MATH 444 HW 7

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3.4.10

x_n is bounded, so by the Bolzano-Weierstrass Theorem, x_n has a convergent subsequence. This comes from the monotone subsquence theorem. We realize that we have 2 cases for s_n:

1. s_n is a constant sequence.
In this case, because $\forall\, k \in \mathbb{N}\ \sup x_k:\ k > n = L$ means that even after taking finitely many x_i out of the sequence, that infinitely many x_{k+i} are epsilon close to s_n, thus converging to $\inf s_n = s_n$.

2. s_n is a decreasing sequence.
In this case, we can find a natural k such that $s_k > s_{k+1}$. Now consider the s_{k+1} tail of the sequence s_n. We can keep repeating this process and s_n will converge because s_n is decreasing and bounded. Once we find the limit of s_n, we know that infinitely many x_n are epsilon close to the limit of s_n (because of case 1), so we see that there is definitely a subsequence of x_n which converges to the limit of s_n which is S (because s_n is decreasing).

3.5.2

a.

If $x_n = \left(\frac{n+1}{n}\right)$, show that $\forall\, \epsilon > 0,\ \exists\, N_\epsilon \in \mathbb{N}:\ n > N_\epsilon \Rightarrow |x_n - x_m| < \epsilon$.

$$|x_n - x_m| = \left|\frac{n+1}{n} - \frac{n+p+1}{n+p}\right| = \left|\frac{(n+p)(n+1)}{n(n+p)} - \frac{n(n+p+1)}{n(n+p)}\right| = \left|\frac{n^2+pn+n+1-n^2-np-n}{n(n+p)}\right| =$$

$$\frac{1}{n(n+p)} < \frac{1}{n} < \epsilon \Rightarrow \frac{1}{\epsilon} < n$$

By the Archimedean Property, such n exists, and it follows that x_n is Cauchy.

b.

If $x_n = 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}$, show that $\forall\, \epsilon > 0,\ \exists\, N_\epsilon \in \mathbb{N}:\ n, m > N_\epsilon \Rightarrow |x_n - x_m| < \epsilon$

$$|x_n - x_m| = |x_n - x_{n+p}| = \left|1 + \frac{1}{2!} + \ldots + \frac{1}{n!} - \left(1 + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots + \frac{1}{(n+p)!}\right)\right| =$$

$$\frac{1}{(n+1)!} + \ldots + \frac{1}{(n+p)!} < \frac{p}{n!} \le \frac{p}{n} < \epsilon \Rightarrow \frac{p}{\epsilon} < n$$

By the Archimedean Property, such n exists, and it follows that x_n is Cauchy.

3.5.3c

Suppose $\ln n$ is Cauchy. Then, $\forall\,\epsilon > 0,\ \exists\, N_\epsilon \in \mathbb{N}:\ n, m > N_\epsilon \Rightarrow |x_n - x_m| < \epsilon$.
Let us fix $n > N_\epsilon$. $\ln n$ is well defined. Let us now examine $\lim_{m\to\infty} \ln m = \infty$. We can clearly see that $\lim_{m\to\infty} |x_n - x_m| = |\ln n - \infty| = \infty < \epsilon$, which is a contradiction. Thus, $x_n = \ln n$ must not be Cauchy.

3.5.4

Suppose that x_n and y_n are Cauchy. Then it follows that:

$$\forall\,\epsilon > 0\ \exists\, N_\epsilon :\ n, m > N_\epsilon \Rightarrow |x_n - x_m| < \frac{\epsilon}{2}$$

$$\forall\,\epsilon > 0\ \exists\, N_\epsilon :\ i, k > N'_\epsilon \Rightarrow |y_i - y_k| < \frac{\epsilon}{2}$$

Denote $z_n \coloneqq \{x_n + y_n,\ n \in \mathbb{N}\}$. Let $M = \max\{N_\epsilon, N'_\epsilon\}$. We can see that

$$|z_n - z_m| = |x_n + y_n - x_m - y_m| = |x_n - x_m + y_n - y_m| \le |x_n - x_m| + |y_n - y_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So we see that $\forall\,\epsilon > 0,\ \exists\, M \in \mathbb{N}:\ n, m > M \Rightarrow |z_n - z_m| < \epsilon$, so z_n is Cauchy.

Every Cauchy sequence is bounded, so let $X \coloneqq \sup x_n$ and $Y \coloneqq \sup y_n$. We see that

$$\forall\,\epsilon > 0\ \exists\, N_\epsilon :\ n, m > N_\epsilon \Rightarrow |x_n - x_m| < \frac{\epsilon}{2 \cdot X}$$

$$\forall \epsilon > 0\ \exists\, N'_\epsilon :\ i, k > N'_\epsilon \Rightarrow |y_i - y_k| < \frac{\epsilon}{2 \cdot Y}$$

Denote $z_n \coloneqq \{x_n y_n,\ n \in \mathbb{N}\}$.

$$|z_n - z_m| = |x_n y_n - x_m y_m| = |x_n y_n - x_n y_m + x_n y_m - x_m y_m| = |x_n(y_n - y_m) + y_m(x_n - x_m)| \le$$

$$x_n |y_n - y_m| + y_m |x_n - x_m| < x_n \frac{\epsilon}{2 \cdot X} + y_m \frac{\epsilon}{2 \cdot Y} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So wee see that, taking $M = \max\{N_\epsilon, N'_\epsilon\}$, $\forall\,\epsilon > 0,\ \exists\, M \in \mathbb{N}:\ n, m > M \Rightarrow |z_n - z_m| < \epsilon$, so z_n is Cauchy.

3.5.5

If $x_n = \sqrt{n}$, show that x_n satisfies $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$.

Show that $\forall\,\epsilon > 0,\ \exists\, N_\epsilon \in \mathbb{N}:\ n > N_\epsilon \Rightarrow |x_{n+1} - x_n| < \epsilon^2 + 1$.

$$|x_{n+1} - x_n| = |\sqrt{n+1} - \sqrt{n}| \le |sqrtn + 1| - |\sqrt{n}| = \sqrt{n+1} - \sqrt{n} < \sqrt{n+1} < \epsilon \Rightarrow n > \epsilon^2 + 1$$

We see that such n exists due to the Archimedean property, so it follows that $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$.

Claim: $x_n = \sqrt{n}$ is not Cauchy.
Suppose that x_n is Cauchy. Then $\forall\,\epsilon > 0,\ \exists\, N_\epsilon \in \mathbb{N}:\ n, m > N_\epsilon \Rightarrow |x_n - x_m| < \epsilon$.
Fix $n > N_\epsilon$. We see that $\lim_{m\to\infty} \sqrt{m} = \infty$, so $\lim_{m\to\infty} |x_n - x_m| = |\sqrt{n} - \infty| = \infty < \epsilon$, a contradiction. Thus, $x_n = \sqrt{n}$ is not Cauchy.

3.5.7

x_n is Cauchy, so $\forall\, \epsilon > 0 \;\exists\, N_\epsilon \in \mathbb{N}:\; n, m > N_\epsilon \Rightarrow |x_n - x_m| < \epsilon$.
Fix $\epsilon = 1$. By the above definition, there exist $n, m \in \mathbb{N}$ usch that $|x_n - x_m| < \epsilon$. But x_n and x_m are both integers, and integer addition and subtraction always results in an integer, so $|x_n - x_m| < 1 \Rightarrow x_n = x_m$, which means that $\forall\, i > \min\{n, m\}$, the x_i tail of the sequence is constant.

3.7.3a

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} \Rightarrow 1 = A(n+2) + B(n+1)$$

Let $n = -2$. Then:

$$1 = A(0) + B(-1) \Rightarrow B = -1$$

Let $n = -1$. Then:

$$1 = A(1) + B(0) \Rightarrow A = 1$$

so

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{1}{n+1} - \frac{1}{n+2}$$

.

$$\sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+2} = \left[1 - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \ldots + \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

After canceling values, we observe that $\sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+2} = \left[1 - \frac{1}{n+2}\right]$.

$$\lim_{n\to\infty} \left[1 - \frac{1}{n+2}\right] = 1 - 0 = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$$

3.7.5

No, because $\sum(x_n + y_n) = \sum x_n = \sum y_n$ and for $\sum(x_n + y_n)$ to be convergent, both x_n and y_n must be convergent. $(L + \infty = \infty)$