EC516 HW5 Solutions

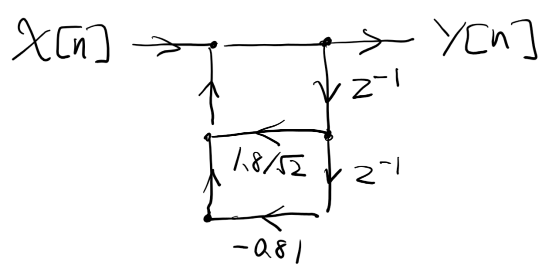
Problem 5.1

(a) Since $H(z) = Y(z)/X(z)$,

$$\begin{aligned}(1-0.9e^{\frac{j\pi}{4}}z^{-1})(1-0.9e^{-\frac{j\pi}{4}}z^{-1})Y(z) &= X(z)\\ (1-1.8\cos(\pi/4)z^{-1}+0.81z^{-2})Y(z) &= X(z)\\ y[n]-1.8\cos(\pi/4)y[n-1]+0.81y[n-2] &= x[n]\\ y[n] &= (1.8/\sqrt{2})y[n-1]-0.81y[n-2]+x[n]\end{aligned}$$

(b) All coefficients are real. Real valued input will generate real valued output signal.

(c) Since it's FIR, Direct II and Direct I is equivalent.

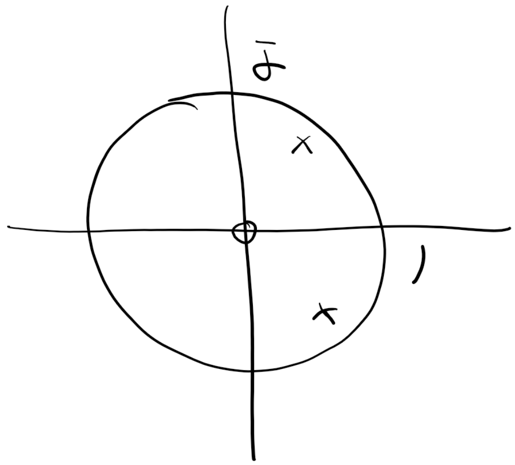


Problem 5.1(c): Direct II Flowgraph

(d) We multiply both numerator and denominator with z's to get rid of z^{-1}'s.

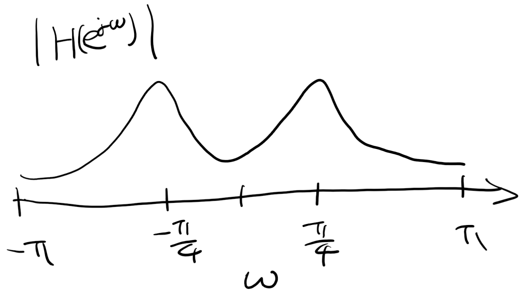
$$\frac{1}{(1-0.9e^{\frac{j\pi}{4}}z^{-1})(1-0.9e^{-\frac{j\pi}{4}}z^{-1})} = \frac{z^2}{(z-0.9e^{\frac{j\pi}{4}})(z-0.9e^{-\frac{j\pi}{4}})}$$

There are poles at $z = 0.9e^{\pm\frac{j\pi}{4}}$ and 2 zeros at $z = 0$.



Problem 5.1(d): Poles (x) and Zeros (o) on a complex plane

(e) Recall that $\left|\frac{ab}{c}\right| = \frac{|a||b|}{|c|}$ for complex numbers a, b, c. Using the zero-pole plot, we can get a sense of the frequency response.



Problem 5.1(e): Approximate $|H(e^{j\omega})|$

Since the zeros do not exist on the unit circle, there are no zeros in $H(e^{j\omega})$.

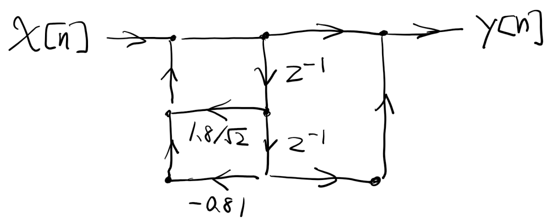
Problem 5.2

(a)

$$
\begin{aligned}
(1 - 0.9e^{\frac{j\pi}{4}}z^{-1})(1 - 0.9e^{-\frac{j\pi}{4}}z^{-1})Y(z) &= (1 + z^{-2})X(z) \\
(1 - 1.8\cos(\pi/4)z^{-1} + 0.81z^{-2})Y(z) &= (1 + z^{-2})X(z) \\
y[n] - 1.8\cos(\pi/4)y[n-1] + 0.81y[n-2] &= x[n] + x[n-2] \\
y[n] &= (1.8/\sqrt{2})y[n-1] - 0.81y[n-2] + x[n] + x[n-2]
\end{aligned}
$$

(b) All coefficients are real. Real valued input will generate real valued output signal.

(c) Flowgraph is shown below.

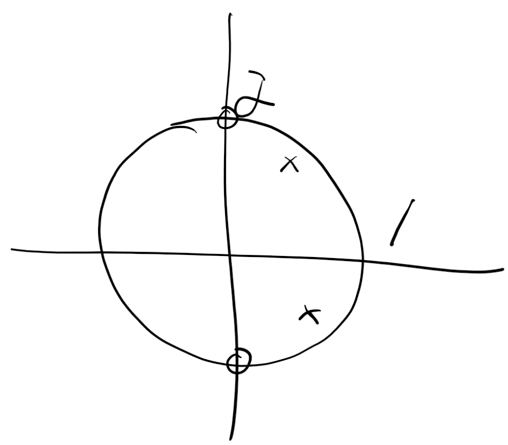


Problem 5.2(c): Direct II Flowgraph

(d)

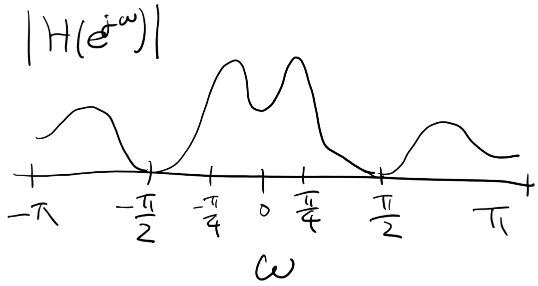
$$\frac{1+z^{-2}}{(1-0.9e^{\frac{j\pi}{4}}z^{-1})(1-0.9e^{-\frac{j\pi}{4}}z^{-1})} = \frac{z^2+1}{(z-0.9e^{\frac{j\pi}{4}})(z-0.9e^{-\frac{j\pi}{4}})}$$

There are poles at $z = 0.9e^{\pm\frac{j\pi}{4}}$ and 2 zeros at $z = \pm j$.



Problem 5.2(d): Poles (x) and Zeros (o) on a complex plane

(e) Approximate frequency response is shown below.



Problem 5.2(e): Approximate $|H(e^{j\omega})|$

$|H(e^{j\omega})| = 0$ at $\omega = \pm\frac{\pi}{2}$.

Problem 5.3

(a) FIR, since it can be written in non-recursive finite sum of input signal.

(b) We factorize $1 - z^{-8}$ into products of second order equation.

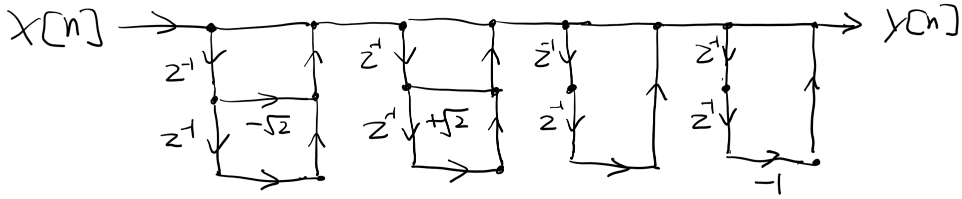
$$\begin{aligned}(1-z^{-8}) &= (1+z^{-4})(1-z^{-4}) \\ &= (1+e^{j\pi/2}z^{-2})(1+e^{-j\pi/2}z^{-2})(1+z^{-2})(1-z^{-2})\end{aligned}$$

We ended up with some complex coefficients. However, if we factorize it further and multiply the conjugates together, we are able to get rid of complex coefficients.

$$\begin{aligned}&= \Big((1+e^{j3\pi/4}z^{-1})(1+e^{-j\pi/4}z^{-1})\Big)\Big((1+e^{-j3\pi/4}z^{-1})(1+e^{j\pi/4}z^{-1})\Big)(1+z^{-2})(1-z^{-2}) \\ &= \Big((1+e^{j3\pi/4}z^{-1})(1+e^{-j3\pi/4}z^{-1})\Big)\Big((1+e^{j\pi/4}z^{-1})(1+e^{-j\pi/4}z^{-1})\Big)(1+z^{-2})(1-z^{-2}) \\ &= (1+2\cos(3\pi/4)z^{-1}+z^{-2})(1+2\cos(\pi/4)z^{-1}+z^{-2})(1+z^{-2})(1-z^{-2}) \\ &= (1-\sqrt{2}z^{-1}+z^{-2})(1+\sqrt{2}z^{-1}+z^{-2})(1+z^{-2})(1-z^{-2})\end{aligned}$$

Now we can write the filter as a composition of 4 second order sections $H(z) = H_1(z)H_2(z)H_3(z)H_4(z)$.

$$\begin{aligned}H_1(z) = (1-\sqrt{2}z^{-1}+z^{-2}) &\rightarrow y_1[n] = x_1[n] - \sqrt{2}x_1[n-1] + x_1[n-2] \\ H_2(z) = (1+\sqrt{2}z^{-1}+z^{-2}) &\rightarrow y_2[n] = x_2[n] + \sqrt{2}x_2[n-1] + x_2[n-2] \\ H_3(z) = (1+z^{-2}) &\rightarrow y_3[n] = x_3[n] + x_3[n-2] \\ H_4(z) = (1-z^{-2}) &\rightarrow y_4[n] = x_4[n] - x_4[n-2]\end{aligned}$$



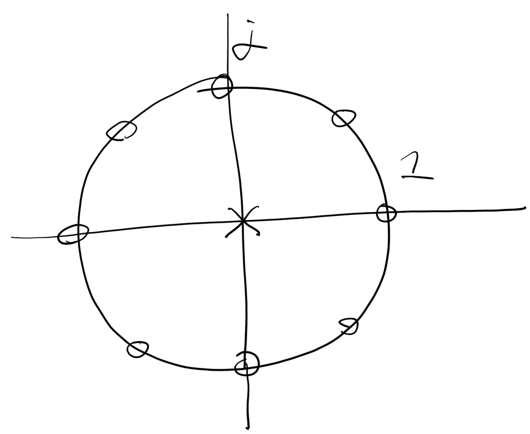
Problem 5.3(b): Direct II Flowgraph of Cascaded Filters

(c) Using the factorization we got from the previous question, we get

$$\frac{z^8(1-z^{-8})}{z^8} = \frac{(z+e^{j3\pi/4})(z+e^{-j\pi/4})(z+e^{-j3\pi/4})(z+e^{j\pi/4})(z^2+1)(z^2-1)}{z^8}$$
$$= \frac{(z+e^{j3\pi/4})(z+e^{-j\pi/4})(z+e^{-j3\pi/4})(z+e^{j\pi/4})(z+j)(z-j)(z+1)(z-1)}{z^8}$$

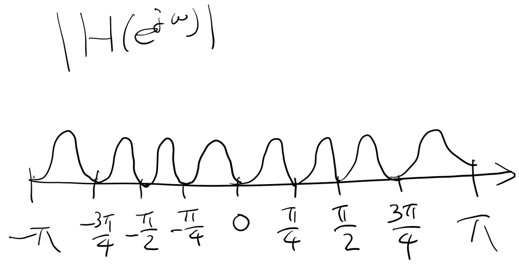
This filter has 8 poles at $z=0$ and zeros at $z = e^{\pm j3\pi/4}, e^{\pm j\pi/4}, \pm j, \pm 1$

Poles and zeros plot shown below.



Problem 5.3(c): Poles (x) and Zeros (o) on a complex plane

(d) Approximate frequency response is shown below.



Problem 5.3(d): Approximate $|H(e^{j\omega})|$

Problem 5.4

(a)

$$H(z) = \frac{(z+1)(z-e^{j5\pi/6})(z-e^{-j5\pi/6})(z-e^{j2\pi/3})(z-e^{-j2\pi/3})}{(z-0.5)(z-0.5e^{j\pi/6})(z-0.5e^{-j\pi/6})(z-0.5e^{j\pi/3})(z-0.5e^{-j\pi/3})}$$

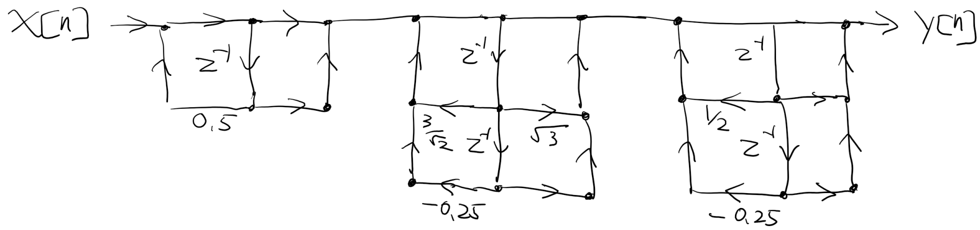
Filter has poles that are not at 0. It's an IIR filter.

(b) In order to ensure the filter coefficients are real valued, we multiply the conjugates. First we multiply the denominator and numerator by z^{-5}.

$$\begin{aligned}
H(z) &= \frac{(1+z^{-1})(1-e^{j5\pi/6}z^{-1})(1-e^{-j5\pi/6}z^{-1})(1-e^{j2\pi/3}z^{-1})(1-e^{-j2\pi/3}z^{-1})}{(1-0.5z^{-1})(1-0.5e^{j\pi/6}z^{-1})(1-0.5e^{-j\pi/6}z^{-1})(1-0.5e^{j\pi/3}z^{-1})(1-0.5e^{-j\pi/3}z^{-1})} \\
&= \frac{(1+z^{-1})(1-2\cos(5\pi/6)z^{-1}+z^{-2})(1-2\cos(2\pi/3)z^{-1}+z^{-2})}{(1-0.5z^{-1})(1-\cos(\pi/6)z^{-1}+0.25z^{-2})(1-\cos(\pi/3)z^{-1}+0.25z^{-2})} \\
&= \frac{1+z^{-1}}{1-0.5z^{-1}} \cdot \frac{1+\sqrt{3}z^{-1}+z^{-2}}{1-(\sqrt{3}/2)z^{-1}+0.25z^{-2}} \cdot \frac{1+z^{-1}+z^{-2}}{1-(1/2)z^{-1}+0.25z^{-2}}
\end{aligned}$$

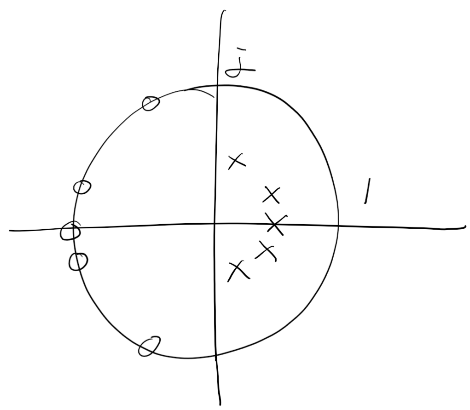
We write the filter as a composition of 3 second order sections $H(z) = H_1(z)H_2(z)H_3(z)H_4(z)$.

$$\begin{aligned}
H_1(z) &= \frac{1+z^{-1}}{1-0.5z^{-1}} \to y_1[n] = x_1[n] + x_1[n-1] + 0.5y_1[n-1] \\
H_2(z) &= \frac{1+\sqrt{3}z^{-1}+z^{-2}}{1-(\sqrt{3}/2)z^{-1}+0.25z^{-2}} \to y_2[n] = x_2[n] + \sqrt{3}x_2[n-1] + x_2[n-2] \\
&\qquad + (\sqrt{3}/2)y_2[n-1] - 0.25y_2[n-2] \\
H_3(z) &= \frac{1+z^{-1}+z^{-2}}{1-(1/2)z^{-1}+0.25z^{-2}} \to y_3[n] = x_3[n] + x_3[n-1] + x_3[n-2] \\
&\qquad + (1/2)y_2[n-1] - 0.25y_2[n-2]
\end{aligned}$$



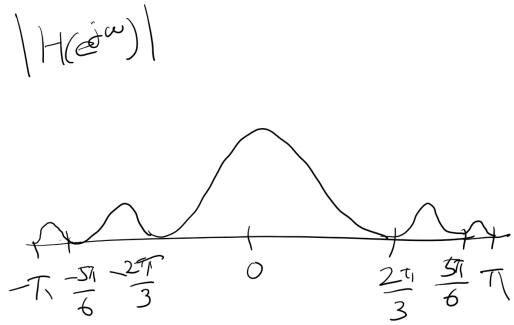
Problem 5.4(b): Direct II Flowgraph of Cascaded Filters

(c) Poles and zeros plot shown below.



Problem 5.4(c): Poles (x) and Zeros (o) on a complex plane

(d) Approximate frequency response is shown below.



Problem 5.4(d): Approximate $|H(e^{j\omega})|$

This filter is an approximation of a low pass filter since the response is higher in the lower frequency than the high frequency.