \sqrt{\tfrac{\hbar}{2m\omega}}(a_+^x + a_-^x)\, i\sqrt{\tfrac{\hbar\omega m}{2}}(a_+^y - a_-^y) | \psi_n\rangle$$

$$= \langle X_n(x)Y_n(y) | i\frac{\hbar}{2}(a_+^x + a_-^x)(a_+^y - a_-^y) | X_n(x)Y_n(y)\rangle$$

$$= i\frac{\hbar}{2}\langle X_n(x)Y(y) | (a_+^x + a_-^x) | X_n(x)\left[\sqrt{n_y+1}\,Y_{n+1}(y) - \sqrt{n_y}\,Y_{n-1}(y)\right]\rangle$$

$$= i\frac{\hbar}{2}\langle X_n(x)Y(y) | \left[\sqrt{n_x+1}\,X_{n+1}(x) + \sqrt{n_x}\,X_{n-1}(x)\right]\left[\sqrt{n_y+1}\,Y_{n+1}(y) - \sqrt{n_y}\,Y_{n-1}(y)\right]\rangle$$

$$= i\frac{\hbar}{2}\langle X_n(x)Y(y) | \left[\sqrt{n_x+1}\,X_{n+1}(x) + \sqrt{n_x}\,X_{n-1}(x)\right]\left[\sqrt{n_y+1}\,Y_{n+1}(y) - \sqrt{n_y}\,Y_{n-1}(y)\right]\rangle$$

$$\psi_{n_1,n_2} \text{ C.O.N. set} \iff \int |\psi_{i,j}\psi_{n,m}|^2\, d\vec{z} = \delta_{in}\delta_{jm}$$

$$\Rightarrow \langle x p_y\rangle = 0$$

$$\langle p_x y\rangle = i\frac{\hbar}{2}\langle X_n(x)Y_n(y) | (a_+^x - a_-^x)(a_+^y + a_-^y) | X_n(x)Y_n(y)\rangle$$

$$= i\frac{\hbar}{2}\left[\langle X_n(x)| (a_+^x - a_-^x) | X_n(x)\rangle\langle Y_n(y)| (a_+^y + a_-^y) | Y_n(y)\rangle\right.$$

$$= i\frac{\hbar}{2}\left[\langle X_n(x)| \sqrt{n+1}\,X_{n+1}(x) - \sqrt{n}\,X_{n-1}(x)\rangle\langle Y_n(y)| \sqrt{n+1}\,Y_{n+1}(y) + \sqrt{n}\,Y_{n-1}(y)\rangle\right.$$

$$= 0$$

- Problem 3

Consider the radial Schroedinger equation:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right] R = l(l+1) R \tag{1}$$

where $R = R(r)$ is the radial solution, l the angular momentum quantum number.

– Perform a transformation of variables $u(r) = r\,R(r)$ and determine an equation for $u(r)$ equivalent to (1).

– For this new equation consider the case $l = 0$ and solve (compute eigenvalues and eigenfunctions) the problem of the infinite spherical quantum well: $V(r) = 0 \quad for \quad r < a$ and $V(r) = \infty \quad for \quad r > a$. Note: what happens at r = 0 ?

– Rewrite your solution in terms of $R(r)$ do you recognize what function this is?

1)

$$u(z) = z R(z)$$

$$\frac{d}{dz}\left(z^2\frac{d}{dz}\left(z^{-1}u(z)\right)\right) = \frac{d}{dz}\left[z^2\left(-z^{-2}u(z) + z^{-1}\frac{du}{dz}\right)\right] = \frac{d}{dz}\left[-u(z) + z\frac{du}{dz}\right]$$

$$= -\frac{du}{dz} + \frac{du}{dz} + z\frac{d^2u}{dz^2} = z\frac{d^2u}{dz^2}$$

$$\Rightarrow z\frac{d^2u}{dz^2} - \frac{2mz^2}{\hbar^2}\left[V(z) - E\right]\frac{u}{\cancel{z}} = l(l+1)\frac{u}{z}$$

$$z^2\frac{d^2u}{dz^2} - \frac{2mz^2}{\hbar^2}\left[V(z) - E\right]u = l(l+1)u$$

$$z^2\frac{d^2u}{dz^2} - \frac{2mz^2}{\hbar^2}V(z)\,u = \left[-\frac{2mz^2}{\hbar^2}E + l(l+1)\right]u$$

2)

$$\cancel{z^2}\frac{d^2u}{dz^2} - \frac{2m\cancel{z^2}}{\hbar^2}V(z)\,u = -\frac{2m\cancel{z^2}}{\hbar^2}E_n\,u$$

$$\frac{d^2u}{dz^2} + \underbrace{\frac{2m}{\hbar^2}E_n}_{k^2}\,u = 0$$

$$\Rightarrow u = A\sin(kz) + B\cos(kz)$$

$$z = a \Rightarrow u = 0$$

$$\Rightarrow u = A\sin\left(\frac{n\pi}{}z\right) + B\cos\left(\frac{n\pi}{}z\right)$$

$$R(z) = \frac{1}{z} A \sin\left(\frac{n\pi}{2a} z\right) + B \cos\left(\frac{n\pi}{2a} z\right)$$

$$z \to 0 \quad R(z) = A + \infty \Rightarrow B = 0$$

$$R(z) = \frac{1}{z} A \sin\left(\frac{n\pi}{2a} z\right)$$

- Problem 4

Consider the ground-state wavefunction of the Hydrogen atom ground state:

$$\Psi_{100}(r) = \frac{1}{\sqrt{4\pi}} \frac{2}{a^{3/2}} e^{-r/a}$$

where a is the Bohr's radius.

- Compute the expectation values $< r >$ and $< r^2 >$.
- Compute the expectation values $< x >$ and $< x^2 >$.

Note: you can take the long way and use $x = r\,sin\theta\,cos\phi$. Otherwise notice $r^2 = x^2 + y^2 + z^2$ and use the symmetry of the ground state.

1)

$$\langle z \rangle = \langle \Psi | z | \Psi \rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} \Psi^* z \Psi z^2 \sin\theta \, dz \, d\theta \, d\varphi$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} \Psi^2 z^3 \sin\theta \, dz \, d\theta \, d\varphi$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{\cancel{4}\pi} \frac{\cancel{4}}{a^3} e^{-2\frac{z}{a}} z^3 \sin\theta \, dz \, d\theta \, d\varphi$$

$$= \frac{1}{\pi a^3} \cdot 2\pi \int_0^\infty e^{-2\frac{z}{a}} z^3 \, dz \int_0^\pi \sin\theta \, d\theta \qquad \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$= \frac{2 \cdot 4}{a^3} \frac{3!}{\left(\frac{2}{a}\right)^4} = a \cdot \frac{3}{2}$$

$$\langle z^2 \rangle = \frac{4}{a^3} \int_0^\infty e^{-2\frac{z}{a}} z^4 \, dz = \frac{4}{a^3} \cdot \frac{4!}{\left(\frac{2}{a}\right)^5} = \frac{2^2 \cdot 2 \cdot 3 \cdot 2^2}{2^5} a^2 = 3a^2$$

$$\langle x \rangle = \langle \Psi | \hat{x} | \Psi \rangle = \langle \Psi | z \sin\theta \cos\varphi | \Psi \rangle =$$

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \Psi^* \Psi z^3 \cos\varphi \sin^2\theta \, dz \, d\theta \, d\varphi$$

$$= \int_0^\infty \Psi^* \Psi z^3 \, dz \int_0^\pi \sin^2\theta \, d\theta \int_0^{2\pi} \cos\varphi \, d\varphi$$

$$\int_0^{\pi} \cos\varphi \, d\varphi = -\sin\varphi \Big|_0^{2\pi} = 0$$

$$\Rightarrow \langle x \rangle = 0$$

$\langle x^2 \rangle$

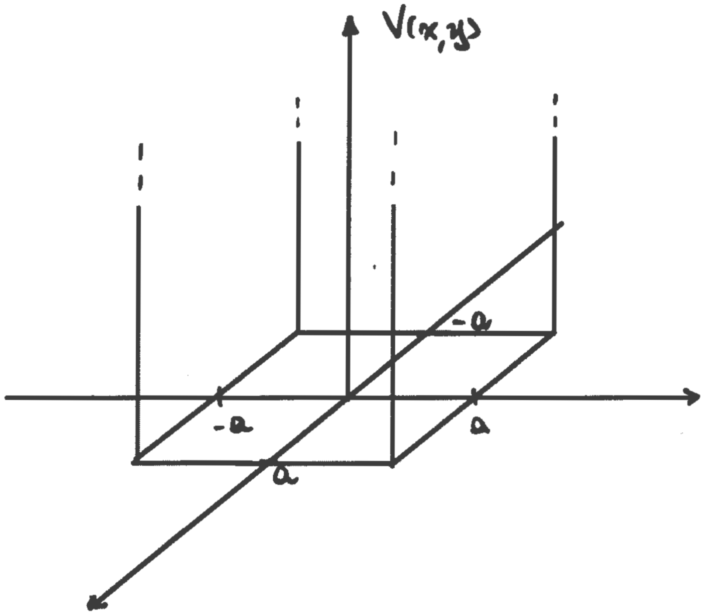
$$z^2 = x^2 + y^2 + z^2 \Rightarrow x^2 = \frac{1}{3} z^2$$

$$\Rightarrow \langle x^2 \rangle = a^2$$

- Problem 5

Consider a infinitely deep two-dimensional quantum well defined as: $V(x, y) = 0$ for $-a < x < a$, $-a < y < a$ and $V(x) = \infty$ elsewhere.

- Write an expression for the energy eigenvalues. Are there any degeneracies?
- Write a general expression for the eigenfunctions



1)

$$\left[-\frac{\hbar^2}{2m}\nabla + V(x,y)\right]\varphi_n(x,y) = E_n\varphi_n(x,y)$$

$$-\frac{\hbar^2}{2m}\left[\frac{\delta}{\delta x}\varphi_n(x,y) + \frac{\delta}{\delta y}\varphi_n(x,y)\right] = E_n\varphi_n(x,y)$$

$$\varphi_n(x,y) = X(x)\,Y(y) \qquad E = \varepsilon_x + \varepsilon_y$$

$$\begin{cases} -\frac{\hbar^2}{2m}\left[\frac{d}{dx}X(x)\right] = \varepsilon_x X(x) \\ -\frac{\hbar^2}{2m}\left[\frac{d}{dy}Y(y)\right] = \varepsilon_y Y(y) \end{cases}$$

$$X(x) = A_x \sin(K_x x) + B_x \cos(K_x x)$$

$$Y(y) = A_y \sin(K_y y) + B_y \cos(K_y y)$$

$$E = \varepsilon_x + \varepsilon_y = \frac{\hbar^2 \pi^2 n_x^2}{8 m a^2} + \frac{\hbar^2 \pi^2 n_y^2}{8 m Q^2} = \frac{\hbar^2 \pi^2}{8m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{Q^2}\right)$$

$$\varphi_n(x,y) = X(x)\,Y(y) = \begin{cases} \frac{1}{\sqrt{ab}} \cos\left(\frac{n_x \pi}{2a} x\right) \cos\left(\frac{n_y \pi}{2Q} y\right) &, n_x, n_y \text{ odd} \\ \frac{1}{\sqrt{ab}} \sin\left(\frac{n_x \pi}{2a} x\right) \sin\left(\frac{n_y \pi}{2Q} y\right) &, n_x, n_y \text{ even} \\ \frac{1}{\sqrt{ab}} \cos\left(\frac{n_x \pi}{2a} x\right) \sin\left(\frac{n_y \pi}{2Q} y\right) &, n_x \text{ odd } n_y \text{ even} \\ \frac{1}{\sqrt{ab}} \sin\left(\frac{n_x \pi}{2a} x\right) \cos\left(\frac{n_y \pi}{2Q} y\right) &. n_x \text{ even } n_y \text{ odd} \end{cases}$$

Degeneracy

$$E_{n_{x_1}, n_{y_1}} = E_{n_{x_2}, n_{y_2}} \qquad n_{x_1} \neq n_{x_2}\,,\; n_{y_1} \neq n_{y_2}$$

$$\cancel{\frac{\hbar^2 \pi^2}{8m}}\left(\frac{n_{x_1}^2}{a^2} + \frac{n_{y_1}^2}{Q^2}\right) = \cancel{\frac{\hbar^2 \pi^2}{8m}}\left(\frac{n_{x_2}^2}{a^2} + \frac{n_{y_2}^2}{Q^2}\right)$$

$$\frac{1}{a^2}\left(n_{x_1}^2 - n_{x_2}^2\right) = \frac{1}{Q^2}\left(n_{y_2}^2 - n_{y_1}^2\right)$$

$$1 = \frac{n_{y_2}^2 - n_{y_1}^2}{n_{x_1}^2 - n_{x_2}^2}$$

$$\exists\; n_{y_2}, n_{y_1}, n_{x_1}, n_{x_2} \in \mathbb{N} : \frac{n_{y_2}^2 - n_{y_1}^2}{n_{x_1}^2 - n_{x_2}^2} = 1$$

$\Rightarrow$ Degeneracy

- Problem 6

Consider an operator defined as $L_\pm = L_x \pm iL_y$.

- Write the eigenvalue problem for the L_z operator and show that the eigenvalue is $m\hbar$ with m integer.
- Compute the value of the commutator $[L_z, L_\pm]$.
- If ϕ_m is an eigenfunction of L_z, compute the value of $L_z(L_\pm \phi_m)$.

$$\boxed{[L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y.}$$

$$L_z = xp_y - yp_x$$

$$L_z \varphi_m(\vec{r}) = m\hbar \varphi_m(\vec{r})$$

$$\begin{aligned}
[L_z, L_\pm] &= [L_z, L_x \pm iL_y] = \\
&= [L_z, L_x] \pm i[L_z, L_y] = \\
&= \pm i \cdot i\hbar L_y + \hbar L_x = \\
&= \hbar(L_x \pm iL_y) = \pm \hbar L_\pm
\end{aligned}$$

$$L_z(L_\pm \phi_m) =$$

$$\begin{aligned}
&= (L_z L_\pm - L_\pm L_z + L_\pm L_z)\phi_m \\
&= [L_z, L_\pm]\phi_m + L_\pm m\hbar = \\
&= \hbar L_\pm \phi_m + L_\pm m\hbar \\
&= (\hbar + m\hbar) L_\pm \phi_m
\end{aligned}$$

- Problem 7

Consider an isotropic two-dimensional harmonic oscillator. The eigenfunction of the one-dimensional harmonic oscillator are given by:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\xi^2}{2}} \quad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{2}{\sqrt{2}} \xi \, e^{-\frac{\xi^2}{2}} \quad \xi = \sqrt{\frac{m\omega}{\hbar}} x$$

$$\int_0^\infty x^m e^{-a x^2} dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}} \quad \Gamma[n+1] = n! \;\; \Gamma[n] = (n-1)! \;\; \Gamma[1/2] = \sqrt{\pi}$$

−1−Compute the expectation values $< xy^2 >$ for ψ_{01} and ψ_{10}.

$$\psi_{01} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\xi_x^2}{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{2}{\sqrt{2}} \xi_y e^{-\frac{\xi_y^2}{2}}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{\xi_x^2}{2}} \cdot \frac{2}{\sqrt{2}} \xi_y e^{-\frac{\xi_y^2}{2}}$$

$$\psi_{10} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{\xi_y^2}{2}} \cdot \frac{2}{\sqrt{2}} \xi_x e^{-\frac{\xi_x^2}{2}}$$

ψ_{01})

$$\langle xy^2\rangle = \underbrace{\langle \psi_0 | x | \psi_0 \rangle}_{\text{Odd}\cdot\text{even}} \langle \psi_1 | y^2 | \psi_1 \rangle = 0$$

ψ_{10})

$$\langle xy^2\rangle = \underbrace{\langle \psi_1 | x | \psi_1 \rangle}_{\text{Odd}\cdot\text{even}} \langle \psi_0 | y^2 | \psi_0 \rangle = 0$$

- Problem 8

Consider a 2D infinite quantum well where the potential $V = 0$ for $0x < a$ and $0 < y < a$, and $V = \infty$ everywhere else.

– Determine the solution of the Schroedinger equation in terms of wavefunctions and eigenvalues. Are there any degeneracies?

– Suppose the system is perturbed with an external potential $V_0\,\delta(x - a/4, y - 3/4a)$ and $V_1\,\delta(x - 3/4a, y - 1/4a)$ and Compute the first order correction to the ground state and first excited state energies.

– Compute the correct zeroth order wavefunctions.

$0 < x < a \qquad 0 < y < a$

$$\psi_{n,n} = \frac{2}{a}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right)$$

$$E_{nn} = \frac{\hbar\pi^2}{2ma^2}(n_x^2 + n_y^2)$$

$\hookrightarrow$ Degenerate

$$\psi_{1,2} = \frac{2}{a}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}y\right)$$

$$\psi_{21} = \frac{2}{a}\sin\left(\frac{2\pi}{a}x\right)\sin\left(\frac{\pi}{a}y\right)$$

$$H' = V_0\left[\delta\left(x - \frac{a}{4}, y - \frac{3}{4}a\right) + \delta\left(x - \frac{3}{4}a, y - \frac{1}{4}a\right)\right]$$

$$\begin{bmatrix} W_{aa} & W_{ab} \\ W_{ba} & W_{bb} \end{bmatrix}$$

$$W_{aa} = \iint \psi_{12} H' \psi_{12}\, dx\, dy =$$

$$= \frac{4}{a^2} V_0 \left[\sin^2\left(\frac{\pi}{4}\right)\sin^2\left(\frac{3}{2}\pi\right) + \sin^2\left(\frac{3}{4}\pi\right)\sin^2\left(\frac{\pi}{2}\right)\right] =$$

$$= \frac{4}{a^2} V_0 \left[\frac{1}{2} + \frac{1}{2}\right] = \frac{4}{a^2} V_0 = W_{BB}$$

$$W_{AB} = \iint \psi_{12} H' \psi_{21}\, dx\, dy$$

$$= \frac{4}{a^2} V_0 \left[\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{3}{2}\pi\right)\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{3}{4}\pi\right) + \sin\left(\frac{3}{4}\pi\right)\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{3}{2}\pi\right)\sin\left(\frac{\pi}{4}\pi\right)\right] =$$

$$= \frac{4}{a^2} V_0 \left[\frac{\sqrt{2}}{2}\cdot -1 \cdot 1 \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot 1 \cdot -1 \cdot \frac{\sqrt{2}}{2}\right] =$$

$$= \frac{4}{a^2} V_0 [-1] = -\frac{4}{a^2} V_0$$

$$\Rightarrow \frac{4}{a^2} V_0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\det\begin{bmatrix} 1-E & -1 \\ -1 & 1-E \end{bmatrix} = (1-E)^2 - 1 = E(E-2) = 0 \Rightarrow E_{1,2} = \begin{cases} 0 \\ 2 \end{cases} \qquad E_{1,2} = 0,\ \frac{8}{a^2}V_0$$

$E_1 = 0$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \qquad \psi_{12} + \psi_{21}$$

$E_1 = -2$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \qquad \psi_{12} - \psi_{21}$$

$$E_1^{(1)} = \begin{cases} E_1 + 0 \\ E_1 + \frac{8}{a^2}V_0 \end{cases} = \begin{matrix} \frac{\hbar^2\pi^2}{2ma^2}5 \\ \frac{\hbar^2\pi^2}{2ma^2}5 + \frac{8}{a^2}V_0 \end{matrix}$$

$$\psi_c = a(\psi_{12} + \psi_{21}) + b(\psi_{12} - \psi_{21})$$

Ground State

$$\iint \psi_{11}(x,y)\, H'\, \psi_{11}(x,y)\, dx\, dy = \qquad H' = V_0\left[\delta(x - \tfrac{a}{4}, y - \tfrac{3}{4}a) + \delta(x - \tfrac{3}{4}a, y - \tfrac{1}{4}a)\right]$$

$$= \frac{4}{a^2}V_0 \iint H' \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{a}y\right) dx\, dy =$$

$$= \frac{4}{a^2}V_0\left[\sin^2\left(\frac{\pi}{4}\right)\sin^2\left(\frac{3}{4}\pi\right) + \sin^2\left(\frac{3}{4}\pi\right)\sin^2\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{4}{a^2}V_0\left[\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\right] = \frac{4}{a^2}\cdot\frac{2}{4} = \frac{2}{a^2}V_0$$

$$E_0^{(0)} = E_0 + \frac{2}{a^2}V_0$$

• Problem 9

Consider the solution of the 1D oscillator problem e studied in class. Compute the correction to the energy eigenvalues caused by the following perturbations:

- $H' = cx$
- $H' = cx^3$
- $H' = cx^4$

where c is a real constant. To help in this process use the knowledge of the wavefunctions symmetry and the results of the equations [4.166] on page 174 of the book.

cx)

$$c\int \left(\frac{\beta}{\sqrt{2}\,2^n n!}\right)^{1/2} e^{-\beta^2\frac{x^2}{2}} H_n(\beta x)\; x \left(\frac{\beta}{\sqrt{2}\,2^n n!}\right)^{1/2} e^{-\beta^2\frac{x^2}{2}} H_n(\beta x)\; dx$$

$$= 0$$

cx³)

$$c\int \left(\frac{\beta}{\sqrt{2}\,2^n n!}\right)^{1/2} e^{-\beta^2\frac{x^2}{2}} H_n(\beta x)\; x^3 \left(\frac{\beta}{\sqrt{2}\,2^n n!}\right)^{1/2} e^{-\beta^2\frac{x^2}{2}} H_n(\beta x)\; dx$$

$$= 0$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$$

cx⁴)

$$c\left(\frac{\hbar}{2m\omega}\right)^2 \int \psi_n\,(a_+ + a_-)^4\,\psi_n\,dx$$

$$c\left(\frac{\hbar}{2m\omega}\right)^2 \int \psi_n\left(\cancel{a_+^4} + 4\cancel{a_+^3}a_- + 6a_+^2a_-^2 + 4a_+\cancel{a_-^3} + \cancel{a_-^4}\right)\psi_n\,dx$$

$$(a_+^2 a_-^2)$$

$$(a_+a_+a_-a_-) = n(n-1)$$

$$(a_+a_-a_+a_-) = n^2$$

$$(a_+a_-a_-a_+) = \sqrt{n+1}\sqrt{n+1}\sqrt{n}\sqrt{n} = n(n+1)$$

$$(a_-a_-a_+a_+) = \sqrt{n+1}\sqrt{n+2}\sqrt{n+2}\sqrt{n+1} = (n+1)(n+2)$$

$$(a_-a_+a_-a_+) = \sqrt{n+1}\sqrt{n+1}\sqrt{n+1}\sqrt{n+1} = (n+1)^2$$

$$(a_-a_+a_+a_-) = \sqrt{n}\sqrt{n}\sqrt{n+1}\sqrt{n+1} = n(n+1)$$

$$n(n-1) + n^2 + n(n+1) + (n+1)(n+2) + (n+1)^2 + n(n+1)$$

$$n^2 - n + n^2 + n^2 + n + n^2 + 2n + n + 2 + n^2 + 2n + 1 + n^2 + n$$

$$= 6n^2 + 6n + 3$$

$$(6n^2 + 6n + 3)\, c \left(\frac{\hbar}{2m\omega}\right)^2 \int \psi_n \psi_n \, dx =$$

$$= (6n^2 + 6n + 3)\, c \left(\frac{\hbar}{2m\omega}\right)^2$$

- Problem 10

Consider a 2D isotropic harmonic oscillator characterized by an hamiltonian of the type:
$H' = \frac{p^2}{2m} + \frac{1}{2}k(x^2 + y^2)$

- Compute the Schroedinger equation and determine the degeneracy of the eigenvalues.
- Compute the first order correction tho the eigenvalues of the second state caused by the perturbing hamiltonian $H' = cx^4y^4$

$$\varphi_{n_x n_y}(x,y) = \left(\frac{\beta}{\sqrt{\pi}\, 2^{n_x} n_x!}\right)^{1/2} e^{-\frac{\beta^2}{2}x^2} H_{n_x}(\beta x) \left(\frac{\beta}{\sqrt{\pi}\, 2^{n_y} n_y!}\right)^{1/2} e^{-\frac{\beta^2}{2}y^2} H_{n_y}(\beta y)$$

$E_{n_x n_y} = (1 + n_x + n_y)\hbar\omega_0$ Degenerate

$$\varphi_{10} = \left(\frac{\beta}{\sqrt{\pi}\,2}\right)^{1/2} e^{-\frac{\beta^2}{2}x^2} \cdot 2\beta x \left(\frac{\beta}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{\beta^2}{2}y^2}$$

$$= \frac{2}{\sqrt{2}} \frac{\beta^2}{\sqrt{\pi}} e^{-\frac{\beta^2}{2}(x^2+y^2)} x$$

$$\varphi_{01} = \frac{2}{\sqrt{2}} \frac{\beta^2}{\sqrt{\pi}} e^{-\frac{\beta^2}{2}(x^2+y^2)} y$$

$H' = cx^4y^4$

$$W_{aa} = 2c\frac{\beta^4}{\pi} \int_{-\infty}^{+\infty} x^4 e^{-\beta^2x^2} x^2\, dx \int_{-\infty}^{+\infty} y^4 e^{-\beta^2y^2}\, dy$$

$$= 4c\frac{\beta^4}{\pi} \int_0^{+\infty} x^6 e^{-\beta^2x^2}\, dx \int_0^{+\infty} y^4 e^{-\beta^2y^2}\, dy$$

$$= \cancel{4}c\frac{\beta^4}{\pi} \frac{\Gamma(7/2)}{\cancel{2}\beta^7} \frac{\Gamma(5/2)}{2\beta^5} =$$

$$= c\frac{\beta^8}{\pi} \cdot \frac{45}{16}\pi = c\beta^8 \cdot \frac{45}{16}$$

$W_{bb} = W_{aa}$

$$\Gamma(5/2) = \left(\tfrac{5}{2}-1\right)\Gamma\left(\tfrac{5}{2}-1\right)$$
$$= \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) =$$
$$= \tfrac{3}{2}\left[\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)\right] =$$
$$= \tfrac{3}{4}\sqrt{\pi}$$
$$\Gamma(7/2) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \tfrac{15}{4}\sqrt{\pi}$$

$$W_{ab} = \iint \varphi_{10} H' \varphi_{01} \, dx \, dy$$

$$= 2C \frac{\beta^4}{\pi} \int x^5 e^{-\beta x^2} dx \int y^5 e^{-\beta y^2} dx = 0$$

$$E_{1,2} = \left(C\beta^8 \frac{45}{16} - E \right)^2 = E^2 - C\beta^8 \frac{45}{8} E + C^2 \beta^{16} \left(\frac{45}{16} \right)^2$$

- Problem 12

Consider the four degenerate eigenfunctions corresponding to the first ($n = 2$) excited state of the hydrogen atom:

$$\psi_{2,0,0}(r,\theta,\phi) = 2\left(\frac{1}{2a_0}\right)^{-3/2}\left(1-\frac{r}{2a_0}\right)e^{-\frac{r}{2a_0}}Y_{0,0}(\theta,\phi)$$

$$\psi_{2,1,1}(r,\theta,\phi) = 3^{-1/2}\left(\frac{1}{2a_0}\right)^{-3/2}\left(\frac{r}{a_0}\right)e^{-\frac{r}{2a_0}}Y_{1,1}(\theta,\phi)$$

$$\psi_{2,1,0}(r,\theta,\phi) = 3^{-1/2}\left(\frac{1}{2a_0}\right)^{-3/2}\left(\frac{r}{a_0}\right)e^{-\frac{r}{2a_0}}Y_{1,0}(\theta,\phi)$$

$$\psi_{2,1,-1}(r,\theta,\phi) = 3^{-1/2}\left(\frac{1}{2a_0}\right)^{-3/2}\left(\frac{r}{a_0}\right)e^{-\frac{r}{2a_0}}Y_{1,-1}(\theta,\phi)$$

where a_0 is the Bohr radius, and $Y_{n,m}(\theta,\phi)$ the spherical harmonics. Consider a perturbing electric field E directed along the z-axis that introduces a perturbing Hamiltonian $H' = -qEz$.

– Using degenerate perturbation theory compute the first order energy correction for the Hydrogen state with $n = 2$.

– Determine the eigenvectors and the proper zeroth order eigenfuctions.

Hints!

Notice that:

$$cos(\theta) = \sqrt{\frac{4\pi}{3}}Y_{1,0}(\theta,\phi)$$

Use the symmetry of the spherical harmonics to simplify your integration.

$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$	$Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm2i\phi}$
$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$	$Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2}(5\cos^3\theta - 3\cos\theta)$
$Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$	$Y_3^{\pm1} = \mp\left(\frac{21}{64\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$
$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1)$	$Y_3^{\pm2} = \left(\frac{105}{32\pi}\right)^{1/2}\sin^2\theta\cos\theta e^{\pm2i\phi}$
$Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi}$	$Y_3^{\pm3} = \mp\left(\frac{35}{64\pi}\right)^{1/2}\sin^3\theta e^{\pm3i\phi}$

$$\int \cos\theta\, Y_{1,\pm1} = 0$$

$$H' = -qEz\cos\theta$$

$\psi_{200}\psi_{200}$ $\psi_{200}\psi_{211}$ $\psi_{200}\psi_{210}$ $\psi_{200}\psi_{21-1}$

$\psi_{211}\psi_{200}$ $\psi_{211}\psi_{211}$ $\psi_{211}\psi_{210}$ $\psi_{211}\psi_{21-1}$

$\psi_{210}\psi_{200}$ $\psi_{210}\psi_{211}$ $\psi_{210}\psi_{210}$ $\psi_{210}\psi_{21-1}$

$\psi_{21-1}\psi_{200}$ $\psi_{21-1}\psi_{211}$ $\psi_{21-1}\psi_{210}$ $\psi_{21-1}\psi_{21-1}$

All orthonormal to $\hat{z}$, except ψ_{210}

$$\psi_{210} = \frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{-3/2}\left(\frac{z}{a_0}\right)e^{-\frac{z}{2a_0}}\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$$

$$= \frac{1}{\cancel{2}\sqrt{\pi}}\cdot\cancel{2a_0}\sqrt{2a_0}\cdot\frac{z}{\cancel{a_0}}\cos\theta\, e^{-\frac{z}{2a_0}}$$

$$= \sqrt{\frac{2a_0}{\pi}}\, z\, e^{-\frac{z}{2a_0}}\cos\theta$$

$$\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$$

$$W_{CA} = W_{AC} = \frac{2a_0}{\pi}\cdot -qE\iiint z^2 e^{-\frac{z}{a_0}}\cos^2\theta\, z\cos\theta\, z\sin\theta\, dz\, d\theta\, d\varphi$$

$$= -q\mathcal{E}\,\frac{2a_0}{\pi}\int_0^\infty r^4 e^{-\frac{r}{a_0}}\,dr \int_0^\pi \cos^3\theta \sin\theta\, dr \cdot 2\pi$$

$$= +q\mathcal{E}\,4a_0\; 4!\,a_0^5\,\frac{\cos^4(\theta)}{4}\Big|_0^\pi =$$

$$= q\mathcal{E}\,a_0^6\,4!$$

$$\det\left(W_{CA}\begin{bmatrix} -E & 0 & 1 & 0 \\ 0 & -E & 0 & 0 \\ 1 & 0 & -E & 0 \\ 0 & 0 & 0 & -E \end{bmatrix}\right) = 0$$

$$E^4 + (-E \cdot -E) = E^4 - E^2 = 0$$

$$E(E^3 - E) = 0 \Rightarrow E_1 = 0$$

$$E(E^2 - 1) = 0 \Rightarrow E_2 = 0$$

$$E^2 = 1 \qquad \Rightarrow E_3 = 1 \quad E_4 = -1$$

$$E_2^{(1)} = \begin{cases} \left.\begin{matrix} E_2 \\ E_2 \end{matrix}\right\} \text{Double degenerate} \\ E_2 + W_{CA} \\ E_2 - W_{CA} \end{cases}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a\left(\psi_{211} + \psi_{21-1}\right) + b\left(\psi_{200} + \psi_{210}\right) + c\left(\psi_{200} - \psi_{210}\right)$$

- Problem 13

Consider a charged particle in the one-dimensional harmonic oscillator potential. Suppose we turn on a weak electric field E so that the system is perturbed by a potential $H' = -qEx$.

- Solve the Schroedinger equation using the following variable transformation: $x' = x - qE/m\omega^2$. Where m is the particle mass and ω the oscillator angular frequency.

- Compute the first and second order corrections and compare them with the exact solutions.

$$H = H_0 + H' = \left[-\frac{\hbar}{2m}\frac{d^2}{dx} + \frac{1}{2}Kx^2 - q\mathcal{E}x\right]$$

$$H\,\varphi_n(x) = E_n\varphi_n(x)$$

$$x' = x - \frac{q\mathcal{E}}{m\omega^2}$$

$$x = x' + \frac{q\mathcal{E}}{m\omega^2}$$

$$-\frac{\hbar}{2m}\frac{d^2}{dx^2} + \frac{1}{2}K\left[x'^2 + 2x'\frac{q\mathcal{E}}{m\omega^2} + \frac{q^2\mathcal{E}^2}{m^2\omega^4}\right] - q\mathcal{E}\left(x' + \frac{q\mathcal{E}}{m\omega^2}\right)$$

$$-\frac{\hbar}{2m}\frac{d^2}{dx^2} + \frac{1}{2}Kx'^2 + \cancel{K}\cancel{x'}\frac{q\mathcal{E}}{\cancel{m\omega^2}} + \frac{1}{2}\cancel{K}\frac{q^2\mathcal{E}^2}{m^{\cancel{2}}\omega^{\cancel{4}2}} - q\mathcal{E}\cancel{x'} - \frac{q^2\mathcal{E}^2}{m\omega^2}$$

$$-\frac{\hbar}{2m}\frac{d^2}{dx} + \frac{1}{2}Kx'^2 - \frac{1}{2}\frac{q^2\mathcal{E}^2}{m\omega^2}$$

$$\left[-\frac{\hbar}{2m}\frac{d^2}{dx} + \frac{1}{2}Kx'^2\right]\varphi_n = \left[E + \frac{1}{2}\frac{q^2\mathcal{E}^2}{m\omega^2}\right]\varphi_n$$

$$E_n^{(0)} = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{1}{2}\frac{q^2\mathcal{E}^2}{m\omega^2}$$