Practice Problems

CS 374 Problem Set 1

Practice #1

1. abab

2. $h(L) = (ab)^*$

3.

Not guaranteed. Take $L_1, L_2 \in \{0,1\}^*$. Let $L_1 = (0)^*$, $L_2 = (01)^*$. $h(L_1) = h(L_2)$, but $L_1 \neq L_2$.

Practice #2

1.

$$\begin{aligned} rev((0+1)^*1(0+1)) &= rev(1(0+1))rev((0+1)^*) \\ &= rev((0+1))rev(1)rev((0+1))^* \\ &= (0+1)1(0+1)^* \end{aligned}$$

2.

$$\begin{aligned} rev((0+1)^*011(0+1)^*) &= rev(011(0+1^*)rev((0+1)^*) \\ &= rev((0+1)^*)rev(011)rev((0+1))^* \\ &= rev((0+1))^*rev(11)rev(0)(0+1)^* \\ &= (0+1)^*rev(1)rev(1)0(0+1)^* \\ &= (0+1)^*110(0+1)^* \end{aligned}$$

3.

This is where it gets a little fuzzy
Let us first define two lemmas that we will need later.

Lemma #1: $(uv)^R = v^R u^R \; \forall \; u, v \in \Sigma^*$.
We will prove this by using induction over $|u|$.

Base: $u = \epsilon$. $(uv)^R = (\epsilon v)^R = v^R = v^R \epsilon = v^R \epsilon^R$. Base case holds.
Inductive: Assume that $\forall \; |u| = 0, \ldots, n-1$, that $(uv)^R = v^R u^R$.
Let $u \in \Sigma^n$. Note that $u = a \cdot u'$, $a \in \Sigma, u' \in \Sigma^{n-1}$. Now we have $(uv)^R = ((au'v)^R$ and by definition of w^R, we see that $(a(u'v))^R = (u'v)^R a$ and from the inductive assumption, it follows that $(u'v)^R a = v^R u'^R a = v^R u^R$ by definition of u'.

$\square$

Lemma #2: $(w^k)^R = (w^R)^k$
Use induction on k.

Base: $k = 0$, $(w^0)^R) = \epsilon^R = (\epsilon^R)^0$. Base case holds.
Inductive: Assume that for $k = 0, \ldots, n-1$, $(w^k)^R = (w^R)^k$.
$w^n = w^{n-1} \cdot w \Rightarrow (w^n)^R = (w^{n-1} \cdot w)^R = (w)^R \cdot (w^{n-1})^R$. by the inductive assumption, we now see that $(w)^R \cdot (w^{n-1})^R = (w)^R \cdot (w^R)^{k-1} = (w^R)^k$.

$\square$

To prove that $L(rev(r)) = (L(r))^R$, we will use induction over the number of operations in r.

Base: By definition of $rev(r)$, we must prove 3 base cases: $r = \emptyset$, $r = \epsilon$, $r = a \in \Sigma$.

<u>Case $r = \emptyset$:</u> $(L(\emptyset))^R = \emptyset^R = \emptyset = rev(\emptyset)$
<u>Case $r = \epsilon$:</u> $(L(\epsilon))^R = \{\epsilon\}^R = \{\epsilon\} = rev(\epsilon)$
<u>Case $r = a \in \Sigma$:</u> $(L(a))^R = \{a\}^R = \{a\} = rev(a)$ Base cases hold.

Inductive hypothesis: Let us assume that $L(rev(r)) = (L(r))^R$ for all regular expressions r with number of operations ($\#_{ops}$) less than n.
Inductive claim: For a regular expressions r with n operations, $L(rev(r)) = (L(r))^R$.

A regular expression r with n operations can be written as one of 3 cases:

1. $r_1 + r_2$, where r_1 is a regular expression with $n-1$ operations and r_2 is a regular expression with 1 operation.

2. $r_1 r_2$, where r_1 is a regular expression with $n-1$ operations and r_2 is a regular expression with 1 operation.

3. r_1^*, where r_1 is a regular expression with $n-1$ operations.

For each case, prove that $L(rev(r)) = (L(r))^R$.

Case $r_1 + r_2$:

$$L(r_1 + r_2) = \{w \mid w \text{ is described by either regular expression } r_1 \text{ or } r_2\}$$

Because $\#_{ops}(r_1) = n - 1$ and $\#_{ops}(r_2) = 1$, they both fall under the inductive assumption so it follows that $\forall\, w \in L(r_1 + r_2)$, $rev(w) = w^R \Rightarrow L(rev(r_1 + r_2)) = (L(r_1 + r_2))^R \Rightarrow L(rev(r)) = (L(r))^R$.

Case $r_1 r_2$:

By definition, $rev(r) = rev(r_1 r_2) = rev(r_2) rev(r_1)$. Since $\#_{ops}(r_1) = n - 1$ and $\#_{ops}(r_2) = 1$, by the inductive assumption,

$$L(rev(r_2))L(rev(r_1)) = (L(r_2))^R (L(r_1))^R$$

Note that $L(r_1) = \{w \mid w \text{ is defined by } r_1\} \Rightarrow (L(r_1))^R = \{w^R \mid w \text{ is defined by } r_1\}$. $L(r_2)$ can be defined similarly. It follows that $(L(r_2))^R (L(r_1))^R = \{w_2^R w_1^R \mid w_1 \text{ is defined by } r_1,\ w_2 \text{ is defined by } r_2\}$.

$L(r_1 r_2) = \{w_1 w_2 \mid w_1 \text{ is defined by } r_1 \text{ and } w_2 \text{ is defined by } w_2\}$. By use of Lemma #1, we can say that $L(r_1 r_2)^R = \{w_2^R w_1^R \mid w_1 \text{ is defined by } r_1 \text{ and } w_2 \text{ is defined by } r_2\}$.

$L(r)^R = L(r_1 r_2)^R = (L(r_2))^R (L(r_1))^R = L(rev(r_2))L(rev(r_1)) = L(rev(r_1 r_2) = L(rev(r))$

Case $(r_1)^*$:

By the inductive assumption, we know that $rev((r_1)^*) = (rev(r_1))^* = \{(w^R)^k \mid w \text{ is defined by } r_1\}$. Also, note that $L((r_1)^*)^R = \{(w^k)^R \mid w \text{ is defined by } r_1\}$. By Lemma #2, we can see that these two sets are equal.

$\square$