EC516 HW2 Solutions

Problem 2.1

Part(A) (a)

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}dt$$
$$= e^{-j\omega t_0}$$

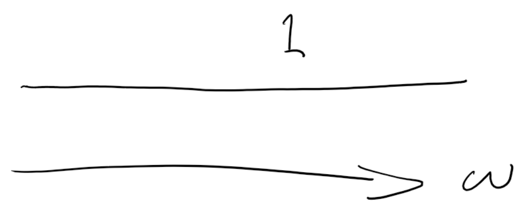

Figure 1: $|X(j\omega)|$ for Part (A) (a). It is a constant ($= 1$)

(b) $u(t+T) - u(t-T)$ is equivalent to rectangular function with length $2T$ centered at 0. Here, we derive that CTFT of rectangular function is a sinc function, but you are welcome to use the properties table.

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} \left(u(t+T) - u(t-T)\right) e^{-j\omega t}dt \\
&= \int_{-T}^{T} e^{-j\omega t}dt \\
&= \left[\frac{e^{-j\omega t}}{-j\omega} \Bigg|_{t=-T}^{T} \right] \\
&= \frac{e^{-j\omega T} - e^{j\omega T}}{-j\omega} \\
&= \frac{-2j\sin(\omega T)}{-j\omega} \\
&= 2T \cdot \frac{\sin(\omega T)}{\omega T}
\end{aligned}$$

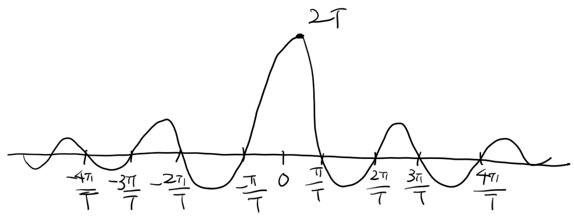

Figure 2: $|X(j\omega)|$ for Part(A) (b). It is a sinc function and becomes 0 at integer multiple of π/T.

Part(B) (a)

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - 100\pi)e^{j\omega t}dt\omega \\
&= e^{j100\pi t}
\end{aligned}
$$

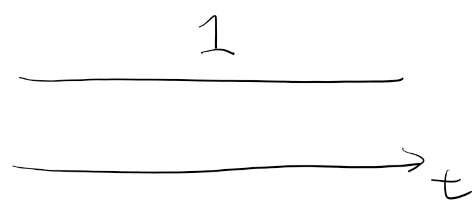

Figure 3: $|x(t)|$ for Part(B) (a). It is a constant ($= 1$)

(b) Inverse CTFT of a rectangular function is also a sinc function but divided by 2π.

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(u(\omega + 2\pi) - u(\omega - 2\pi)\right) e^{j\omega t}dt\omega \\
&= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} e^{j\omega t}dt\omega \\
&= 2 \cdot \frac{sin(2\pi t)}{2\pi t}
\end{aligned}
$$

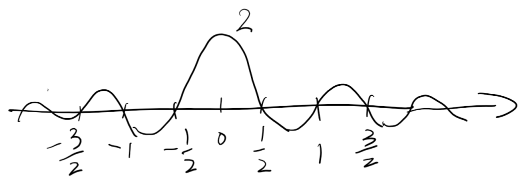

Figure 4: $|x(t)|$ for Part(B) (b). It is a sinc function.

Part(C) By definition of fourier transform,

$$X(j1000\pi) = \int_{-\infty}^{\infty} x(t)e^{-j1000\pi t}dt$$

We take the conjugate of $X(j1000\pi)$ and since $x(t) = (x(t))^*$

$$\begin{aligned}
(X(j1000\pi))^* &= \left(\int_{-\infty}^{\infty} x(t)e^{-j1000\pi t}dt\right)^* \\
&= \int_{-\infty}^{\infty} \left(x(t)e^{-j1000\pi t}\right)^* dt \\
&= \int_{-\infty}^{\infty} x(t)e^{j1000\pi t}dt \\
&= X(-j1000\pi)
\end{aligned}$$

Therefore,

$$X(-j1000\pi) = 1 - j$$

<u>Problem 2.2</u>

Part(A)

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \delta[n-3]e^{-j\omega n} \\
&= e^{-j3\omega}
\end{aligned}$$

Part(B) (a) Note that $u[n+2] - u[n-3]$ is a discrete version of rect function. Again, properties table are welcome, but we derive the DTFT by using finite sum formula for geometric series.

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \left(u[n+2]-u[n-3]\right) e^{-j\omega n} \\
&= \sum_{n=-2}^{2} e^{-j\omega n} \\
&= e^{j2\omega} \cdot \sum_{n=0}^{4} e^{-j\omega n} \\
&= e^{j2\omega} \cdot \frac{1-e^{-j5\omega}}{1-e^{-j\omega}} \\
&= \frac{e^{j2\omega}-e^{-j3\omega}}{1-e^{-j\omega}} \\
&= \frac{e^{-j\omega/2} \cdot \left(e^{j\omega(2+1/2)}-e^{-j\omega(2+1/2)}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)} \\
&= \frac{j2\sin(\omega(2+1/2))}{j2\sin(\omega/2)} \\
&= \frac{\sin(\omega(2+1/2))}{\sin(\omega/2)}
\end{aligned}
$$

(b) Note that $u[n]-u[n-5]$ is a shifted version of the function from (a). Using the time shift property of DTFT, we can simply multiply the result from (a) by $e^{-j\omega n_0}$ where n_0 is the time shift.

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \left(u[n]-u[n-5]\right) e^{-j\omega n} \\
&= \sum_{n=0}^{4} e^{-j\omega n} \\
&= e^{-j2\omega} \cdot \sum_{n=-2}^{2} e^{-j\omega n} \\
&= e^{-j2\omega} \cdot \frac{\sin(\omega(2+1/2))}{\sin(\omega/2)}
\end{aligned}
$$

(c) Be careful that the rectangular function here has a different length from the previous two questions.

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \left(u[n] - u[n-4]\right) e^{-j\omega n} \\
&= \sum_{n=0}^{3} e^{-j\omega n} \\
&= \frac{1 - e^{-j4\omega}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j2\omega} \cdot \left(e^{j2\omega} - e^{-j2\omega}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} \\
&= e^{-j\omega(2-1/2)} \frac{j2\sin(2\omega)}{j2\sin(\omega/2)} \\
&= e^{-j\omega(3/2)} \frac{\sin(2\omega)}{\sin(\omega/2)}
\end{aligned}$$

Part(C) (a)

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 0.5\pi - 2\pi k) e^{j\omega n} d\omega &= \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \delta(\omega - 0.5\pi - 2\pi k) e^{j\omega n} d\omega \\
&= \int_{-\pi}^{\pi} \delta(\omega - 0.5\pi) e^{j\omega n} d\omega \\
&= e^{j(\pi/2)n}
\end{aligned}$$

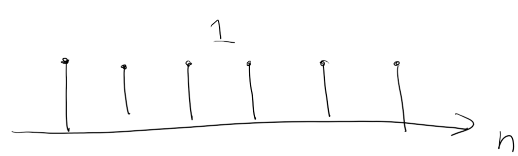

Figure 5: $|x[n]|$ for Part(C) (a)

(b)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty}\left(\pi\delta(\omega+0.5\pi-2\pi k)+\pi\delta(\omega-0.5\pi-2\pi k)\right)e^{j\omega n}d\omega$$
$$=\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}\left(\frac{1}{2}\delta(\omega+0.5\pi-2\pi k)+\frac{1}{2}\delta(\omega-0.5\pi-2\pi k)\right)e^{j\omega n}d\omega$$
$$=\int_{-\pi}^{\pi}\left(\frac{1}{2}\delta(\omega+0.5\pi)+\frac{1}{2}\delta(\omega-0.5\pi)\right)e^{j\omega n}d\omega$$
$$=\frac{1}{2}e^{-j(\pi/2)n}+\frac{1}{2}e^{j(\pi/2)n}$$
$$=\cos(\pi n/2)$$

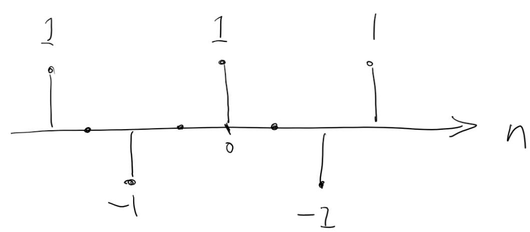

Figure 6: $|x[n]|$ for Part(C) (b)

Problem 2.3

(a)

$$y[n]=0.5y[n-1]+x[n]$$

We transform the equation into Fourier domain. By linearity and time shift property,

$$Y(e^{j\omega})=0.5e^{j\omega}\cdot Y(e^{j\omega})+X(e^{j\omega})$$

By moving the terms around, we get

$$Y(e^{j\omega})=\frac{X(e^{j}\omega)}{1-0.5e^{j\omega}}$$

(b) By definition of DTFT,

$$X(e^{j0.25\pi})=\sum_{n=-\infty}^{\infty}x[n]e^{-j0.25\pi n}$$

We apply conjugate to $X(e^{j0.25\pi})$. Since $x[n] = (x[n])^*$, we get $\left(X(e^{j0.25\pi})\right)^* = X(e^{-j0.25\pi})$.

$$\begin{aligned}
\left(X(e^{j0.25\pi})\right)^* &= \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j0.25\pi n}\right)^* \\
&= \sum_{n=-\infty}^{\infty} \left(x[n]e^{-j0.25\pi n}\right)^* \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{j0.25\pi n} \\
&= X(e^{-j0.25\pi}) \\
&= 1 - j
\end{aligned}$$

Problem 2.4

(a) When $T = 0.0001$, sample rate is 20000π rads/s. 20000π in CTFT's ω-axis corresponds to 2π in DTFT's ω-axis. 10000π in CTFT's ω-axis corresponds to π in DTFT's ω-axis.

In the interval of $[-\pi, \pi]$, $|X(e^{j\omega}| = 0$ when ω is $-\pi$ or π.

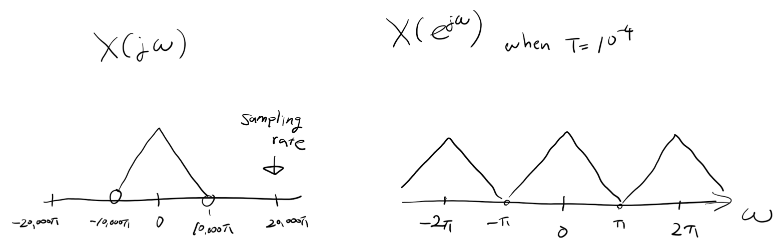

Figure 7: (left) CTFT of $x(t)$. (right) DTFT of $x[n]$ when $T = 0.0001$.

(b) When $T = 0.00005$, sample rate is 40000π rads/s. 40000π in CTFT's ω-axis corresponds to 2π in DTFT's ω-axis. 10000π in CTFT's ω-axis corresponds to $\frac{\pi}{2}$ in DTFT's ω-axis.

In the interval of $[-\pi, \pi]$, $|X(e^{j\omega})| = 0$ when ω is in $\left[-\pi, -\frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \pi\right]$.

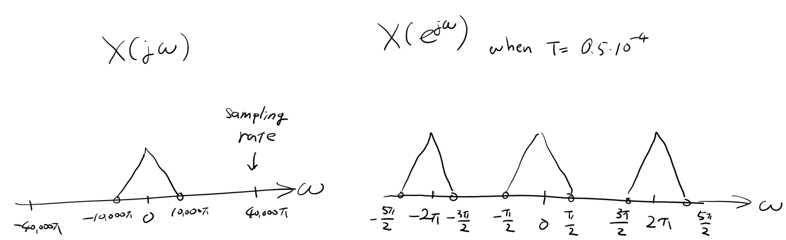

Figure 8: (left) CTFT of $x(t)$. (right) DTFT of $x[n]$ when $T = 0.00005$.

(c) When $T = 0.00001$, sample rate is 200000π rads/s. 200000π in CTFT's ω-axis corresponds to 2π in DTFT's ω-axis. 10000π in CTFT's ω-axis corresponds to $\frac{\pi}{10}$ in DTFT's ω-axis.

In the interval of $[-\pi, \pi]$, $|X(e^{j\omega})| = 0$ when ω is in $\left[-\pi, -\frac{\pi}{10}\right]$ or $\left[\frac{\pi}{10}, \pi\right]$.

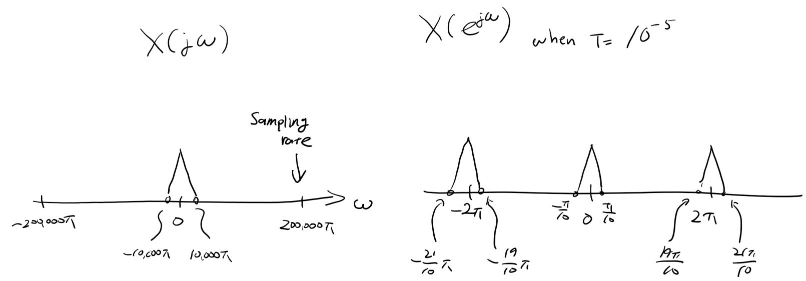

Figure 9: (left) CTFT of $x(t)$. (right) DTFT of $x[n]$ when $T = 0.00001$.