EC 574 PS 0.

Problem 1

1. Feyhman's Techinque.

$$I(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

$$\frac{dI(\alpha)}{d\alpha} = \int_{-\infty}^{\infty} \frac{\partial}{\partial\alpha} e^{-\alpha x^2} dx = \int_{-\infty}^{\infty} (-x^2) e^{-\alpha x^2} dx$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$\frac{dI(\alpha)}{d\alpha} = \frac{-1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$I(\alpha) = C\sqrt{\frac{\pi}{\alpha}}$$ C is a contstant

$\lim_{\alpha\to\infty} I(\alpha)$ Vanishes, so $C=1$

$$I(\alpha) = \sqrt{\frac{\pi}{\alpha}}$$

2. Polar

$$I^2 \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \int_{-\infty}^{\infty} e^{-\alpha y^2} dy = \left(\int_{-\infty}^{\infty} e^{-\alpha x^2} dx\right)^2$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dx\, dy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-\alpha r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \underbrace{\int_0^{\infty} e^{-\alpha r^2} r\, dr\, d\theta}_{\frac{1}{2\alpha}} \qquad u = \alpha r^2 \quad du = 2\alpha r\, dr$$

$$= 2\pi \cdot \frac{1}{2\alpha} = \frac{\pi}{\alpha}$$

$$I = \sqrt{\frac{\pi}{\alpha}}$$

Problem 2:

1) $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$ $\qquad \cos(nx)\cos(mx) = \frac{1}{2}[\cos((n-m))x + \cos(n+m)x]$

$= \int_{-\pi}^{\pi} \cos(n-m)x\,dx \begin{cases} 2\pi & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$

2) $\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx$ $\qquad \cos(nx)\sin(mx) = \frac{1}{2}[\sin((n+m)x) - \sin((n-m)x)]$

This is always zero on $[-\pi,\pi]$ range as it consist only of sin which is odd on $[-\pi,\pi]$.

3) $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx$ $\qquad \frac{1}{2}[\cos((n-m)x) - \cos((n+m)x)]$

the integral is 2π if $n=m$

0 if $n\neq m$,

as $\int_{-\pi}^{\pi} \cos((n-m)x)\,dx = 0$

and $\int_{-\pi}^{\pi} \cos(n+m)x\,dx = 0$

Problem 3:

$\frac{dy}{dt} = ay + b$

$y(t) = Ce^{at} - \frac{b}{a}$ is general solution.

this is because for homogeneous part of ODE

① $\frac{dy}{dt} = ay$ $\qquad \frac{1}{y}\frac{dy}{dt} = a$ $\qquad \int \frac{1}{y}dy = \int a\,dt$ $\qquad y(t) = C_1 e^{at}$

$\ln|y| = at + C_1$

② as b is constant, we assume $y_p(t)$ is constant.

$\frac{dy_p}{dt} = ay_p + b$ $\qquad y_p = -\frac{b}{a}$

$0 = ay_p + b$

So general solution is $Ce^{at} - \frac{b}{a}$.

2. $y(0)=1$ $\boxed{a=1,\ b=4 \quad y(t)=Ce^{t}-4}$

So $C-4=1 \quad C=5$

$y(t)=5e^{t}-4$

Problem 4

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+\boxed{cy}=0$$

First solve characteristic equation:

1) $a=1,\ b=4,\ c=3$

$r^2+4r+3=0 \Rightarrow (r+3)(r+1)=0 \quad r_1=-3 \quad r_2=-1$

$y(x)=C_1e^{-3x}+C_2e^{-x}$

2) $a=1,\ b=4,\ c=5$

$r^2+4r+5=0$

$r=-2\pm i$

$y(x)=e^{-2x}(C_1e^{-ix}+C_2e^{+ix})$ cos(x) and sin(x)

$=e^{-2x}(C_1\cos(x)+C_2\sin(x))$

3) $y(0)=1 \quad y(1)=3,$

① $C_1e^{-3x}+C_2e^{-x}$

$$\begin{cases} C_1+C_2=1 \\ C_1e^{-3}+C_2e^{-1}=3 \end{cases} \Rightarrow \begin{cases} C_2=\dfrac{3-e^{-3}}{e^{-1}-e^{-3}} \\ C_1=1-C_2 \end{cases}$$

② $\begin{cases} C_1=1 \text{ from } y(0) \\ e^{-2}(\cos(1)+C_2\sin(1))=3 \end{cases}$

$$C_2=\frac{3e^2-\cos(1)}{\sin(1)}$$