Problem Set II
EC574A1, Fall 2024
Assigned October 22, 2024
Due November 30, 2024

The solution of the assigned problem set is **mandatory** and it is your responsibility. If you do not work on the homework you will not be able to solve the exam problems. It is strongly suggested that you start solving the homework sets immediately without waiting the day before the exam. You are required to turn in the solution of Problem Set II the day of the exam.

- Problem 1

  Consider a infinitely deep 2D quantum well of size $-a < x < a$ and $-b < y < b$, where $b \neq a$.

  $-1-$Determine the eigenvalues and eigenfunctions of the confined states. Determine the degeneracy.

  $-2-$Write a state function for the system that has equal probability in the first three states.

- Problem 2

  You have a harmonic oscillator with $\alpha = (mk/\hbar^2)^{1/4}$, where k is the oscillator spring constant and $\omega = (k/m)^{1/2}$ the corresponding frequency. The eigenfunctions solution of the Schroedinger equation are given by:

  $$\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi}\, 2^n\, n!}\right)^{1/2} e^{-\dfrac{\alpha^2 x^2}{2}} H_n(\alpha\, x)$$

  and the corresponding generating function of the Hermite polynomia is:

  $$H_n(\xi) = (-1)^n\, e^{\xi^2}\, \frac{d^n\, e^{-\xi^2}}{d\xi^n}$$

  Consider a 2D harmonic oscillator where k_x is twice k_y.

  $-$ Determine the solution of the Schroedinger equation by writing an explicit expression of the eigenfuntion and eigenvalues. Are there any degeneracy in the system? What kind?

  $-$ Using the lowering and raising operators technique compute the expectation values of $<x\, p_y>$ and $<p_x\, y>$ for the harmonic oscillator states. NOTE - watch for the coordinates on which a^+ and a^- operate on.

- Problem 3

Consider the radial Schroedinger equation:

$$\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right)\;-\;\frac{2mr^2}{\hbar^2}\,[V(r)\;-\;E]\;R\;=\;l(l+1)\,R \tag{1}$$

where $R = R(r)$ is the radial solution, l the angular momentum quantum number.

– Perform a transformation of variables $u(r) = r\,R(r)$ and determine an equation for $u(r)$ equivalent to (1).

– For this new equation consider the case $l = 0$ and solve (compute eigenvalues and eigenfunctions) the problem of the infinite spherical quantum well: $V(r)\;=\;0\quad for\quad r\;<\;a$ and $V(r)\;=\;\infty\quad for\quad r\;>\;a$. Note: what happens at r = 0 ?

– Rewrite your solution in terms of $R(r)$ do you recognize what function this is?

- Problem 4

Consider the ground-state wavefunction of the Hydrogen atom ground state:

$$\Psi_{100}(r)\;=\;\frac{1}{\sqrt{4\pi}}\frac{2}{a^{3/2}}\,e^{-r/a}$$

where a is the Bohr's radius.

– Compute the expectation values $< r >$ and $< r^2 >$.

– Compute the expectation values $< x >$ and $< x^2 >$.

Note: you can take the long way and use $x\;=\;r\,sin\theta\,cos\phi$. Otherwise notice $r^2\;=\;x^2\;+\;y^2\;+\;z^2$ and use the symmetry of the ground state.

- Problem 5

Consider a infinitely deep two-dimensional quantum well defined as: $V(x,y) = 0$ for $-a < x < a$, $-a < y < a$ and $V(x) = \infty$ elsewhere.

– Write an expression for the energy eigenvalues. Are there any degeneracies?

– Write a general expression for the eigenfunctions

- Problem 6

Consider an operator defined as $L_\pm\;=\;L_x\;\pm\;iL_y$.

– Write the eigenvalue problem for the L_z operator and show that the eigenvalue is $m\hbar$ with m integer.

– Compute the value of the commutator $[L_z, L_\pm]$.

– If ϕ_m is an eigenfunction of L_z, compute the value of $L_z(L_\pm \phi_m)$.

- Problem 7
Consider an isotropic two-dimensional harmonic oscillator. The eigenfunction of the one-dimensional harmonic oscillator are given by:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\xi^2}{2}} \quad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{2}{\sqrt{2}}\,\xi\, e^{-\frac{\xi^2}{2}} \quad \xi = \sqrt{\frac{m\omega}{\hbar}}x$$

$$\int_0^\infty x^m\, e^{-a\,x^2}\, dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}} \;\; \Gamma[n+1] = n! \;\; \Gamma[n] = (n-1)! \;\; \Gamma[1/2] = \sqrt{\pi}$$

$-1-$Compute the expectation values $< xy^2 >$ for ψ_{01} and ψ_{10}.

- Problem 8

Consider a 2D infinite quantum well where the potential $V = 0$ for $0\,x < a$ and $0 < y < a$, and $V = \infty$ everywhere else.

– Determine the solution of the Schroedinger equation in terms of wavefunctions and eigenvalues. Are there any degeneracies?

– Suppose the system is perturbed with an external potential $V_0\,\delta(x - a/4, y - 3/4a)$ and $V_1\,\delta(x - 3/4a, y - 1/4a)$ and Compute the first order correction to the ground state and first excited state energies.

– Compute the correct zeroth order wavefunctions.

- Problem 9

Consider the solution of the 1D oscillator problem e studied in class. Compute the correction to the energy eigenvalues caused by the following perturbations:

– $H' = cx$

– $H' = cx^3$

– $H' = cx^4$

where c is a real constant. To help in this process use the knowledge of the wavefunctions symmetry and the results of the equations [4.166] on page 174 of the book.

- Problem 10

Consider a 2D isotropic harmonic oscillator characterized by an hamiltonian of the type:
$H' = \frac{p^2}{2m} + \frac{1}{2}k(x^2 + y^2)$

- Compute the Schroedinger equation and determine the degeneracy of the eigenvalues.

- Compute the first order correction tho the eigenvalues of the second state caused by the perturbing hamiltonian $H' = cx^4y^4$

- Problem 11

Consider the 3D isotropic quantum box of size $-a < x < a$, $-a < y < a$,$-a < z < a$ we studied in class. Determine the first order correction to the second energy eigenvalue when a perturbing hamiltonian $H' = cx^2z^4$ is applied to the system.

- Problem 12

Consider the four degenerate eigenfunctions corresponding to the first ($n = 2$) excited state of the hydrogen atom:

$$\psi_{2,0,0}(r,\theta,\phi) = 2\left(\frac{1}{2\,a_0}\right)^{-3/2}\left(1 - \frac{r}{2\,a_0}\right) e^{-\frac{r}{2\,a_0}}\, Y_{0,0}(\theta,\phi)$$

$$\psi_{2,1,1}(r,\theta,\phi) = 3^{-1/2}\left(\frac{1}{2\,a_0}\right)^{-3/2}\left(\frac{r}{a_0}\right) e^{-\frac{r}{2\,a_0}}\, Y_{1,1}(\theta,\phi)$$

$$\psi_{2,1,0}(r,\theta,\phi) = 3^{-1/2}\left(\frac{1}{2\,a_0}\right)^{-3/2}\left(\frac{r}{a_0}\right) e^{-\frac{r}{2\,a_0}}\, Y_{1,0}(\theta,\phi)$$

$$\psi_{2,1,-1}(r,\theta,\phi) = 3^{-1/2}\left(\frac{1}{2\,a_0}\right)^{-3/2}\left(\frac{r}{a_0}\right) e^{-\frac{r}{2\,a_0}}\, Y_{1,-1}(\theta,\phi)$$

where a_0 is the Bohr radius, and $Y_{n,m}(\theta,\phi)$ the spherical harmonics. Consider a perturbing electric field E directed along the z-axis that introduces a perturbing Hamiltonian $H' = -q\,E\,z$.

– Using degenerate perturbation theory compute the first order energy correction for the Hydrogen state with $n = 2$.

– Determine the eigenvectors and the proper zeroth order eigenfuctions.

Hints!

Notice that:

$$cos(\theta) = \sqrt{\frac{4\pi}{3}}Y_{1,0}(\theta,\phi)$$

Use the symmetry of the spherical harmonics to simplify your integration.

- Problem 13

  Consider a charged particle in the one-dimensional harmonic oscillator potential. Suppose we turn on a weak electric field E so that the system is perturbed by a potential $H' = -q\,E\,x$.

  - Solve the Schroedinger equation using the following variable transformation: $x' = x - q\,E/m\omega^2$. Where m is the particle mass and ω the oscillator angular frequency.

  - Compute the first and second order corrections and compare them with the exact solutions.