CS 374 Spring 2018
Homework 2

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Problem 1

Let us first define some functions and notation that will simplify this problem:

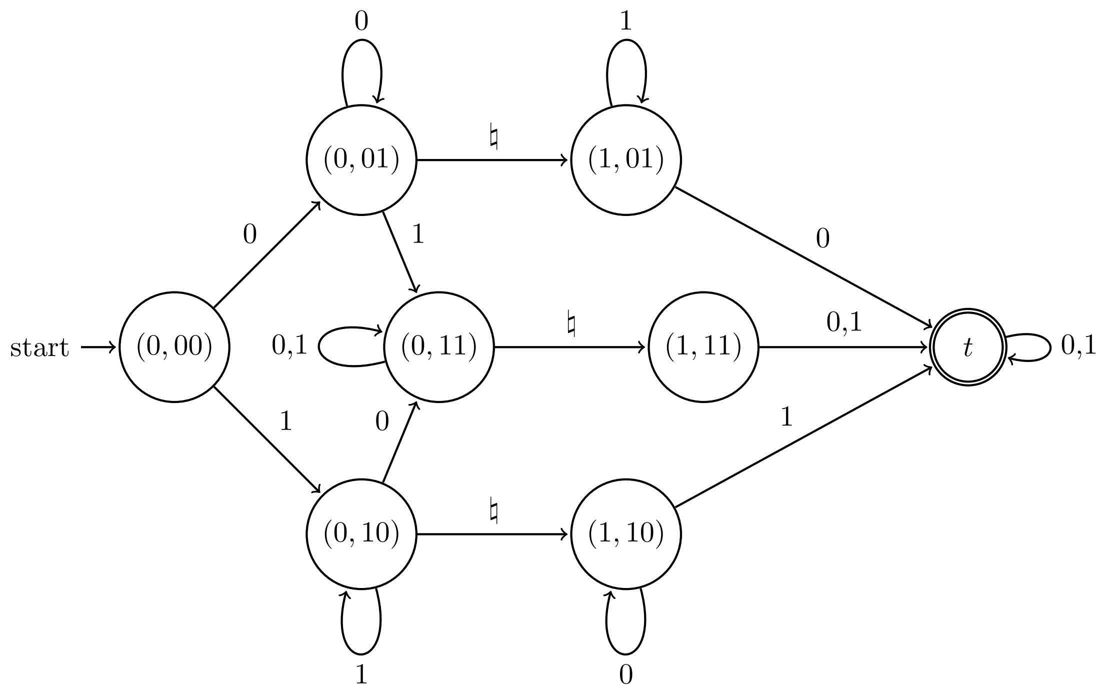
- Let s be a binary string. Let us indtroduce notation s_i, which defines the value of the i^{th} character from the end of the string (zero indexed). i.e. $s = 1110,\ s_0 = 0,\ s_1 = 1$
- Define $b_k : 2^k \to \mathcal{P}(\Sigma_k)$ such that for a given binary string s of length k, $b_k(s) = \{a \in \Sigma_k \mid s_a = 1\}$. Notice that $b_k(s) \subseteq \Sigma_k$.

(a)

Let us define $M = (Q, \Sigma, \delta, s, A)$.

- $Q = \{0, 1\} \times 2^2 \cup \{t, g\}$, where 2^2 is all binary strings of length 2.
- $\Sigma = \Sigma_k \cup \{\natural\} = \{0, 1\} \cup \{\natural\}$
- $s = (0, 00)$
- $A = \{t\}$
- q is defined by the following. Note that for some case3s we use numerical values fo the state names to determine the next state. Also note that $q \in Q = q(i, j)$, where $i \in \{0, 1\}$ and $j \in 2^2$. Note that we also use $|$ symbol to represent bitwise OR between two binary strings.

$$\delta(q, a) = \delta\big((i, j), a\big) = \begin{cases} (0, j|j') & \text{if } i = 0,\ a \in \{0, 1\}, \text{where } j'_a = 1, \text{ and all other positions are zeroes.} \\ (i + 1, j) & \text{if } i = 1,\ |j| \geq 1 \text{ and } a = \natural \\ t & \text{if } i = 1 \text{ and } a \in b_k(j) \\ (1, j) & \text{if } i = 1 \text{ and } a \notin b_k(j) \\ g & \text{otherwise} \end{cases}$$


start
(0, 00)
(0, 01)
(0, 11)
(0, 10)
(1, 01)
(1, 11)
(1, 10)
t
0
1
0,1
♮

(b)

We will prove the following statement by induction on $|w|$. For every string w,

(a) $\forall\, w \in \Sigma_k^*,\ \delta^*(s, w) = (0, j)$ iff $\natural \notin w$ and $w \in b_k(j)^*$

(b) $\forall\, w \in \Sigma_k^*,\ \delta^*(s, w) = (1, j)$ iff $w \in \{u\natural v, u \in b_k(j)^*,\ v \in \Sigma_k^* \cup \{\epsilon\}$ and $\text{set}(u) \cap \text{set}(v) = \emptyset\}$

(c) $\forall\, w \in \Sigma_k^*,\ \delta^*(s, w) = t$ iff $w \in \{u\natural v,\ u, v \in \Sigma_k^*$ and $\text{set}(u) \cap \text{set}(v) \neq \emptyset\}$

(d) $\forall\, w \in \Sigma_k^*,\ \delta^*(s, w) = g$ iff $w \in \{u\natural v,\ u, v \in \Sigma_k^* \cup \{\epsilon, \natural\}$ such that either

- $u = \epsilon$
- $\text{set}(v) \cap \text{set}(\{\natural\}) \neq \emptyset\}$

i.e. w starts with a $\natural$ symbol, or contains more than one $\natural$ symbol.

We want to prove that all four conditions (a)-(d) hold for every string w.

Base case: $|w| = 0,\ w = \epsilon$. By our definition of *delta*, we see that $delta^*(s, w) = \delta^*(s, \epsilon) = (0, 00)$, which is in the form $(0, j)$, where $j = 00$. We see that $\natural \notin w$ and $w \in b_k(j)^*$ vacuously. The base case holds.

Inductive hypothesis: Assume that for any string w of length less than i, conditions (a)-(d) hold.

Induction Step: Consider w of length i, where $i > 0$. Without loss of generality, we can say that $w = ua$, where $u \in \Sigma^*$ and $a \in \Sigma$. We see that

$$\delta^*\big((0, 00), ua\big) = \delta^*\Big(\delta^*\big(0, 00\big), u\big), a\Big) = \delta\Big(\delta^*\big((0, 00), u\big), a\Big)$$

Using the inductive assumption, let us prove cases (a)-(d).

Case (a): $\delta^*\big((0, 00), u\big) = (0, j) \Rightarrow \natural \notin u$ and $u \in b_k(j)^*$

Let us consider all inputs:
$a \in \{0, 1\}$:
Then $\delta\Big(\delta^*\big((0, 00), u\big), a\Big) = (0, j|j')$, where j' is defined above. This implies that $\delta^*\big(0, 00), w\big)$ is still in the form $(0, j)$, which implies that $\natural \notin w$ and $w \in b_k(j)^*$. We see that the first condition ($\natural \notin w$) trivially holds, and that $w \in b_k(j|j')$ by our construction of j'.

$\underline{a = \natural:}$

We see that if $|u| = 0$, $\delta\Big(\delta^*\big((0,00),u\big),\natural\Big) = g$ which implies that $w = ua$ must have a $\natural$ in the first position. this is, in fact the case because $|u| = 0$. Let us now assume that $|u| \geq 1$. by our definition of δ, $\delta\Big(\delta^*\big((0,00),u\big)\natural\Big) = (1,j)$ such that $u \in b_k(j)^*$. To be in state $(1,j)$, our string $ua = w$ must be in the set

$$L_{(1,j)} = \{u\natural v \mid u \in b_k(j), v \in \Sigma_k^* \cup \{\epsilon\} \text{ and } \operatorname{set}(u) \cap \operatorname{set}(v) = \emptyset\}$$

By inductive assumption, $u \in b_k(j) \Rightarrow u\natural = u\natural\epsilon = u\natural v \in L_{(1,j)}$ because $\operatorname{set}(u) \cap \operatorname{set}(v) = \operatorname{set}(u) \cap \operatorname{set}(\epsilon) = \operatorname{set}(u) \cap \emptyset = \emptyset$.

Case (b): $\delta\big((0,00),u\big) = (1,j) \Rightarrow u \in \{u'\natural v' \mid u' \in b_k(j), v' \in \Sigma_k^* \cup \{\epsilon\}, \operatorname{set}(u') \cap \operatorname{set}(v') = \emptyset\}$.

Let us consider all inputs:

$\underline{a \in \{0,1\}:}$

By our definition, if $a \in b_k(j)$, then $\delta\Big(\delta^*\big((0,00),u\big),a\Big) = t$, which implies that $w = ua \in \{u\natural v \mid u, v \in \Sigma_k^*, \operatorname{set}(u) \cap \operatorname{set}(v) \neq \emptyset\}$. By inductive assumption, $u \in b_k(j) \subseteq \Sigma_k^*$, so $u \in \Sigma_k^*$. We also see that $v \in \Sigma_k^* \cup \{\epsilon\} \Rightarrow v \cdot a \in \Sigma_k^*$ because $\{0,1\} \subset \Sigma_k^*$. It follows that $a \in b_k(j) \Rightarrow \operatorname{set}(u) \cap \operatorname{set}(v'a) \neq \emptyset$ because a is a common symbol in both strings.

$\underline{a = \natural:}$ By our definition of δ, if $a = \natural$, then $\delta\Big(\delta^*\big((0,00),u\big),\natural\Big) = g$, which implies that $u\natural$ is in the set $\{u\natural v \mid u, v \in \Sigma_k^* \cup \{\epsilon\}, u = \epsilon \text{ or } \operatorname{set}(v) \cap \operatorname{set}(\{\natural\}) \neq \emptyset\}$ which means that either $u\natural$ starrts with a $\natural$ or contains more than one $\natural$ symbo. $u\natural$ cannot star5t with a $\natural$ because, by inductive assumption, u does not start with $\natural$ becuase u is in a state with form $(1,j)$. Now, we see that $u\natural$ must contain 2 $\natural$ symbols because, by inductive assumption, u contains one $\natural$ symbol, so $u\natural$ must contain 2 $\natural$ symbols.

Case (c): $\delta^*\big((0,00),u\big) = t \Rightarrow u \in T_k = \{u'\natural v' \mid u', v' \in \Sigma_k^*, \operatorname{set}(u') \cap \operatorname{set}(v') \neq \emptyset\}$

Let us consider all inputs:

$\underline{a \in \{0,1\}:}$

By inductive assumption, we trivially see that if $a \in \{0,1\} \subseteq \Sigma_k^*$, $u \in T_k \Rightarrow ua \in T_k$ which agrees with our definition of $\delta:\ \delta(t,0) = \delta(t,1) = t$.

$\underline{a = \natural:}$

By our definition of δ, $\delta(t,\natural) = g$, so $\delta^*\big((0,00),ua\big) = \delta\Big(\delta^*\big((0,00),u\big),\natural\Big) = g$. We must now prove that either $w = u\natural$ starts with a $\natural$ symbol or contains 2 $\natural$ symbols. By the inductive assumption, we see that $u = u'\natural v'$ contains one $\natural$ symbol which implies that $u\natural$ contains 2 $\natural$ symbols.

Case (d): $\delta^*\big((0,00),u\big) = g \Rightarrow u \in L_g = \{u'\natural v' \mid u', v' \in \Sigma_k^* \cup \{\epsilon, \natural\},\ \mathrm{set}(v') \cap \mathrm{set}(\{\natural\}) \neq \emptyset\}$

Let us consider all inputs:
$a \in \{0, 1, \natural\}$:
$\delta(g,a) = g \Rightarrow] \delta\Big(\delta^*\big((0,00),u\big),a\Big) = g$. It is trivial to show that $u \in L_g \Rightarrow ua \in L_g$ because $u = u'\natural v' \Rightarrow ua = u'\natural v'a = u'\natural v$ holds because $a \in \Sigma_k^* \cup \{\epsilon, \natural\}$ and $\mathrm{set}(v') \cap \mathrm{set}(\{\natural\}) \neq \emptyset \Rightarrow \mathrm{set}(v'a) \cap \mathrm{set}(\{\natural\}) \neq \emptyset$.

$\square$