$$y = f(\mathbf{x}) = f(x_1, \cdots, x_N)$$

$$\mathbf{x}^* = [x_1^2, \cdots, x_N^*]^T \in \mathbb{R}^N$$

$f(\mathbf{x}^*)$

$f(\mathbf{x})$

$$-f(\mathbf{x})$$

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} f(\mathbf{x}),$$

$$f(\mathbf{x}^*) = \min_{\mathbf{x}} f(\mathbf{x})$$

X

$\mathbb{R}^N$

$$y = f(\mathbf{x}, \mathbf{a})$$

$$\mathcal{D} = \{(\mathbf{x}_n, y_n), \ \ n = 1, \cdots, N\}$$

$$\mathbf{a} = [a_1, \cdots, a_M]^T$$

$$y = f(x) = ax + b$$

a

b

$$o(\mathbf{a}) = \sum_{n=1}^{N} [y_n - f((\mathbf{x}_n, \mathbf{a}))]^2$$

a

$$o(\mathbf{a}) = L(\mathbf{a}|\mathcal{D}) = p(\mathcal{D}|\mathbf{a}) = \prod_{n=1}^{N} p(y_n|\mathbf{x}_n, \mathbf{a})$$

$\mathbf{a}^*$

$$\mathbf{x}^*$$

$$\mathbf{g}_f = \bigtriangledown_{\mathbf{x}} f(\mathbf{x}) = \frac{df(\mathbf{x})}{d\mathbf{x}} = \left[\frac{\partial f(\mathbf{x})}{\partial x_1}, \cdots, \frac{\partial f(\mathbf{x})}{\partial x_N} \right]^T = \mathbf{0}$$

$$\mathbf{g}_f = \bigtriangledown_{\mathbf{x}} f(\mathbf{x}) = d\, f(\mathbf{x})/d\mathbf{x}$$

$$\mathbf{x} = [\mathbf{x}_1, \cdots, \mathbf{x}_N]^T$$

$$\mathbf{g}_f = \mathbf{0}$$

$$\mathbf{H}_f = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_N \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_N^2} \end{bmatrix}_{\mathbf{x}=\mathbf{x}^*}$$

$\mathbf{H}_f > 0$

$$\mathbf{H}_f < 0$$

$$\mathbf{H}_f$$

$$f_1(x, y) = x^2 + y^2$$

$$f_2(x, y) = -x^2 - y^2$$

$$f_3(x, y) = x^2 - y^2$$

$$\mathbf{g}_{f_1} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}, \quad \mathbf{g}_{f_2} = \begin{bmatrix} -2x \\ -2y \end{bmatrix}, \quad \mathbf{g}_{f_3} = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$

$$\mathbf{H}_{f_1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} > 0; \quad \mathbf{H}_{f_2} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} < 0; \quad \mathbf{H}_{f_3} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\mathbf{x}^* = [0,\ 0]^T$$

$$f_1(x, y) = f_2(x, y) = f_3(x, y) = 0$$

$$f_1(\mathbf{x}^*)$$

$f_2(\mathbf{x}^*)$

$$f_3(\mathbf{x}^*)$$

x

y

$$\mathbf{g}_f(\mathbf{x}) = \triangledown_{\mathbf{x}} f(\mathbf{x}) = \frac{df(\mathbf{x})}{d\mathbf{x}} = \left[\begin{array}{c} \partial f/\partial x_1 \\ \vdots \\ \partial f/\partial x_N \end{array} \right] = \mathbf{0}$$

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{J}_f^{-1}(\mathbf{x}_n)\,\mathbf{f}(\mathbf{x}_n)$$

$$\mathbf{g}_f(\mathbf{x}) = \mathbf{0}$$

$$\mathbf{J}_g(\mathbf{x}) = \mathbf{H}_f(\mathbf{x})$$

$$\mathbf{g}_f(\mathbf{x})$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{J}_g^{-1}(\mathbf{x}_n)\,\mathbf{g}_f(\mathbf{x}_n) = \mathbf{x}_n - \mathbf{H}_g^{-1}(\mathbf{x}_n)\,\mathbf{g}_f(\mathbf{x}_n)$$

$$\mathbf{g}_f$$

$$\mathbf{f}(\mathbf{x})$$

$$\mathbf{J}_g = \mathbf{H}_f$$

$$\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}), \cdots, g_N(\mathbf{x})]^T = \mathbf{0}$$

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{g}^T(\mathbf{x})\mathbf{g}(\mathbf{x}) = \frac{1}{2}||\mathbf{g}(\mathbf{x})||^2 = \frac{1}{2}\sum_{i=1}^{N}|g_i(\mathbf{x})|^2$$

$$\frac{\partial f(\mathbf{x})}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{2}\sum_{i=1}^{N}(g_i(\mathbf{x}))^2\right) = \sum_{i=1}^{N} g_i(\mathbf{x})\frac{\partial g_i(\mathbf{x})}{\partial x_j} = 0, \qquad (j = 1, \cdots, N)$$

N

$$f(\mathbf{x}) = \mathbf{g}^T \mathbf{g}$$

$$\bigtriangledown_{\mathbf{x}} f(\mathbf{x}) = \frac{d}{d\mathbf{x}} \left[\frac{1}{2} \mathbf{g}^T(\mathbf{x}) \mathbf{g}(\mathbf{x}) \right] = \frac{d}{d\mathbf{x}} \mathbf{g}(\mathbf{x}) \;\; \mathbf{g}(\mathbf{x}) = \mathbf{J}_{\mathbf{g}}(\mathbf{x})\, \mathbf{g}(\mathbf{x}) = \mathbf{0}$$

$$\mathbf{J_g(x)}$$

$$\mathbf{g}(\mathbf{x})$$

$$\mathbf{g}(\mathbf{x}) = \mathbf{0}$$

$$f(x)$$

$$f(x) = 0$$

$$x_0 < x_1 < x_2$$

$$f(x_0) > f(x_1) < f(x_2)$$

x_0

x_2

$$x_3$$

x_1

$$f(x_3) = f_{3a} > f(x_1)$$

$$x_3 - x_0 = a + c$$

$$x_0 < x_1 < x_3$$

$$f(x_3) = f_{3b} < f(x_1)$$

$$x_2 - x_1 = b$$

$$x_1 < x_3 < x_2$$

$$x_2 - x_0 = a + b$$

$$a + c$$

$a + c = b,$

$$c = b - a$$

$$\frac{a+b}{b} = \frac{a+c}{a} = \frac{b}{a}$$

$$\frac{a+b}{b} = \frac{b}{b-c} = \frac{b}{a}$$

$$a^2 + ab - b^2 = 0$$

$$a_{1,2} = \frac{-1 \pm \sqrt{5}}{2}\ b = \begin{cases} 0.618\, b \\ -1.618\, b \end{cases}$$

$$\frac{a}{b} = 0.618, \qquad \frac{b}{a+b} = \frac{b}{0.618b+b} = \frac{1}{1.618} = 0.618$$

$$|x_2 - x_0|$$

$$10^{-9}$$

$$|x_2 - x_0| < \epsilon(|x_1| + |x_3|)$$

ϵ

$$|x_1| + |x_3|$$

$$f(x_0) < f(x_1) < f(x_2)$$

$$f(x_0) > f(x_1) > f(x_2)$$

$$x_0^{new} = x_1^{old}, \quad x_1^{new} = x_2^{old}, \quad x_2^{new} = x_2^{old} + c(x_2^{old} - x_1^{old})$$

$c > 1$

$$c = 1.618$$

$$f(x_2) > f(x_1)$$

$$d = x_2 - x_1$$

$$d \propto c^n$$

$$a < b < c$$

$$f(a) > f(b) < f(c)$$

$q(x)$

$$q(x) = f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)} + f(b)\frac{(x-c)(x-a)}{(b-c)(b-a)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)}$$

$q(x_{min})$

$$q'(x) = f(a)\frac{(x-b)+(x-c)}{(a-b)(a-c)} + f(b)\frac{(x-c)+(x-a)}{(b-c)(b-a)} + f(c)\frac{(x-a)+(x-b)}{(c-a)(c-b)} = 0$$

$$(a-b)(b-c)(c-a)$$

$$f(a)(c-b)(2x-b-c)+f(b)(a-c)(2x-c-a)+f(c)(b-a)(2x-a-b)$$

$=$

$$2x[f(a)(c-b)+f(b)(a-c)+f(c)(b-a)]-[f(a)(c^2-b^2)+f(b)(a^2-c^2)+f(c)(b^2-a^2)]=0$$

x_{min}

$$\frac{1}{2}\,\frac{f(a)(c^2-b^2)+f(b)(a^2-c^2)+f(c)(b^2-a^2)}{f(a)(c-b)+f(b)(a-c)+f(c)(b-a)}$$

$$b + \frac{1}{2} \frac{f(a)(c-b)(c+b-2b) + f(b)(a-c)(a+c-2b) + f(c)(b-a)(b+a-2b)}{f(a)(c-b) + f(b)(a-c) + f(c)(b-a)}$$

$$b+\frac{1}{2}\frac{f(a)(c-b)^2+f(b)(a-c)(a+c-2b)-f(c)(b-a)^2}{f(a)(c-b)+f(b)(a-c)+f(c)(b-a)}$$

$$b + \frac{1}{2}\frac{[f(a)-f(b)](c-b)^2-[f(c)-f(b)](b-a)^2}{[f(a)-f(b)](c-b)+[f(c)-f(b)](b-a)}$$

$$f(x_{min})$$

$f(b)$

x_{min}

$$\{a, x_{min}, b\}$$

$x_{min} < b$

$$\{b, x_{min}, c\}$$

$$x_{min} > b$$

$$[f(a) - f(b)](c - b)^2 = [f(c) - f(b)](b - a)^2$$

$$x_{min} = b$$

$f(\mathbf{x}$

$N + 1$

$$\mathbf{x}_0, \cdots, \mathbf{x}_N$$

$$\mathbf{x}_i = \mathbf{x}_0;\ (i = 1, \cdots, N)$$

$$S = \left\{ \sum_{i=0}^{N} c_i \mathbf{x}_i \;\middle|\; \sum_{i=0}^{N} c_i = 1, \quad c_i \geq 0 \right\}$$

$\mathbf{x}_0$

$$\{\mathbf{e}_1, \cdots, \mathbf{e}_N\}$$

$$\mathbf{x}_i = \mathbf{x}_0 + c\mathbf{e}_i$$

$$\alpha > 0$$

$$\alpha = 1$$

$$0 < \beta < 1$$

$$\beta = 1/2$$

$$\gamma > 1$$

$$\gamma = 2$$

$0 < \delta < 1$

$$\delta = 1/2$$

$$f(\mathbf{x}_0) \leq f(\mathbf{x}_1) \leq \cdots \leq f(\mathbf{x}_N)$$

$$\mathbf{x}_0 = \mathbf{x}_l$$

$$\mathbf{x}_N = \mathbf{x}_h$$

$$\mathbf{x}_{N-1} = \mathbf{x}_s$$

$$\mathbf{c} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i$$

$$\mathbf{x}_r = \mathbf{c} + \alpha(\mathbf{c} - \mathbf{x}_h)$$

$$f(\mathbf{x}_r)$$

$$f(\mathbf{x}_s < f(\mathbf{x}_r) < f(\mathbf{x}_l)$$

$\mathbf{x}_h$

$$\mathbf{x}_r$$

$$f(\mathbf{x}_r) < f(\mathbf{x}_l)$$

$$\mathbf{x}_e = \mathbf{c} + \gamma(\mathbf{x}_r - \mathbf{c})$$

$\mathbf{x}_e$

$$f(\mathbf{x}_r) > f(\mathbf{x}_s)$$

$$bfx_r - \mathbf{c}$$

$$\mathbf{x}_c = \mathbf{c} + \beta(\mathbf{x}_h - \mathbf{c})$$

$$f(\mathbf{x}_c) < f(\mathbf{x}_h)$$

$$\mathbf{x}_c$$

$$f(x)$$

$$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + \cdots$$

$\approx$

$$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 = q(x)$$

$$f(x) = q(t)$$

$$f(x) = q(x)$$

$q'(x)$

$$\frac{d}{dx}q(x) = \frac{d}{dx}\left[f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2\right]$$

$$f'(x_0) + f''(x_0)(x - x_0) = 0$$

$$x^* = x = x_0 - \frac{f'(x_0)}{f''(x_0)} = x_0 + \Delta x_0$$

$$\Delta x_0 = -f'(x_0)/f''(x_0)$$

x^*

$f(x^*)$

$$f''(x^*) > 0$$

$$f''(x^*) < 0$$

$$f(x) \neq q(x)$$

$$x_{n+1} = x_n + \Delta x_n = x_n - \frac{f'(x_n)}{f''(x_n)}$$

x_n

$$q(x_n)$$

x_{n+1}

$$q(x_{n+1})$$

$n \to \infty$

$$x_{n+1} = x_n - f'(x_n)/f''(x_n)$$

$$f'(x) = 0$$

$$q(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

$$q(x) = \frac{a}{2}(x - b)^2 + c$$

$$\left\{ \begin{array}{l} a = f''(x_0) \\ b = x_0 - f'(x_0)/f''(x_0) \\ c = f(x_0) - f'^2(x_0)/2f''(x_0) \end{array} \right.$$

c

$$x = b$$

$$q(x) = q(b) = c$$

$$f(x) = 2x^3 - 4x^2 + x, \qquad f'(x) = 6x^2 - 8x + 1, \qquad f''(x) = 12x - 8$$

$$x^* = 1.333$$

$$f(x^*) = -1.037$$

$$x_0 = 3$$

$$f(x_0) = f(3) = 21, \qquad f'(x_0) = f'(3) = 31, \qquad f''(x_0) = f''(3) = 28$$

$$f''(x_0) > 0$$

$$x(x)$$

$$\left\{\begin{array}{l} a = f''(3) = 28 \\ b = 3 - f'(3)/f''(3) = 1.8929 \\ c = f(3) - f'^2(3)/2f''(3) = 3.8393 \end{array}\right.$$

$$x_1 = b = 1.8929$$

$$f(x_1) = f(1.8929) = 1.1251$$

$$f(x_0) = f(3) = 21$$

$$x = b = x_0 - f'(x_0)/f''(x_0)$$

$$x_1 = x_0 - f'(x_0)/f''(x_0)$$

$$f(\mathbf{x}) = f(x_1, \cdots, x_N)$$

$$q(\mathbf{x})$$

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \mathbf{g}_0^T(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T \mathbf{H}_0 (\mathbf{x} - \mathbf{x}_0) = q(\mathbf{x})$$

$\mathbf{g}_0$

$$\mathbf{H}_0$$

$\mathbf{g}_0$

$$\mathbf{g}_f(\mathbf{x}_0) = \frac{d}{d\mathbf{x}} f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f(\mathbf{x}_0)}{\partial x_1} \\ \vdots \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_N} \end{bmatrix},$$

$\mathbf{H}_0$

$$\mathbf{H}_f(\mathbf{x}_0) = \frac{d}{d\mathbf{x}}\mathbf{g}(\mathbf{x}_0) = \frac{d^2}{d\mathbf{x}^2}f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_1 \partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_N \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_N^2} \end{bmatrix}$$

$$f(\mathbf{x}) = q(\mathbf{x})$$

$$\frac{d}{d\mathbf{x}}\left[f(\mathbf{x}_0)+\mathbf{g}_0^T(\mathbf{x}-\mathbf{x}_0)+\frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\mathbf{H}_0\left(\mathbf{x}-\mathbf{x}_0\right)\right]=\mathbf{g}_0+\mathbf{H}_0\left(\mathbf{x}-\mathbf{x}_0\right)=\mathbf{0}$$

$$\mathbf{x}^* = \mathbf{x} = \mathbf{x}_0 - \mathbf{H}(\mathbf{x}_0)^{-1}\mathbf{g}(\mathbf{x}_0)$$

$$f''(\mathbf{x}^*)$$

$$\mathbf{H}^* = \mathbf{H}(\mathbf{x}^*)$$

$$\mathbf{H}^* > 0$$

$$\mathbf{H}^* < 0$$

H*

$$f(\mathbf{x}) \neq q(\mathbf{x})$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta \mathbf{x}_n = \mathbf{x}_n - \mathbf{H}_n^{-1} \mathbf{g}_n = \mathbf{x}_n + \mathbf{d}_n$$

$$\mathbf{g}_n = \mathbf{g}(\mathbf{x}_n)$$

$$\mathbf{H}_n = \mathbf{H}(\mathbf{x}_n)$$

$$\mathbf{d}_n = \Delta \mathbf{x}_n = -\mathbf{H}_n^{-1} \mathbf{g}_n$$

$$\mathbf{g}_n = f'(x_n)$$

$$\mathbf{H}_n = f''(x_n)$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{H}_n^{-1}\,\mathbf{g}_n$$

f

$\delta > 0$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \delta_n \, \Delta\mathbf{x}_n = \mathbf{x}_n - \delta_n \, \mathbf{H}_n^{-1}\mathbf{g}_n$$

$$q(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{b})^T \mathbf{A}(\mathbf{x} - \mathbf{b}) + c$$

$$
\begin{cases}
\mathbf{A} = \mathbf{H}_0 \\
\mathbf{b} = \mathbf{x}_0 - \mathbf{H}^{-1}\mathbf{g}_0 \\
c = f(\mathbf{x}_0) - \mathbf{g}_0^T \mathbf{H}_0^{-1} \mathbf{g}_\mathbf{0}/2
\end{cases}
$$

$$\mathbf{g}_0 = \mathbf{g}_f(\mathbf{x}_0) = \mathbf{A}(\mathbf{x}_0 - \mathbf{b}) = \mathbf{A}\mathbf{e}_0, \qquad \mathbf{H}_0 = \mathbf{H}_f(\mathbf{x}_0) = \mathbf{A}$$

$$\mathbf{e}_0 = \mathbf{x}_0 - \mathbf{b}$$

$$||\mathbf{e}_0||$$

$$\mathbf{x} = \mathbf{b}$$

$$q(\mathbf{b}) = c$$

H

$$q(x,y) = \frac{1}{2}[x,\ y]\left[\begin{array}{cc} a & b/2 \\ b/2 & c \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \frac{1}{2}(ax^2 + bxy + cy^2)$$

$$\mathbf{g} = \begin{bmatrix} ax + by/2 \\ bx/2 + cy \end{bmatrix}, \qquad \mathbf{H} = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \qquad \det \mathbf{H} = ac - b^2/4$$

$$a = 1$$

$$c = 2$$

$$b = -2$$

$$\lambda_1 = 2.618$$

$$\lambda_2 = 0.382$$

$$\det \mathbf{H} = \lambda_1 \lambda_2 = 1$$

$\mathbf{H} > 0$

$$f(0,\ 0) = 0$$

$$b = 0$$

$$\lambda_1 = 2$$

$$\lambda_2 = 1$$

$$\det \mathbf{H} = \lambda_1 \lambda_2 = 2$$

$$b = 2$$

$$b = 4$$

$$\lambda_1 = 3.562$$

$$\lambda_2 = -0.562$$

$$\det \mathbf{H} = \lambda_1 \lambda_2 = -2$$

$\delta > 1$

δ

$\delta < 1$

$\mathbf{x}_n$

$\mathbf{g}_n$

$$\mathbf{x}_{n+1}$$

$$N = 3$$

$$\left\{\begin{array}{l} f_1(x_1,\, x_2,\, x_3) = 3x_1 - (x_2x_3)^2 - 3/2 \\ f_2(x_1,\, x_2,\, x_3) = 4x_1^2 - 625\, x_2^2 + 2x_2 - 1 \\ f_3(x_1,\, x_2,\, x_3) = exp(-x_1x_2) + 20x_3 + 9 \end{array}\right.$$

$$(x_1 = 0.5,\ x_2 = 0,\ x_3 = -0.5)$$

$$o(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\mathbf{f}(\mathbf{x})$$

$$\mathbf{x}_0 = \mathbf{0}$$

n	$\mathbf{x} = [x_1,\, x_2,\, x_3]$	$\|\|\mathbf{f}(\mathbf{x})\|\|$
0	$0.000000,\quad 0.000000,\quad 0.000000$	$1.016120e+01$
1	$0.500000,\quad 0.500000,\quad -0.500000$	$1.552502e+02$
2	$0.499550,\quad 0.250800,\quad -0.493801$	$3.881300e+01$
3	$0.500096,\quad 0.126206,\quad -0.496852$	$9.702208e+00$
4	$0.500025,\quad 0.063914,\quad -0.498405$	$2.425198e+00$
5	$0.500010,\quad 0.032778,\quad -0.499181$	$6.059054e-01$
6	$0.500005,\quad 0.017231,\quad -0.499570$	$1.510777e-01$
7	$0.500003,\quad 0.009498,\quad -0.499763$	$3.737330e-02$
8	$0.500002,\quad 0.005712,\quad -0.499857$	$8.959365e-03$
9	$0.500001,\quad 0.003968,\quad -0.499901$	$1.900145e-03$
10	$0.500001,\quad 0.003326,\quad -0.499917$	$2.577603e-04$
11	$0.500001,\quad 0.003206,\quad -0.499920$	$8.932714e-06$
12	$0.500001,\quad 0.003202,\quad -0.499920$	$1.238536e-08$
13	$0.500001,\quad 0.003202,\quad -0.499920$	$2.371437e-14$

$$o(\mathbf{x}) = ||\mathbf{f}(\mathbf{x}^*)||^2 \approx 10^{-28}$$

$$\mathbf{x}^* = \begin{bmatrix} 0.5000008539707297 \\ 0.0032017070323056 \\ -0.4999200212218281 \end{bmatrix}$$

$\mathbf{H}_f(\mathbf{x})$

$$\mathbf{H}_f^{-1}$$

$$x_1 = x_0 + \Delta x_0 = x_0 - \delta f'(x_0),$$

$$\Delta x_0 = -\delta f'(x_0)$$

$$f'(x_0)$$

$$f(x_1)$$

$$f(x_1) \approx f(x_0) + f'(x_0)\Delta x_0 = f(x_0) - |f'(x_0)|^2\delta < f(x_0)$$

$$x_{n+1} = x_n - \delta_n \; f'(x_n)$$

$$f'(x^*) = 0$$

$$\mathbf{g}(\mathbf{x}) = df(\mathbf{x})/d\mathbf{x}$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta\mathbf{x}_n$$

$$\Delta \mathbf{x} = -\delta \mathbf{g} \ (\delta > 0)$$

$$f(\mathbf{x}_1) = f(\mathbf{x}_0 + \Delta\mathbf{x}) \approx f(\mathbf{x}_0) + \mathbf{g}_0^T \Delta\mathbf{x} = f(\mathbf{x}_0) - \delta \mathbf{g}_0^T \mathbf{g}_0 = f(\mathbf{x}_0) - \delta ||\mathbf{g}||^2 < f(\mathbf{x}_0)$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \delta_n\,\mathbf{g}_n = \left(\mathbf{x}_{n-1} - \delta_{n-1}\,\mathbf{g}_{n-1}\right) - \delta_n\,\mathbf{g}_n = \cdots = \mathbf{x}_0 - \sum_{i=0}^{n} \delta_n\,\mathbf{g}_i$$

$$\mathbf{g} = \mathbf{0}$$

$$\mathbf{d}_n = -\mathbf{g}_n$$

$$\mathbf{d}_n = -\mathbf{H}_n^{-1}\mathbf{g}_n$$

$\mathbf{H}_n$

$$\mathbf{H} = \mathbf{I}$$

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c = [x_1 \; x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [b_1 \; b_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c$$

A

$$a_{12} = a_{21}$$

$$\mathbf{A} \geq 0$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad c = 0$$

$$f(x_1, x_2) = \frac{1}{2}[x_1 \ x_2] \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}(2x_1^2 + 2x_1x_2 + x_2^2)$$

$$f(x_1,\, x_2) = 0$$

$$x = y = 0$$

$$\mathbf{x}_0 = [1,\ 2]^T$$

$$\mathbf{g}_0 = [4,\ 3]^T$$

d

$$\mathbf{g} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}, \quad \mathbf{H} = \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{H}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{d}_0 = -\mathbf{H}^{-1}\mathbf{g}_0 = -[1,\ 2]^T$$

$$\mathbf{x}_1 = \mathbf{x}_0 - \mathbf{H}^{-1}\mathbf{g}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{d}_0 = -\mathbf{g}_0 = -[4,\ 3]^T$$

$$\mathbf{x}_1 = \mathbf{x}_0 - \delta \mathbf{g}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \delta \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 - \delta 4 \\ 2 - \delta 3 \end{bmatrix}$$

$\mathbf{x}_1$

g

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \delta_n \mathbf{d}_n$$

$\mathbf{d}_n$

$$f(\mathbf{x}_{n+1})$$

$f(\mathbf{x}_n)$

θ

$\pi/2$

$$\theta = \cos^{-1} \frac{\mathbf{d}_n^T \mathbf{g}_n}{||\mathbf{d}_n|| \; ||\mathbf{g}_n||} > \frac{\pi}{2} \qquad i.e., \qquad \mathbf{d}_n^T \mathbf{g}_n < 0$$

$$\mathbf{d}_n = -\mathbf{H}_n^{-1}\mathbf{g}_n$$

$$\mathbf{d}_n^T \mathbf{g}_n = -\mathbf{g}_n^T \mathbf{H}_n^{-1} \mathbf{g}_n < 0, \qquad \mathbf{d}_n^T \mathbf{g}_n = -\mathbf{g}_n^T \mathbf{g}_n = -||\mathbf{g}_n||^2 < 0$$

δ_n

$$f(\mathbf{x}_{n+1}) = f(\mathbf{x}_n + \delta_n \mathbf{d}_n)$$

$$\frac{d}{d\delta_n} f(\mathbf{x}_{n+1}) = \frac{d}{d\delta_n} f(\mathbf{x}_n + \delta_n \mathbf{d}_n) = \left(\frac{d\, f(\mathbf{x}_{n+1})}{d\mathbf{x}} \right)^T \frac{d(\mathbf{x}_n + \delta_n \mathbf{d}_n)}{d\delta_n} = \mathbf{g}_{n+1}^T \mathbf{d}_n = 0$$

$$\mathbf{g}_{n+1} = f'(\mathbf{x}_{n+1})$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \delta \mathbf{d}_n$$

$$\mathbf{g}_{n+1}$$

$$f(\mathbf{x}_{n+1}) = f(\mathbf{x}_n + \delta \mathbf{d}_n)$$

$\delta = 0$

$$f(\mathbf{x}_{n+1}) = f(\mathbf{x}_n + \delta \mathbf{d}_n) \approx [f(\mathbf{x}_n + \delta \mathbf{d}_n)]_{\delta=0} + \delta \left[\frac{d}{d\delta} f(\mathbf{x}_n + \delta \mathbf{d}_n)\right]_{\delta=0} + \frac{\delta^2}{2} \left[\frac{d^2}{d\delta^2} f(\mathbf{x}_n + \delta \mathbf{d}_n)\right]_{\delta=0}$$

$$\left[f(\mathbf{x}_n + \delta \mathbf{d}_n) \right]_{\delta=0}$$

$$f(\mathbf{x}_n)$$

$$\left[\frac{d}{d\delta}f(\mathbf{x}_n + \delta\mathbf{d}_n)\right]_{\delta=0}$$

$$\mathbf{g}(\mathbf{x}_n + \delta\mathbf{d}_n)^T\mathbf{d}_n\Big|_{\delta=0} = \mathbf{g}_n^T\mathbf{d}_n$$

$$\left[\frac{d^2}{d\delta^2} f(\mathbf{x}_n + \delta \mathbf{d}_n) \right]_{\delta=0}$$

$$\left[\frac{d}{d\delta}\mathbf{g}(\mathbf{x}_n+\delta\mathbf{d}_n)^T\right]_{\delta=0}\mathbf{d}_n=\left[\frac{d}{d\mathbf{x}}\mathbf{g}(\mathbf{x})\ \frac{d}{d\delta}(\mathbf{x}_n+\delta\mathbf{d}_n)\right]_{\delta=0}^T\mathbf{d}_n$$

$$(\mathbf{H}_n \mathbf{d}_n)^T \mathbf{d}_n = \mathbf{d}_n^T \mathbf{H}_n \mathbf{d}_n$$

$$\mathbf{H}_n = \mathbf{H}_n^T$$

$$f(\mathbf{x}_n + \delta\mathbf{d}_n) \approx f(\mathbf{x}_n) + \delta\, \mathbf{g}_n^T\, \mathbf{d}_n + \frac{\delta^2}{2}\, \mathbf{d}_n^T \mathbf{H}_n\, \mathbf{d}_n$$

$$f(\mathbf{x}_n + \delta\mathbf{d}_n)$$

$$\frac{d}{d\delta} f(\mathbf{x}_n + \delta \mathbf{d}_n) \approx \frac{d}{d\delta} \left(f(\mathbf{x}_n) + \delta \mathbf{g}_n^T \mathbf{d}_n + \frac{\delta^2}{2} \mathbf{d}_n^T \mathbf{H}_n \mathbf{d}_n \right) = \mathbf{g}_n^T \mathbf{d}_n + \delta \, \mathbf{d}_n^T \mathbf{H}_n \mathbf{d}_n = 0$$

$$\delta_n = -\frac{\mathbf{g}_n^T \mathbf{d}_n}{\mathbf{d}_n^T \mathbf{H}_n \mathbf{d}_n}$$

$$\delta_n = -\frac{\mathbf{g}_n^T \mathbf{d}_n}{\mathbf{d}_n^T \mathbf{H}_n \mathbf{d}_n} = \frac{\mathbf{g}_n^T (\mathbf{H}_n^{-1} \mathbf{g}_n)}{(\mathbf{H}_n^{-1} \mathbf{g}_n)^T \mathbf{H}_n (\mathbf{H}_n^{-1} \mathbf{g}_n)} = \frac{\mathbf{g}_n^T (\mathbf{H}_n^{-1} \mathbf{g}_n)}{(\mathbf{H}_n^{-1} \mathbf{g}_n)^T \mathbf{g}_n} = 1$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \delta_n \mathbf{d}_n = x_n - \delta_n \mathbf{H}_n^{-1} \mathbf{g}_n = \mathbf{x}_n - \mathbf{H}_n^{-1} \mathbf{g}_n$$

$$\mathbf{g}_{n+1}^T \mathbf{d}_n = -\mathbf{g}_{n+1}^T \mathbf{g}_n = 0,$$

$$\mathbf{g}_{n+1} \perp \mathbf{g}_n$$

$$\mathbf{d}_{n+1} = -\mathbf{g}_{n+1}$$

$$\delta_n = -\frac{\mathbf{g}_n^T \mathbf{d}_n}{\mathbf{d}_n^T \mathbf{H}_n \mathbf{d}_n} = \frac{\mathbf{g}_n^T \mathbf{g}_n}{\mathbf{g}_n^T \mathbf{H}_n \mathbf{g}_n} = \frac{||\mathbf{g}_n||^2}{\mathbf{g}_n^T \mathbf{H}_n \mathbf{g}_n}$$

$$f''(\delta)\Big|_{\delta=0}$$

$$\delta = \sigma$$

σ

$$\left[\frac{d^2}{d\delta^2} f(\mathbf{x} + \delta \mathbf{d})\right]_{\delta=0}$$

$$\left[\frac{d}{d\delta}f'(\mathbf{x}+\delta\mathbf{d})\right]_{\delta=0} = \lim_{\sigma\to 0}\frac{f'(\mathbf{x}+\sigma\mathbf{d})-f'(\mathbf{x})}{\sigma}$$

$$\frac{\mathbf{g}^T(\mathbf{x}+\sigma\mathbf{d})\,\mathbf{d}-\mathbf{g}^T(\mathbf{x})\,\mathbf{d}}{\sigma}=\frac{\mathbf{g}_\sigma^T\mathbf{d}-\mathbf{g}^T\mathbf{d}}{\sigma}$$

$$\mathbf{g} = \mathbf{g}(\mathbf{x})$$

$$\mathbf{g}_\sigma = \mathbf{g}(\mathbf{x} + \sigma \mathbf{d})$$

$$\mathbf{d}_n^T \mathbf{H}_n \mathbf{d}_n$$

$$\frac{d}{d\delta} f(\mathbf{x}_n + \delta \mathbf{d}_n) \approx \mathbf{g}_n^T\, \mathbf{d}_n + \frac{\delta}{\sigma} \left(\mathbf{g}_{\sigma n}^T \mathbf{d}_n - \mathbf{g}_n^T \mathbf{d}_n \right) = 0$$

$$\delta_n = -\frac{\sigma\, \mathbf{g}_n^T \mathbf{d}_n}{(\mathbf{g}_{\sigma n}^T \mathbf{d}_n - \mathbf{g}_n^T \mathbf{d}_n)} = -\frac{\sigma\, \mathbf{g}_n^T \mathbf{d}_n}{(\mathbf{g}_{\sigma n} - \mathbf{g}_n)^T \mathbf{d}_n}$$

$$\delta = ||\mathbf{g}_n||^2 / \mathbf{g}_n^T \mathbf{H}_n \mathbf{g}_n$$

$$\delta_n = -\frac{\sigma\, \mathbf{g}_n^T \mathbf{d}_n}{(\mathbf{g}_{\sigma n} - \mathbf{g}_n)^T \mathbf{d}_n} = -\frac{\sigma\, \mathbf{g}_n^T \mathbf{g}_n}{(\mathbf{g}_{\sigma n} - \mathbf{g}_n)^T \mathbf{g}_n} = \frac{\sigma\, ||\mathbf{g}_n||^2}{||\mathbf{g}_n||^2 - \mathbf{g}_{\sigma n}^T \mathbf{g}_n}$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \delta_n \mathbf{g}_n = \mathbf{x}_n + \frac{\sigma \left|\left|\mathbf{g}_n\right|\right|^2}{\mathbf{g}_{\sigma n}^T \mathbf{g}_n - \left|\left|\mathbf{g}_n\right|\right|^2} \, \mathbf{g}_n$$

$$\sigma = 10^{-6}$$

n	$\mathbf{x} = [x_1,\ x_2,\ x_3]$	$\|\|\mathbf{f}(\mathbf{x})\|\|$
0	$0.0000,\ \ 0.0000,\ \ 0.0000$	$1.032500e+02$
10	$0.4246,\ \ -0.0073,\ \ -0.5002$	$1.535939e-01$
20	$0.5015,\ \ 0.0064,\ \ -0.4998$	$2.448241e-05$
30	$0.5009,\ \ 0.0057,\ \ -0.4998$	$9.178209e-06$
40	$0.5006,\ \ 0.0052,\ \ -0.4998$	$4.17587e-06$
50	$0.5004,\ \ 0.0049,\ \ -0.4999$	$2.122594e-06$
60	$0.5003,\ \ 0.0047,\ \ -0.4999$	$1.182466e-06$
100	$0.5001,\ \ 0.0043,\ \ -0.4999$	$1.805871e-07$
150	$0.5000,\ \ 0.0041,\ \ -0.4999$	$2.720744e-08$
200	$0.5000,\ \ 0.0041,\ \ -0.4999$	$4.934027e-09$
250	$0.5000,\ \ 0.0040,\ \ -0.4999$	$9.630085e-10$
300	$0.5000,\ \ 0.0040,\ \ -0.4999$	$1.939747e-10$
350	$0.5000,\ \ 0.0040,\ \ -0.4999$	$3.961646e-11$
400	$0.5000,\ \ 0.0040,\ \ -0.4999$	$8.140393e-12$
450	$0.5000,\ \ 0.0040,\ \ -0.4999$	$1.677968e-12$
500	$0.5000,\ \ 0.0040,\ \ -0.4999$	$3.462695e-13$
550	$0.5000,\ \ 0.0040,\ \ -0.4999$	$7.136628e-14$
600	$0.5000,\ \ 0.0040,\ \ -0.4999$	$1.474205e-14$

$$o(\mathbf{x}) = ||\mathbf{f}(\mathbf{x}^*)||^2 \approx 10^{-14}$$

$$o(\mathbf{x}) \approx 10^{-28}$$

$$\mathbf{x}^* = \begin{bmatrix} 0.5000013623816102 \\ 0.0040027495837189 \\ -0.4999000311539049 \end{bmatrix}$$

$$f(\mathbf{x}_{n+1}) = f(\mathbf{x}_n + \delta \mathbf{d}_n) \leq f(\mathbf{x}_n) + c_1\, \delta \mathbf{d}^T \mathbf{g}_n$$

$$\mathbf{g}_{n+1}^T \mathbf{d}_n \geq c_2 \ \mathbf{g}_n^T \mathbf{d}_n$$

$$\mathbf{g}_n^T \mathbf{d}_n < 0$$

$$|\mathbf{g}_{n+1}^T \mathbf{d}_n| < |c_2 \ \mathbf{g}_n^T \mathbf{d}_n|$$

c_1

c_2

$$0 < c_1 < c_2 < 1$$

$$\phi(\delta) = f(\mathbf{x}_n + \delta \mathbf{d}_n)$$

$$L_0(\delta) = a + b\delta$$

$$a = L_0(0) = \phi(\delta)\Big|_{\delta=0} = f(\mathbf{x}_n)$$

$$b = \frac{d}{d\delta}\ f(\mathbf{x}_n + \delta \mathbf{d}_n)\Big|_{\delta=0} = \mathbf{g}_n^T \mathbf{d}_n < 0$$

$$f(\mathbf{x}_n + \delta \mathbf{d}_n) < f(\mathbf{x}_n)$$

$$L_0(\delta) = a + b\delta = f(\mathbf{x}_n) + \mathbf{g}_n^T \mathbf{d}_n \delta$$

$$L_1(\delta) = f(\mathbf{x}_n)$$

$L_0(\delta)$

$L_1(\delta)$

$$L(\delta) = f(\mathbf{x}_n) + c_1 \, \mathbf{g}_n^T \mathbf{d}_n \delta$$

$$0 < c_1 < 1$$

$$0 < c_1 \mathbf{g}_n^T \mathbf{d}_n < \mathbf{g}_n^T \mathbf{d}_n$$

$$f(\mathbf{x}_{n+1}) = f(\mathbf{x}_n + \delta \mathbf{d}_n) \leq f(\mathbf{x}_n) + c_1 \, \mathbf{g}_n^T \mathbf{d}_n \delta < f(\mathbf{x}_n)$$

$$\mathbf{g}_{n+1} \perp \mathbf{g}_n$$

$$-\mathbf{g}_n$$

$$f(\delta_n) = f(\mathbf{x}_n - \delta_n \mathbf{g}_n)$$

$f(\delta_n)$

$$\mathbf{g}_{n+1}^T \mathbf{g}_n = 0$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \delta_n \mathbf{g}_n + \alpha_n (\mathbf{x}_n - \mathbf{x}_{n-1})$$

α_m

$$f(x, y) = (a - x)^2 + b(y - x^2)^2$$

typically $a = 1,\ b = 100$

$$f(1,\ 1) = 0$$

$(1,\ 1)$

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \delta \mathbf{J}(\mathbf{x}_n)^{-1} \mathbf{f}(\mathbf{x}_n) = \mathbf{x}_n - \delta\ \mathbf{J}_n^{-1} \mathbf{f}_n$$

$$\mathbf{J}_n = \mathbf{J}(\mathbf{x}_n)$$

$$\mathbf{f}(\mathbf{x}_n)$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \delta \mathbf{H}(\mathbf{x}_n)^{-1} \mathbf{g}(\mathbf{x}_n) = \mathbf{x}_n - \delta\ \mathbf{H}_n^{-1} \mathbf{g}_n$$

$$\mathbf{J}(\mathbf{x})$$

$\mathbf{H}(\mathbf{x})$

$O(N^3)$

$$O(N^2)$$

$$f(\mathbf{x}) = f(\mathbf{x}_{n+1}) + (\mathbf{x} - \mathbf{x}_{n+1})^T \mathbf{g}_{n+1} + \frac{1}{2}(\mathbf{x} - \mathbf{x}_{n+1})^T \mathbf{H}_{n+1}(\mathbf{x} - \mathbf{x}_{n+1}) + O(||\mathbf{x} - \mathbf{x}_{n+1}||^3)$$

$$\frac{d}{d\mathbf{x}}f(\mathbf{x}) = \mathbf{g}(\mathbf{x}) = \mathbf{g}_{n+1} + \mathbf{H}_{n+1}(\mathbf{x} - \mathbf{x}_{n+1}) + O(||\mathbf{x} - \mathbf{x}_{n+1}||^2)$$

$$\mathbf{x} = \mathbf{x}_n$$

$$\mathbf{g}(\mathbf{x}_n) = \mathbf{g}_n$$

$$\mathbf{g}_{n+1} - \mathbf{g}_n$$

$$\mathbf{H}_{n+1}(\mathbf{x}_{n+1} - \mathbf{x}_n) + O(||\mathbf{x}_{n+1} - \mathbf{x}_n||^2)$$

$$\mathbf{B}_{n+1}(\mathbf{x}_{n+1} - \mathbf{x}_n)$$

$\mathbf{B}_n$

$$\mathbf{s}_n = \mathbf{x}_{n+1} - \mathbf{x}_n, \qquad \mathbf{y}_n = \mathbf{g}_{n+1} - \mathbf{g}_n$$

$$\mathbf{B}_{n+1}\mathbf{s}_n = \mathbf{y}_n,$$

$$\mathbf{B}_{n+1}^{-1}\mathbf{y}_n = \mathbf{s}_n$$

$$\mathbf{B}_n^{-1}$$

$\mathbf{B}_0$

$$n = 0$$

$$\mathbf{d}_n = -\mathbf{B}_n^{-1}\mathbf{g}_n$$

$$\mathbf{B}_{n+1} = \mathbf{B}_n + \Delta\mathbf{B}_n$$

$$\mathbf{B}_{n+1}^{-1} = \mathbf{B}_n^{-1} + \Delta\mathbf{B}_n^{-1}$$

$$n = n + 1$$

$$\mathbf{B}_n = \mathbf{I}$$

$$\mathbf{B}_n = \mathbf{H}_n$$

$\mathbf{y}_n$

$\mathbf{s}_m$

$$\mathbf{B}_{n+1} = \mathbf{B}_n + \mathbf{u}\mathbf{u}^T$$

u

$$\mathbf{u}\mathbf{u}^T$$

$$\mathbf{B}_{n+1}\mathbf{s}_n = \mathbf{B}_n\mathbf{s}_n + \mathbf{u}\mathbf{u}^T\mathbf{s}_n = \mathbf{y}_n,$$

$$\mathbf{u}(\mathbf{u}^T\mathbf{s}_n) = \mathbf{y}_n - \mathbf{B}_n\mathbf{s}_n$$

$$\mathbf{w} = \mathbf{y}_n - \mathbf{B}_n \mathbf{s}_n$$

$$\mathbf{u} = c\mathbf{w}$$

$$\mathbf{u}\mathbf{u}^T\mathbf{s}_n = c^2\mathbf{w}\mathbf{w}^T\mathbf{s}_n = \mathbf{w}$$

$$c = 1/(\mathbf{w}^T \mathbf{s}_n)^{1/2}$$

$$\mathbf{u} = c\mathbf{w} = \frac{\mathbf{y}_n - \mathbf{B}_n \mathbf{s}_n}{(\mathbf{w}^T \mathbf{s}_n)^{1/2}} = \frac{\mathbf{y}_n - \mathbf{B}_n \mathbf{s}_n}{((\mathbf{y}_n - \mathbf{B}_n \mathbf{s}_n)^T \mathbf{s}_n)^{1/2}}$$

$$\mathbf{B}_{n+1} = \mathbf{B}_n + \mathbf{u}\mathbf{u}^T = \mathbf{B}_n + \frac{(\mathbf{y}_n - \mathbf{B}_n\mathbf{s}_n)(\mathbf{y}_n - \mathbf{B}_n\mathbf{s}_n)^T}{(\mathbf{y}_n - \mathbf{B}_n\mathbf{s}_n)^T\mathbf{s}_n}$$

$$\mathbf{B}_{n+1}$$

$$\mathbf{B}_{n+1}^{-1}$$

$$\left(\mathbf{B}_n + \mathbf{u}\mathbf{u}^T\right)^{-1} = \mathbf{B}_n^{-1} - \frac{\mathbf{B}_n^{-1}\mathbf{u}\mathbf{u}^T\mathbf{B}_n^{-1}}{1 + \mathbf{u}^T\mathbf{B}_n^{-1}\mathbf{u}}$$

$$\mathbf{B}_n^{-1} + \frac{(\mathbf{s}_n - \mathbf{B}_n^{-1}\mathbf{y}_n)(\mathbf{s}_n - \mathbf{B}_n^{-1}\mathbf{y}_n)^T}{(\mathbf{s}_n - \mathbf{B}_n^{-1}\mathbf{y}_n)^T\mathbf{y}_n}$$

$$\mathbf{B}_{n+1}^{-1}$$

$$\mathbf{B}_{n+1}^{-1} = \mathbf{B}_n^{-1} + \mathbf{u}\mathbf{u}^T$$

$$\mathbf{B}_{n+1}^{-1}\mathbf{y}_n = \mathbf{B}_n^{-1}\mathbf{y}_n + \mathbf{u}\mathbf{u}^T\mathbf{y}_n = \mathbf{s}_n,$$

$$\mathbf{w}^T\mathbf{y}_n = (\mathbf{s}_n - \mathbf{B}_n^{-1}\mathbf{y}_n)^T\mathbf{y}_n = \mathbf{s}_n^T\mathbf{y}_n - \mathbf{y}_n^T\mathbf{B}_n^{-1}\mathbf{y}_n > 0$$

$$\mathbf{B}_{n+1} = \mathbf{B}_n + \alpha \mathbf{u}\mathbf{u}^T + \beta \mathbf{v}\mathbf{v}^T$$

$$\mathbf{B}_{n+1}\mathbf{s}_n = (\mathbf{B}_n + \alpha\mathbf{u}\mathbf{u}^T + \beta\mathbf{v}\mathbf{v}^T)\mathbf{s}_n = \mathbf{B}_n\mathbf{s}_n + \mathbf{u}(\alpha\mathbf{u}^T\mathbf{s}_n) + \mathbf{v}(\beta\mathbf{v}^T\mathbf{s}_n) = \mathbf{y}_n$$

$$\mathbf{u}(\alpha \mathbf{u}^T \mathbf{s}_n) + \mathbf{v}(\beta \mathbf{v}^T \mathbf{s}_n) = \mathbf{y}_n - \mathbf{B}_n \mathbf{s}_n$$

$$\mathbf{u} = \mathbf{y}_n, \qquad \alpha = \frac{1}{\mathbf{s}_n^T \mathbf{y}_n}, \qquad \mathbf{v} = \mathbf{B}_n \mathbf{s}_n, \qquad \beta = -\frac{1}{\mathbf{v}^T \mathbf{s}_n} = -\frac{1}{\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n}$$

$$\mathbf{B}_{n+1} = \mathbf{B}_n + \alpha \mathbf{u}\mathbf{u}^T + \beta \mathbf{v}\mathbf{v}^T$$

$$\mathbf{B}_{n+1} = \mathbf{B}_n + \alpha \mathbf{u}\mathbf{u}^T + \beta \mathbf{v}\mathbf{v}^T = \mathbf{B}_n + \frac{\mathbf{y}_n \mathbf{y}_n^T}{\mathbf{y}_n^T \mathbf{s}_n} - \frac{\mathbf{B}_n \mathbf{s}_n \mathbf{s}_n^T \mathbf{B}_n}{\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n}$$

$$\mathbf{U} = [\mathbf{u}_1 \ \mathbf{u}_2]$$

$$\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2]$$

$$\mathbf{u}_1 = \mathbf{v}_1 = \frac{\mathbf{y}_n}{(\mathbf{s}_n^T \mathbf{y}_n)^{1/2}}, \qquad \mathbf{u}_2 = -\mathbf{v}_2 = \frac{\mathbf{B}_n \mathbf{s}_n}{(\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n)^{1/2}}$$

$$\mathbf{B}_{n+1} = \mathbf{B}_n + \mathbf{u}_1\mathbf{v}_1^T + \mathbf{u}_2\mathbf{v}_2^T = (\mathbf{B}_n + \mathbf{U}\mathbf{V}^T)^{-1}$$

$$(\mathbf{B}_n + \mathbf{U}\mathbf{V}^T)^{-1} = \mathbf{B}_n^{-1} - \mathbf{B}_n^{-1}\mathbf{U}(\mathbf{I} + \mathbf{V}^T\mathbf{B}_n^{-1}\mathbf{U})^{-1}\mathbf{V}^T\mathbf{B}_n^{-1}$$

$$\mathbf{B}_n^{-1} - \mathbf{B}_n^{-1}\mathbf{U}\mathbf{C}^{-1}\mathbf{V}^T\mathbf{B}_n^{-1} = \mathbf{B}_n^{-1} - \mathbf{B}_n^{-1}[\mathbf{u}_1 \ \mathbf{u}_2]\mathbf{C}^{-1}(\mathbf{B}_n^{-1}[\mathbf{v}_1 \ \mathbf{v}_2])^T$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \mathbf{I} + \mathbf{V}^T \mathbf{B}_n^{-1} \mathbf{U} = \mathbf{I} + [\mathbf{v}_1 \ \mathbf{v}_2]^T \mathbf{B}_n^{-1} [\mathbf{u}_1 \ \mathbf{u}_2]$$

$$c_{11} = 1 + \mathbf{v}_1^T \mathbf{B}_n^{-1} \mathbf{u}_1 = 1 + \frac{\mathbf{y}_n^T \mathbf{B}_n^{-1} \mathbf{y}_n}{\mathbf{s}_n^T \mathbf{y}_n}$$

$$c_{22} = 1 + \mathbf{v}_2 \mathbf{B}_n^{-1} \mathbf{u}_2 = 1 - \frac{\mathbf{s}_n^T \mathbf{B}_n \mathbf{B}_n^{-1} \mathbf{B}_n \mathbf{s}_n}{\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n} = 0$$

$$c_{12} = \mathbf{v}_1^T \mathbf{B}_n^{-1} \mathbf{u}_2 = \frac{\mathbf{y}_n^T \mathbf{B}_n^{-1} \mathbf{B}_n \mathbf{s}_n}{(\mathbf{s}_n^T \mathbf{y}_n)^{1/2} (\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n)^{1/2}} = \frac{(\mathbf{s}_n^T \mathbf{y}_n)^{1/2}}{(\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n)^{1/2}}$$

$$c_{21} = \mathbf{v}_2^T \mathbf{B}_n^{-1} \mathbf{u}_1 = -c_{12}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ -c_{12} & 0 \end{bmatrix}, \qquad \mathbf{C}^{-1} = \begin{bmatrix} 0 & -1/c_{12} \\ 1/c_{12} & c_{11}/c_{12}^2 \end{bmatrix}$$

$$\mathbf{C}^{-1}$$

$$\mathbf{B}_n^{-1} - [\mathbf{B}_n^{-1}\mathbf{u}_1 \ \mathbf{B}_n^{-1}\mathbf{u}_2] \left[\begin{array}{cc} 0 & -1/c_{12} \\ 1/c_{12} & c_{11}/c_{12}^2 \end{array} \right] [\mathbf{B}_n^{-1}\mathbf{v}_1 \ \mathbf{B}_n^{-1}\mathbf{v}_2]^T$$

$$\mathbf{B}_n^{-1} - \frac{1}{c_{12}}[\mathbf{B}_n^{-1}\mathbf{u}_2\mathbf{v}_1^T\mathbf{B}_n^{-1} - \mathbf{B}_n^{-1}\mathbf{u}_1\mathbf{v}_2^T\mathbf{B}_n^{-1}] - \frac{c_{11}}{c_{12}^2}\mathbf{B}_n^{-1}\mathbf{u}_2\mathbf{v}_2^T\mathbf{B}_n^{-1}$$

$$\frac{1}{c_{12}}\mathbf{B}_n^{-1}\mathbf{u}_2\mathbf{v}_1^T\mathbf{B}_n^{-1} = \frac{(\mathbf{s}_n^T\mathbf{B}_n\mathbf{s}_n)^{1/2}}{(\mathbf{s}_n^T\mathbf{y}_n)^{1/2}}\mathbf{B}_n^{-1}\frac{\mathbf{B}_n\mathbf{s}_n}{(\mathbf{s}_n^T\mathbf{B}_n\mathbf{s}_n)^{1/2}}\frac{\mathbf{y}_n^T}{(\mathbf{s}_n^T\mathbf{y}_n)^{1/2}}\mathbf{B}_n^{-1} = \frac{\mathbf{s}_n\mathbf{y}_n^T\mathbf{B}_n^{-1}}{\mathbf{s}_n^T\mathbf{y}_n}$$

$$-\frac{1}{c_{12}}\mathbf{B}_n^{-1}\mathbf{u}_1\mathbf{v}_2^T\mathbf{B}_n^{-1} = \frac{(\mathbf{s}_n^T\mathbf{B}_n\mathbf{s}_n)^{1/2}}{(\mathbf{s}_n^T\mathbf{y}_n)^{1/2}}\mathbf{B}_n^{-1}\frac{\mathbf{y}_n}{(\mathbf{s}_n^T\mathbf{y}_n)^{1/2}}\frac{\mathbf{s}_n^T\mathbf{B}_n}{(\mathbf{s}_n^T\mathbf{B}_n\mathbf{s}_n)^{1/2}}\mathbf{B}_n^{-1} = \frac{\mathbf{B}_n^{-1}\mathbf{y}_n\mathbf{s}_n^T}{\mathbf{s}_n^T\mathbf{y}_n}$$

$$\frac{c_{11}}{c_{12}^2}\mathbf{B}_n^{-1}\mathbf{u}_2\mathbf{v}_2^T\mathbf{B}_n^{-1}$$

$$- \left(1 + \frac{\mathbf{y}_n^T \mathbf{B}_n^{-1} \mathbf{y}_n}{\mathbf{s}_n^T \mathbf{y}_n}\right) \frac{\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n}{\mathbf{s}_n^T \mathbf{y}_n} \mathbf{B}_n^{-1} \frac{\mathbf{B}_n \mathbf{s}_n}{(\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n)^{1/2}} \frac{\mathbf{s}_n^T \mathbf{B}_n}{(\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n)^{1/2}} \mathbf{B}_n^{-1}$$

$$-\left(1+\frac{\mathbf{y}_n^T\mathbf{B}_n^{-1}\mathbf{y}_n}{\mathbf{s}_n^T\mathbf{y}_n}\right)\frac{\mathbf{s}_n\mathbf{s}_n^T}{\mathbf{s}_n^T\mathbf{y}_n}$$

$$\mathbf{B}_{n+1}^{-1} = \mathbf{B}_n^{-1} - \frac{\mathbf{B}_n^{-1}\mathbf{y}_n\mathbf{s}_n^T + \mathbf{s}_n\mathbf{y}_n^T\mathbf{B}_n^{-1}}{\mathbf{s}_n^T\mathbf{y}_n} + \left(1 + \frac{\mathbf{y}_n^T\mathbf{B}_n^{-1}\mathbf{y}_n}{\mathbf{s}_n^T\mathbf{y}_n}\right)\frac{\mathbf{s}_n\mathbf{s}_n^T}{\mathbf{s}_n^T\mathbf{y}_n}$$

$$\mathbf{B}_n^{-1}$$

$$\mathbf{B}_{n+1}^{-1} = \mathbf{B}_n^{-1} + \alpha \mathbf{u}\mathbf{u}^T + \beta \mathbf{v}\mathbf{v}^T$$

α

β

V

$$\mathbf{B}_{n+1}^{-1}\mathbf{y}_n = (\mathbf{B}_n^{-1} + \alpha\mathbf{u}\mathbf{u}^T + \beta\mathbf{v}\mathbf{v}^T)\mathbf{y}_n = \mathbf{B}_n^{-1}\mathbf{y}_n + \mathbf{u}(\alpha\mathbf{u}^T\mathbf{y}_n) + \mathbf{v}(\beta\mathbf{v}^T\mathbf{y}_n) = \mathbf{s}_n$$

$$\mathbf{u}(\alpha \mathbf{u}^T \mathbf{y}_n) + \mathbf{v}(\beta \mathbf{v}^T \mathbf{y}_n) = \mathbf{s}_n - \mathbf{B}_n^{-1} \mathbf{y}_n$$

$$\mathbf{u} = \mathbf{s}_n, \qquad \alpha = \frac{1}{\mathbf{s}^T \mathbf{y}_n}, \qquad \mathbf{v} = \mathbf{B}_n^{-1} \mathbf{y}_n, \qquad \beta = -\frac{1}{\mathbf{v}^T \mathbf{y}_n} = -\frac{1}{\mathbf{y}_n^T \mathbf{B}_n^{-1} \mathbf{y}_n}$$

$$\mathbf{B}_{n+1}^{-1} = \mathbf{B}_n^{-1} + \alpha \mathbf{u}\mathbf{u}^T + \beta \mathbf{v}\mathbf{v}^T$$

$$\mathbf{B}_{n+1}^{-1} = \mathbf{B}_n^{-1} + \alpha \mathbf{u}\mathbf{u}^T + \beta \mathbf{v}\mathbf{v}^T = \mathbf{B}_n^{-1} + \frac{\mathbf{s}_n \mathbf{s}_n^T}{\mathbf{s}_n^T \mathbf{y}_n} - \frac{\mathbf{B}_n^{-1} \mathbf{y}_n \mathbf{y}_n^T \mathbf{B}_n^{-1}}{\mathbf{y}_n^T \mathbf{B}_n^{-1} \mathbf{y}_n}$$

$$\mathbf{B}_{n+1} = \mathbf{B}_n - \frac{\mathbf{B}_n \mathbf{s}_n \mathbf{y}_n^T + \mathbf{y}_n \mathbf{s}_n^T \mathbf{B}_n}{\mathbf{y}_n^T \mathbf{s}_n} + \left(1 + \frac{\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n}{\mathbf{y}_n^T \mathbf{s}_n}\right) \frac{\mathbf{y}_n \mathbf{y}_n^T}{\mathbf{y}_n^T \mathbf{s}_n}$$

$$\mathbf{z}^T \mathbf{B}_n \mathbf{z} > 0$$

$$\mathbf{z} \neq \mathbf{0}$$

$$\mathbf{g}_{n+1}^T \mathbf{d}_n \geq c_2 \ \mathbf{g}_n^T \mathbf{d}_n$$

$$\left(\mathbf{g}_{n+1} - c_2\ \mathbf{g}_n\right)^T \mathbf{d}_n \geq 0$$

$$\delta \mathbf{d}_n = \mathbf{x}_{n+1} - \mathbf{x}_n = \mathbf{s}_n$$

$$(\mathbf{g}_{n+1} - c_2 \mathbf{g}_n)^T \mathbf{s}_n = \mathbf{g}_{n+1}^T \mathbf{s}_n - c_2 \mathbf{g}_n^T \mathbf{s}_n \geq 0$$

$$\mathbf{y}_n^T \mathbf{s}_n = (\mathbf{g}_{n+1} - \mathbf{g}_n)^T \mathbf{s}_n = \mathbf{g}_{n+1}^T \mathbf{s}_n - \mathbf{g}_n^T \mathbf{s}_n$$

$$\mathbf{y}_n^T \mathbf{s}_n - (\mathbf{g}_{n+1}^T \mathbf{s}_n - c_2 \mathbf{g}_n^T \mathbf{s}_n) = (c_2 - 1)\mathbf{g}_n^T \mathbf{s}_n > 0$$

$$\mathbf{g}_n^T \mathbf{s}_n < 0$$

$c_2 < 1$

$$\mathbf{y}_n^T \mathbf{s}_n \geq \mathbf{g}_{n+1}^T \mathbf{s}_n - c_2 \mathbf{g}_n^T \mathbf{s}_n \geq 0$$

$$\mathbf{y}_n^T \mathbf{s}_n > 0$$

$$\mathbf{B}_n = \mathbf{L}\mathbf{L}^T$$

$$\mathbf{a} = \mathbf{L}^T \mathbf{z}$$

$$\mathbf{b} = \mathbf{L}^T \mathbf{s}$$

$$\mathbf{a}^T\mathbf{a} = \mathbf{z}^T\mathbf{B}_n\mathbf{z}, \quad \mathbf{b}^T\mathbf{b} = \mathbf{s}^T\mathbf{B}_n\mathbf{s}, \quad \mathbf{a}^T\mathbf{b} = \mathbf{z}^T\mathbf{B}_n\mathbf{s}$$

$$\mathbf{z}^T \left[\mathbf{B}_n - \frac{\mathbf{B}_n \mathbf{s}_n \mathbf{s}_n^T \mathbf{B}_n}{\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n} \right] \mathbf{z} = \mathbf{z}^T \mathbf{B}_n \mathbf{z} - \frac{(\mathbf{z}^T \mathbf{B}_n \mathbf{s}_n)^2}{\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n} = \mathbf{a}^T \mathbf{a} - \frac{(\mathbf{a}^T \mathbf{b})^2}{\mathbf{b}^T \mathbf{b}} \geq 0$$

$$\mathbf{s}_n^T \mathbf{y}_n \geq 0$$

$$\mathbf{z}^T \left(\frac{\mathbf{s}_n \mathbf{s}_n^T}{\mathbf{s}_n^T \mathbf{y}_n} \right) \mathbf{z} \geq 0$$

$$\mathbf{z}^T \mathbf{B}_{n+1} \mathbf{z} = \mathbf{z}^T \left(\mathbf{B}_n - \frac{\mathbf{B}_n \mathbf{s}_n \mathbf{s}_n^T \mathbf{B}_n}{\mathbf{s}_n^T \mathbf{B}_n \mathbf{s}_n} \right) \mathbf{z} + \mathbf{z}^T \left(\frac{\mathbf{y}_n \mathbf{y}_n^T}{\mathbf{s}_n^T \mathbf{y}_n} \right) \mathbf{z} \geq 0$$

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{x} + c$$

$$\mathbf{A} = \mathbf{A}^T$$

$$\mathbf{g}(\mathbf{x}) = \frac{d}{d\mathbf{x}} f(\mathbf{x}) = \frac{d}{d\mathbf{x}} \left(\frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} - \mathbf{b}^T \mathbf{x} + c \right) = \mathbf{A}\mathbf{x} - \mathbf{b}$$

$$\mathbf{H}(\mathbf{x}) = \frac{d^2}{d\mathbf{x}^2} f(\mathbf{x}) = \frac{d}{d\mathbf{x}} \mathbf{g} = \frac{d}{d\mathbf{x}} (\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{A}$$

$$\mathbf{g}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}$$

$$\mathbf{x}^* = \mathbf{A}^{-1}\mathbf{b}$$

$$f(\mathbf{x}^*) = \frac{1}{2}(\mathbf{A}^{-1}\mathbf{b})^T\mathbf{A}(\mathbf{A}^{-1}\mathbf{b}) - \mathbf{b}^T(\mathbf{A}^{-1}\mathbf{b}) + c = -\frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} + c$$

$$\mathbf{g}(\mathbf{x}^*) = \mathbf{A}\mathbf{x}^* - \mathbf{b} = \mathbf{0}$$

$$\varepsilon = ||\mathbf{g}_n||^2$$

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{u}^T \mathbf{A} \mathbf{v} = (\mathbf{A} \mathbf{v})^T \mathbf{u} = \mathbf{v}^T \mathbf{A} \mathbf{u} = 0$$

$$\mathbf{A} = \mathbf{I}$$

$$\mathbf{v}^T \mathbf{u} = 0$$

$$\{\mathbf{d}_0, \cdots, \mathbf{d}_{N-1}\}$$

$$\mathbf{d}_i^T \mathbf{A} \mathbf{d}_j = 0$$

$i \neq j$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{u}_1 = \mathbf{v}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right], \qquad \mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{u}_1^T \mathbf{A} \mathbf{v}_2}{\mathbf{u}_1^T \mathbf{A} \mathbf{u}_1} \mathbf{u}_1 = \left[\begin{array}{c} -1/3 \\ 1 \end{array} \right]$$

$$\mathbf{x} = [2,\ 3]^T$$

$$\mathbf{v}_1$$

$\mathbf{v}_2$

$$\mathbf{p}_{\mathbf{v}_1}(\mathbf{x}) = \frac{\mathbf{v}_1^T \mathbf{x}}{\mathbf{v}_1^T \mathbf{v}_1} \mathbf{v}_1 = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad \mathbf{p}_{\mathbf{v}_2}(\mathbf{x}) = \frac{\mathbf{v}_2^T \mathbf{x}}{\mathbf{v}_2^T \mathbf{v}_2} \mathbf{v}_2 = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\mathbf{u}_1$$

$\mathbf{u}_2$

$$\mathbf{p}_{\mathbf{u}_1}(\mathbf{x}) = \frac{\mathbf{u}_1^T \mathbf{A} \mathbf{x}}{\mathbf{u}_1^T \mathbf{A} \mathbf{u}_1} \mathbf{u}_1 = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \mathbf{p}_{\mathbf{u}_2}(\mathbf{x}) = \frac{\mathbf{u}_2^T \mathbf{A} \mathbf{x}}{\mathbf{u}_2^T \mathbf{A} \mathbf{u}_2} \mathbf{u}_2 = 3 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\mathbf{v}_1 + 3\mathbf{v}_2 = \mathbf{p}_{\mathbf{v}_1}(\mathbf{x}) + \mathbf{p}_{\mathbf{v}_2}(\mathbf{x})$$

$$3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1/3 \\ 1 \end{bmatrix} = 3\mathbf{u}_1 + 3\mathbf{u}_2 = \mathbf{p}_{\mathbf{u}_1}(\mathbf{x}) + \mathbf{p}_{\mathbf{u}_2}(\mathbf{x})$$

$$\{-\mathbf{g}_0, \cdots, -\mathbf{g}_n, \cdots\}$$

$$\mathbf{d}_i^T \mathbf{A} \mathbf{d}_j = 0 \ (i \neq j)$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \delta_n \mathbf{d}_n = \cdots = \mathbf{x}_0 + \sum_{i=0}^{n} \delta_i \mathbf{d}_i$$

$$\mathbf{x}^* = \mathbf{x}_{n+1} - \mathbf{e}_{n+1} = \mathbf{x}_n - \mathbf{e}_n$$

$$\mathbf{e}_{n+1} = \mathbf{e}_n + \delta_n \mathbf{d}_n = \cdots = \mathbf{e}_0 + \sum_{i=0}^{n} \delta_i \mathbf{d}_i$$

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{A}\mathbf{x}/2 - \mathbf{b}^T \mathbf{x} + c$$

$\mathbf{x}_i$

$$\mathbf{g}_i = f'(\mathbf{x}_i) = \mathbf{A}\mathbf{x}_i - \mathbf{b} = \mathbf{A}(\mathbf{x}^* + \mathbf{e}_i) - \mathbf{b} = \mathbf{A}\mathbf{x}^* - \mathbf{b} + \mathbf{A}\mathbf{e}_i = \mathbf{A}\mathbf{e}_i$$

δ_i

$$-\frac{\mathbf{g}_i^T\mathbf{d}_i}{\mathbf{d}_i^T\mathbf{A}\mathbf{d}_i} = -\frac{\mathbf{e}_i^T\mathbf{A}\mathbf{d}_i}{\mathbf{d}_i^T\mathbf{A}\mathbf{d}_i} = -\frac{\mathbf{d}_i^T\mathbf{A}\mathbf{e}_i}{\mathbf{d}_i^T\mathbf{A}\mathbf{d}_i} = -\frac{\mathbf{d}_i^T\mathbf{A}\left(\mathbf{e}_0 + \sum_{j=0}^{i-1}\delta_j\mathbf{d}_j\right)}{\mathbf{d}_i^T\mathbf{A}\mathbf{d}_i}$$

$$-\frac{\mathbf{d}_i^T \mathbf{A} \mathbf{e}_0 + \sum_{j=0}^{i-1} \delta_j \mathbf{d}_i^T \mathbf{A} \mathbf{d}_j}{\mathbf{d}_i^T \mathbf{A} \mathbf{d}_i} = -\frac{\mathbf{d}_i^T \mathbf{A} \mathbf{e}_0}{\mathbf{d}_i^T \mathbf{A} \mathbf{d}_i}$$

$$\mathbf{d}_i$$

$$\mathbf{d}_j$$

$$\mathbf{d}_i^T \mathbf{A} \mathbf{d}_j = 0$$

$$\mathbf{e}_{n+1} = \mathbf{e}_n + \delta_n \mathbf{d}_n = \mathbf{e}_n - \left(\frac{\mathbf{d}_n^T \mathbf{A} \mathbf{e}_0}{\mathbf{d}_n^T \mathbf{A} \mathbf{d}_n} \right) \mathbf{d}_n$$

$$\mathbf{e}_0 = \mathbf{x}_0 - \mathbf{x}^*$$

$$\mathbf{e}_0 = \sum_{i=0}^{N-1} c_i \mathbf{d}_i = \sum_{i=0}^{N-1} \mathbf{p}_{\mathbf{d}_i}(\mathbf{e}_0) = \sum_{i=0}^{N-1} \left(\frac{\mathbf{d}_i^T \mathbf{A} \mathbf{e}_0}{\mathbf{d}_i^T \mathbf{A} \mathbf{d}_i} \right) \mathbf{d}_i$$

$$\mathbf{p}_{\mathbf{d}_i}(\mathbf{e}_0)$$

$\mathbf{e}_0$

$$\mathbf{p}_{\mathbf{d}_i}(\mathbf{e}_0) = c_i \mathbf{d}_i = \left(\frac{\mathbf{d}_i^T \mathbf{A} \mathbf{e}_0}{\mathbf{d}_i^T \mathbf{A} \mathbf{d}_i} \right) \mathbf{d}_i$$

c_i

$$c_i = \frac{\mathbf{d}_i^T \mathbf{A} \mathbf{e}_0}{\mathbf{d}_i^T \mathbf{A} \mathbf{d}_i} = -\delta_i$$

$$\mathbf{e}_{n+1}$$

$$\mathbf{e}_{n+1} = \mathbf{e}_0 + \sum_{i=0}^{n} \delta_i \mathbf{d}_i = \sum_{i=0}^{N-1} c_i \mathbf{d}_i - \sum_{i=0}^{n} c_i \mathbf{d}_i = \sum_{i=n+1}^{N-1} c_i \mathbf{d}_i = \sum_{i=n+1}^{N-1} \mathbf{p}_{\mathbf{d}_i}(\mathbf{e}_0)$$

n

$$n + 1$$

$$\mathbf{p}_{\mathbf{d}_n}(\mathbf{e}_0)$$

$$\mathbf{e}_n$$

$$\mathbf{e}_N = \mathbf{0}$$

$$\mathbf{x}_N = \mathbf{x}^* + \mathbf{e}_N = \mathbf{x}^*$$

$$\mathbf{d}_k^T \mathbf{A} \ (k \leq n)$$

$$\mathbf{d}_k^T \mathbf{A} \mathbf{e}_{n+1} = \mathbf{d}_k^T \mathbf{g}_{n+1} = \sum_{i=n+1}^{N-1} c_i \mathbf{d}_k^T \mathbf{A} \mathbf{d}_j = 0$$

$$\mathbf{d}_0, \cdots, \mathbf{d}_n$$

$N = 2$

$$-\mathbf{g}_0$$

$\mathbf{d}_1$

$\mathbf{d}_0$

$\mathbf{e}_1$

$-\mathbf{g}_1$

$$\left\{\mathbf{v}_0, \cdots, \mathbf{v}_{N-1}\right\}$$

$$\mathbf{d}_n = \mathbf{v}_n - \sum_{j=0}^{n-1} \mathbf{p}_{\mathbf{d}_j}(\mathbf{v}_n) = \mathbf{v}_n - \sum_{m=0}^{n-1} \left(\frac{\mathbf{d}_m^T \mathbf{A} \mathbf{v}_n}{\mathbf{d}_m^T \mathbf{A} \mathbf{d}_m} \right) \mathbf{d}_m = \mathbf{v}_n - \sum_{m=0}^{n-1} \beta_{nm} \mathbf{d}_m$$

$$\beta_{nm} = \mathbf{d}_m^T \mathbf{A} \mathbf{v}_n / (\mathbf{d}_m^T \mathbf{A} \mathbf{d}_m)$$

$\beta_{nm}\mathbf{d}_m$

$$\mathbf{v}_n$$

$$\mathbf{d}_m$$

$$\mathbf{v}_n = -\mathbf{g}_n$$

$$\mathbf{d}_n = -\mathbf{g}_n - \sum_{m=0}^{n-1} \beta_{nm} \mathbf{d}_m$$

$$\beta_{nm} = \frac{\mathbf{d}_m^T \mathbf{A} \mathbf{v}_n}{\mathbf{d}_m^T \mathbf{A} \mathbf{d}_m} = -\frac{\mathbf{d}_m^T \mathbf{A} \mathbf{g}_n}{\mathbf{d}_m^T \mathbf{A} \mathbf{d}_m} \qquad (m < n)$$

$$\mathbf{g}_n = \sum_{i=0}^{n} \alpha_i \mathbf{d}_i$$

$$\mathbf{g}_{n+1}^T$$

$$\mathbf{g}_{n+1}^T \mathbf{g}_k = \mathbf{g}_{n+1}^T \left(\sum_{i=0}^{k} \alpha_i \mathbf{d}_i \right) = \sum_{i=0}^{k} \alpha_i \ \mathbf{g}_{n+1}^T \mathbf{d}_i = 0$$

$$\mathbf{g}_{n+1}^T \mathbf{d}_k = 0$$

$\mathbf{g}_0, \cdots, \mathbf{g}_n$

$$\mathbf{g}_k^T (k \geq n)$$

$$\mathbf{g}_k^T \mathbf{d}_n = -\mathbf{g}_k^T \mathbf{g}_n - \sum_{m=0}^{n-1} \beta_{mn}\, \mathbf{g}_k^T \mathbf{d}_m = -\mathbf{g}_k^T \mathbf{g}_n = \begin{cases} -\|\mathbf{g}_n\|^2 & n = k \\ 0 & n < k \end{cases}$$

$$\mathbf{g}_k^T \mathbf{d}_m = 0$$

$$m < n \leq k$$

$$\mathbf{g}_n^T \mathbf{d}_n = -||\mathbf{g}_n||^2$$

$$\delta_n = -\frac{\mathbf{g}_n^T \mathbf{d}_n}{\mathbf{d}_n^T \mathbf{A} \mathbf{d}_n} = \frac{||\mathbf{g}_n||^2}{\mathbf{d}_n^T \mathbf{A} \mathbf{d}_n}$$

$$\mathbf{g}_{m+1} = \mathbf{A}\mathbf{x}_{m+1} - \mathbf{b} = \mathbf{A}(\mathbf{x}_m + \delta_m \mathbf{d}_m) - \mathbf{b} = (\mathbf{A}\mathbf{x}_m - \mathbf{b}) + \delta_m \mathbf{A}\mathbf{d}_m = \mathbf{g}_m + \delta_m \mathbf{A}\mathbf{d}_m$$

$$\mathbf{g}_n^T$$

$n > m$

$$\mathbf{g}_n^T \mathbf{g}_{m+1} = \mathbf{g}_n^T \mathbf{g}_m + \delta_m \mathbf{g}_n^T \mathbf{A} \mathbf{d}_m = \delta_m \mathbf{g}_n^T \mathbf{A} \mathbf{d}_m$$

$$\mathbf{g}_n^T \mathbf{g}_m = 0$$

$m \neq n$

$$\mathbf{g}_n^T \mathbf{A} \mathbf{d}_m$$

$$\mathbf{g}_n^T \mathbf{A} \mathbf{d}_m = \frac{1}{\delta_m} \mathbf{g}_n^T \mathbf{g}_{m+1} = \begin{cases} ||\mathbf{g}_n||^2/\delta_{n-1} & m = n-1 \\ 0 & m < n-1 \end{cases}$$

$$\beta_{nm} = -\frac{\mathbf{d}_m^T \mathbf{A} \mathbf{g}_n}{\mathbf{d}_m^T \mathbf{A} \mathbf{d}_m} = \begin{cases} -||\mathbf{g}_n||^2/\delta_{n-1}\mathbf{d}_{n-1}^T \mathbf{A} \mathbf{d}_{n-1} & m = n-1 \\ 0 & m < n-1 \end{cases}$$

$$m = n - 1$$

m

β_{nm}

$$\delta_{n-1} = ||\mathbf{g}_{n-1}||^2 / \mathbf{d}_{n-1}^T \mathbf{A} \mathbf{d}_{n-1}$$

$$\beta_{nm} = \beta_m$$

$$\beta_n = -\frac{||\mathbf{g}_n||^2}{||\mathbf{g}_{n-1}||^2}$$

$$\mathbf{d}_0 = -\mathbf{g}_0$$

$$\delta_n = \frac{||\mathbf{g}_n||^2}{\mathbf{d}_n^T \mathbf{A} \mathbf{d}_n}, \qquad \mathbf{x}_{n+1} = \mathbf{x}_n + \delta_n \mathbf{d}_n$$

$$\mathbf{g}_{n+1} = \frac{d}{d\mathbf{x}} f(\mathbf{x}_{n+1})$$

$$\beta_{n+1} = \frac{||\mathbf{g}_{n+1}||^2}{||\mathbf{g}_n||^2}$$

$$\mathbf{d}_{n+1} = -\mathbf{g}_{n+1} + \beta_{n+1}\mathbf{d}_n$$

$$f(x, y) = \mathbf{x}^T \mathbf{A} \mathbf{x} = [x_1,\ x_2] \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{x}_0 = [1.5,\ -0.75]^T$$

n	$\mathbf{x} = [x_1,\, x_2]$	$f(\mathbf{x})$
0	$1.500000, -0.750000$	2.812500
1	$0.250000, -0.750000$	$0.468750e-01$
2	$0.250000, -0.125000$	$7.812500e-02$
3	$0.041667, -0.125000$	$1.302083e-02$
4	$0.041667, -0.020833$	$2.170139e-03$
5	$0.006944, -0.020833$	$3.616898e-04$
6	$0.006944, -0.003472$	$6.028164e-05$
7	$0.001157, -0.003472$	$1.004694e-05$
8	$0.001157, -0.000579$	$1.674490e-06$
9	$0.000193, -0.000579$	$2.790816e-07$
10	$0.000193, -0.000096$	$4.651361e-08$
11	$0.000032, -0.000096$	$7.752268e-09$
12	$0.000032, -0.000016$	$1.292045e-09$
13	$0.000005, -0.000016$	$2.153408e-10$

n	$\mathbf{x} = [x_1,\ x_2]$	$f(\mathbf{x})$
0	$1.500000, -0.750000$	$2.812500e+00$
1	$0.250000, -0.750000$	$4.687500e-01$
2	$0.000000, -0.000000$	$1.155558e-33$

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$$

$$\mathbf{A} = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\mathbf{x}_0 = [1,\ 2,\ 3]^T$$

$$\mathbf{x}_{41} = [3.5486e-06,\ -7.4471e-06,\ 4.6180e-06]^T$$

$$f(\mathbf{x}) = 8.5429e - 11$$

n	$\mathbf{x} = [x_1,\ x_2,\ x_3]$	$f(\mathbf{x})$
0	$1.000000, 2.000000, 3.000000$	$4.500000e+01$
1	$-0.734716, -0.106441, 1.265284$	$2.809225e+00$
2	$0.123437, -0.209498, 0.136074$	$3.584736e-02$
3	$-0.000000, 0.000000, 0.000000$	$3.949119e-31$

$N = 9$

$$||\mathbf{e}|| \approx 10^{-10}$$

$$||\mathbf{e}|| \approx 10^{-16}$$

$$x_1 = 0.5,\ x_2 = 0,\ x_3 = -0.5$$

n	$\mathbf{x} = [x_1,\ x_2,\ x_3]$	$o(\mathbf{x})$
0	$0.0000, 0.0000, 0.0000$	$1.032500e+02$
1	$0.0113, 0.0050, -0.5001$	$3.160163e+00$
2	$0.0188, -0.0021, -0.5004$	$3.095894e+00$
3	$0.5009, -0.0018, -0.5004$	$7.268252e-05$
4	$0.5009, -0.0017, -0.5000$	$1.051537e-05$
5	$0.5008, -0.0012, -0.5000$	$6.511151e-06$
6	$0.5001, -0.0005, -0.5000$	$6.365321e-07$
7	$0.5001, -0.0005, -0.5000$	$5.667357e-07$
8	$0.5002, -0.0004, -0.5000$	$2.675128e-07$
9	$0.5001, -0.0003, -0.5000$	$1.344218e-07$
10	$0.5001, -0.0002, -0.5000$	$1.241196e-07$
11	$0.5000, -0.0001, -0.5000$	$2.120969e-08$
12	$0.5000, -0.0001, -0.5000$	$1.541814e-08$
13	$0.5000, -0.0001, -0.5000$	$7.282025e-09$
14	$0.5000, -0.0001, -0.5000$	$4.801781e-09$
15	$0.5000, -0.0000, -0.5000$	$4.463926e-09$

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}/2 - \mathbf{b}^T \mathbf{x} + c$$

$$d\,f(\mathbf{x})/d\mathbf{x} = \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}$$

$$\mathbf{Ax} - \mathbf{b} = \mathbf{0}$$

$$\mathbf{d}_i,\ (i = 1, \cdots, N)$$

$$\mathbf{d}_i^T \mathbf{A} \mathbf{d}_j = 0 \ (i \neq j)$$

$$\mathbf{x} = \sum_{i=1}^{N} c_i \mathbf{d}_i$$

$$\mathbf{b} = \mathbf{A}\mathbf{x} = \mathbf{A}\left[\sum_{i=1}^{N} c_i \mathbf{d}_i\right] = \sum_{i=1}^{N} c_i \mathbf{A}\mathbf{d}_i$$

$$\mathbf{d}_j^T$$

$$\mathbf{d}_j^T \mathbf{b} = \sum_{i=1}^{N} c_i \mathbf{d}_j^T \mathbf{A} \mathbf{d}_i = c_j \mathbf{d}_j^T \mathbf{A} \mathbf{d}_j$$

c_j

$$c_j = \frac{\mathbf{d}_j^T \mathbf{b}}{\mathbf{d}_j^T \mathbf{A} \mathbf{d}_j}$$

$$\mathbf{x} = \sum_{i=1}^{N} c_i \mathbf{d}_i = \sum_{i=1}^{N} \left(\frac{\mathbf{d}_i^T \mathbf{b}}{\mathbf{d}_i^T \mathbf{A} \mathbf{d}_i} \right) \mathbf{d}_i$$

$$\mathbf{b} = \mathbf{A}\mathbf{x}$$

$$\mathbf{p}_{\mathbf{d}_i}(\mathbf{x}) = \left(\frac{\mathbf{d}_i^T \mathbf{A}\mathbf{x}}{\mathbf{d}_i^T \mathbf{A}\mathbf{d}_i}\right)\mathbf{d}_i$$

M

$$R = N < M$$

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{A}^T\mathbf{A}$$

$$\Delta E = f(\mathbf{x}_{n+1}) - f(\mathbf{x}_n)_0$$

Th

P

$$P(\Delta E, T) = e^{-\Delta E/T} > Th$$

ΔE

T

$$\mathbf{x} = [x_1, \cdots, x_N]^T$$

$$
\begin{array}{ll}
\text{optimize} & f(\mathbf{x}) = f(x_1, \cdots, x_N) \\[2ex]
\text{subject to:} & \left\{ \begin{array}{ll} h_i(\mathbf{x}) = 0, & (i = 1, \cdots m) \\ g_j(\mathbf{x}) \le 0, & (j = 1, \cdots n) \end{array} \right.
\end{array}
$$

$$m + n$$

$$g_i(\mathbf{x})$$

$$h_j(\mathbf{x})$$

$$\begin{array}{ll} \text{maximize/minimize} & f(\mathbf{x}) = f(x_1, \cdots, x_N) \\ \text{subject to:} & h_i(\mathbf{x}) = h_i(x_1, \cdots, x_N) = 0, \qquad (i = 1, \cdots m) \end{array}$$

$$h_i(\mathbf{x}) = 0$$

$m \leq N$

$m = 1$

$h(\mathbf{x})$

$$h(\mathbf{x}) = 0$$

$$f(\mathbf{x}) = d$$

d

$$\bigtriangledown_{\mathbf{x}} f(\mathbf{x}^*) = \lambda^* \ \bigtriangledown_{\mathbf{x}} h(\mathbf{x}^*)$$

λ^*

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda\, h(\mathbf{x})$$

λ

$$\bigtriangledown_{\mathbf{x},\lambda} L(\mathbf{x}, \lambda) = \bigtriangledown_{\mathbf{x},\lambda}[f(\mathbf{x}) - \lambda\, h(\mathbf{x})] = \mathbf{0}$$

$$\begin{cases} \bigtriangledown_{\mathbf{x}} f(\mathbf{x}) = \lambda \bigtriangledown_{\mathbf{x}} h(\mathbf{x}) \\ \bigtriangledown_{\lambda} L(\mathbf{x}, \lambda) = \partial L(\mathbf{x}, \lambda)/\partial \lambda = -h(\mathbf{x}) = 0 \end{cases}$$

$$N + m = 2 + 1 = 3$$

$x_1,\, x_2,\, \lambda$

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = \lambda \frac{\partial h(\mathbf{x})}{\partial x_i} \quad (i = 1, 2), \qquad h(\mathbf{x}) = 0$$

$$\mathbf{x}^* = [x_1^*,\, x_2^*]^T$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*) = \lambda^* \nabla_{\mathbf{x}} h(\mathbf{x}^*)$$

$$h(\mathbf{x}^*) = 0$$

$$\lambda^* = 0$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*) = \lambda^* \nabla_{\mathbf{x}} h(\mathbf{x}^*) = 0$$

$\lambda^* \neq 0$

$$\bigtriangledown_{\mathbf{x}} f(\mathbf{x}^*) \neq 0$$

$$\nabla_{\mathbf{x}} h(\mathbf{x}^*) \neq \mathbf{0}$$

$N > 2$

$$h_i(\mathbf{x}) = 0 \ (i = 1, \cdots, m)$$

$$h_1(\mathbf{x}^*) = \cdots = h_m(\mathbf{x}^*) = 0$$

$$f(\mathbf{x}^*) = d$$

$$\nabla f(\mathbf{x}^*)$$

$$\bigtriangledown h_i(\mathbf{x}^*)$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*) = \sum_{i=1}^{m} \lambda_i^* \nabla_{\mathbf{x}} h_i(\mathbf{x}^*)$$

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x})$$

$$\lambda = [\lambda_1, \cdots, \lambda_m]^T$$

$$\bigtriangledown_{\mathbf{x},\lambda} L(\mathbf{x}, \lambda) = \bigtriangledown_{\mathbf{x},\lambda} \left[f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) \right] = \mathbf{0}$$

$$\triangledown_{\mathbf{x}} f(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i \triangledown_{\mathbf{x}} h_i(\mathbf{x})$$

$$\frac{\partial f(\mathbf{x})}{\partial x_j} = \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(\mathbf{x})}{\partial x_j} \quad (j = 1, \cdots, N),$$

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda_i} = h_i(\mathbf{x}) = 0 \quad (i = 1, \cdots, m)$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*)$$

$$\bigtriangledown_{\mathbf{x}} h_i(\mathbf{x}^*)$$

$\lambda_1, \cdots, \lambda_m$

$$\lambda_i = 0$$

$i = 1, \cdots, m$

$$\bigtriangledown_{\mathbf{x}} f(\mathbf{x}^*) = \sum_{i=1}^{m} \lambda_i \bigtriangledown_{\mathbf{x}} h_i(\mathbf{x}) = \mathbf{0}$$

$$
\begin{array}{ll}
\text{maximize/minimize} & f(\mathbf{x}) = f(x_1, \cdots, x_N) \\
\text{subject to:} & g_j(\mathbf{x}) = g_j(x_1, \cdots, x_N) \leq \text{ or } \geq 0, \qquad (j = 1, \cdots n)
\end{array}
$$

$$n = 1$$

$$g(\mathbf{x}) > 0$$

$$\mathbf{x}^* = [x_1^*,\, x_2^*]^T$$

$$g(x_1^*, x_2^*) = 0$$

$$\bigtriangledown_{\mathbf{x}} f(\mathbf{x}) = 0$$

$$g(\mathbf{x}^*) = 0$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*) \neq \mathbf{0}$$

$$g(\mathbf{x})$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*) = \mu^* \ \nabla_{\mathbf{x}}\, g(\mathbf{x}^*)$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*) - \mu^* \ \nabla_{\mathbf{x}}\, g(\mathbf{x}^*) = 0$$

$\mu > 0$

$\mu < 0$

$$g(\mathbf{x}) \geq 0$$

$$g(\mathbf{x}) \leq 0$$

μ

	$g(\mathbf{x}) \geq 0$	$g(\mathbf{x}) \leq 0$
$\max\ f(\mathbf{x})$	$\bigtriangledown f(\mathbf{x}_1) = \mu \bigtriangledown g(\mathbf{x}_1),\ \mu < 0$	$\bigtriangledown f(\mathbf{x}_3) = \mu \bigtriangledown g(\mathbf{x}_3),\ \mu > 0$
$\min\ f(\mathbf{x})$	$\bigtriangledown f(\mathbf{x}_2) = \mu \bigtriangledown g(\mathbf{x}_2),\ \mu > 0$	$\bigtriangledown f(\mathbf{x}_4) = \mu \bigtriangledown g(\mathbf{x}_4),\ \mu < 0$

$$\mu\, g(\mathbf{x}) \begin{cases} \leq 0 & \text{for maximization} \\ \geq 0 & \text{for minimization} \end{cases}$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*) = \mathbf{0}$$

$$g(\mathbf{x}^*) \neq 0$$

$$\bigtriangledown_{\mathbf{x}} f(\mathbf{x}) = \mu \ \bigtriangledown_{\mathbf{x}} g(\mathbf{x}) = \mathbf{0}$$

$$\mu = 0$$

$\mu^* \neq 0$

$$\mu^* = 0$$

$$\mu^* \, g(\mathbf{x}^*) = 0$$

$$g_i(\mathbf{x}) = 0 \ (i = 1, \cdots, n)$$

$$L(\mathbf{x}, \mu) = f(\mathbf{x}) - \sum_{i=1}^{n} \mu_i \, g_i(\mathbf{x})$$

$$\triangledown_{\mathbf{x},\mu} L(\mathbf{x},\mu) = \triangledown_{\mathbf{x},\mu} \left[f(\mathbf{x}) - \sum_{i=1}^{n} \mu_i \, g_i(\mathbf{x}) \right] = \mathbf{0}$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \sum_{i=1}^{n} \mu_i \nabla_{\mathbf{x}} g_i(\mathbf{x})$$

$$\frac{\partial L(\mathbf{x}, \mu)}{\partial \mu_i} = g_i(\mathbf{x}) = 0 \quad (i = 1, \cdots, n)$$

$$\nabla g(\mathbf{x})$$

$g(\mathbf{x}^*)$

μ^*

$\boldsymbol{\mu}$

$$
\begin{array}{ll}
\text{maximize/minimize} & f(\mathbf{x}) = f(x_1, \cdots, x_N) \\
\text{subject to:} & \begin{cases} h_i(\mathbf{x}) = 0, \quad (i = 1, \cdots m) \\ g_j(\mathbf{x}) \le 0 \ \text{ or } \ g_j(\mathbf{x}) \ge 0, \qquad (j = 1, \cdots n) \end{cases}
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize/minimize} & f(\mathbf{x}) \\
\text{subject to:} & \begin{cases} \mathbf{h}(\mathbf{x}) = \mathbf{0} \\ \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \ \text{ or } \ \mathbf{g}(\mathbf{x}) \geq \mathbf{0} \end{cases}
\end{array}
$$

$$\mathbf{h}(\mathbf{x}) = [h_1(\mathbf{x}), \cdots, h_m(\mathbf{x})]^T$$

$$\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}), \cdots, g_n(\mathbf{x})]^T$$

$$L(\mathbf{x}, \lambda, \mu) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) - \sum_{j=1}^{n} \mu_j g_j(\mathbf{x}) = f(\mathbf{x}) - \lambda^T \mathbf{h}(\mathbf{x}) - \mu^T \mathbf{g}(\mathbf{x})$$

$$\mu = [\mu_1, \cdots, \mu_n]^T$$

λ

μ_j

$$g_j(\mathbf{x}) \leq 0$$

$$g_j(\mathbf{x}) \geq 0$$

$$\mu_j \, g_j(\mathbf{x}^*) = 0, \qquad (j = 1, \cdots, n)$$

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$x_1 + x_2 = 1$$

$$x_1 + x_2 \leq 1$$

$$x_1 + x_2 \geq 1$$

$$h(x_1, x_2) = x_1 + x_2 - 1 = 0$$

$$\text{minimize/maximize:} \quad f(x_1, x_2) = x_1^2 + x_2^2$$
$$\text{subject to:} \quad h(x_1, x_2) = x_1 + x_2 - 1 = 0$$

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2) = x_1^2 + x_2^2 - \lambda(x_1 + x_2 - 1)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda = 0, \quad \frac{\partial L}{\partial x_2} = 2x_2 - \lambda = 0, \quad \frac{\partial L}{\partial \lambda} = x_1 + x_2 - 1 = 0$$

$$\lambda^* = 1$$

$$x_1^* = x_2^* = 0.5$$

$$x_1^* + x_2^* = 1$$

$$f(0.5,\ 0.5) = 0.5$$

$$f(x_1, x_2)$$

$$g(x_1, x_2)$$

$$\bigtriangledown f(0.5, 0.5) = \bigtriangledown h(0.5, 0.5) = [1, 1]^T$$

$$\begin{aligned} &\text{minimize:} && f(x_1, x_2) = x_1^2 + x_2^2 \\ &\text{subject to:} && g(x_1, x_2) = x_1 + x_2 - 1 \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{maximize:} && f(x_1, x_2) = x_1^2 + x_2^2 \\ &\text{subject to:} && g(x_1, x_2) = x_1 + x_2 - 1 \leq 0 \end{aligned}$$

$$L(x_1, x_2, \mu) = f(x_1, x_2) - \mu\, g(x_1, x_2) = x_1^2 + x_2^2 - \mu(x_1 + x_2 - 1)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - \mu = 0, \quad \frac{\partial L}{\partial x_2} = 2x_2 - \mu = 0, \quad \frac{\partial L}{\partial \mu} = x_1 + x_2 - 1 = 0$$

$$\mu^* = 1 > 0$$

$$x_1^* = x_2^* = \mu^*/2 = 0.5$$

$$
\begin{array}{ll}
\text{minimize:} & f(x_1, x_2) = x_1^2 + x_2^2 \\
\text{subject to:} & g(x_1, x_2) = x_1 + x_2 - 1 \leq 0
\end{array}
$$

$$\begin{aligned} \text{maximize:} \quad & f(x_1, x_2) = x_1^2 + x_2^2 \\ \text{subject to:} \quad & g(x_1, x_2) = x_1 + x_2 - 1 \geq 0 \end{aligned}$$

$$\frac{\partial L}{\partial x_1} = 2x_1 = 0, \quad \frac{\partial L}{\partial x_2} = 2x_2 = 0$$

$$x_1^* = x_2^* = 0$$

$$
\begin{array}{ll}
\text{minimize} & f_p(\mathbf{x}) \\
\text{subject to:} & \begin{cases} \mathbf{h}(\mathbf{x}) = \mathbf{0} \\ \mathbf{g}(\mathbf{x}) \le \mathbf{0} \ \text{ or } \ \mathbf{g}(\mathbf{x}) \ge \mathbf{0} \end{cases}
\end{array}
$$

$$L(\mathbf{x}, \lambda, \mu) = f_p(\mathbf{x}) - \lambda^T \mathbf{h}(\mathbf{x}) - \mu^T \mathbf{g}(\mathbf{x})$$

$$\mu^T g(\mathbf{x}) \geq 0$$

$$f_d(\lambda, \mu) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) = \inf_{\mathbf{x}} \left[f_p(\mathbf{x}) - \lambda^T \mathbf{h}(\mathbf{x}) - \mu^T \mathbf{g}(\mathbf{x}) \right]$$

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) = \mathbf{0}$$

$f_d(\lambda, \mu)$

$$
\begin{array}{ll}
\text{maximize} & f_d(\lambda, \mu) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) \\
\text{subject to:} & \mu \leq \mathbf{0} \ \text{ or } \ \mu \geq \mathbf{0}
\end{array}
$$

$$\mu^T \mathbf{g}(\mathbf{x}) \geq 0$$

$$\mathbf{h}(\mathbf{x}) = \mathbf{0}$$

$$\mathbf{g}(\mathbf{x}) \geq \mathbf{0}$$

$$f_d(\lambda, \mu) \leq L(\mathbf{x}, \lambda, \mu) = f_p(\mathbf{x}) - \lambda^T \mathbf{h}(\mathbf{x}) - \mu^T \mathbf{g}(\mathbf{x}) \leq f_p(\mathbf{x}) - \mu^T \mathbf{g}(\mathbf{x}) \leq f_p(\mathbf{x})$$

$$d^* = f_d(\lambda^*, \mu^*)$$

$$f_p(\mathbf{x}^*) = p^*$$

$$d^* = f_d(\lambda^*, \mu^*) \leq L(\mathbf{x}, \lambda, \mu) \leq f_p(\mathbf{x}^*) = p^*$$

$$p^* = f(\mathbf{x}^*) \geq f(\mathbf{x})$$

$\lambda^*,\ \mu^*$

$$d^* = f_d(\lambda^*,\ \mu^*) \geq f_p(\mathbf{x}^*) = p^*$$

$$p^* - d^* \geq 0$$

$$d^* - p^* \geq 0$$

$$d^* \neq p^*$$

d^*

$$f_d(\lambda^*, \mu^*) = L(\mathbf{x}^*, \lambda^*, \mu^*) = f_p(\mathbf{x}^*) - \lambda^{*T}\mathbf{h}(\mathbf{x}^*) - \mu^{*T}\mathbf{g}(\mathbf{x}^*)$$

$$f_p(\mathbf{x}^*) - \mu^{*T}\mathbf{g}(\mathbf{x}^*) = f_p(\mathbf{x}^*) = p^*$$

$$
\begin{cases}
\triangledown_{\mathbf{x}} L(\mathbf{x}^*, \lambda, \mu) = \triangledown_{\mathbf{x}} f(\mathbf{x}^*) - \sum_{i=1}^{m} \lambda_i \triangledown_{\mathbf{x}} h_i(\mathbf{x}^*) - \sum_{j=1}^{n} \mu_j \triangledown_{\mathbf{x}} g_j(\mathbf{x}^*) = \mathbf{0} & \text{(stationarity)} \\
\begin{cases} h_i(\mathbf{x}^*) = 0, \quad (i = 1, \cdots, m) \\ g_j(\mathbf{x}^*) \le 0 \ \text{ or } \ g_j(\mathbf{x}^*) \ge 0, \quad (j = 1, \cdots, n) \end{cases} & \text{(primal feasibility)} \\
\mu_j^* \le 0, \ \text{ or } \ \mu_j^* \ge 0, \quad (j = 1, \cdots, n) & \text{(dual feasibility)} \\
\mu_j^* g_j(\mathbf{x}^*) = 0, \quad (j = 1, \cdots, n) & \text{(complementarity)}
\end{cases}
$$

$$\mu_j^*$$

$$g_j(\mathbf{x}^*)$$

$$g_j(\mathbf{x}^*) \neq 0$$

$$\mu_j^* = 0$$

$\mu_j^* \neq 0$

$$g_j(\mathbf{x}^*) = 0$$

$$f(x_1, \cdots, x_N) = \sum_{i=1}^{N} c_i x_i$$

$$x_1, \cdots, x_N$$

$$f'(x_1, \cdots, x_N) = -\sum_{i=1}^{N} c_i x_i$$

$$
\begin{array}{ll}
\text{maximize} & f(x_1, \cdots, x_N) = \sum_{i=1}^{N} c_i x_i \\
\text{subject to:} & \left\{ \begin{array}{l} \sum_{i=1}^{n} a_{1i} x_i \leq b_1 \\ \cdots\cdots\cdots\cdots \\ \sum_{i=1}^{N} a_{Mi} x_i \leq b_M \\ x_1 \geq 0, \cdots, x_N \geq 0 \end{array} \right.
\end{array}
$$

$c_1, \cdots, c_N$

a_{ij}

b_i

$$f(x_1, \cdots, x_N)$$

$b_1, \cdots, b_M$

$$
\begin{aligned}
&\text{maximize} \quad f(x,y,z) = 2x - y + 3z \\
&\text{subject to:} \quad \begin{cases} x - 2y + z = 3 \\ 3x - y + 4z = 10 \\ y \geq 0, \;\; z \geq 0 \end{cases}
\end{aligned}
$$

$$x = 3 + 2y - z$$

$$2x - y + z = 3y + z + 6$$

$$5y + z = 1$$

$$
\begin{array}{ll}
\text{maximize} & f(y,z) = 3y + z + 6 \\
\text{subject to:} & \begin{cases} 5y + z = 1 \\ y \geq 0, \ \ z \geq 0 \end{cases}
\end{array}
$$

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} \mathbf{A}\mathbf{x} - \mathbf{b} \leq \mathbf{0} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MN} \end{bmatrix}$$

$$L(\mathbf{x}) = \mathbf{c}^T\mathbf{x} - \mathbf{y}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = (\mathbf{c} - \mathbf{A}^T\mathbf{y})^T\mathbf{x} + \mathbf{y}^T\mathbf{b}$$

$$\mathbf{y} = [y_1, \cdots, y_M]^T$$

$$\mathbf{Ax} - \mathbf{b} \leq \mathbf{0}$$

$$\mathbf{y} \geq \mathbf{0}$$

$$L(\mathbf{x})$$

$$f_d(\mathbf{y}) = \max_{\mathbf{x}} L(\mathbf{x}) = \max_{\mathbf{x}} [\mathbf{c}^T \mathbf{x} - \mathbf{y}^T (\mathbf{A}\mathbf{x} - \mathbf{b})] = \max_{\mathbf{x}} [(\mathbf{c} - \mathbf{A}^T \mathbf{y})^T \mathbf{x} + \mathbf{y}^T \mathbf{b}]$$

$$\bigtriangledown_{\mathbf{x}} L(\mathbf{x}) = \bigtriangledown_{\mathbf{x}}[(\mathbf{c} - \mathbf{A}^T\mathbf{y})^T\mathbf{x} + \mathbf{y}^T\mathbf{b}] = \mathbf{c} - \mathbf{A}^T\mathbf{y} = \mathbf{0}$$

$$L_d(\mathbf{y}) = \max_{\mathbf{x}} L(\mathbf{x})$$

$$f_d(\mathbf{y}) = \mathbf{y}^T \mathbf{b}$$

$$L(\mathbf{x}, \mathbf{y})$$

$$\mathbf{c} - \mathbf{A}^T\mathbf{y} \leq 0$$

$$(\mathbf{c} - \mathbf{A}^T\mathbf{y})^T\mathbf{x} \le 0$$

$$\mathbf{x} \geq 0$$

$$f_d(\mathbf{y})$$

$$f_p(\mathbf{x})$$

$$f_d(\mathbf{y}) = \mathbf{b}^T\mathbf{y} \geq L(\mathbf{x}) = \mathbf{c}^T\mathbf{x} - \mathbf{y}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) \geq \mathbf{c}^T\mathbf{x} = f_p(\mathbf{x})$$

$$\begin{array}{ll} \text{minimize} & f_d(\mathbf{y}) = \mathbf{b}^T \mathbf{y} \\ \text{subject to} & \begin{cases} \mathbf{A}^T \mathbf{y} - \mathbf{c} \geq \mathbf{0} \\ \mathbf{y} \geq \mathbf{0} \end{cases} \end{array}$$

$$\begin{cases} \max: & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} - \mathbf{b} \leq \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0} \end{cases} \Longleftrightarrow \begin{cases} \min & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}\mathbf{y} - \mathbf{c} \geq \mathbf{0}, \quad \mathbf{y} \geq \mathbf{0} \end{cases}$$

$$\left\{\begin{array}{ll} \min: & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x}-\mathbf{b}\leq\mathbf{0},\quad \mathbf{x}\geq\mathbf{0} \end{array}\right. \Longleftrightarrow \left\{\begin{array}{ll} \max & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}\mathbf{y}-\mathbf{c}\leq\mathbf{0},\quad \mathbf{y}\leq\mathbf{0} \end{array}\right. \Longleftrightarrow \left\{\begin{array}{ll} \max & -\mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}\mathbf{y}+\mathbf{c}\geq\mathbf{0},\quad \mathbf{y}\geq\mathbf{0} \end{array}\right.$$

$$d^*$$

p^*

$$
\begin{array}{ll}
\text{minimize} & \mathbf{b}^T\mathbf{y} \\
\text{subject to} & \mathbf{A}^T\mathbf{y} - \mathbf{c} \geq \mathbf{0}
\end{array}
$$

$$\mathbf{s} \geq \mathbf{0}$$

$$\begin{array}{ll} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & \begin{cases} \mathbf{A}^T\mathbf{y} - \mathbf{s} = \mathbf{c} \\ \mathbf{s} \geq \mathbf{0} \end{cases} \end{array}$$

$$\mathbf{c} \leq \mathbf{A}^T \mathbf{y}$$

$$\mathbf{b} \geq \mathbf{A}\mathbf{x}$$

$$\mathbf{c}^T\mathbf{x} \leq (\mathbf{A}^T\mathbf{y})^T\mathbf{x} = \mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{A}^T\mathbf{y} = (\mathbf{A}\mathbf{x})^T\mathbf{y} \leq \mathbf{b}^T\mathbf{y}$$

$$\mathbf{b}^T\mathbf{y} \geq \mathbf{c}^T\mathbf{x}$$

$$\sum_{i=1}^{N} a_i x_i \leq b \quad \Longrightarrow \quad \sum_{i=1}^{N} a_i x_i + s = b, \quad (s \geq 0)$$

$$
\begin{array}{ll}
\text{maximize} & z = f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} = \sum_{j=1}^{N} c_j x_j \\
\text{subject to:} & \left\{ \begin{array}{llllll}
\sum_{i=1}^{N} a_{1i}x_i & +s_1 & & & & = b_1 \\
\sum_{i=1}^{N} a_{2i}x_i & & +s_2 & & & = b_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\sum_{i=1}^{N} a_{Mi}x_i & & & & +s_M & = b_M \\
x_1 \geq 0, & \cdots, & x_N \geq 0, & s_1 \geq 0, & \cdots & s_M \geq 0
\end{array} \right.
\end{array}
$$

$$N + M$$

$\{s_1, \cdots, s_M\}$

$$\{x_1, \cdots, x_N\}$$

$$\mathbf{x} = \left[x_1,\ x_2, \cdots, x_N,\ s_1,\ s_2, \cdots, s_M\right]^T$$

$$M \times (N + M)$$

$$\mathbf{A} = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1N} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{1N} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} & 0 & \cdots & 0 & 1 \end{array} \right]_{M\times(N+M)} = \left[\ \mathbf{A}_{M\times N} \mid \mathbf{I}_{M\times M}\ \right]$$

$$\mathbf{A}_{M \times N}$$

$$\mathbf{I}_{M \times M}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} \mathbf{A}\mathbf{x} + \mathbf{s} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0},\ \mathbf{s} \geq \mathbf{0} \end{cases} \end{array} \qquad \text{or} \qquad \begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \end{array}$$

C

$$||\mathbf{c}|| = 1$$

$$f(\mathbf{x}) = \mathbf{c}^T \mathbf{x}$$

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{a}_j = [a_{i1}, \cdots, a_{IN}]^T$$

$$\sum_{j=1}^{N} a_{ij} x_j = \mathbf{a}_j^T \mathbf{x} = b_i, \qquad (i = 1, \cdots, M)$$

$$x_j \geq 0$$

$$\mathbf{x} \geq \mathbf{0}$$

$\mathbf{e}_j$

$$C_{M+N}^{N} = \frac{(M+N)!}{N!\,M!}$$

$$C_{M+N}^{N}$$

$\mathbf{x}^* \in P$

$\epsilon > 0$

$$\mathbf{x}' = \mathbf{x}^* + \epsilon\ \mathbf{c}/||\mathbf{c}|| \in P$$

$$f(\mathbf{x}') = \mathbf{c}^T\mathbf{x}' = \mathbf{c}^T(\mathbf{x}^* + \epsilon\ \mathbf{c}/||\mathbf{c}||) = \mathbf{c}^T\mathbf{x}^* + \epsilon\ \mathbf{c}^T\mathbf{c}/||\mathbf{c}|| = \mathbf{c}^T\mathbf{x}^* + \epsilon\ ||\mathbf{c}|| > \mathbf{c}^T\mathbf{x}^* = f(\mathbf{x}^*)$$

$$C_{N+M}^{N}$$

$$\begin{array}{ll} \text{maximize} & f_p(x_1, x_2) = 2x_1 + 3x_2 \\ \text{subject to:} & \begin{cases} 2x_1 + x_2 \leq 18 \\ 6x_1 + 5x_2 \leq 60 \\ 2x_1 + 5x_2 \leq 40 \\ x_1 \geq 0, \ \ x_2 \geq 0 \end{cases} \end{array} \qquad \text{or} \qquad \begin{array}{ll} \text{maximize} & f_p(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} & \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 18 \\ 60 \\ 40 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 5 \end{bmatrix}$$

$$M = 3$$

$$n + m = 5$$

$$C_{N+M}^{N} = C_5^2 = 10$$

$$f_p(x_1, x_2) = \mathbf{c}^T \mathbf{x} = c_1 x_1 + c_2 x_2$$

$$\mathbf{x} = [x_1,\ x_2]^T$$

$$\mathbf{c} = [2,\ 3]^T$$

$$C_5^2 = 10$$

$$\mathbf{x}^* = (5,\ 6)$$

$$f_p(x_1, x_2) = 2x_1 + 3x_2 = 2 \times 5 + 3 \times 6 = 28$$

$$\mathbf{c}^T\mathbf{x}^*/||\mathbf{c}|| = 28/3.61 = 7.77$$

	$(x_1,\ x_2)$		objective function (normalized)
1	5.00, 6.00	feasible	28 (7.77)
2	6.25, 5.50	infeasible	29 (8.04)
3	7.50, 3.00	feasible	24 (6.66)
4	20.00, 0.00	infeasible	40 (11.1)
5	10.00, 0.00	infeasible	20 (5.55)
6	9.00, 0.00	feasible	18 (4.10)
7	0.00, 8.00	feasible	24 (6.66)
8	0.00, 12.00	infeasible	36 (9.98)
9	0.00, 18.00	infeasible	54 (14.99)
10	0.00, 0.00	feasible	0 (0.00)

$$
\begin{array}{ll}
\text{maximize} & f_p(x_1, x_2) = 2x_1 + 3x_2 \\
\text{subject to:} & \left\{ \begin{array}{l} 2x_1 + x_2 \le 18 \\ 6x_1 + 5x_2 \le 60 \\ 2x_1 + 5x_2 \le 40 \\ x_1 \ge 0, \ \ x_2 \ge 0 \end{array} \right.
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ll}
\text{maximize} & f_p(x_1, x_2) = 2x_1 + 3x_2 \\
\text{subject to:} & \left\{ \begin{array}{l} 2x_1 + x_2 + s_1 = 18 \\ 6x_1 + 5x_2 + s_2 = 60 \\ 2x_1 + 5x_2 + s_3 = 40 \\ x_1 \ge 0, \ x_2 \ge 0, \ s_1 \ge 0, \ s_2 \ge 0, \ s_3 \ge 0 \end{array} \right.
\end{array}
$$

$$\mathbf{c}^T \mathbf{x} = z$$

$$\begin{array}{ll} \text{minimize} & f_d(y_1, y_2, y_3) = 18y_1 + 60y_2 + 40y_3 \\ \text{subject to:} & \begin{cases} 2y_1 + 6y_2 + 2y_3 \geq 2 \\ y_1 + 5y_2 + 5y_3 \geq 3 \\ y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0 \end{cases} \end{array} \quad \text{or} \quad \begin{array}{ll} \text{minimize} & f_d(\mathbf{y}) = \mathbf{b}^T\mathbf{y} \\ \text{subject to:} & \begin{cases} \mathbf{A}^T\mathbf{y} \leq \mathbf{b} \\ \mathbf{y} \geq \mathbf{0} \end{cases} \end{array}$$

$m = 2$

$n = 3$

$$y_1 \leq 0$$

$$y_2 \leq 0$$

$$y_3 \leq 0$$

10

	(y_1, y_2, y_3)	objective function	
1	$(1, 0, 0)$	infeasible	18
2	$(3, 0, 0)$	feasible	54
3	$(0, 1/3, 0)$	infeasible	20
4	$(0, 3/5, 0)$	feasible	36
5	$(0, 0, 1)$	feasible	40
6	$(0, 0, 3/5)$	infeasible	24
7	$(0, 1/5, 2/5)$	feasible	28
8	$(1/2, 0, 1/2)$	feasible	29
9	$(-2, 1, 0)$	infeasible	24
10	$(0, 0, 0)$	infeasible	0

$$(0, 1/5, 2/5)$$

$$f_d(y_1, y_2, y_3) = 18y_1 + 60y_2 + 40y_3$$

$$f_p(x_1, x_2) = 2x_1 + 3x_2$$

$$(5, 6)$$

$$\mathbf{A}_{M \times N} \mathbf{x}_{N \times 1} \leq \mathbf{b}_{M \times 1}$$

$$\mathbf{A}\mathbf{x} = [\mathbf{A}_{M \times N} \mid \mathbf{I}_{M \times M}]\mathbf{x} = \mathbf{b}$$

$$rank(\mathbf{A}) = M$$

I

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A}\mathbf{x} = [\mathbf{A}_n \;\; \mathbf{I}] \left[\begin{array}{c} \mathbf{x}_n \\ \mathbf{x}_b \end{array} \right] = \mathbf{A}_n \mathbf{x}_n + \mathbf{I}\,\mathbf{x}_b = \mathbf{b}$$

$\mathbf{x}_b$

$\mathbf{A}_n$

$$\mathbf{x}_n = \mathbf{0}$$

$$\mathbf{x}_b = \mathbf{b}$$

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_n \\ \mathbf{x}_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix}$$

$$\mathbf{Ax} \leq \mathbf{b}$$

$$\{\mathbf{e}_1, \cdots, \mathbf{e}_M\}$$

$$C_{N+M}^{M}$$

$$N + M + 1$$

$M + 1$

$$\sum_{j=1}^{N} a_{ij} x_j + s_i = b_i$$

$i = 1, \cdots, M$

$(M+1)st$

$$f(\mathbf{x}) = \sum_{j=1}^{N} c_j x_j$$

$$x_1 = \cdots = x_N = 0$$

$$z - c_1 x_1 - \cdots - c_N x_N = 0$$

$$(N + M + 1)st$$

$$\mathbf{b} = [b_1, \cdots, b_M]^T$$

$$z = f(\mathbf{x})$$

basic variables	x_1	x_2	$\cdots$	x_N	s_1	s_2	$\cdots$	s_M	basic solution
s_1	a_{11}	a_{12}	$\cdots$	a_{1N}	1	0	$\cdots$	0	b_1
s_2	a_{21}	a_{22}	$\cdots$	a_{2N}	0	1	$\cdots$	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
s_M	a_{M1}	a_{M2}	$\cdots$	a_{MN}	0	0	$\cdots$	1	b_M
z	$-c_1$	$-c_2$	$\cdots$	$-c_M$	0	0	$\cdots$	0	0

$$\mathbf{x}_n = [x_1, \cdots, x_N]^T$$

$$\mathbf{x}_b = [s_1, \cdots, s_M]^T$$

$$M \times M$$

$$\mathbf{x}_b = [s_1, \cdots, s_M]^T = \mathbf{b}, \quad \mathbf{x}_n = [x_1, \cdots, x_N]^T = \mathbf{0}$$

$$\mathbf{A}\mathbf{x} = \mathbf{A}_n\mathbf{x}_n + \mathbf{I}\mathbf{x}_b = \mathbf{b}$$

x_j

$-c_j$

$$c_j = \max\{c_1, \cdots, c_N\}$$

$$z = \sum_{j=1}^{N} c_j x_j$$

$x_k \quad (k \neq j)$

$$\sum_{j=1}^{N} a_{kj} x_j \leq b_k$$

$$x_j \leq b_k/a_{kj}$$

$$a_{kj} > 0$$

$$b_i / a_{ij} \le b_k / a_{kj}$$

$k = 1, \cdots, M$

s_i

$$a_{kj} < 0$$

$\mathbf{e}_i$

$$\mathbf{r}_i \leftarrow \mathbf{r}_i / a_{ij}$$

$$a_{ij} = 1$$

a_{kj}

$$\mathbf{r}_k \leftarrow \mathbf{r}_k - a_{kj}\mathbf{r}_i$$

$$a_{kj} = 0$$

$k \neq i$

$$s_i = 0$$

b

$$M > N$$

$$M < N$$

$$C_{M+N}^{M} = C_{10}^{3} = 10$$

$$M + N = 5$$

	x_1	x_2	s_1	s_2	s_3		objective function (normalized)
1	5	6	2	0	0	feasible	28 (7.77)
2	6.25	5.5	0	-5	0	infeasible	29 (8.04)
3	7.5	3	0	0	10	feasible	24 (6.66)
4	20	0	-2	-60	20	infeasible	40 (11.1)
5	10	0	-2	0	20	infeasible	20 (5.55)
6	9	0	0	6	22	feasible	18 (4.10)
7	0	8	10	20	0	feasible	24 (6.66)
8	0	12	6	0	-20	infeasible	36 (9.98)
9	0	18	0	-30	-50	infeasible	54 (14.99)
10	0	0	18	60	40	feasible	0 (0.00)

$$
\begin{array}{ll}
\text{maximize} & f(\mathbf{x}) = 2x_1 + 3x_2 \\
\text{subject to:} & \begin{cases} 2x_1 + x_2 + s_1 = 18 \\ 6x_1 + 5x_2 + s_2 = 60 \\ 2x_1 + 5x_2 + s_3 = 40 \\ x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0 \end{cases}
\end{array}
$$

u

v

w

b_1

b_2

b_3

	x_1	x_2	s_1	s_2	s_3	b
s_1	2	1	1	0	0	18
s_2	6	5	0	1	0	60
s_3	2	5	0	0	1	40
z	-2	-3	0	0	0	0

$$s_1 = 18$$

$$s_2 = 60$$

$$s_3 = 40$$

$$x_1 = x_2 = 0$$

$$j = 2$$

$$-c_2 = -3$$

$i = 3$

$$b_3/a_{32} = 40/5 = 8$$

b_k/a_{k2}

$18/1,\ 60/5,\ 40/5$

$$\mathbf{r}_i = \mathbf{r}_3 = [2,\ 5,\ 0,\ 0,\ 1,\ 40]$$

$$a_{ij} = a_{32} = 5$$

$$\mathbf{r}_3 = [0.4,\ 1,\ 0,\ 0,\ 0.2,\ 8]$$

$$a_{kj} = a_{k2}$$

$\mathbf{r}_k$

$$a_{kj} = a_{k2} = 0, \ (k = 1, 2)$$

$$a_{4j}\mathbf{r}_i = a_{42}\mathbf{r}_3$$

$$\mathbf{r}_4$$

	x_1	x_2	s_1	s_2	s_3	b
s_1	1.6	0	1	0	-0.2	10
s_2	4	0	0	1	-1	20
x_2	0.4	1.0	0	0	0.2	8
z	-0.8	0	0	0	0.6	24

$$x_1 = 0$$

$$x_2 = b_3 = 8$$

$$s_1 = 10$$

$$s_2 = 20$$

$$s_3 = 0$$

$$z = 24$$

$$j = 1$$

$$-c_1 = -0.8$$

$i = 2$

$$b_2/a_{21} = 20/4 = 5$$

b_k/a_{k1}

$$10/1.6 = 6.25,\ 20/4 = 5,\ 8/0.4 = 20$$

$$\mathbf{r}_i = \mathbf{r}_2 = [4,\ 0,\ 0,\ 1,\ -1,\ 20]$$

$$a_{21} = 4$$

$$\mathbf{r}_2 = [1,\ 0,\ 0,\ 0.25,\ -0.25,\ 5]$$

a_{k1}

$$a_{k1} = 0, (k = 1, 3)$$

$$a_{4j}\mathbf{r}_i = a_{41}\mathbf{r}_4$$

	x_1	x_2	s_1	s_2	s_3	b
s_1	0	0	1	-0.4	0.2	2
x_1	1	0	0	0.25	-0.25	5
x_2	0	1	0	$-0,1$	0.3	6
z	0	0	0	0.2	0.4	28

$$x_1 = b_2 = 5$$

$$y_2 = b_3 = 6$$

$$s_1 = 2$$

$$s_2 = s_3 = 0$$

$$z = 28$$

$$C_{N+M}^{M} = C_{5}^{3} = 10$$

$$x_1 = s_2 = 0$$

$$z = 0$$

$$x_2 = 8$$

$$x_1 = 5$$

$$x_2 = 6$$

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) = \frac{1}{2}[\mathbf{x}-\mathbf{m}]^T\mathbf{Q}[\mathbf{x}-\mathbf{m}] = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{c}^T\mathbf{x} + c \\ \text{subject to:} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array}$$

$$\mathbf{Q} = \mathbf{Q}^T$$

$$\mathbf{c} = -\mathbf{Q}\mathbf{m}, \ c = \mathbf{m}^T \mathbf{Q}\mathbf{m}/2$$

Q

$N \times N$

$m \times N$

$$\mathbf{x}^T \mathbf{Q} \mathbf{x}$$

m

$$\mathbf{Ax} - \mathbf{b} = \mathbf{0}$$

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{c}^T\mathbf{x} + \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b})$$

$\nabla_{\mathbf{x}} L(\mathbf{x}, \lambda)$

$$\nabla_{\mathbf{x}} f(\mathbf{x}) + \nabla_{\mathbf{x}} \lambda^T (\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{Q}\mathbf{x} + \mathbf{c} + \mathbf{A}^T \lambda = \mathbf{0}$$

$$\nabla_\lambda L(\mathbf{x}, \lambda)$$

$$\mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{Q} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} -\mathbf{c} \\ \mathbf{b} \end{bmatrix}$$

$m + N$

λ^*

$$f(x_1, x_2) = x_1^2 + x_2^2 = \frac{1}{2}[x_1,\ x_2] \left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = \frac{1}{2}\mathbf{x}^T \mathbf{Q} \mathbf{x}$$

$$\mathbf{A}\mathbf{x} = [1,\ 1] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = x_1 + x_2 = \mathbf{b} = 1$$

$$\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{A} = [1,\ 1], \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{b} = 1;$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1^* = x_2^* = 0.5$$

$$\lambda^* = -1$$

$$f(x_1^*, x_2^*) = 0.5$$

$$m = N$$

$m > N$

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ \text{subject to:} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array}$$

$$
\begin{aligned}
&\text{minimize} \quad && f(\mathbf{x}) \\
&\text{subject to:} \quad && \begin{cases} \mathbf{h}(\mathbf{x}) = \mathbf{0} \\ \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \quad \text{or} \quad \mathbf{g}(\mathbf{x}) \geq \mathbf{0} \\ \mathbf{x} \geq \mathbf{0} \end{cases}
\end{aligned}
$$

$$\mathbf{g}(\mathbf{x}) \leq \mathbf{0} \implies \mathbf{g}(\mathbf{x}) + \mathbf{s} = \mathbf{0}, \qquad \mathbf{g}(\mathbf{x}) \geq \mathbf{0} \implies \mathbf{g}(\mathbf{x}) - \mathbf{s} = \mathbf{0}$$

S

$$
\begin{aligned}
&\text{minimize} \quad && f(\mathbf{x}) \\
&\text{subject to:} \quad && \begin{cases} \mathbf{h}(\mathbf{x}) = \mathbf{0} \\ \mathbf{x} \geq \mathbf{0} \end{cases}
\end{aligned}
$$

$$L(\mathbf{x}, \lambda, \mu) = f(\mathbf{x}) + \lambda^T \mathbf{h}(\mathbf{x}) - \mu^T \mathbf{x}$$

$$\left\{\begin{array}{ll}\bigtriangledown_{\mathbf{x}} L(\mathbf{x},\lambda,\mu)=\mathbf{g}_f(\mathbf{x})+\mathbf{J}_{\mathbf{h}}^T(\mathbf{x})\lambda-\mu=\mathbf{0} & \text{(stationarity)}\\ \mathbf{h}(\mathbf{x})=\mathbf{0},\quad \mathbf{x}\geq\mathbf{0} & \text{(primal feasibility)}\\ \mu\geq\mathbf{0} & \text{(dual feasibility)}\\ \mu_j x_j=0,\quad (j=1,\cdots,N)\quad \text{or}\quad \mathbf{XM1}=\mathbf{0} & \text{(complementarity)}\end{array}\right.$$

$$\mathbf{X} = diag(x_1, \cdots, x_N)$$

$$\mathbf{M} = diag(\mu_1, \cdots, \mu_N)$$

$$\mathbf{J_f(x)}$$

$$\mathbf{h}(\mathbf{x})$$

$$\mathbf{X} = \begin{bmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{bmatrix}_{N\times N}, \quad \mathbf{M} = \begin{bmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_N \end{bmatrix}_{N\times N}, \quad \mathbf{J_h}(\mathbf{x}) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}_{M\times N}$$

$$I(x) = \begin{cases} 0 & x \geq 0 \\ \infty & x < 0 \end{cases}$$

$$x \geq 0$$

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) + \sum_{i=1}^{N} I(x_i) \\ \text{subject to} & \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{array}$$

$I(x)$

$t > 0$

$$-\frac{1}{t}\ln(x) \quad \overset{t\to\infty}{\Longrightarrow} \quad I(x)$$

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) - \frac{1}{t}\sum_{i=1}^{N} \ln x_i \\ \text{subject to} & \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{array}$$

$$L(\mathbf{x}, \lambda, \mu) = f(\mathbf{x}) + \lambda^T \mathbf{h}(\mathbf{x}) - \frac{1}{t} \sum_{j=1}^{N} \ln x_j$$

$$\bigtriangledown_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) = \mathbf{g}_f(\mathbf{x}) + \mathbf{J}_{\mathbf{h}}^T(\mathbf{x})\lambda - \frac{1}{t}\mathbf{X}^{-1}\mathbf{1}$$

$$\mu = \frac{\mathbf{x}}{t} = \frac{1}{t}\mathbf{X}^{-1}\mathbf{1},$$

$$\mathbf{X}\mu = \mathbf{X}\mathbf{M}\mathbf{1} = \frac{\mathbf{1}}{t}$$

$$\begin{cases} \mathbf{g}_f(\mathbf{x}) + \mathbf{J}_{\mathbf{h}}^T(\mathbf{x})\lambda - \mu = \mathbf{0} & \text{(stationarity)} \\ \mathbf{h}(\mathbf{x}) = \mathbf{0} & \text{(primal feasibility)} \\ \mathbf{XM1} - \mathbf{1}/t = \mathbf{0} & \text{(complementarity)} \end{cases}$$

$$\mu \geq \mathbf{0}$$

$$1/t$$

$t \to \infty$

$$\mathbf{F}(\mathbf{x}, \lambda, \mu) = \mathbf{0}$$

$\mathbf{J_F}$

$$\mathbf{F}(\mathbf{x}, \lambda, \mu)$$

$$\mathbf{F}(\mathbf{x}, \lambda, \mu) = \begin{bmatrix} \mathbf{g}_f(\mathbf{x}) + \mathbf{J}_{\mathbf{h}}^T(\mathbf{x})\lambda - \mu \\ \mathbf{h}(\mathbf{x}) \\ \mathbf{XM1} - \mathbf{1}/t \end{bmatrix}, \qquad \mathbf{J}_{\mathbf{F}} = \begin{bmatrix} \mathbf{W}(\mathbf{x}) & \mathbf{J}_{\mathbf{h}}^T(\mathbf{x}) & -\mathbf{I} \\ \mathbf{J}_{\mathbf{h}}(\mathbf{x}) & \mathbf{0} & \mathbf{0} \\ \mathbf{M} & \mathbf{0} & \mathbf{X} \end{bmatrix}$$

$$\mathbf{W}(\mathbf{x})$$

$$\nabla_{\mathbf{x}} \left[\mathbf{g}_f(\mathbf{x}) + \mathbf{J}_{\mathbf{h}}^T(\mathbf{x})\lambda - \mu \right]$$

$$\bigtriangledown_{\mathbf{x}} \mathbf{g}_f(\mathbf{x}) - \bigtriangledown_{\mathbf{x}} \left[\sum_{i=1}^{M} \lambda_i \frac{\partial h_i}{\partial x_1}, \cdots, \sum_{i=1}^{M} \lambda_i \frac{\partial h_i}{\partial x_N} \right]^T = \mathbf{H}_f(\mathbf{x}) - \sum_{i=1}^{M} \lambda_i \mathbf{H}_{h_i}(\mathbf{x})$$

$$\mathbf{H}_{h_i},\ (i = 1, \cdots, M)$$

$$\begin{bmatrix} \mathbf{x}_{n+1} \\ \lambda_{n+1} \\ \mu_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{n} \\ \lambda_{n} \\ \mu_{n} \end{bmatrix} + \alpha \begin{bmatrix} \delta\mathbf{x}_{n} \\ \delta\lambda_{n} \\ \delta\mu_{n} \end{bmatrix}$$

$$[\delta \mathbf{x}_n, \delta \lambda_n, \delta \mu_n]^T$$

$$\mathbf{J_F}(\mathbf{x}_n, \lambda_n, \mu_n) \begin{bmatrix} \delta\mathbf{x}_n \\ \delta\lambda_n \\ \delta\mu_n \end{bmatrix} = \begin{bmatrix} \mathbf{W}(\mathbf{x}_n) & \mathbf{J}_\mathbf{h}^T(\mathbf{x}_n) & -\mathbf{I} \\ \mathbf{J_h}(\mathbf{x}_n) & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_n & \mathbf{0} & \mathbf{X}_n \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_n \\ \delta\lambda_n \\ \delta\mu_n \end{bmatrix}$$

$$-\mathbf{F}(\mathbf{x}_n, \lambda_n, \mu_n) = -\left[\begin{array}{c} \mathbf{g}_f(\mathbf{x}_n) + \mathbf{J}_{\mathbf{h}}^T(\mathbf{x}_n)\lambda_n - \mu_n \\ \mathbf{h}(\mathbf{x}_n) \\ \mathbf{X_n M_n 1} - \mathbf{1}/t \end{array}\right]$$

$$\mu_0 = \mathbf{x}_0 / t$$

λ_0

$$\mathbf{X}^{-1}$$

$$\mathbf{M}\delta\mathbf{x} + \mathbf{X}\delta\mu = -\mathbf{X}\mathbf{M}\mathbf{1} + \mathbf{1}/t$$

$$\mathbf{X}^{-1}\mathbf{M}\delta\mathbf{x} + \delta\mu = -\mathbf{M}\mathbf{1} + \mathbf{X}^{-1}\mathbf{1}/t = -\mu + \mathbf{X}^{-1}\mathbf{1}/t$$

$$\mathbf{W}\delta\mathbf{x} + \mathbf{J}_{\mathbf{h}}^{T}(\mathbf{x})\delta\lambda - \delta\mu = -\mathbf{g}_f(\mathbf{x}) - \mathbf{J}_{\mathbf{h}}^{T}(\mathbf{x})\lambda + \mu$$

$$(\mathbf{W} + \mathbf{X}^{-1}\mathbf{M})\delta\mathbf{x} + \mathbf{J}_{\mathbf{h}}^T(\mathbf{x})\delta\lambda = -\mathbf{g}_f(\mathbf{x}) - \mathbf{J}_{\mathbf{h}}^T(\mathbf{x})\lambda + \mathbf{X}^{-1}\mathbf{1}/t = - \bigtriangledown_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu)$$

$\delta \mathbf{x}$

$\delta\lambda$

$$\begin{bmatrix} \mathbf{W} + \mathbf{X}^{-1}\mathbf{M} & \mathbf{J}_{\mathbf{h}}^T(\mathbf{x}) \\ \mathbf{J}_{\mathbf{h}}(\mathbf{x}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\lambda \end{bmatrix} = - \begin{bmatrix} \bigtriangledown_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) \\ \mathbf{h}(\mathbf{x}) \end{bmatrix}$$

$\delta\mu$

$$\mathbf{M}\delta\mathbf{x} + \mathbf{X}\delta\mu = -\mathbf{X}\mathbf{M}\mathbf{1} - \mathbf{1}/t$$

$$\delta\mu = \mathbf{X}^{-1}\mathbf{1}/t - \mathbf{X}^{-1}\mathbf{M}\delta\mathbf{x} - \mu$$

$$
\begin{aligned}
&\text{minimize} \quad && f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\
&\text{subject to:} \quad && \mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

$$\mathbf{g}_f(\mathbf{x}) = \bigtriangledown_{\mathbf{x}} f(\mathbf{x}) = \mathbf{c}$$

$$\mathbf{J_h}(\mathbf{x}) = \bigtriangledown_{\mathbf{x}}(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{A}$$

$$\mathbf{W}(\mathbf{x}) = \bigtriangledown^2_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) = \mathbf{0}$$

$$L(\mathbf{x}, \lambda, \mu) = \mathbf{c}^T \mathbf{x} + \lambda^T (\mathbf{A}\mathbf{x} - \mathbf{b}) - \mu^T \mathbf{x}$$

$$\left\{\begin{array}{l} \bigtriangledown_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) = \mathbf{g}_f(\mathbf{x}) + \mathbf{J}_{\mathbf{h}}^T(\mathbf{x})\lambda - \mu = \mathbf{c} + \mathbf{A}^T\lambda - \mu = \mathbf{0} \\ \mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0} \\ \mathbf{X}\mathbf{M}\mathbf{1} - \mathbf{1}/t = \mathbf{0} \end{array}\right.$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{A}^T & -\mathbf{I} \\ \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{M} & \mathbf{0} & \mathbf{X} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x} \\ \delta \lambda \\ \delta \mu \end{bmatrix} = - \begin{bmatrix} \mathbf{c} + \mathbf{A}^T \lambda - \mu \\ \mathbf{A}\mathbf{x} - \mathbf{b} \\ \mathbf{X}\mathbf{M}\mathbf{1} - \mathbf{1}/t \end{bmatrix}$$

$$\mathbf{c} + \mathbf{A}^T \lambda - \mu = \mathbf{0}$$

$$
\begin{array}{ll}
\text{maximize} & f(x_1, x_2) = 2x_1 + 3x_2 \\
\text{subject to:} & \begin{cases} 2x_1 + x_2 \leq 18 \\ 6x_1 + 5x_2 \leq 60 \\ 2x_1 + 5x_2 \leq 40 \\ x_1 \geq 0, \ \ x_2 \geq 0 \end{cases}
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ll}
\text{minimize} & f(x_1, x_2, x_3, x_4, x_5) = -2x_1 - 3x_2 \\
\text{subject to:} & \begin{cases} 2x_1 + x_2 + x_3 = 18 \\ 6x_1 + 5x_2 + x_4 = 60 \\ 2x_1 + 5x_2 + x_5 = 40 \\ x_i \geq 0, \ \ (i = 1, \cdots, 5) \end{cases}
\end{array}
$$

$$
\begin{array}{ll}
\text{minimize} & f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\
\text{subject to} & \mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}, \;\; \mathbf{x} \geq \mathbf{0}
\end{array}
$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 6 & 5 & 0 & 1 & 0 \\ 2 & 5 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 18 \\ 60 \\ 40 \end{bmatrix}$$

$x_3,\ x_4,\ x_5$

$$\mathbf{x}_0 = [1,\ 2,\ 1,\ 1,\ 1]^T$$

$t = 9$

9

$$\mathbf{x}^* = [5,\ 6]^T$$

$$f(\mathbf{x}^*) = 2x_1 + 3x_2 = 28$$

	$(x_1$	$x_2)$	$f(\mathbf{x})$	error
1	$(1.000000e+00$	$2.000000e+00)$	-8.000000	52.413951
2	$(4.654514e+00$	$6.346354e+00)$	-28.348090	3.080204
3	$(5.040828e+00$	$5.946973e+00)$	-27.922575	0.213509
4	$(4.997282e+00$	$6.001764e+00)$	-27.999856	0.004262
5	$(4.999906e+00$	$5.999962e+00)$	-27.999697	0.000303
6	$(4.999989e+00$	$5.999996e+00)$	-27.999966	0.000034
7	$(4.999999e+00$	$6.000000e+00)$	-27.999996	0.000004
8	$(5.000000e+00$	$6.000000e+00)$	-28.000000	0.000000

$$x_3 = 2,\ x_4 = x_5 = 0$$

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ \text{subject to:} & \mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0} \end{array}$$

$$\mathbf{g}_f(\mathbf{x}) = \bigtriangledown_{\mathbf{x}} f(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \mathbf{c}$$

$$\mathbf{W}(\mathbf{x}) = \bigtriangledown^2_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) = \mathbf{Q}$$

$$L(\mathbf{x}, \lambda, \mu) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{c}^T\mathbf{x} + \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b}) - \mu^T\mathbf{x}$$

$$\begin{bmatrix} \mathbf{Q} & \mathbf{A}^T & -\mathbf{I} \\ \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{M} & \mathbf{0} & \mathbf{X} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\lambda \\ \delta\mu \end{bmatrix} = - \begin{bmatrix} \mathbf{Q}\mathbf{x} + \mathbf{c} + \mathbf{A}^T\lambda - \mu \\ \mathbf{A}\mathbf{x} - \mathbf{b} \\ \mathbf{X}\mathbf{M}\mathbf{1} - \mathbf{1}/t \end{bmatrix}$$

$$
\begin{array}{ll}
\text{minimize} & f(\mathbf{x}) = [\mathbf{x}-\mathbf{m}]^T \mathbf{Q}[\mathbf{x}-\mathbf{m}] = \mathbf{x}^T \mathbf{Q}\mathbf{x} + \mathbf{c}^T \mathbf{x} + d \\
\text{subject to} & \mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}, \;\; \mathbf{x} \geq \mathbf{0}
\end{array}
$$

$$\mathbf{Q} = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\mathbf{m} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$$\mathbf{x}_0 = [2,\ 1]^T$$