**STAT 243 HW#4**

**1.** We use the Monte Carlo method to evaluate each of the following integrals:

**a.**

$$\int_0^1 x^2 dx$$

Here, we let $f(x) = \frac{1}{1-0}$, a Uniform(0,1) distribution, and we let $h(x) = x^2$. We take 1,000,000 samples from a Uniform(0,1), and using the Monte Carlo method we obtain an estimate of the evaluated integral as .333328, or equivalenty $\frac{1}{3}$.

**b.**

$$\int_0^1 \int_{-2}^2 x^2 \cos(xy) \quad dxdy$$

**c.**

$$\int_0^\infty \frac{3}{4} e^{-\frac{x^3}{4}} dx$$

**2.** We let

$$I = \frac{1}{\sqrt{2\pi}} \int_1^2 e^{-\frac{x^3}{2}} dx$$

and estimate $I$ using importance sampling. We take $g$ to be $N(1.5, \nu^2)$ with $\nu = 0.1$ and 10.

**3.** We will approximate the following integral using both the simple Monte Carlo integration and the control variate method:

$$I = \int_0^1 \frac{1}{1+x} dx$$

**a.** We let $h(x) = \frac{1}{1+x}$ and $U_1, \dots U_n$ be iid Unif[0,1].

**b.**

**c.**

**d.**

**4.** We consider a common application in statistics: three different treatments are to be compared by applying them to a randomly selected experimental units. However; instead of the usualy assumption associated with $e_{ij}$ in the typical analysis of variance methods of comparison, we assume that the $e_{ij}$ have independent and identical double exponential distributions centered on zero.

a.

b.

**5.** We consider the zero-inflated Poisson( ZIP) model, in which random data $X_1, \ldots, X_n$ are assumed to be of the form $X_i = R_i Y_i$, where $Y_i$'s have a Poisson($\lambda$) distribution and the $R_i$'s have a Bernoulli($p$) distribution, all independent of each other. Given an outcome $x = (x_1, \ldots, x_n)$, the objective is to estimate both $\lambda$ and $p$. We consider the following hierarchical Bayes model:

- $p \sim$ Uniform(0,1) (prior for $p$)
- $(\lambda|p) \sim$ Gamma$(a, b)$ (prior for $\lambda$)
- $(r_i|p, \lambda) \sim$ Bernoulli($p$) independently (from the model above)
- $(x_i|r, \lambda, p) \sim$ Poisson($\lambda r_i$) independently (from the model above)

where $a$ and $b$ are known parameters, and $r = (r_1, \ldots, r_n)$. It follows that:

$$f(x, r, \lambda, p) = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} \prod_{i=1}^{n} \frac{e^{-\lambda r_i} (\lambda r_i)^{x_i}}{x_i!} p^{r_i} (1-p)^{1-r_i}$$

We wish to sample from the posterior pdf $f(\lambda, p, r|x)$ using the Gibbs sampler.

**a.**

**b.**

**c.**

**6.** The Independence - Metropolis - Hastings Algorithm is an importance-sampling version of MCMC. We draw the proposal from a fixed distribution $g$. Generally, $g$ is chosen to be an approximation to $f$. The acceptance probability becomes:

$$r(x, y) = min\left\{\frac{f(y)}{f(x)} \cdot \frac{g(x)}{g(y)}, 1\right\}$$

A random variable Z has an inverge Gaussian distribution it it has density

$$f(z) \propto z^{-3/2} \text{exp}\left\{-\theta_1 z - \frac{\theta_2}{z} + 2\sqrt{\theta_1\theta_2} + \log\sqrt{2\theta_2}\right\}, \quad z > 0$$

where $\theta_1 0$ and $\theta_2 > 0$ are parameters. It can be shown that

$$E(Z) = \sqrt{\frac{\theta_2}{\theta_1}} \quad \text{and} \quad E\left(\frac{1}{Z}\right) = \sqrt{\frac{\theta_1}{\theta_2}} + \frac{1}{2\theta_2}$$

We let $\theta_1 = 1.5$ and $\theta_2 = 2$, and draw a sample of size 1,000 using the independence-Metropolis-Hastings algroithm. We use a Gamma distribution as the proposal density.