CS 374 Spring 2018
Homework 3

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Problem 1

1.

$\Sigma = \{a, b\}$ and $\Delta = \{0, 1\}$. Define homomorphism $h : \Sigma \to \Delta^*$ where $h(a) = 01$ and $h(b) = 10$. We see that $h^{-1}(01) = a$ and $h^{-1}(10) = b$.

- $h^{-1}(\{0101\}) = \{aa\}$
- $h^{-1}(\{00\}) = \emptyset$
- $h^{-1}(\{001\}) = \emptyset$
- $h^{-1}(\{1001\}) = \{ba\}$

Notice that the second and third are equivalent to $\emptyset$ because

$$h^{-1}(L) = \{w \in \Sigma^* \mid h(w) \in L\} \Rightarrow h^{-1}(w) = \{u \in \Sigma^* \mid h(u) = w\}$$

and no such $u \in \Sigma^*$ exists such that $h(u) \in \{00, 001\}$.

Let $L = L\big((00 + 1)^*\big)$.

- $h^{-1}(L) = (ab)^*$
- $h(h^{-1}(L)) = (1001)^*$

$h^{-1}((00 + 1)^*) = (ab)^*$ because for every $w \in h^{-1}((00 + 1)^*)$, w must start with a 1, then must be followed by 00, then must be followed by another 1.

2.

We will use the property that $h(uv) = h(u)h(v)$.

(a)

For every $w \in \Sigma^*$, $\delta_N^*(s', w) = \delta_M^*(s, h(w))$.

(b)

<u>Claim:</u> $\forall\, w \in \Sigma^*$, $\delta_N^*(s', w) = \delta_M^*(s, h(w))$.
<u>Proof:</u>

<u>Base</u>: $|w| = 0$, $w = \epsilon$. $\delta_N^*(s', w) = \delta_N^*(s', \epsilon) = \delta(s', \epsilon) = s' = s = \delta'(s, \epsilon) = \delta'(s, h(\epsilon)) = \delta_M^*(s, h(\epsilon)) = \delta_M^*(s, h(w))$.

<u>Inductive Hypothesis</u>: Assume that $\forall\, w \in \Sigma^*$, $|w| < k \Rightarrow \delta_N^*(s', w) = \delta_M^*(s, h(w))$.

<u>Inductive Case</u>: Let $w \in \Sigma^*$ such that $|w| = k$. w can be written as $w = ua$, where $u \in \Sigma^{k-1}$ and $u \in \Sigma$. We see that:

$$\delta_N^*(s', w) = \delta_N^*(s', ua) = \delta_N^*\big(\delta_N^*(s', u), a\big) = \delta'\big(\delta_N^*(s', u), a\big)$$

$$= \delta'\big(\delta_M^*(s, h(u)), a\big) = \delta_M^*\big(\delta_M^*(s, h(u)), h(a)\big) = \delta_M^*(s, h(u)h(a))$$

Because $h(uv) = h(u)h(v)$, $u, v \in \Sigma^*$, it follows that

$$\delta_M^*(s, h(u)h(a)) = \delta_M^*(s, h(w))$$

(c)

Prove $L(N) = h^{-1}(L)$.

Fix $w \in L(M)$. We want to show that $h^{-1}(w) \in L(N)$.
From the definition of $h^{-1}(L) = \{w \in \Sigma^* \mid h(w) \in L\}$, we see that it suffices to fix $w \in L(N)$ and show that $h(w) \in L(M)$.

Because $Q' = Q$ and $A' = A$, we see that

$$\delta_N^*(s', w) = \delta_M^*(s, h(w)), \;\; \forall\, w \in \Sigma^*$$

which means that $\delta_N^*(s', w) \in A \Rightarrow \delta_M^*(s, h(w)) \in A$.

It follows that $L(N) = h^{-1}(L)$.