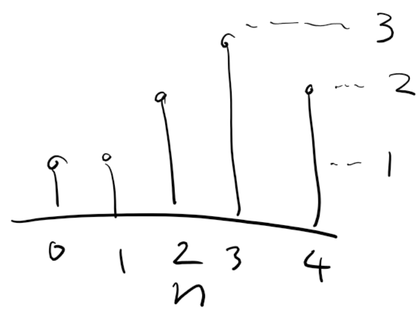
EC516 HW11 Solutions

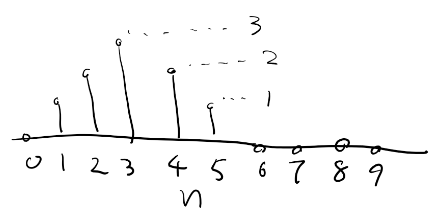
Problem 11.1

(a) $g[(n)_N]$ means $g[n]$ that repeats every N.

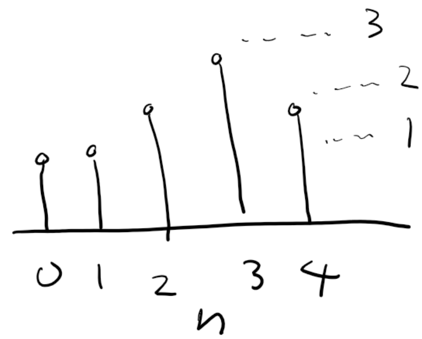
$$g[(n)_N] = \sum_{k=-\infty}^{\infty} g[n-kN]$$



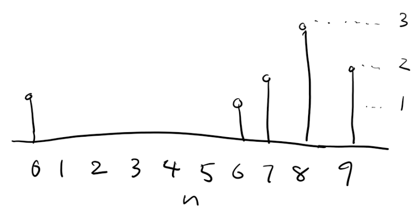
(b)



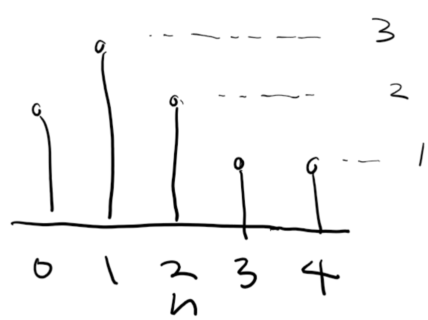
(c)



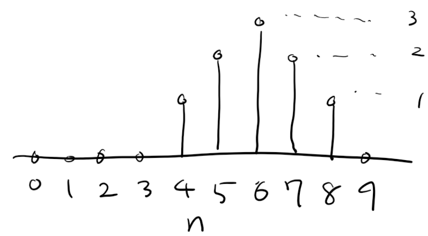
(d)



(e)



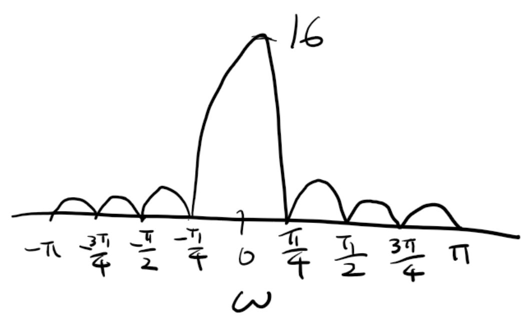
(f)



Problem 11.2

(a)

$$X(e^{j\omega}) = 2e^{-j7\omega/2}\frac{\sin(4\omega)}{\sin(\omega/2)}$$

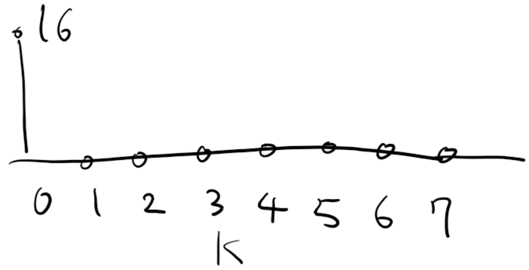


(b) Recall that

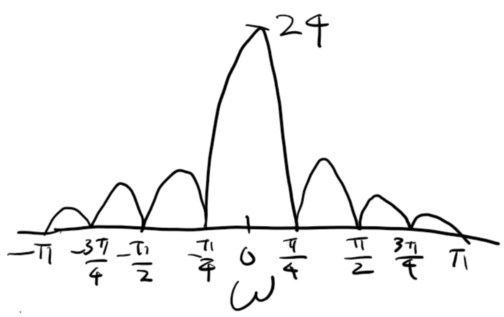
$$X[k]_N = X(e^{j\frac{2\pi k}{N}})$$

Note that except for $k = 0$, DFT is sampled at the zero-crossings.

$$X[k]_8 = [16, 0, 0, 0, 0, 0, 0, 0]$$

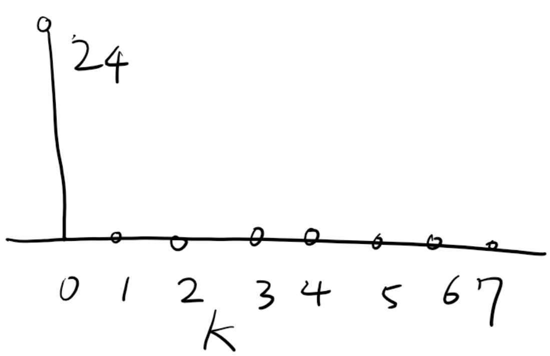


(c)

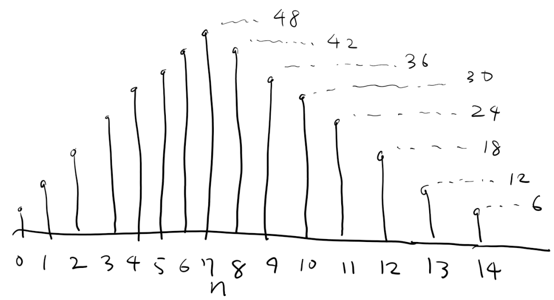


(d)

$$G[k]_8 = [24, 0, 0, 0, 0, 0, 0, 0]$$

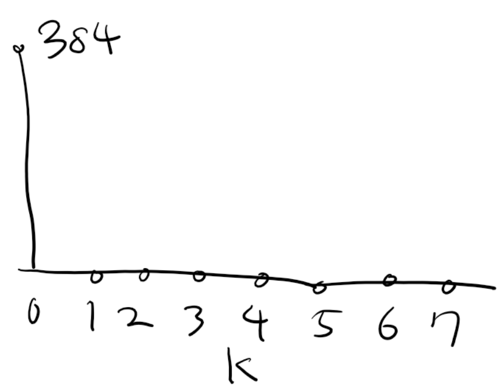


(e) $y[n]$ is simply the convolution of $x[n]$ and $g[n]$.



(f)

$$Q[k]_8 = [384, 0, 0, 0, 0, 0, 0, 0]$$



(g) Take the DTFT of $q[n] = 48(u[n] - u[n-8])$.

$$Q(e^{j\omega}) = 48e^{-j7\omega/2}\frac{\sin(4\omega)}{\sin(\omega/2)}$$

Evaluating the above expression at $\omega = \frac{2\pi k}{8}$ gives us,

$$Q[k]_8 = [384, 0, 0, 0, 0, 0, 0, 0]$$

(h) No, the results are different. Multiplication of DFT is equivalent to circular convolution of the two signals. Although $x[n]$ and $g[n]$ are 8-point signals, its convolution results in signal longer than length 8, which tells us the results of the circular convolution is different from the regular convolution, and hence the difference.

(i)

$$
\begin{aligned}
q[0] &= x[0]g[0] + x[1]g[7] + \cdots + x[7]g[1] \\
q[1] &= x[0]g[1] + x[1]g[0] + \cdots + x[7]g[2] \\
&\vdots \\
q[7] &= x[0]g[7] + x[1]g[6] + \cdots + x[7]g[0]
\end{aligned}
$$

Since $x[n]$ and $g[n]$ are constants, it is easy to see that

$$q[n] = 48(u[n] - u[n-7])$$

which is the same result from part(g).

<u>Problem 11.3</u>

(a) Assuming $x[n], g[n]$ are non-zero at $n = 0, 1, 2, \ldots, 7$, from the convolution equation,

$$y[0] = x[0]g[0]$$

and

$$y[6] = x[3]g[3] = x[3]g[6-3]$$

at $n < 0$ or $n > 6$, there are no overlaps between the two signals. Therefore, the length of $y[n]$ is 7.

(b) Since $g[(n-k)_N]$ repeats every N,

$$\sum_{k=0}^{N-1} x[k]g[(n-k)_N](u[k] - u[k-N]) = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{N-1} x[k]g[n-k-mN] \quad \text{for } 0 \le n \le N$$

If $N \ge 4$, we can write the circular convolution as a sum of shifted versions of regular convolution.

$$
\begin{aligned}
&\sum_{k=0}^{N} x[k]g[(n-k)_N](u[k] - u[k-N]) \\
&\qquad = \sum_{m=-\infty}^{\infty} (x[k] * g[k-mN]) \\
&\qquad = \sum_{m=-\infty}^{\infty} (x[k] * g[k])[n-mN] \quad \text{for } 0 \le n \le N
\end{aligned}
$$

Since $(x[k] * g[k])[n]$ is a 7-point signal (non-zero for $0 \le n \le 7$), when $N \ge 7$, circular convolution becomes equal to the regular convolution.

<u>Problem 11.4</u>

(a)

$$
\begin{aligned}
Q[k]_N &= \sum_{n=-\infty}^{\infty} p[n] e^{-j\frac{2\pi k}{N} n} \\
&= \sum_{n=0}^{N} \sum_{m=-\infty}^{\infty} p[n-mN] e^{-j\frac{2\pi k}{N}(n-mN)} \\
&= \sum_{n=0}^{N} \underbrace{\sum_{m=-\infty}^{\infty} p[n-mN]}_{q[n]} e^{-j\frac{2\pi k}{N} n}
\end{aligned}
$$

$q[n]$ is periodic with N.

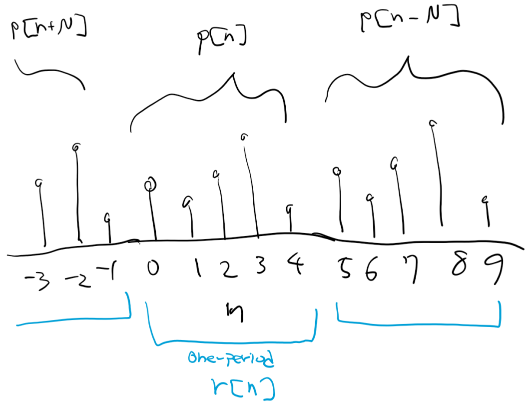
$$
\begin{aligned}
q[n+N] &= \sum_{m=-\infty}^{\infty} p[(n+N)-mN] \\
&= \sum_{m=-\infty}^{\infty} p[n-(m-1)N] \\
&= \sum_{m=-\infty}^{\infty} p[n-mN] \\
&= q[n]
\end{aligned}
$$

Same cannot be same for shift smaller than N and bigger than 1. Thus, the smallest period is N.

(b) Because $p[n]$ is an N-point signal,

$$
\begin{aligned}
q[n] &= \sum_{m=-\infty}^{\infty} p[n-mN] \\
&= p[n-mN] \quad \text{for } mN \le n < (m+1)N
\end{aligned}
$$

An example for $N = 5$ is shown below.



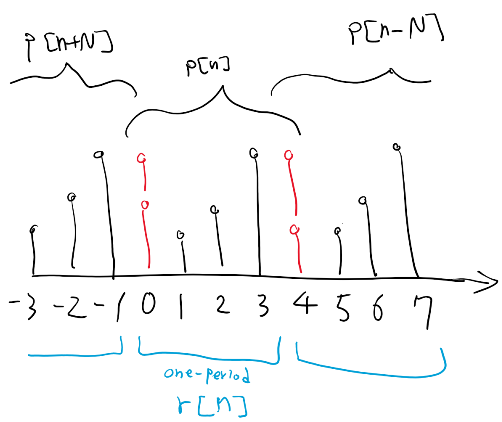
Since the $p[n]$ is an N-point signal and the shifted counterparts are also separated by N, there are no overlaps. Taking the first period of the signal would equal to $p[n]$.

(c)

$$Q[k]_{N-1} = \sum_{n=0}^{N-1} \underbrace{\sum_{m=-\infty}^{\infty} p[n - m(N-1)]\, e^{-j\frac{2\pi k}{N-1}n}}_{q[n]}$$

Using the same logic from part (a), we can see that the period of this signal is $N-1$.

(d) An example of $q[n] = \sum_{k=-\infty}^{\infty} p[n - m(N-1)]$ when $N = 5$ is shown below.



Unlike previously, there are overlaps. Taking the first signal would give us

$$r[n] = \begin{cases} p[0] + p[N-1] & \text{for } n = 0 \\ p[n] & \text{for } 1 \leq n < N-1 \\ 0 & \text{otherwise} \end{cases}$$

(e)

$$Q[k]_{N-2} = \sum_{n=0}^{N-2} \underbrace{\sum_{m=-\infty}^{\infty} p[n - m(N-2)]}_{q[n]} e^{-j\frac{2\pi k}{N-2}n}$$

Smallest period is $N-2$.

(f) $q[n] = \sum_{k=-\infty}^{\infty} p[n - m(N-2)]$ will now cause 2 samples from $p[n]$ to be contaminated. Taking the first period will yield

$$r[n] = \begin{cases} p[0] + p[N-2] & \text{for } n = 0 \\ p[1] + p[N-1] & \text{for } n = 1 \\ p[n] & \text{for } 2 \leq n < N-2 \\ 0 & \text{otherwise} \end{cases}$$

(g) M-point DFT of $q[n] = \sum_{m=-\infty}^{\infty} p[n - mM]$ is equal to $r[n]$ where $r[n]$ is a M-point signal and the first period of $q[n]$.

$$\begin{aligned} Q[k]_M &= \sum_{n=-\infty}^{\infty} p[n] e^{-j\frac{2\pi k}{M}n} \\ &= \sum_{n=0}^{M} \sum_{m=-\infty}^{\infty} p[n - mM] e^{-j\frac{2\pi k}{M}(n-mM)} \\ &= \sum_{n=0}^{M} \underbrace{\sum_{m=-\infty}^{\infty} p[n - mM]}_{q[n]} e^{-j\frac{2\pi k}{M}n} \\ &= \sum_{n=0}^{M} \underbrace{r[n]}_{\text{first period of q[n]}} e^{-j\frac{2\pi k}{M}n} \end{aligned}$$

In general, $r[n]$ will have the following expression.

$$r[n] = \begin{cases} p[0] + p[M] & \text{for } n = 0 \\ p[1] + p[M+1] & \text{for } n = 1 \\ \quad\vdots & \\ p[N-M-1] + p[N-1] & \text{for } n = N-M-1 \\ p[n] & \text{for } N-M \leq n < M \\ 0 & \text{otherwise} \end{cases}$$

$r[n]$ is the same as $p[n]$ except for samples from 0 to $N - M - 1$.

(h) Let $x[n]$ and $g[n]$ be some signals such that $p[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-k]$ is a N-point signal. Let $r[n]$ be a M-point signal whose M-point DFT $R[k]_M$ matches $P(e^{j\omega})$ at $\omega = \frac{2\pi k}{M}$. Since $P(e^{j\omega}) = X(e^{j\omega})G(e^{j\omega})$, $R[k]_M = X[k]_M G[k]_M$. Multiplication of M-point DFTs implies M-point circular convolution, so $r[n]$ is a M-point circular convolution of $x[n]$ and $g[n]$.

As we saw in part(b) and from part(g), When $M \geq N$, $r[n] = p[n]$ for $0 \leq n < M$. This means when $M \geq N$, regular convolution and circular convolutions are the same.

However, when $M < N$, $r[n]$ is equal to $p[n]$ except for the first $N - M$ samples. This means when $M < N$, the circular convolution equals to regular convolution except for the first $N-M$ samples.