

# EC574A1 Supporting Information Fall-24 V.1

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## 1 Introduction

In Section 4 this document we outline the proof of the Ehrenfest's Theorems and in Section 5 their generalization.

## 2 Some Useful Mathematical Tools

### 2.1 Schrödinger Equation and its Conjugate

It is useful to write the Schrödinger equation and its conjugate.

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}, t) \right] \Psi(\vec{r}, t) \quad (1)$$

and its conjugate

$$-i\hbar \frac{\partial}{\partial t} \Psi^*(\vec{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}, t) \right] \Psi^*(\vec{r}, t) \quad (2)$$

Note that on the LHS of both equations we have the Hamiltonian operator in the time dependent form and on the RHS the Hamiltonian in the spatial dependent form. These two equations are useful to exchange time derivative of the wavefunction with spatial derivative of the same wavefunction. For example we can write:

$$\frac{d}{dt} \Psi(\vec{r}, t) = \frac{1}{i\hbar} H \Psi(\vec{r}, t) \quad (3)$$

and

$$\frac{d}{dt} \Psi^*(\vec{r}, t) = \frac{1}{-i\hbar} H \Psi^*(\vec{r}, t) \quad (4)$$

## 2.2 Green's Identities

Consider two scalar functions  $v(\vec{r}) = v(x, y, z)$  and  $u(\vec{r}) = u(x, y, z)$  defined on a volume  $V$  with surface  $S$ . We also assume that  $u, v \in C^2$  (note that for the first identity it can only be  $u \in C^1$ ). The first Green's identity is given by:

$$\boxed{\int_V [u \nabla^2 v + \vec{\nabla} u \cdot \vec{\nabla} v] d\vec{r} = \int_S u \vec{\nabla} v \cdot d\vec{S}} \quad (5)$$

The integral on the RHS is a volume integral while on the LHS the integral is on the surface  $S$  of the volume  $V$ . Note that we can also write:

$$\int_V u \nabla^2 v d\vec{r} = \int_S u \vec{\nabla} v \cdot d\vec{S} - \int_V \vec{\nabla} u \cdot \vec{\nabla} v d\vec{r}$$

and

$$\int_V \vec{\nabla} u \cdot \vec{\nabla} v d\vec{r} = \int_S u \vec{\nabla} v \cdot d\vec{S} - \int_V u \nabla^2 v d\vec{r} \quad (6)$$

The second Green's identity is given by:

$$\boxed{\int_V [u \nabla^2 v - v \nabla^2 u] d\vec{r} = \int_S [u \vec{\nabla} v - v \vec{\nabla} u] \cdot d\vec{S}} \quad (7)$$

The integral on the RHS is a volume integral while on the LHS the integral is on the surface  $S$  of the volume  $V$ .

## 2.3 Differentiation under the Sign of Integral The Liebnitz Integral Rule

Let  $f(x, t)$  be a function such that both  $f(x, t)$  and its partial derivative  $f_x(x, t)$  are continuous in  $t$  and  $x$  in some region of the  $xt$ -plane, including  $a(x) \leq t \leq b(x)$   $x_0 \leq x \leq x_1$ . Also suppose that the functions  $a(x)$  and  $b(x)$  are both continuous and both have continuous derivatives for  $x_0 \leq x \leq x_1$ . Then, for  $x_0 \leq x \leq x_1$ ,

$$\begin{aligned} \frac{d}{dx} \left( \int_{a(x)}^{b(x)} f(x, t) dt \right) &= \\ &= f(x, b(x)) \cdot \frac{d}{dx} b(x) - f(x, a(x)) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt \end{aligned} \quad (8)$$

The right hand side may also be written using Lagrange's notation as:

$$f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} f_x(x, t) dt \quad (9)$$

### 3 First Ehrerferst Theorem

Considier a QM system described by a wavefunction  $\Psi(\vec{r}, t)$ , defined on a volume  $V$  with surface  $S$ . We also know that:

$$\int |\Psi(\vec{r}, t)|^2 d\vec{r} = 1$$

which implies that  $\lim_{|\vec{r}| \rightarrow \infty} \Psi(\vec{r}, t) = 0$ , the first Ehrerferst Theorem states that:

$$\frac{d}{dt} \langle x \rangle = \frac{1}{m} \langle p_x \rangle$$

Consider first the expression:

$$\frac{d}{dt} \langle x \rangle = \frac{d}{dt} \int \Psi(\vec{r}, t)^* x \Psi(\vec{r}, t) d\vec{r}$$

Using the the Liebnitz Integral Rule, and remebersing that  $x$  is an operator and it is time-independedet we can write:

$$\frac{d}{dt} \langle x \rangle = \int_V \left[ \frac{d}{dt} \Psi^*(\vec{r}, t) x \Psi(\vec{r}, t) + \Psi^*(\vec{r}, t) x \frac{d}{dt} \Psi(\vec{r}, t) \right] d\vec{r}$$

Using Eq. 3 and Eq. 4 we have can write:

$$\frac{d}{dt} \langle x \rangle = \int_V \left[ \frac{1}{-i\hbar} H \Psi^*(\vec{r}, t) x \Psi(\vec{r}, t) + \Psi^*(\vec{r}, t) x \frac{1}{i\hbar} H \Psi(\vec{r}, t) \right] d\vec{r}$$

expanding the Hamiltonian operator we have:

$$\begin{aligned} \frac{d}{dt} \langle x \rangle = \frac{1}{i\hbar} \left[ \int_V \Psi^*(\vec{r}, t) x \left( -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}, t) + V(\vec{r}, t) \Psi(\vec{r}, t) \right) d\vec{r} + \right. \\ \left. - \int_V \left( -\frac{\hbar^2}{2m} \nabla^2 \Psi^*(\vec{r}, t) + V(\vec{r}, t) \Psi^*(\vec{r}, t) \right) x \Psi(\vec{r}, t) d\vec{r} \right] \end{aligned}$$

or

$$\begin{aligned} \frac{d}{dt} \langle x \rangle = \frac{1}{i\hbar} \left[ \int_V [\Psi^*(\vec{r}, t) x V(\vec{r}, t) \Psi(\vec{r}, t) d\vec{r} - V(\vec{r}, t) \Psi^*(\vec{r}, t) x \Psi(\vec{r}, t)] d\vec{r} \right. \\ \left. + \int_V \Psi^*(\vec{r}, t) x \left( -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}, t) \right) - \left( -\frac{\hbar^2}{2m} \nabla^2 \Psi^*(\vec{r}, t) \right) x \Psi(\vec{r}, t) d\vec{r} \right] \end{aligned} \quad (10)$$

the first intrgral vanishes, and we have:

$$\frac{d}{dt} \langle x \rangle = \frac{1}{i\hbar} \frac{\hbar^2}{2m} \left[ \int_V -\Psi^*(\vec{r}, t) \mathbf{x} \nabla^2 \Psi(\vec{r}, t) + \nabla^2 \Psi^*(\vec{r}, t) \mathbf{x} \Psi(\vec{r}, t) d\vec{r} \right]$$

we then use Eq. 5 with  $u = \mathbf{x}\Psi$  and  $v = \Psi^*$ , then:

$$\begin{aligned} \int_V \nabla^2 \Psi^*(\vec{r}, t) \mathbf{x} \Psi(\vec{r}, t) d\vec{r} &= \int_V \mathbf{x} \Psi(\vec{r}, t) \nabla^2 \Psi^*(\vec{r}, t) d\vec{r} = \\ &= \int_S \mathbf{x} \Psi(\vec{r}, t) \overrightarrow{\nabla \Psi^*(\vec{r}, t)} \cdot d\vec{S} - \int_V \overrightarrow{\nabla \Psi^*(\vec{r}, t)} \cdot \overrightarrow{\nabla (\mathbf{x} \Psi(\vec{r}, t))} d\vec{r} \end{aligned}$$

Since we know that  $\lim_{|\vec{r}| \rightarrow \infty} \Psi(\vec{r}, t) = 0$ , then the surface integral (first term on the RHS) is zero, and we have:

$$\frac{d}{dt} \langle x \rangle = \frac{1}{i\hbar} \frac{\hbar^2}{2m} \left[ \int_V -\Psi^*(\vec{r}, t) \mathbf{x} \nabla^2 \Psi(\vec{r}, t) - \overrightarrow{\nabla \Psi^*(\vec{r}, t)} \cdot \overrightarrow{\nabla (\mathbf{x} \Psi(\vec{r}, t))} d\vec{r} \right]$$

$$\frac{d}{dt} \langle x \rangle = -\frac{1}{i\hbar} \frac{\hbar^2}{2m} \left[ \int_V \Psi^*(\vec{r}, t) \mathbf{x} \nabla^2 \Psi(\vec{r}, t) + \overrightarrow{\nabla \Psi^*(\vec{r}, t)} \cdot \overrightarrow{\nabla (\mathbf{x} \Psi(\vec{r}, t))} d\vec{r} \right]$$

using again Eq. 6 on the second term in the integral on the RHS, we have:

$$\int_V \overrightarrow{\nabla \Psi^*(\vec{r}, t)} \cdot \overrightarrow{\nabla (\mathbf{x} \Psi(\vec{r}, t))} d\vec{r} = \int_S \Psi^*(\vec{r}, t) \overrightarrow{\nabla (\mathbf{x} \Psi(\vec{r}, t))} \cdot d\vec{S} - \int_V \Psi^*(\vec{r}, t) \nabla^2 (\mathbf{x} \Psi(\vec{r}, t)) d\vec{r}$$

Once again, since we know that  $\lim_{|\vec{r}| \rightarrow \infty} \Psi(\vec{r}, t) = 0$ , then the surface integral (first term on the RHS) is zero, and we have:

$$\int_V \overrightarrow{\nabla \Psi^*(\vec{r}, t)} \cdot \overrightarrow{\nabla (\mathbf{x} \Psi(\vec{r}, t))} d\vec{r} = - \int_V \Psi^*(\vec{r}, t) \nabla^2 (\mathbf{x} \Psi(\vec{r}, t)) d\vec{r}$$

and

$$\frac{d}{dt} \langle x \rangle = -\frac{1}{i\hbar} \frac{\hbar^2}{2m} \left[ \int_V \Psi^*(\vec{r}, t) \mathbf{x} \nabla^2 \Psi(\vec{r}, t) - \int_V \Psi^*(\vec{r}, t) \nabla^2 (\mathbf{x} \Psi(\vec{r}, t)) d\vec{r} \right]$$

or

$$\frac{d}{dt} \langle x \rangle = -\frac{1}{i\hbar} \frac{\hbar^2}{2m} \int_V \Psi^*(\vec{r}, t) [\mathbf{x} \nabla^2 \Psi(\vec{r}, t) - \nabla^2 (\mathbf{x} \Psi(\vec{r}, t))] d\vec{r}$$

now note that:

$$\begin{aligned}\frac{\partial^2}{\partial x^2} [x\Psi(\vec{r}, t)] &= \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} (x\Psi(\vec{r}, t)) \right] = \frac{\partial}{\partial x} \left[ \Psi(\vec{r}, t) + x \frac{\partial}{\partial x} \Psi(\vec{r}, t) \right] = \\ &= \frac{\partial}{\partial x} \Psi(\vec{r}, t) + x \frac{\partial^2}{\partial x^2} \Psi(\vec{r}, t) + \frac{\partial}{\partial x} \Psi(\vec{r}, t) = 2 \frac{\partial}{\partial x} \Psi(\vec{r}, t) + x \frac{\partial^2}{\partial x^2} \Psi(\vec{r}, t)\end{aligned}$$

note also that:

$$\begin{aligned}\frac{\partial^2}{\partial y^2} [x\Psi(\vec{r}, t)] &= x \frac{\partial^2}{\partial y^2} \Psi(\vec{r}, t) \\ \frac{\partial^2}{\partial z^2} [x\Psi(\vec{r}, t)] &= x \frac{\partial^2}{\partial z^2} \Psi(\vec{r}, t)\end{aligned}$$

it follows that:

$$\nabla^2(x\Psi(\vec{r}, t)) = 2 \frac{\partial}{\partial x} \Psi(\vec{r}, t) + x \nabla^2 \Psi(\vec{r}, t)$$

consequently:

$$\frac{d}{dt} \langle x \rangle = -\frac{1}{i\hbar} \frac{\hbar^2}{2m} \int_V \Psi^*(\vec{r}, t) \left[ x \nabla^2 \Psi(\vec{r}, t) - 2 \frac{\partial}{\partial x} \Psi(\vec{r}, t) - x \nabla^2 (\Psi(\vec{r}, t)) \right] d\vec{r}$$

it follows that:

$$\frac{d}{dt} \langle x \rangle = -\frac{1}{i\hbar} \frac{\hbar^2}{2m} \int_V -\Psi^*(\vec{r}, t) 2 \frac{\partial}{\partial x} \Psi(\vec{r}, t) d\vec{r}$$

rearranging:

$$\frac{d}{dt} \langle x \rangle = \frac{-i\hbar}{m} \int_V \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} \Psi(\vec{r}, t) d\vec{r}$$

or:

$$\frac{d}{dt} \langle x \rangle = \frac{1}{m} \int_V \Psi^*(\vec{r}, t) (-i\hbar) \frac{\partial}{\partial x} \Psi(\vec{r}, t) d\vec{r}$$

$$\frac{d}{dt} \langle x \rangle = \frac{1}{m} \int_V \Psi^*(\vec{r}, t) p_x \Psi(\vec{r}, t) d\vec{r} = \frac{1}{m} \langle p_x \rangle$$

That is first Ehrenferst theorem.

## 4 Second Ehrerferst Theorem

Considier a QM system described by a wavefunction  $\Psi(\vec{r}, t)$ , defined on a volume  $V$  with surface  $S$ . We also know that:

$$\int |\Psi(\vec{r}, t)|^2 d\vec{r} = 1$$

which implies that  $\lim_{|\vec{r}| \rightarrow \infty} \Psi(\vec{r}, t) = 0$ , the second Ehrerferst Theorem states that:

$$\frac{d}{dt} \langle p_x \rangle = - \left\langle \frac{\partial V(\vec{r}, t)}{\partial x} \right\rangle$$

Consider first the expression:

$$\begin{aligned} \frac{d}{dt} \langle p_x \rangle &= \frac{d}{dt} \int_V \Psi(\vec{r}, t)^* \left( -i\hbar \frac{\partial}{\partial x} \right) \Psi(\vec{r}, t) d\vec{r} \\ \frac{d}{dt} \langle p_x \rangle &= -i\hbar \frac{d}{dt} \int_V \Psi(\vec{r}, t)^* \frac{\partial}{\partial x} \Psi(\vec{r}, t) d\vec{r} \end{aligned}$$

Using the the Liebnitz Integral Rule, and the chain integration rule, we can write:

$$\frac{d}{dt} \langle p_x \rangle = -i\hbar \left[ \int_V \frac{\partial}{\partial t} \Psi(\vec{r}, t)^* \frac{\partial}{\partial x} \Psi(\vec{r}, t) d\vec{r} + \int_V \Psi(\vec{r}, t)^* \frac{\partial}{\partial t} \frac{\partial}{\partial x} \Psi(\vec{r}, t) d\vec{r} \right]$$

exchanging the derivatives in the second integral on the RHS:

$$\frac{d}{dt} \langle p_x \rangle = -i\hbar \left[ \int_V \Psi(\vec{r}, t)^* \frac{\partial}{\partial x} \frac{\partial}{\partial t} \Psi(\vec{r}, t) d\vec{r} + \int_V \frac{\partial}{\partial t} \Psi(\vec{r}, t)^* \frac{\partial}{\partial x} \Psi(\vec{r}, t) d\vec{r} \right]$$

Using Eq. 3 and Eq. 4 and expanding the Hamiltonian operator we have:

$$\begin{aligned} \frac{d}{dt} \langle p_x \rangle &= \left[ \int_V -\Psi^*(\vec{r}, t) \frac{\partial}{\partial x} \left( -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}, t) + V(\vec{r}, t) \Psi(\vec{r}, t) \right) d\vec{r} + \right. \\ &\quad \left. \int_V \left( -\frac{\hbar^2}{2m} \nabla^2 \Psi^*(\vec{r}, t) + V(\vec{r}, t) \Psi^*(\vec{r}, t) \right) \frac{\partial}{\partial x} \Psi(\vec{r}, t) d\vec{r} \right] \end{aligned}$$

rearranging the terms associated to the potential  $V(\vec{r}, t)$ , we have:

$$\begin{aligned} \frac{d}{dt} \langle p_x \rangle &= \frac{\hbar^2}{2m} \int_V \left[ \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} (\nabla^2 \Psi(\vec{r}, t)) - \nabla^2 \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} \Psi(\vec{r}, t) \right] d\vec{r} + \\ &\quad \int_V \left[ V(\vec{r}, t) \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} \Psi(\vec{r}, t) - \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} (V(\vec{r}, t) \Psi(\vec{r}, t)) \right] d\vec{r} \end{aligned}$$

We note that:

$$\begin{aligned}\frac{\partial}{\partial x} \nabla^2 \Psi &= \frac{\partial}{\partial x} \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \Psi = \left[ \frac{\partial^3}{\partial x^3} + \frac{\partial}{\partial x} \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial x} \frac{\partial^2}{\partial z^2} \right] \Psi \\ \frac{\partial}{\partial x} \nabla^2 \Psi &= \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \frac{\partial}{\partial x} \Psi = \nabla^2 \left[ \frac{\partial}{\partial x} \Psi \right]\end{aligned}$$

consequently

$$\begin{aligned}\frac{d}{dt} \langle p_x \rangle &= \frac{\hbar^2}{2m} \int_V \left[ \Psi^*(\vec{r}, t) \nabla^2 \left( \frac{\partial}{\partial x} \Psi(\vec{r}, t) \right) - \nabla^2 \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} \Psi(\vec{r}, t) \right] d\vec{r} + \\ &\quad \int_V \left[ V(\vec{r}, t) \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} \Psi(\vec{r}, t) - \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} (V(\vec{r}, t) \Psi(\vec{r}, t)) \right] d\vec{r}\end{aligned}\tag{11}$$

using Eq. 7 with  $u = \Psi^*$  and  $v = \frac{\partial}{\partial x} \Psi$ , we have modify the first integral on the RHS of the equation above:

$$\begin{aligned}&\int_V \left[ \Psi^*(\vec{r}, t) \nabla^2 \left( \frac{\partial}{\partial x} \Psi(\vec{r}, t) \right) - \nabla^2 \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} \Psi(\vec{r}, t) \right] d\vec{r} \\ &= \\ &\int_S \left[ \Psi^*(\vec{r}, t) \overrightarrow{\nabla \left( \frac{\partial}{\partial x} \Psi(\vec{r}, t) \right)} - \frac{\partial}{\partial x} \Psi(\vec{r}, t) \overrightarrow{\nabla \Psi^*(\vec{r}, t)} \right] d\vec{r} \\ &= 0\end{aligned}$$

Since we know that  $\lim_{|\vec{r}| \rightarrow \infty} \Psi(\vec{r}, t) = 0$  and  $\lim_{|\vec{r}| \rightarrow \infty} \frac{\partial}{\partial x} \Psi(\vec{r}, t) = 0$ , then the surface integral is zero. Consequently Eq. 11 becomes:

$$\begin{aligned}\frac{d}{dt} \langle p_x \rangle &= \int_V \left[ V(\vec{r}, t) \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} \Psi(\vec{r}, t) - \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} (V(\vec{r}, t) \Psi(\vec{r}, t)) \right] d\vec{r} \\ &= - \int_V \Psi^*(\vec{r}, t) \left[ \frac{\partial}{\partial x} (V(\vec{r}, t) \Psi(\vec{r}, t)) - V(\vec{r}, t) \frac{\partial}{\partial x} \Psi(\vec{r}, t) \right] d\vec{r}\end{aligned}\tag{12}$$

Since

$$\frac{\partial}{\partial x} (V(\vec{r}, t) \Psi(\vec{r}, t)) = V(\vec{r}, t) \frac{\partial}{\partial x} \Psi(\vec{r}, t) + \Psi(\vec{r}, t) \frac{\partial}{\partial x} V(\vec{r}, t)$$

then

$$\frac{d}{dt} \langle p_x \rangle = - \int_V \Psi^*(\vec{r}, t) \frac{\partial}{\partial x} V(\vec{r}, t) \Psi(\vec{r}, t) d\vec{r} \quad (13)$$

that is the final result:

$$\frac{d}{dt} \langle p_x \rangle = - \left\langle \frac{\partial V(\vec{r}, t)}{\partial x} \right\rangle \quad (14)$$

## 5 Generalization of the Ehrenferst Theorems

Consider an observable  $\mathcal{A}$  represented by an operator  $A$ , we can evaluated in Dirac's notation as:

$$\begin{aligned} \frac{d}{dt} \langle A \rangle &= \frac{d}{dt} \langle \Psi | A | \Psi \rangle \\ &= \left\langle \frac{d}{dt} \Psi \middle| A \middle| \Psi \right\rangle + \left\langle \Psi \middle| \frac{d}{dt} A \middle| \Psi \right\rangle + \left\langle \Psi \middle| A \middle| \frac{d}{dt} \Psi \right\rangle \\ &= \int_V \frac{d}{dt} \Psi^* A \Psi d\vec{r} + \int_V \Psi^* \frac{d}{dt} A \Psi d\vec{r} + \int_V \Psi^* A \frac{d}{dt} \Psi d\vec{r} \end{aligned} \quad (15)$$

The expression above contains the time derivative of the operator  $A$ . The value of this derivative depends on the *representation* used.

- – **Schrödinger Representation** In the Schrödinger Representation the time dependence of the system is embedded in the wavefunctions  $\Psi(\vec{r}, t)$  and the operators  $A$  are time-independent.
- – **Heisemberg Representation** In the Heisemberg Representation the time dependence of the system is embedded in the operators  $A(t)$  and wavefunctions  $\Psi(\vec{r})$  are time-independent.
- – **Interaction Representation** In the Interaction Representation the time dependence of the system is embedded both in the operators  $A(t)$  and the wavefunctions  $\Psi(\vec{r}, t)$ .

Using Eq. 3 and Eq. 4 we can write Eq.18 as:

$$\begin{aligned} \frac{d}{dt} \langle A \rangle &= -\frac{1}{i\hbar} \langle H \Psi | A | \Psi \rangle + \left\langle \Psi \middle| \frac{d}{dt} A \middle| \Psi \right\rangle + \frac{1}{i\hbar} \langle \Psi | A | H \Psi \rangle \\ &= \left\langle \Psi \middle| \frac{d}{dt} A \middle| \Psi \right\rangle + \frac{1}{i\hbar} [\langle \Psi | A | H \Psi \rangle - \langle H \Psi | A | \Psi \rangle] \end{aligned} \quad (16)$$

since the Hamiltonia operator  $H$  is hermitian:

$$\langle H \Psi | A | \Psi \rangle = \langle \Psi | H A | \Psi \rangle \quad (17)$$

we can write:



$$\frac{d}{dt} \langle A \rangle = \left\langle \Psi \left| \frac{d}{dt} A \right| \Psi \right\rangle + \frac{1}{i\hbar} [\langle \Psi | A H | \Psi \rangle - \langle \Psi | H A | \Psi \rangle] \quad (18)$$

or:

$$\frac{d}{dt} \langle A \rangle = \left\langle \frac{d}{dt} A \right\rangle + \frac{1}{i\hbar} \langle [A, H] \rangle \quad (19)$$