Problem 1

For harmonic oscillator,

$P_{cl}(x) = \frac{t(x)}{T_0} \cdot 2$

$T_0 = \frac{2\pi}{\omega_0}$

The reason we add $\cdot 2$ is because within a period, it will pass same point twice with same velocity. → same amount of time in same dx.

$t(x)$ represents the time it takes to pass the point at position x

Now, lets calculate it.

$k = m\omega_0^2$

$\frac{1}{2} m V(x)^2 = E - \frac{1}{2} k x^2$

$m V(x)^2 = 2E - kx^2$

$V(x) = \sqrt{\frac{2E - kx^2}{m}} = \frac{dx}{dt}$

$\frac{dt}{dx} = \sqrt{\frac{m}{2E - kx^2}} = t(x)$

$P_{cl}(x) = \sqrt{\frac{m}{2E - m\omega_0^2 x^2}} \cdot \cancel{2} \cdot \frac{\omega_0}{\cancel{2}\pi}$

$= \frac{1}{\sqrt{2E - m\omega_0^2 x^2}} \cdot \frac{\sqrt{m\omega_0^2}}{\pi} = \frac{\sqrt{m\omega_0^2}}{\pi} \cdot \frac{1}{\sqrt{2E - m\omega_0^2 x^2}}$

Problem 2

$h\nu = E = 13.6\,eV$ $\quad h = 4.1357 \times 10^{-15}\, eV \cdot s$ $\quad C=$

frequency: $\nu = \frac{13.6\,eV}{4.1357 \times 10^{-15}\, eV\cdot s} \approx 3.29 \times 10^{15}\, Hz$

wavelength: $\lambda = \frac{c}{\nu} = \frac{3 \times 10^8\, m/s}{3.29 \times 10^{15}\, Hz} \approx 0.912 \times 10^{-7}\, m = 9.12 \times 10^{-8}\, m$

$= 91.2\,nm$

wavenumber: $\frac{1}{\lambda} = \frac{1}{9.12 \times 10^{-8}\, m} = \frac{1}{9.12 \times 10^{-8}}\, m^{-1}$

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Problem 3:

According to the uncertainty rule:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = 10nm$$

$$\Delta p \geq \frac{\hbar}{2\Delta x} = \frac{1.0546\times10^{-34}\ kg\cdot m^2/s}{2\cdot 10\times10^{-9}m}$$

-34-

$$= 0.5273\times10^{-26}\ \underbrace{kgm^2/s}_{unit}$$

Problem 4:

suppose in the set ψ_n and ψ_m where $n \neq m$

① Normalized

for any n in set: $\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = 1$

$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$$\int_{-a}^{a}\left(\sqrt{\frac{1}{a}}\sin\left(\frac{n\pi x}{a}\right)\right)^2 dx = \frac{1}{a}\int_{-a}^{a}\sin^2\left(\frac{n\pi x}{a}\right)dx \qquad \theta = \frac{n\pi x}{a}$$

$$= \frac{1}{a}\int_{-a}^{a}\frac{1}{2}(1-\cos(2\theta))dx$$

$$= \frac{1}{a}\left[\int_{-a}^{a}\frac{1}{2}dx - \int_{-a}^{a}\cos\frac{2n\pi x}{a}dx\right]$$

$$= \frac{1}{a}\left[a - \left[\frac{2n\pi x}{a}\cdot\underbrace{\left(-\sin\left(\frac{2n\pi y}{a}\right)\right)}_{0}\right]_{-a}^{a}\right.$$

$$= \frac{1}{a}\cdot a = \boxed{1}$$

② orthogonal:

$$\int_{-\infty}^{\infty}\psi_n^*(x)\,\psi_m(x)\,dx = 0\ ?$$

Since $|x| \geq a = 0$

$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$\int_{-\infty}^{\infty}\psi^*(x)\psi_m(x)dx = \int_{-\infty}^{\infty}\psi_n(x)\,\psi_m(x)\,dx = \int_{-a}^{a}\sqrt{\frac{1}{a}}\sin\left(\frac{n\pi x}{a}\right)\cdot\sqrt{\frac{1}{a}}\sin\left(\frac{m\pi x}{a}\right)dx$$

because of being real

$$= \frac{1}{a}\int_{-a}^{a}\frac{1}{2}\left[\underbrace{\cos\left(\frac{(n-m)\pi x}{a}\right)}_{0} - \underbrace{\cos\left(\frac{(n+m)\pi x}{a}\right)}_{0}\right]dx$$

$= 0$ So they are orthogonal.

Problem 5:

$$\int \psi_1^* \psi_2 \, dx = d$$

a) $\int \psi_1^* \phi \, dx = 0$ $\quad \int_{-\infty}^{\infty} \psi_1^* \{ c_1 \psi_1 + c_2 \psi_2 \} \, dx$

$$= c_1 \underbrace{\int_{-\infty}^{\infty} \psi_1^* \psi_1 \, dx}_{\text{normalized}} + c_2 \int_{-\infty}^{\infty} \psi_1^* \psi_2 \, dx$$

$$= c_1 + c_2 \cdot d = 0$$

$$c_1 = -d \cdot c_2$$

as we want new wave to be also normalized.

$\int |\phi|^2 dx = 1$ $\quad \int (-d c_2 \psi_1 + c_2 \psi_2)^* (-d c_2 \psi_1 + c_2 \psi_2) \, dx$

$= 1$

$$c_2^2 (d^2 + 1 - 2d) = 1 \qquad \begin{cases} c_2 = \frac{1}{1-d} \\ c_1 = -\frac{d}{1-d} \end{cases}$$

b) $\psi_1 + \psi_2$

$$\int (\psi_1 + \psi_2)^* \phi \, dx = 0$$

$$\int (\psi_1^* + \psi_2^*) (c_1 \psi_1 + c_2 \psi_2) \, dx = 0$$

$$c_1 \int \psi_1^* \psi_1 \, dx + c_1 \int \psi_2^* \psi_1 \, dx + c_2 \int \psi_1^* \psi_2 \, dx + c_2 \int \psi_2^* \psi_2 \, dx = 0$$

$$c_1 + c_1 d + c_2 d + c_2 = 0$$

$$\Downarrow$$

$$(c_1 + c_2) = 0$$

$$c_1 = -c_2$$

$$\int |\phi|^2 dx = 1 \Rightarrow c_2^2 \int (-\psi_1 + \psi_2)^* (-\psi_1 + \psi_2) \, dx$$

$$c_2^2 (2 - 2d) = 1$$

$$\begin{cases} c_1 = \frac{1}{\sqrt{2(1-d)}} \\ c_2 = -\frac{1}{\sqrt{2(1-d)}} \end{cases}$$

Problem 6

$$J = \frac{\hbar}{2mi}\left[\psi^*(r,t)\,\nabla\psi(r,t) - \psi(r,t)\,\nabla\psi^*(r,t)\right]$$

Free Particle, $\psi(r,t) = Ae^{i(k\cdot r - \omega t)}$

gradient is with respect to r.

$\nabla\psi(r,t) = ik\,\psi(r,t)$

$\nabla\psi^*(r,t) = -ik\psi^*(r,t)$

$$\vec{j} = \frac{\hbar}{2mi}\left[\psi^*(ik\psi) - \psi(-ik\psi^*)\right]$$

$$J = \frac{h}{2mi}\left[ik(\psi^*\psi) + ik\psi^*\psi)\right]$$

$$= \frac{\hbar}{\cancel{2}m\cancel{i}}\,\cancel{2i}\,k|\psi|^2$$

$$= \frac{\hbar}{m}k|\psi|^2$$

momentum:

$\hat{p}\psi = -i\hbar\nabla\psi(r,t)$

$= -i\hbar\nabla(Ae^{-ik\cdot r - \omega t})$

$= \hbar k\,\psi(r,t)$

Now Let's say velocity is kind of

$$v = \frac{p}{m} = \underbrace{\frac{\hbar k}{m}\psi(r,t)}$$

$$\langle v\rangle = \frac{\hbar k}{m}$$

$$J = \frac{\hbar}{m}k|\psi|^2$$

$= \underbrace{\langle v\rangle}_{\text{expect } v \text{ which is } v \text{ observable.}}|\psi|^2$ ↗ particle density.

$\& \; c e^{-\left[\frac{(P_x - P_0)^2}{2 \Delta P_x^2}\right]} = \Phi(P_x)$

$\int_{-\infty}^{\infty} (\Phi_{P(x)})^2 \, dx$ as everything is real

$= \int_{-\infty}^{\infty} \left(c e^{-\frac{(P_x - P_0)^2}{2 \Delta P_x^2}} \right)^2 dx$

$= c^2 \int_{-\infty}^{\infty} e^{-\frac{(P_x - P_0)^2}{\Delta P_x^2}} dx$

$= c^2 \int_{-\infty}^{\infty} e^{-\frac{u^2}{\Delta P_x^2}} du$ $\quad$ $u = P_x - P_0$, $du = dP_x$

$= c^2 \cdot \sqrt{\pi \cdot \Delta P_x^2} = 1$

$c^2 = \frac{1}{\sqrt{\pi \Delta P_x^2}}$ $\quad$ $c = \sqrt{\frac{1}{\sqrt{\pi} \Delta P_x}}$

Problem 7 addition $\quad$ Let $u = P_x - P_0$ $\; du = dP_x$

$= \underbrace{e^{\frac{i P_0 x}{\hbar}} \frac{c}{\sqrt{2\pi\hbar}}}_{A} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2 \Delta P_x^2}} \cdot e^{\frac{i u x}{\hbar}} du$

$= A \cdot \int_{-\infty}^{\infty} e^{-\frac{u^2}{2 \Delta P_x^2}} \cdot e^{\frac{i u x}{\hbar}} du$

Gaussian Integral with exponent

$= A \cdot \sqrt{2\pi (\Delta P_x)^2} \, e^{-\frac{x^2 \cdot \Delta P_x^2}{2\hbar^2}} = \frac{1}{\sqrt{\sqrt{\pi} \Delta P_x}} e^{\frac{i P_0 x}{\hbar}} e^{-\frac{x^2 (\Delta P_x)^2}{2\hbar^2}}$

ΔP is ΔP

$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

$\langle x \rangle = 0$ as x makes function odd.

$|\psi(x)|^2 = \underbrace{\boxed{\frac{1}{\sqrt{\pi} \Delta P_x}}}_{W} e^{-\frac{x^2 (\Delta P_x)^2}{\hbar^2}}$

$\langle x^2 \rangle \int_{-\infty}^{\infty} x^2 \cdot W \cdot e^{-\frac{(\Delta P_x)^2}{\hbar^2} x^2} dx = \frac{\hbar}{\sqrt{2} \Delta P_x}$

will according to guassian format

the Δ real $= \frac{\Delta P_x}{\sqrt{2}}$

So $\Delta x \cdot \Delta$ real $= \frac{\hbar}{\sqrt{2} \Delta P_x} \frac{\Delta P_x}{\sqrt{2}}$

$= \boxed{\frac{\hbar}{2}}$

$\psi(x$

$-P_0$

dP_x

$P_x = u + P_0$

$\sqrt{\pi}$

$|c|^2$

Problem 8.

a) $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

$$|C|^2 \int_{-\infty}^{\infty} e^{\frac{-|x|}{\Delta x}} dx$$

$$= |C|^2 \left[\int_{-\infty}^{0} e^{\frac{x}{\Delta x}} dx + \int_{0}^{\infty} e^{\frac{-x}{\Delta x}} dx \right]$$

$$= |C|^2 \left[\Delta x \, e^{\frac{x}{\Delta x}} \Big|_{x=-\infty}^{x=0} + \cancel{(-\Delta x) e^{-\frac{x}{\Delta x}}} \Big|_{0}^{\infty} \right]$$

$$= |C|^2 \left[\Delta x (1-0) - \Delta x (0-1) \right] = 2\Delta x |C|^2$$

So $|C| = \frac{1}{\sqrt{2\Delta x}}$

b) $$\phi(p_x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{ip_x x}{\hbar}} \psi(x) dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{-ip_x}{\hbar}} \frac{1}{\sqrt{2\Delta x}} e^{\frac{ip_0 x}{\hbar}} e^{-\frac{|x|}{2\Delta x}} dx$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{\pi \Delta x}} \left[\frac{1}{\frac{\hbar}{2\Delta x} + i(-p_x + p_0)} e^{\frac{\frac{\hbar}{2\Delta x} + i(-p_x+p_0)x}{\hbar}} \Big|_{-\infty}^{0} + \right.$$

$$\left. \frac{\hbar}{\frac{-\hbar}{2\Delta x} + i(-p_x+p_0)} e^{\frac{[\frac{-\hbar}{2\Delta x} + i(-p_x+p_0)]x}{\hbar}} \Big|_{0}^{\infty} \right]$$

$$= \frac{1}{2}\sqrt{\frac{\hbar^3}{\Delta x^3 \pi}} \left[\frac{1}{\frac{\hbar^2}{4\Delta x^2} + (-p_x+p_0)^2} \right]$$

$$\int \left| \frac{1}{2}\sqrt{\frac{\hbar^3}{\Delta x^3 \pi}} \frac{1}{\frac{\hbar^2}{4\Delta x^2} + (-p_x+p_0)^2} \right|^2 dp_x$$

$$= \frac{\hbar^3}{4\Delta x^3 \pi} \int_{-\infty}^{\infty} \frac{1}{\left(\frac{\hbar^2}{4\Delta x^2} + (-p_x+p_0)^2\right)^2} dp_x$$

$$= \frac{\hbar^3}{4\Delta x^3 \pi} \cdot \left[\frac{4\Delta x^3}{\hbar^3} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) \right] = 1$$ so normalized.

The function has peak at $P_x = P_0$

$$\phi(P_0) = \frac{1}{2}\sqrt{\frac{\hbar^3}{\Delta x^3 \pi}}\left[\frac{1}{\frac{\hbar^2}{4\Delta x^2} + (-P_x + P_0)^2}\right]$$

$$= \frac{1}{2}\sqrt{\frac{\hbar^3}{\Delta x^3 \pi}}\frac{1}{\hbar^2/4\Delta x^2}$$

$$= 2\sqrt{\frac{\Delta x}{\hbar \pi}}$$

For what p_x the function is $\sqrt{\frac{\Delta x}{\hbar\pi}}$

$$\frac{1}{2}\sqrt{\frac{\hbar^3}{\Delta x^3 \pi}}\left[\frac{1}{\frac{\hbar^2}{4\Delta x^2} + (-P_x + P_0)^2}\right] = \sqrt{\frac{\Delta x}{\hbar\pi}}$$

$$\Downarrow$$

$$\frac{4\Delta x^2}{\hbar^2}(-P_x + P_0)^2 = 1 \qquad P_x = P_0 \pm \frac{\hbar}{2\Delta x}$$

So we could define

$$(\Delta P_x)_{define} = \frac{\hbar}{\Delta x} \quad \text{such that} \quad \Delta P_{defin} \Delta x = \hbar .$$

Problem 9:

① 10 Å 2eV potential barrier. $E = 1.5eV$

In potential Barrier.

$$k = \sqrt{\frac{2m}{\hbar^2} E} \qquad k' = \sqrt{\frac{2m}{\hbar^2}(E - V_0)}$$

$E = 1.5eV$

$V_0 = 2eV$

$\Rightarrow k \approx 6.25 \times 10^9 m^{-1}$

$k' \approx 3.62 \times 10^9 m^{-1}$

$$R = \left|\frac{k - k'}{k + k'}\right|^2 \qquad T = \left|\frac{2k}{k + k'}\right|^2 \times e^{-2k'a}$$

$a = 10 \times 10^{-10} m$

from ①

So $R \approx 0.072$

$T \approx 0.00115$

Problem 10.

$$(\hat{T} + \hat{V})\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + qEx\psi = E_n\psi$$

I am assuming E is the field strength, a constant.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + (qEx - E_n)\psi = 0$$

An Airy differential equation is solution to $y'' - xy = 0$.

$$\frac{d^2\psi}{dx^2} = \frac{2mqE}{\hbar^2}\psi_n(x)\left(x - \frac{E_n}{qE}\right)$$

Let $z = \beta\left(x - \frac{E_n}{qE}\right)$ β is for normalization

$$\frac{dz}{dx} = \beta \qquad \text{so } \beta = \sqrt[3]{\frac{2mqE}{\hbar^2}}$$

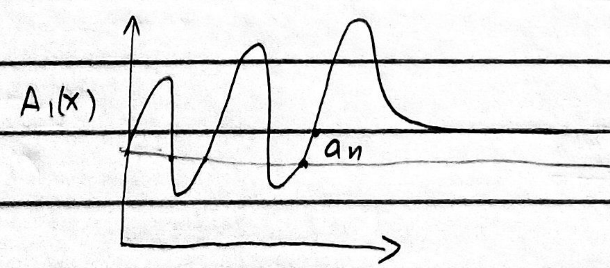
$$\frac{d^2\psi_n(z)}{dz^2} = \frac{2mqE}{\hbar^2}\psi_n(z)\, z\beta^{-3}$$

$$\psi_n(z) = A\,Ai(z) + \boxed{B\,Bi(z)}$$ → goes away as it cause infinite.

⇓ plug in x back

$$\psi_n(x) = Ai\left[\sqrt[3]{\frac{2mqE}{\hbar^2}}\left(x - \frac{E_n}{qE}\right)\right]$$

$\psi_n(0) = 0$ as the other side is infinite well



an is zeros of Ai function.

$$a_n = -\sqrt[3]{\frac{2mqE}{\hbar^2}}\frac{E_n}{qE} \qquad E_n = -a_n\sqrt[3]{\frac{\hbar^2q^2E^2}{2m}}$$

ψ_n after normalized is:

$$\left(\frac{\sqrt[3]{\frac{2mqE}{\hbar^2}}}{Ai'^2\left[-\sqrt[3]{\frac{2mqE}{\hbar^2}}\frac{E_n}{qE}\right]}\right)^{\frac{1}{2}} Ai\left[\sqrt[3]{\frac{2mqE}{\hbar^2}}\left(x - \frac{E_n}{qE}\right)\right]$$

Problem 11: $\psi_m(x,t)$ E_n

a) $$\psi(x,t) = C_1\psi_1(x)e^{-\frac{iE_1t}{\hbar}} + C_2\psi_2(x)e^{-\frac{iE_2t}{\hbar}}$$

$c_1 = c_2 = \frac{1}{\sqrt{2}}$ for first question.

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-\frac{iE_1t}{\hbar}} + \psi_2(x)e^{-\frac{iE_2t}{\hbar}}\right]$$

b) $$\langle E\rangle = |C_1|^2E_1 + |C_2|^2E_2$$
$$= \frac{E_1+E_2}{2}$$

c) $$\langle \Delta E\rangle = \sqrt{\langle E^2\rangle - \langle E\rangle^2} \qquad \langle E^2\rangle = |C_1|^2E_1^2 + |C_2|^2E_2^2$$
$$= \sqrt{\frac{E_1^2+E_2^2}{2} - \left(\frac{E_1+E_2}{2}\right)^2} \qquad = \frac{E_1^2+E_2^2}{2}$$
$$= \frac{|E_1-E_2|}{2}$$

d) $$\langle x(t)\rangle = \int_{-\infty}^{\infty} x\,|\psi(x,t)|^2\,dx$$
$$= \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-\frac{iE_1t}{\hbar}} + \psi_2(x)e^{-\frac{iE_2t}{\hbar}}\right]\frac{1}{\sqrt{2}}\left[\psi_1^*(x)e^{+\frac{iE_1t}{\hbar}} + \psi_2^*(x)e^{+\frac{iE_2t}{\hbar}}\right]dx$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\left[x\psi_1^2 + x\psi_1\psi_2e^{\frac{i(E_2-E_1)t}{\hbar}} + x\psi_2\psi_1e^{\frac{i(E_1-E_2)t}{\hbar}} + \psi_2^2\cdot x\right]dx$$
$$= \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} x(\psi_1^2+\psi_2^2)\,dx}_{x_0} + \underbrace{\frac{1}{2}\int_{-\infty}^{\infty}\psi_1\psi_2^*e^{-\frac{i(E_1-E_2)t}{\hbar}} + \psi_2\psi_1^*e^{\frac{i(E_1-E_2)t}{\hbar}}\,dx}_{a\cos\left(\frac{E_2-E_1}{\hbar}t\right)}$$
$$= \qquad x_0 \qquad\qquad a\cos\left(\frac{E_2-E_1}{\hbar}t\right)$$

①

① takes the form since

$$C_1^*C_2e^{\frac{i(E_2-E_1t)}{\hbar}} + C_2^*C_1e^{\frac{-i(E_2-E_1)t}{\hbar}} = 2\,\mathrm{Re}(C_1^*C_2)\cos\left(\frac{E_2-E_1}{\hbar}t\right)$$

P

Problem 12

$$\psi(x,t) = Ae^{-a\left(\frac{mx^2}{\hbar}+it\right)}$$

the time part gets canceled.

a) $\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$

$$= \int_{-\infty}^{\infty} A^2 e^{-\frac{2amx^2}{\hbar}} dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{\frac{-2am}{\hbar}x^2} dx$$

$$= A^2 \sqrt{\frac{\pi}{\frac{2am}{\hbar}}} = A^2 \sqrt{\frac{\pi\hbar}{2am}} = 1 \qquad A^2 = \sqrt{\frac{2am}{\pi\hbar}}$$

$$A = \left(\frac{2am}{\pi\hbar}\right)^{\frac{1}{4}}$$

b) $\psi(x,t)$

$$\underbrace{i\hbar\frac{\partial}{\partial t}\psi(x,t)}_{①} = (\hat{T}+\hat{V})\psi$$

$$= -\frac{\hbar^2}{2m}\underbrace{\frac{\partial^2}{\partial x^2}\psi(x,t)}_{②} + \hat{V}\psi \qquad u = \frac{amx^2}{\hbar}$$

① $\frac{\partial}{\partial t}\psi(x,t) = Ae^{-a\left(\frac{mx^2}{\hbar}+it\right)} \cdot (-ai)$

$$\frac{d\psi}{dx} = \frac{d\psi}{du} \cdot \frac{du}{dx} \qquad \frac{du}{dx} = \frac{2amx}{\hbar}$$

② $\frac{\partial^2}{\partial x^2}\psi(x,t) =$

$$\frac{\partial}{\partial x}\psi(x,t) = \frac{\partial}{\partial x}Ae^{-a\left(\frac{mx^2}{\hbar}+it\right)} \qquad e^{-u} = e^{-u}$$

$$= Ae^{-ait}\left(\frac{-2amx}{\hbar}\right) \cdot e^{-\frac{amx^2}{\hbar}}$$

$$\frac{\partial^2}{\partial x^2}\psi(x,t) = Ae^{-ait}\left\{\frac{-2am}{\hbar} \cdot e^{\frac{-amx^2}{\hbar}} + \left(\frac{-2amx}{\hbar}\right)^2 \cdot e^{\frac{-amx^2}{\hbar}}\right\}$$

$$-\frac{\hbar^2}{2m}\cancel{Ae^{-ait}}\left[\frac{-2am}{\hbar} \cdot e^{-\frac{amx^2}{\hbar}} + \left(\frac{2amx}{\hbar}\right)^2 e^{-\frac{amx^2}{\hbar}}\right] + \hat{V}\psi$$

$$= i\hbar\,\cancel{Ae^{-ait}}\,e^{\frac{-amx^2}{\hbar}}(-ai)$$

$$-\frac{\hbar^2}{2m}\left[\frac{-2am}{\hbar}\cdot \cancel{e^{-\frac{amx^2}{\hbar}}} + \left(\frac{2amx}{\hbar}\right)^2 \cancel{e^{-\frac{amx^2}{\hbar}}}\right) + V_{(x)}$$

$$= i\hbar\, \cancel{e^{-\frac{amx^2}{\hbar}}}\,(-ai)$$

$$a\,\cancel{e^{-\frac{amx^2}{\hbar}}}$$

$$-\frac{\hbar^2}{2m}\left[\frac{-2am}{\hbar} + \left(\frac{2amx}{\hbar}\right)^2\right] + V_{(x)} = a\hbar$$

$$-\frac{\hbar^2}{2m}\left(\frac{4a^2m^2x^2}{\hbar^2} - \frac{2am}{\hbar}\right) + V_{(x)} = a\hbar$$

$$-\frac{\cancel{\hbar^2}}{\cancel{2m}}\cdot\frac{{}^2\cancel{4}a^2 \cancel{m^2}^{m} x^2}{\cancel{\hbar^2}} + \frac{\cancel{\hbar^2}^{\hbar}}{\cancel{2m}}\cdot\frac{\cancel{2}am}{\cancel{\hbar}} \cancel{+V(x) = a\hbar}$$

$$-\ 2a^2mx^2 + \cancel{a\hbar} + V(x) \cancel{+V(x) = a\hbar}$$

$$V(x) = 2a^2mx^2$$

c) 1) $\langle x\rangle = \int x\psi^*\psi\, dx$

$$= \int A^2 \boxed{e^{-\frac{2amx^2}{\hbar}}\cdot x}\, dx \quad f$$

$= 0$ as $f(x) = -f(-x)$

2) $\langle x^2\rangle = \int x^2\psi^*\psi\, dx$

$$= \int x^2 A^2 e^{-\frac{2amx^2}{\hbar}}\, dx = |A|^2 \frac{\hbar}{2am}\sqrt{\frac{\pi\hbar}{2am}} \qquad |A|^2 = \sqrt{\frac{2am}{\pi\hbar}}$$

$$= \frac{\hbar}{4am}$$

d) $\langle P\rangle = \int -i\hbar\frac{\partial}{\partial x}\psi^*\psi\, dx$ → the t term will anyhow gets canceled.

$$= \int \left(-i\hbar \frac{\partial}{\partial x} A e^{-a\left(\frac{mx^2}{\hbar}\right)}\right) e^{-\left(a\frac{mx^2}{\hbar}\right)}$$

$$\cancel{= \int -i\hbar\,(} \quad \frac{\partial}{\partial x} A e^{-\frac{amx^2}{\hbar}} = A\left(-\frac{2amx}{\hbar}\right) e^{-\frac{amx^2}{\hbar}}$$

$$\int (-i\hbar)\cdot A\left(\frac{-2amx}{\hbar}\right) A e^{-\frac{2amx^2}{\hbar}}\, dx$$

$$= (-i\hbar A^2)\left(\frac{-2am}{\hbar}\right)\int x e^{-\frac{2amx^2}{\hbar}}\, dx$$ ↗ same as x so = 0

$$= \cancel{\psi}\ 0$$

$\langle P^2\rangle = \int \cancel{\psi}\psi^* \left(\hbar^2 \frac{\partial^2}{\partial x^2}\right)\psi\, dx$ time factor is canceled

$$= \hbar^2 \int |\psi(x)|^2 \left(\frac{2am}{\hbar} - \frac{4a^2m^2x^2}{\hbar^2}\right) dx \qquad \frac{\partial^2}{\partial x^2} = \psi(x)\left(\frac{4a^2m^2x^2}{\hbar^2} - \frac{2am}{\hbar}\right)$$

$$= \hbar^2 \left\{\left[\frac{2am}{\hbar}\right] - \int |\psi(x)|^2 \frac{4a^2m^2x^2}{\hbar^2}\, dx\right\}$$

$$= \hbar^2 \left\{\frac{2am}{\hbar} - \frac{4a^2m^2}{\hbar^2}\cdot\frac{\hbar}{4am}\right. = a\hbar$$

Problem 13

$$\psi(x,0) = \begin{cases} \psi(x,0) = A(a^2-x^2) & -a < x \le a \\ \psi(x,0) = 0 & |x| > a \end{cases}$$

1) $\int |\psi(x,0)|^2 dx = 1$

$$\int_{-a}^{a} [A(a^2-x^2)]^2 dx = \int_{-a}^{a} A^2 (a^2-x^2)^2 dx \qquad a^4$$

$$= A^2 \int_{-a}^{a} [a^4 - 2a^2x^2 + x^4]\, dx \qquad \int_{-a}^{a} a^4 dx = 2a^5$$

$$= A^2\left(2a^5 - 2a^2 \cdot \frac{2a^3}{3} + \frac{2}{5}a^5\right) = 1 \qquad \int_{-a}^{a} x^2 dx = \frac{2}{3}a^3$$

$$A^2 2a^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 1 \qquad \int_{-a}^{a} x^4 dx = \frac{2}{5}a^5$$

$$A^2 2a^5 \cdot \frac{8}{15} = 1 \qquad \frac{4a^5}{3}$$

$$A^2 \frac{16}{15} a^5 = 1$$

$$A = \sqrt{\frac{15}{16a^5}} = \frac{(15)^{\frac{1}{2}}}{4a^{\frac{5}{2}}}$$

2) $\langle x \rangle = \int \psi \cdot x \psi^* dx$

$$= \int A(a^2-x^2) \cdot x \cdot A(a^2-x^2)\, dx$$

$$= A^2 \int_{-a}^{a} \underbrace{(a^2-x^2)^2 x}_{f}\, dx \qquad f(x) = -f(x)$$

$$= 0$$

$$\langle x^2 \rangle = \int \psi \cdot x^2 \psi^* dx \qquad \int (a^2-x^2)^2 = \int a^4 - 2a^2x^2 + x^4$$

$$= \int A^2(a^2-x^2)^2 \cdot x^2 dx$$

$$= A^2 \int_{-a}^{a} x^2(a^4 - 2a^2x^2 + x^4)\, dx = A^2\Big(a^4 \overbrace{\int_{-a}^{a} x^2 dx}^{\frac{2}{3}a^3} - 2a^2 \overbrace{\int_{-a}^{a} x^4 dx}^{\frac{2}{5}a^5} + \underbrace{\int_{-a}^{a} x^6 dx}_{\frac{2}{7}a^7}\Big)$$

$$= A^2\left(\frac{2}{3}a^7 - \frac{4}{5}a^7 + \frac{2}{7}a^7\right)$$

$$= A^2 a^7\left(\frac{2}{7} - \frac{2}{15}\right) = a^7 \frac{16}{105} \cdot \frac{15}{16a^5}$$

$$= \frac{a^2}{7}$$

3) $\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \, i\hbar \frac{\partial}{\partial x} \psi(x)\, dx \qquad \frac{\partial}{\partial x}\psi = \frac{\partial}{\partial x}(Aa^2 - Ax^2) = -2Ax$

$$= -i\hbar \int_{-\infty}^{\infty} -2Ax \cdot A(a^2-x^2)\, dx$$

$$= 0 \qquad \frac{\partial^2}{\partial x^2}\psi = -2A$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* \hbar^2 \frac{\partial^2}{\partial x^2} \psi(x)\, dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} A(a^2-x^2) \cdot (-2A)\, dx$$ as original is zero else where.

$$= -\hbar^2(-2A^2) \int_{-a}^{a} a^2 - x^2 dx \qquad a^2 \cdot 2a$$

$$= 2\hbar^2 A^2\left(2a^3 - \frac{2}{3}a^3\right)$$

$$= 2\hbar^2 \cdot \frac{15}{16a^5} a^3 \left(2 - \frac{2}{3}\right) \qquad \frac{4}{3}$$

$$= \frac{5\hbar^2}{2a^2}$$

4) $\sigma_x = \sqrt{\frac{a^2}{7}} \qquad \sigma_p = \sqrt{\frac{5\hbar^2}{2a^2}} \qquad \frac{20}{14\times 4} \quad \sqrt{\frac{14}{14\times 4}}$

$$\sigma_x \sigma_p = \sqrt{\frac{a^2}{7}} \cdot \sqrt{\frac{5\hbar^2}{2a^2}} = \sqrt{\frac{a^2}{7} \cdot \frac{5\hbar^2}{2a^2}}$$

$$= \sqrt{\frac{5}{14}}\, \hbar > \frac{\hbar}{2} \qquad \sqrt{\frac{5}{14}} > \frac{1}{2} = \sqrt{\frac{1}{4}}$$

So uncertainty is true

Problem 14:

1. $\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi}\, 2^n n!}\right)^{\frac{1}{2}} e^{-\frac{\alpha^2 x^2}{2}} H_n(\alpha x)$

$\psi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-\frac{\alpha^2 x^2}{2}}$

$\alpha = \left(\frac{mk}{\hbar^2}\right)^{\frac{1}{4}}$

2. $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \int_{-\infty}^{\infty} x^2 |\psi_0(x)|^2 dx$ $\langle x\rangle = 0$ as ψ_0 is symmetric.

$= \frac{1}{2\alpha^2}$ $\Delta x = \frac{1}{2\alpha^2}$

$\Delta p_x = \sqrt{\langle p_x^2\rangle - \langle p_x\rangle^2}$ $\langle p_x\rangle = 0$ for ground state.

$\langle p_x^2\rangle = \frac{\hbar^2}{a} \langle \alpha^2\rangle$

$\Delta p_x = \frac{\hbar\alpha}{\sqrt{2}}$

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3. $\Delta x \Delta p_x = \frac{1}{\alpha\sqrt{2}} \frac{\hbar\alpha}{\sqrt{2}} = \frac{\hbar}{2}$

the result satisfied the uncertainty principle and is in minimum uncertainty state

Problem 15.

1) $\hat{Q} = 1 \quad [H, 1] = 0$ for real numbers.

$$\frac{d(1)}{dt} = 0$$

So $\frac{d}{dt}\langle Q \rangle = 1$

this mean the expectation of a operator constant over time, the rate it change with time is of course zero.

2) $\hat{Q} = H$

$\hat{Q} = H \quad [H, H] = 0$ as operator is commute to itself.

$$\frac{d}{dt}\langle H \rangle = \left\langle \frac{d}{dt} H \right\rangle$$

This means average of total energy is change rate with time. and when there is no explicit time dependence, we could see the energy is conserved.

3) $\hat{Q} = x \quad [H, x] = \frac{-i\hbar\hat{p}}{m}$

$$\frac{d}{dt}\langle x \rangle = \frac{1}{m}\langle \hat{p} \rangle$$

The rate of change of expectation value of position is proportional to $\frac{\langle \hat{p} \rangle}{m}$ which would say like something v in classical.

4) $\hat{Q} = P \quad [H, P] = -i\hbar\frac{\partial V}{\partial x}$

$$\frac{d}{dt}\langle P \rangle = -\left\langle \frac{\partial V}{\partial x} \right\rangle$$

The rate of change of expectation of momentum. This is like force in classical with $F = -\frac{dV}{dx}$.

Problem 16

~~a)~~ $\cancel{x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)}$

$\cancel{V(x) = \frac{1}{2}mx^2 = \frac{1}{2}m(a_+ + a_-)^2 = \frac{1}{4}m(a^{+2} + a^+a^- + a^-a^+ + a^{-2})}$

$\cancel{\langle V(x)\rangle = \int \psi^*(x)\, V(x)\, \psi(x)\,dx}$

$\cancel{= \int \psi^* \frac{1}{2}}$

a) $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a^+ + a^-)$

$V(x) = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 \frac{\hbar}{2m\omega}(a^+ + a^-)^2$

$= \frac{\hbar\omega}{4}(a^{+2} + a^+a^- + a^-a^+ + a^{-2})$

$\langle V(x)\rangle = \int \psi_n(x)\, V(x)\, \psi(x)\, dx$

$= \frac{\hbar\omega}{4}\int \psi_n^* (a^{+2} + a^+a^- + a^-a^+ + a^{-2})\,\psi_n\, dx$

① $\psi_n^* a_+^2 \psi_n = \psi_n a_+(a_+\psi_n) = \psi_n a_+ \psi_{n+1}\sqrt{n+1} = 0$

② $\psi^* a_-^2 \psi_n = 0$ same as above

③ $\psi^* a_+ a_- \psi_n = n$ and also $\psi^* a_- a_+ \psi_n = n+1$

$\cancel{a_-\psi_n = \sqrt{n}\,\psi_{n-1}}$

$\cancel{a_+\psi_n / \sqrt{n} = 1}$

$\langle V(x)\rangle = \frac{\hbar\omega}{4}\cdot(2n+1)\,dx$

$= \frac{\hbar\omega}{2}(n+\frac{1}{2})$

b) $\langle T(x)\rangle = \int \psi_n^*(x)\, T(x)\, \psi_n^*(x)\, dx$ $\quad T(x) = \frac{-m\hbar\omega}{4m}(a_- - a_+)^2$

similar ~~to~~ befor containing $\quad = -\frac{\hbar\omega}{4}(a_- - a_+)^2$

a_+a_- and a_-a_+ except they are negative

$\langle T(x)\rangle = -\frac{\hbar\omega}{4}\cdot -(2n+1) = \frac{\hbar\omega}{2}(n+\frac{1}{2})$

c) It correspond to half of E_n which is what harmonic periodic change betwen T and V.

Problem 17.

a) ψ_0:

$\langle x \rangle = \int \psi_0^*(x) \cdot \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-) \psi_0(x)\, dx = 0$

$\psi_0 \; a_+ \; \psi_0(x) = \psi_0 \sqrt{n+1}\, \psi_1 = 0$

$\psi_0 \; a_- \; \psi_0(x) = 0$

similarly

$\langle p \rangle = 0$

$\langle x^2 \rangle = \int \psi_0 \frac{\hbar}{2m\omega} (a_+ + a_-)^2 \psi_0 \, dx$ $\quad 2n+1 \quad n=0 \quad 1$

$= \frac{\hbar\omega}{2}$ $\quad$, remember from 16, apply $n=0$

$\langle p^2 \rangle = \int \psi_0 \frac{\hbar}{2m\omega} (a_- - a_+)^2 \psi_0 \, dx$

$= \frac{m\hbar\omega}{2}$

ψ_1 is just

$\langle x \rangle = 0$ $\quad \langle p \rangle = 0$ $\quad$ just as previous how a_+ a_-

$\langle x^2 \rangle = \frac{3}{2}\hbar\omega$ $\quad$ as this just add $\frac{1}{2}$ than n for n's increase.

$\langle p^2 \rangle = \frac{3}{2} m\hbar\omega$

b) $\langle T(x) \rangle$ from previous 16, we know $\frac{1}{2}(n+1)\hbar\omega$

" $\langle V(x) \rangle$ from previous 16, $\frac{1}{2}\hbar\omega(n+1)$

$\begin{cases} \langle T(x) \rangle \psi_0 = \frac{1}{2}\hbar\omega \\ \langle V(x) \rangle \psi_0 = \frac{1}{2}\hbar\omega \end{cases}$ $\quad$ $\begin{cases} \langle T(x) \rangle \psi_1 = \frac{3}{2}\hbar\omega \\ \langle V(x) \rangle \psi_1 = \frac{3}{2}\hbar\omega \end{cases}$

Yes this is the same.

Problem 18

a) $\psi_1(x) = a_+ \psi_0(x)$ $\qquad \psi_0 = \left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{mwx^2}{2\hbar}}$

$\psi_2(x) = \frac{1}{\sqrt{2}} a_+ \psi_1(x) = \frac{1}{\sqrt{2}} a_+^2 \psi_0(x)$

b) $\int_{-\infty}^{\infty} \psi_0^*(x) \cdot \psi_1(x)\, dx$

$= \int_{-\infty}^{\infty} \left(\left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{mwx^2}{2\hbar}} \right) \left(\left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{2}\, x\, e^{-\frac{mwx^2}{2\hbar}} \right) dx$

$= \int_{-\infty}^{\infty} \left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{2}} \underbrace{\sqrt{2}\, x\, e^{-\frac{mwx^2}{\hbar}}}_{odd} dx$

So

$= 0.$

$\int \psi_0 \cdot \psi_2 dx$

$= \int \left(\left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{4}} \cdot e^{-\frac{mwx^2}{2\hbar}} \right) \left(\left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{4}} (2x^2 - 1)\, e^{-\frac{mwx^2}{2\hbar}} \right) dx$

$= \underbrace{\int \left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{2}} (2x^2 - 1)\, e^{-\frac{mwx^2}{\hbar}}}_{odd} dx$

$= 0.$

Problem 19.

① $\langle x \rangle = 0 \qquad \langle p_x \rangle = 0$

$x = \sqrt{\frac{\hbar}{2mw}} (a_+ + a_-)$

$P = -i \sqrt{\frac{m\hbar w}{2}} (a_+ - a_-)$

$\langle x \rangle_n = \langle n | x | n \rangle = \langle n | \sqrt{\frac{\hbar}{2mw}} (a_+ + a_-) | n \rangle$

$a_- |n\rangle = \sqrt{n} |n-1\rangle$

$\langle x \rangle_n = \sqrt{\frac{\hbar}{2mw}} (\underbrace{\langle n | a_+ | n \rangle}_{0} + \underbrace{\langle n | a_- | n \rangle}_{0})$

$= 0$

$\langle p_x \rangle = \langle n | P | n \rangle = \langle n | -i \sqrt{\frac{m\hbar w}{2}} (a_+ - a_-) | n \rangle$

$= -i \sqrt{\frac{m\hbar w}{2}} (\langle n | a_+ | n \rangle - \langle n | a_- | n \rangle)$

$\langle p_x \rangle = 0$

$\langle x^2 \rangle = \frac{\hbar}{2mw} (2n+1)$

$\langle x^2 \rangle = \langle n | x^2 | n \rangle = \langle n | \left(\sqrt{\frac{\hbar}{2mw}} (a_+ + a_-) \right)^2 | n \rangle$

$= \frac{\hbar}{2mw} \langle n | (a_+ + a_-)^2 | n \rangle = \frac{\hbar}{2mw} \langle n | a_+^2 + a_-^2 + a_+ a_- + a_- a_+ | n \rangle$

$= \frac{\hbar}{2mw} (2n+1)$

$\langle p_x^2 \rangle = \frac{m\hbar w}{2} \langle n | (-i)(a_+ - a_-)^2 | n \rangle$

$= m \hbar w (2n+1)$

② $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{(2n+1)\hbar}{2mw}}$

$\Delta p_x = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2} = \sqrt{m\hbar w (2n+1)}$

$\Delta x \Delta p_x = \frac{(2n+1)\hbar}{2} \geq \frac{\hbar}{2}$ So true!

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