Some Useful Commutator Results

For all operators $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and scalar k,

1 $\boldsymbol{[A,B] = -[B,A]}$

Proof $\quad [A,B] = AB - BA = -(BA - AB) = -[B,A]$

2 $\boldsymbol{[A,B+C] = [A,B] + [A,C]}$

Proof $\quad [A,B+C] = A(B+C) - (B+C)A = AB - BA + AC - CA = [A,B] + [A,C]$

3 $\boldsymbol{[A+B,C] = [A,C] + [B,C]}$

Proof $\quad [A+B,C] = (A+B)C - C(A+B) = AC - CA + BC - CB = [A,C] + [B,C]$

4 $\boldsymbol{[A,BC] = [A,B]C + B[A,C]}$

Proof $\quad [A,B]C + B[A,C] = (AB - BA)C + B(AC - CA) = ABC - BCA = [A,BC]$

Let $B = C$, we have $\boldsymbol{[A,B^2] = [A,B]B + B[A,B]}$

5 $\boldsymbol{[AB,C] = A[B,C] + [A,C]B}$

Proof $\quad A[B,C] + [A,C]B = A(BC - CB) + (AC - CA)B = ABC - CAB = [AB,C]$

Let $A = B$, we have $\boldsymbol{[B^2,C] = B[B,C] + [B,C]B}$

6 $\boldsymbol{[A-B,C-D] = [A,C] - [A,D] - [B,C] + [B,D]}$

Proof

$$\begin{aligned}[A-B,C-D] = (A-B)(C-D) - (C-D)(A-B) = (AC - AD - BC + BD) - (CA - CB - DA + DB)\\ = [A,C] - [A,D] - [B,C] + [B,D]\end{aligned}$$

7 $\boldsymbol{[kA,B] = [A,kB] = k[A,B]}$

Proof $\quad [kA,B] = kAB - BkA = \begin{cases} AkB - kBA & = [A,kB] \\ k(AB - BA) & = k[A,B] \end{cases}$

8 $\boldsymbol{[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0}$ **(Jacobi Identity)**

Proof

$$\begin{aligned}&[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = [A,BC-CB] + [B,CA-AC] + [C,AB-BA]\\ &= A(BC-CB) - (BC-CB)A + B(CA-AC) - (CA-AC)B + C(AB-BA) - (AB-BA)C = 0\end{aligned}$$

Note Observe that Jacobi Identity is cyclic.

9 $[A,B] = [B,A] = 0$ if A and B are operators of independent variables.

10 $[A,f(A)] = 0$ as $[A,f(A)] = Af(A) - f(A)A = 0$

11 **Canonical commutator** $\boxed{[x,p] = i\hbar}$

12 **Angular momentum operators**

(i) $\boxed{[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y}$

Notice that angular momentum operators commutators are cyclic.

(ii) $[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$ where $L^2 = L_x^2 + L_y^2 + L_z^2$

12 **Pauli matrices**

$$[\sigma_1, \sigma_2] = 2i\sigma_3 \qquad [\sigma_2, \sigma_3] = 2i\sigma_1 \qquad [\sigma_3, \sigma_1] = 2i\sigma_2$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Observe that commutators of Pauli matrices are cyclic.