MATH 444 HW 8

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4.1.1

a.

$$|x^2 - 1| = |x+1||x-1|$$

$|x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 1 < x+1 < 3$, so clearly, $x+1 < 3$ if $|x-1| < 1$. Now fix $|x-1| < \frac{1}{6}$. Since $|x-1| < \frac{1}{6} < 1 \Rightarrow |x+1| < 3$, we achieve $|x^2+1| = |x+1||x-1| < \left(\frac{1}{6}\right)(3) = \frac{1}{2}$. It suffices to say that $|x-1| < \frac{1}{6} \Rightarrow |x^2-1| < \frac{1}{2}$.

d.

$$|x^3 - 1| = |x-1||x^2+x+1|$$

$$|x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$$

The x-coordinate of the vertex of the parabola x^2+x+1 is $-\frac{b}{2a} = -\frac{1}{2}$, so on the interval $0 < x < 2$, x^2+x+1 s strictly increasing.
$0 < x < 2 \Rightarrow 1 < x^2+x+1 < 7$, so clearly, if $|x-1| < 1$, $x^2+x+1 < 7$.

Now take $|x-1| < \frac{1}{7n}$. We see that $|x^3-1| = |x-1||x^2+x+1| < \left(\frac{1}{7n}\right)(7) = \frac{1}{n}$.

4.1.9

a.

$$\lim_{x\to 2} \frac{1}{1-x} = -1 \Leftrightarrow \forall\, \epsilon > 0\, \exists\, \delta > 0 : 0 < |x-2| < \delta \Rightarrow \left|\frac{1}{1-x} + 1\right| < \epsilon$$

$$\left|\frac{1}{1-x} + 1\right| < \epsilon \Rightarrow \left|\frac{1+1-x}{1-x}\right| = \left|\frac{x-2}{x-1}\right| = \frac{|x-2|}{|x-1|} < \epsilon$$

Let us assume that $|x-2| < \frac{1}{2}$.

$$|x-2| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x-2 < \frac{1}{2} \Rightarrow \frac{3}{2} < x < \frac{5}{2} \Rightarrow x-1 > \frac{3}{2} - 1 = \frac{1}{2}$$

Thus, $\left|\frac{x-2}{x-1}\right| < \frac{1}{2}|x-2| < \epsilon = |x-2| < 2\epsilon$, so set $\delta = \min\{\frac{1}{2}, 2\epsilon\}$.

d.

$$\lim_{x\to 1}\frac{x^2-x+1}{x+1}=\frac{1}{2}=\frac{1}{2}\Rightarrow\ \forall\epsilon>0\,\exists\,\delta>0:\ 0<|x-1|<\delta\Rightarrow\left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right|<\epsilon$$

$$\left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right|=\left|\frac{2(x^2-x+1)}{2(x+1)}-\frac{x+1}{2(x+1)}\right|=\left|\frac{2x^2-2x+2-x-1}{2(x+1)}\right|=$$

$$\left|\frac{2x^2-x+1}{2(x+1)}\right|=\left|\frac{(2x-1)(x-1)}{2(x+1)}\right|=\frac{1}{2}\frac{|2x-1||x-1|}{|x+1|}=\frac{|2x-1||x-1|}{|x+1|}<2\epsilon$$

Assume that $|x-1|<1$. Then,

$$-1<x-1<1\Rightarrow 0<x<1\Rightarrow -1<2x-1<1\Rightarrow 1<x+1<2$$

So clearly, $|x-1|<1\Rightarrow 2x-1<1$ and $x+1>1$.

$\left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right|<\epsilon\Rightarrow\frac{|2x-1||x-1|}{|x+1|}<\epsilon\Rightarrow|x-1|<\epsilon$, so set $\delta=\min\{1,\epsilon\}$.

4.1.12

a.

Let $x_n=\frac{1}{n^2}$, $n\in\mathbb{N}$ ($x_n\neq 0$ and $\lim_{n\to\infty}x_n=0$). Because $(f(x_n)=n)_{n=1}^{\infty}$ is unbounded, $\lim_{n\to\infty}f(n)$ does not exist $\Rightarrow\lim_{x\to 0}f(x)$ does not exists by the divergence criterion.

d.

Let $\sin\left(\frac{1}{x^2}\right)=1\Rightarrow\frac{1}{x^2}=\frac{\pi}{2}+2\pi n$ ($n\in\mathbb{Z}$).
Let $\sin\left(\frac{1}{x^2}\right)=-1\Rightarrow\frac{1}{x^2}=-\frac{\pi}{2}+2\pi n$ ($n\in\mathbb{Z}$).

Define the following sequence: $x_n:=\frac{1}{\sqrt{\frac{\pi}{2}+2\pi n}}$, $y_n:=\frac{1}{\sqrt{-\frac{\pi}{2}+2\pi n}}$. Clearly, $0<-\frac{\pi}{2}+2\pi n<\frac{\pi}{2}+2\pi n$, $n>0$ so x_n and y_n are well defined and $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=0$. But $\lim_{n\to\infty}\sin(x_n)=1\neq -1=\lim_{n\to\infty}\sin(y_n)$. By the divergence criterion, $\lim_{x\to 0}\sin\left(\frac{1}{x^2}\right)$ does not exist.

4.1.15

a.

f has a limit at $x=0$ because by the density of real numbers, $\forall\ \epsilon>0$, every V_ϵ neighborhood of f around 0 contains at least one rational and one irrational. the rational, x_0 and irrational, x_1 are clearly defined inside V_ϵ. $|x-x_0|=|0-x_0|<\epsilon$ because x_0 is in the ϵ neighborhood of f around 0, and $|x-x_1|=|0-0|=0<\epsilon$, so all points in V_ϵ are ϵ close to 0. So 0 is a limit of f.

b.

Given $c \neq 0 \in \mathbb{R}$, there exist subsequences x_n and y_n in $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$, respectively that converge to c due to the density property of real numbers for rational and irrational numbers. So $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c$. But $\lim_{n\to\infty} f(x_n) = c$ and $\lim_{n\to\infty} f(y_n) = 0$. Thus, by the divergence criterion, c is not a limit point of f.

4.2.1

a.

$$\lim_{x\to 1}(x+1)(2x+3) = \left(\lim_{x\to 1}(x+1)\right) = 2 \cdot 5 = 10$$

c.

$$\lim_{x\to 2}\left(\frac{1}{x+1} - \frac{1}{2x}\right) = \lim_{x\to 2}\left(\frac{1}{x+1}\right) - \lim_{x\to 2}\left(\frac{1}{2x}\right) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

5.1.11

$$|x-y| < \delta$$

$$|f(x) - f(y)| \leq K|x-y| < K\delta < \epsilon \Rightarrow \delta < \frac{\epsilon}{K}$$

We have found that such δ exists for every $y \in \mathbb{R}$, so it must be true that f is continuous at every point $c \in \mathbb{R}$.