CS 361 Spring 2018
Homework 4

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4.1

Assuming that the roulette wheel has slots numbered 0-36,

$$P(X = x) = \begin{cases} \frac{1}{37} & \text{if } 0 \leq x \leq 36 \\ 0 & \text{otherwise} \end{cases}$$

4.2

(a)

$$P(X \geq 2) = P(X = 1) + P(X = 2) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

(b)

$$P(X \geq 10) = P(X = 10) + P(X = 11) = \frac{1}{13} + \frac{3}{13} = \frac{4}{13}$$

(c)

$$\begin{aligned} P(X \geq Y) &= P(X \geq X + 1 \cap X \text{ is black}) + P(X \geq X - 1 \cap X \text{ is red}) \\ &= P(X \geq X - 1)P(X \text{ is red}) \quad \text{because of independence} \\ &= 1 \cdot P(\text{X is red}) \\ &= \frac{1}{2} \end{aligned}$$

(d)

$P(Y - X)$ can take on 2 values: $\{-1, 1\}$.

$$Y - X = \begin{cases} (X + 1) - X = 1 & \text{if } X \text{ is black} \\ (X - 1) - X = -1 & \text{if } X \text{ is red} \end{cases}$$

which implies

$$P(Y - X = x) = \begin{cases} P(X \text{ is black}) = \frac{1}{2} & \text{if } x = 1 \\ P(X \text{ is red}) = \frac{1}{2} & \text{if } x = -1 \\ 0 & \text{otherwise} \end{cases}$$

(e)

$$\begin{aligned} P(Y \geq 12) &= P(X + 1 \geq 12 \cap X \text{ is black}) + P(X - 1 \geq 12 \cap X \text{ is red}) \\ &= P(X \geq 11 \cap X \text{ is black}) + P(X \geq 13 \cap X \text{ is red}) \\ &= P(X \geq 11)P(X \text{ is black}) + 0 \quad \text{because of independence} \\ &= \left(\frac{3}{13}\right)\left(\frac{1}{2}\right) = \frac{3}{26} \end{aligned}$$

4.3

Let X be the random variable described in the problem. Let us define two more random variables Y to be the random variable of the coin flip and Z to be the random variable of the die roll.

We will define some preliminary probabilities:

- $P(X = 1) = P(Y = \text{heads}) = \frac{1}{2}$
- $P(X = 2) = P(Y = \text{tails} \cap (Z = 2 \cup Z = 3))$
  $P(X = 2) = P(Y = \text{tails})(P(Z = 2) + P(Z = 3))$ because of independence
  $P(X = 2) = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$
- $P(X = 3) = 1 - P(X = 1) - P(X = 2) = 1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

(a)

$$P(X = x) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{6} & \text{if } x = 2 \\ \frac{1}{3} & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$P(X \leq x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{2}{3} & \text{if } x = 2 \\ 1 & \text{if } x > 2 \end{cases}$$

4.6

(a)

$S = L_1 + L_2 = 0 \Rightarrow L_1 = L_2 = 0$ because $L_1, L_2 > 0$. Thus,

$$P(S = 0) = P(L_1 = 0 \cap L_2 = 0)$$

$$= P(L_1 = 0)P(L_2 = 0) \quad \text{because of independence between the decks}$$

$$= \left(\frac{\binom{30}{7}}{\binom{40}{7}}\right)\left(\frac{\binom{20}{7}}{\binom{40}{7}}\right) = (0.1092)(0.00416) = 4.54 \times 10^{-4}$$

(b)

$D = L_1 - L_2 = 0 \Rightarrow L_1 = L_2$. Thus,

$$P(D = 0) = P(L_1 = 0 \cap L_2 = 0) + P(L_1 = 1 \cap L_2 = 1) + \ldots + P(L_1 = 7 \cap L_2 = 7)$$

$$= P(L_1 = 0)P(L_2 = 0) + P(L_1 = 1)P(L_2 = 1) + \ldots + P(L_1 = 7)P(L_2 = 7) \quad \text{because of independence}$$

$$= \sum_{i=0}^{7} P(L_1 = i)P(L_2 = i)$$

$$= \sum_{i=0}^{7} \left(\frac{\binom{10}{i}\binom{30}{7-i}}{\binom{40}{7}}\right)\left(\frac{\binom{20}{i}\binom{20}{7-i}}{\binom{40}{7}}\right) \approx 0.1348$$

(c)

$$P(L_1 = x) = \frac{\binom{10}{x}\binom{30}{7-x}}{\binom{40}{7}}$$

which implies that

$$P(L_1 = x) = \begin{cases} 0.1092 & \text{if } x = 0 \\ 0.3185 & \text{if } x = 1 \\ 0.344 & \text{if } x = 2 \\ 0.1764 & \text{if } x = 3 \\ 0.0457 & \text{if } x = 4 \\ 0.00588 & \text{if } x = 5 \\ 3.379 \times 10^{-4} & \text{if } x = 6 \\ 6.437 \times 10^{-6} & \text{if } x = 7 \end{cases}$$

(d)

We notice that L_1 may take on values x such that $0 \leq x \leq 5$.

$$P(L_1 = x \mid L_t = 5) = \frac{\binom{5}{x}\binom{25}{7-x}}{\binom{30}{7}}$$

which implies that

$$P(L_1 = x \mid L_t = 5) = \begin{cases} 0.2361 & \text{if } x = 0 \\ 0.435 & \text{if } x = 1 \\ 0.261 & \text{if } x = 2 \\ 0.0621 & \text{if } x = 3 \\ 0.00565 & \text{if } x = 4 \\ 1.474 \times 10^{-4} & \text{if } x = 5 \end{cases}$$

4.7

Alternatively,

$$g(x) = \begin{cases} \cos(x) & \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

(a)

We know that $\int_{-\infty}^{\infty} p(x)dx = 1$, so $p(x) = cg(x) \Rightarrow \int_{-\infty}^{\infty} cg(x)dx = c\int_{-\infty}^{\infty} g(x)dx = 1$

We know that

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)dx = \sin(x)|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2}) = 1 - (-1) = 2$$

It follows that

$$1 = c\int_{-\infty}^{\infty} g(x)dx = c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)dx = 2 \cdot c \Rightarrow c = \frac{1}{2}$$

$$p(x) = \begin{cases} \frac{1}{2}\cos(x) & \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

(b)

By looking at a graph, we can see that on the interval $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos(x) \geq 0$. Therefore, $P(X \geq 0) = 1$.

(c)

From our knowledge of the cosine function, we know that the magnitude is never greater than 1 which implies that the magnitude of $\frac{1}{2}\cos(x)$ can never be greater than $\frac{1}{2}$. Thus, $P(|X| \leq 1) = 1$.

4.9

(a)

$$\mathbb{E}[L_1] = \sum_{i=0}^{7} P(L_1 = i) \cdot i = \sum_{i=0}^{7} \left(\frac{\binom{10}{i}\binom{30}{7-i}}{\binom{40}{i}} \right) \cdot i = \frac{7}{4} = 1.75$$

(b)

$$\mathbb{E}[L_2] = \sum_{i=0}^{7} P(L_2 = i) \cdot i = \sum_{i=0}^{7} \left(\frac{\binom{20}{i}\binom{20}{7-i}}{\binom{40}{7}} \right) \cdot i = \frac{7}{2} = 3.5$$

(c)

$\mathrm{var}[L_1] = \mathbb{E}[L_1^2] - \mathbb{E}[L_1]^2$.

$$\mathbb{E}[L_1^2] = \sum_{i=0}^{7} P(L_1 = i) \cdot i^2 = \sum_{i=0}^{7} \left(\frac{\binom{10}{i}\binom{30}{7-i}}{\binom{40}{7}} \right) \cdot i^2 = \frac{217}{52}$$

$$\mathrm{var}[L_1] = \mathbb{E}[L_1^2] - \mathbb{E}[L_1]^2 = \frac{217}{52} - \left(\frac{7}{4}\right)^2 = \frac{217}{52} - \frac{49}{16} = \frac{231}{208}$$

4.12

(a)

Let X_3 be a random variable that three people play one round.

$$P(X_3 = 1\text{T, } 2\text{H} \ \cup \ X_3 = 1\text{H, } 2\text{T}) = P(X_3 = 1\text{T, } 2\text{H}) + P(X_3 = 1\text{H, } 2\text{T})$$

$$\frac{\binom{3}{1}}{2^3} + \frac{\binom{3}{1}}{2^3} = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$$

(b)

Let X_4 be a random variable that four people play one round.

$$P(X_4 = 1\text{T, } 3\text{H} \ \cup \ X_4 = 1\text{H, } 3\text{T}) = P(X_4 = 1\text{T, } 3\text{H}) + P(X_4 = 1\text{H, } 3\text{T})$$

$$\frac{\binom{4}{1}}{2^4} + \frac{\binom{4}{1}}{2^4} = \frac{4}{16} + \frac{4}{16} = \frac{1}{2}$$

(c)

Let X_5 be a random variable that five people play one round.

$$P(X_5 = 1\text{T, } 4\text{H} \ \cup \ X_5 = 1\text{H, } 4\text{T}) = P(X_5 = 1\text{T, } 4\text{H}) + P(X_5 = 1\text{H, } 4\text{T})$$

$$\frac{\binom{5}{1}}{2^5} + \frac{\binom{5}{1}}{2^5} = \frac{5}{16}$$

Let Y_5 be the number of rounds are played until there is an odd person out.

$$P(Y_5 = x) = \left(\frac{11}{16}\right)^{x-1}\left(\frac{5}{16}\right)$$

$$\mathbb{E}[Y_5] = \left(\frac{5}{16}\right)^{-1} = \textbf{3.2 rounds} \text{ (Expectation of Bernoulli)}$$

4.14

Let us assume that X and Y are independent random variables.

$$\begin{aligned}
\text{var}[X+Y] &= \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 \\
&= \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]^2 \\
&= \big(\mathbb{E}[X^2] - \mathbb{E}[X]^2\big) + \big(\mathbb{E}[Y^2] - \mathbb{E}[Y]^2\big) \\
&= \text{var}[X] + \text{var}[Y]
\end{aligned}$$

□

4.16

We know that $\mathbb{E}[|X|] = 0.2$ and $P(|X| = 0) = P(X = 0)$.

$$P(X = 0) = 1 - \big(P(|X| = 1) + P(|X| + 2)\big) = 1 - P(|X| \geq 1)$$

Finding an upper bound for $P(|X| \geq 1)$ will find a lower bound for $1 - P(|X| \geq 1)$.

$$P(|X| \geq 1) \leq \frac{\mathbb{E}[|X|]}{1}$$

$$P(|X| \geq 1) \leq 0.2$$

It follows that a lower bound for $P(X = 0) = 1 - P(|X| \geq 1) = 1 - 0.2 = 0.8$

4.17

$$\mathbb{E}[X] = 2 \Rightarrow P(X = 2) = 1 - P(|X - \mathbb{E}[X]| > 0)$$

and, since X is a discrete random variable,

$$1 - P(|X - \mathbb{E}[X]| > 0) = 1 - P(|X - \mathbb{E}[X]| \geq 1)$$

We clearly see that finding an upper bound for $P(|X - \mathbb{E}[X]| \geq 1)$ will result in a lower bound for $1 - P(|X - \mathbb{E}[X]| \geq 1)$. Using Chebyshev's inequality,

$$P(|X - \mathbb{E}[X]| \geq 1) \leq \frac{\text{var}[X]}{1}$$

$$P(|X - \mathbb{E}[X]| \geq 1) \leq 0.01$$

It follows that

$$1 - P(|X - \mathbb{E}[X]| \geq 1) = 1 - 0.01 = 0.99 \Rightarrow P(X = 2) \geq 0.99$$

4.22

(a)

Let X be a rondom variable for the number of passengers that show up to the flight.

$$\mathbb{E}[X] = \sum_{i=0}^{6} P(X = x) \cdot i = \sum_{i=0}^{6} \binom{6}{i} \left(p^i (1-p)^{6-i}\right) \cdot i = \sum_{i=0}^{6} \binom{6}{i} \left((0.9)^i (0.1)^{6-i}\right) \cdot i = 5.4$$

(b)

We want $\mathbb{E}[X] \geq 6$. The following program simulates $\mathbb{E}[X]$ given that n tickets are sold. Wee see that the smallest value of n to give an expectation less than 6 is $n = 10$ tickets.

```python
import math

def choose(n,k):
    f = math.factorial
    return f(n) / (f(k)*f(n-k))

def summation(n):
    return sum([choose(n,i)*(((0.7)**i) * ((0.3)**(n-i))) * i for i in range(0,n)])

if __name__ == '__main__':
    count = 0
    while summation(count) < 6:
        count += 1
    print(count)
```

4.23

(a)

Let X be the random variable that represents how many passengers show up to their flight. Let n represent the number of tickets sold.

$$P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if } 0 \leq x \leq n \\ 0 & \text{otherwise} \end{cases}$$

We see that random variable X follows a binomial distribution, and thus, we know that $\mathbb{E}[X] = np$, where p is the probability of a pasenger showing upt to his/her flight. We want

$$\mathbb{E}[X] = np \geq 10$$

It follows that when $n \geq \frac{10}{p}$ tickets are sold, the expected number of passengers that show up is greater than 10.

(b)

Upon running the following simulation with various values of p, I found that

$$\mathbb{E}[(X \mid \text{flight takes off})] = 10 \cdot p$$

Which follows from the event that someone shows up to their flight and the event that the passenger is a woman are independent events.

```python
import collections
import numpy as np
import random

random.seed(0)

def simulate(p):
    pass_list = []
    for i in range(10):
        r = random.random()
        if r < p:
            pass_list.append('w' if r < 0.5 else 'm')
    counter = collections.Counter(pass_list)
    return len(pass_list) if counter['w'] >= 2 else None

if __name__ == '__main__':
    pass_results = []
    for i in range(100000):
        sim_res = simulate(0.9)
        if sim_res is not None:
            pass_results.append(sim_res)

    print(sum(pass_results) / len(pass_results))
```