CS 361 Spring 2018
Homework 8

Nathaniel Murphy (njmurph3)

9.1

(a)

Show that MLE of a dataset following the normal distibution is mean($\{x\}$).

$$\mathcal{L}(\Theta) = P(x_1, \ldots, x_N \mid \Theta,\ \sigma)$$

$$= P(x_1 \mid \Theta,\ \sigma)P(x_2 \mid \Theta,\ \sigma)\ldots P(x_N \mid \Theta,\ \sigma)$$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x_i-\Theta)^2}{2\sigma^2}}$$

$$\log\ \mathcal{L}(\Theta) = \sum_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} + \sum_{i=1}^{N} -\frac{(x_i-\Theta)^2}{2\sigma^2}$$

$$= \frac{N}{\sqrt{2\pi}\sigma} - \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\Theta)^2$$

$$\frac{\partial}{\partial\Theta}\log\ \mathcal{L}(\Theta) = 0 - \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\Theta)$$

Set the derivative equal to zero.

$$0 = \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\Theta)$$

$$0 = \sum_{i=1}^{N}(x_i) - N\Theta$$

$$\Theta = \frac{\sum_{i=1}^{N} x_i}{N} = \text{mean}(\{x\})$$

(b)

Show that the MLE for the standard deviation is std($\{x\}$).

$$\mathcal{L}(\Theta) = P(x_1, \ldots, x_N \mid \Theta,\ \sigma)$$

$$= P(x_1 \mid \Theta,\ \mu)P(x_2 \mid \Theta,\ \mu) \ldots P(x_N \mid \Theta,\ \mu)$$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\Theta} e^{\frac{-(x_i-\mu)^2}{2\Theta^2}}$$

$$\log\, \mathcal{L}(\Theta) = \sum_{i=1}^{N} \log\left(\frac{1}{\sqrt{2\pi}\Theta}\right) + \sum_{i=1}^{N} -\frac{(x_i-\mu)^2}{2\Theta^2}$$

$$= N\log(1) - N\log(\sqrt{2\pi}) - N\log(\Theta) - \frac{1}{\Theta^2}\sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2}$$

$$\frac{\partial}{\partial\Theta}\log\, \mathcal{L}(\Theta) = -\frac{N}{\Theta} + \frac{2}{\Theta^3}\sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2}$$

Set derivative equal to zero.

$$0 = -\frac{N}{\Theta} + \frac{2}{\Theta^3}\sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2}$$

$$\frac{N}{\Theta} = \frac{2}{\Theta^3}\sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2}$$

$$\Theta^2 = \frac{2}{N}\sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2}$$

$$\Theta = \sqrt{\frac{\sum_{i=1}^{N}(x_i-\mu)^2}{N}} = \text{std}(\{x\})$$

(c)

If all numbers $x_1, \ldots, x_N$ take the same value, we have mean($\{x\}$) $= x_i\ \forall\ i \in [1, N]$.

$$\text{std}(\{x\}) = \sqrt{\frac{\sum_{i=1}^{N}(x_i-x_i)^2}{N}} = \sqrt{0} = 0$$

9.3

MLE for Poisson distribution is $\frac{\sum_{i=1}^{N} n_i}{N}$.

(a)

Day 1: $\frac{(3+1+4+2)}{4} = \frac{10}{4} = 2.5$ pop-ups per hour.
Day 2: $\frac{(2+1+2)}{3} = \frac{5}{3} \approx 1.67$ pop-ups per hour.
Day 3: $\frac{(3+2+2+1+4)}{5} = \frac{12}{5} = 2.4$ pop-ups per hour.

(b)

Day 4 MLE: $\frac{13}{6} \approx 2.1$ pop-ups per hour.

(c)

MLE for all 4 days:

$$\frac{\#\text{ total pop-ups}}{\#\text{ total hours}} = \frac{40}{18} \approx 2.22 \text{ pop-ups per hour}$$

9.4

(a)

$$36 + \frac{36}{r}$$

(b)

Not very reliable because if on the very first spin the ball lands in a 0 slot, you would assume that all the slots are zeroes.

(c)

$$\frac{\left(\frac{36}{r_1} + \frac{36}{r_2} + \frac{36}{r_3}\right)}{3}$$

9.5

MLE for Binomial model is $\Theta = \frac{k}{N}$, where k is number of successes and N is number of draws.

(a)

$$\Theta_{blue} = \frac{0}{1} = 0$$

(b)

$$\Theta_{blue} = \frac{3}{10} = 0.3$$

9.7

(a)

$$\log \frac{P(y=1 \mid X,\ \Theta)}{P(y=0 \mid X,\ \Theta)} = X^T\Theta$$