A METHOD OF ANALYSIS OF THE STABILITY OF EMBANKMENTS ASSUMING PARALLEL INTER-SLICE FORCES

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SYNOPSIS

A method of analysis is described for determining the factor of safety of an embankment against failure on a cylindrical slip surface. The analysis is in terms of effective stress and leads to two equations of equilibrium, the first in respect of forces and the second in respect of moments.

Using this method of analysis, the factors affecting the accuracy of Bishop's simplified method have been investigated. Charts have been obtained for three different values of the pore pressure coefficient (r_u) which identify the position of the critical slip circle. A set of stability charts is also given.

On décrit une méthode d'analyse pour déterminer le facteur de sécurité d'un remblai contre la rupture sur une surface de glissement cylindrique. L'analyse est faite d'après la contrainte effective et amène à deux équations d'équilibre, la première se rapportant aux forces et la deuxième aux moments.

En utilisant cette méthode d'analyse, on a étudié les facteurs qui affectent la précision de la méthode simplifiée de Bishop. On est arrivé à des diagrammes pour les trois valeurs différentes de coefficient de pression interstitielle (r_u) qui identifient la position du cercle de glissement critique. On donne aussi un groupe de diagrammes de stabilité.

NOTATION

The symbols in the following list marked thus * are illustrated in Fig. 1.

α	Slope of base of slice*	${F}_{\mathtt{mo}}$	Value of factor of safety which satis-		
β	Slope of embankment*		fies moment equation when $\theta = 0$		
ϕ'	Angle of shearing resistance (with	H	Height of embankment*		
	respect to effective stress)	h	Mean height of slice*		
$\phi'_{ m m}$	Mobilized angle of shearing resistance	$N_{\mathbf{x}}$	Factor giving position of lower end of		
	$\left(\tan\phi'_{\mathrm{m}} = \frac{\tan\phi'}{F}\right)^*$		slip surface*		
	$ \tan \phi_{\rm m} = \overline{F} $	$N_{\mathtt{d}}$	Depth factor*		
γ	Bulk density	P	Total reaction normal to base of		
$\dot{\theta}$	Slope of resultant (Q) of pair of inter-		slice*		
	slice forces*	P'	Reaction (due to effective stress)		
$ heta_{ ext{i}}$	Value of θ which satisfies both force		normal to base of slice*		
-	and moment equations	Q	Resultant of pair of inter-slice forces*		
A, B	Angles defining position of centre of	u	Pore-pressure		
	slip circles (see Fig. 4)	r_u	Pore-pressure coefficient		
b	Width of slice*	S	Total shear force available		
c'	Cohesion (with respect to effective	$S_{\mathbf{m}}$	Total shear force mobilized*		
	stress)	W	Weight of slice*		
F	Factor of safety	x, y	Co-ordinates of centre of slip circle*		
$F_{\mathbf{i}}$	Value of factor of safety which satis-	\boldsymbol{Z}	Inter-slice force*		
	fies both force and moment equa-	$rac{c'}{F\gamma H}$	Stability number		

INTRODUCTION

In a method of determining the stability of an embankment developed by Fellenius (1927, 1936) a cylindrical slip surface is assumed and the earth mass within this surface is divided into vertical slices. In each slice, the resultant of the forces and the sum of the moments of the

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forces must both be zero. Furthermore, any residual force or moment must be equal to the external force or moment on the embankment—and normally these will both be zero. This method is extremely flexible and can be applied in different ways. The result given, using a particular method of analysis, will depend to some extent upon the way in which the factor of safety is defined and also upon the assumptions which are made about the inter-slice forces which act across the vertical boundaries between the slices.

In a definition by Bishop (1955), which has gained fairly wide acceptance, the factor of safety (F) is taken as the ratio of the total shear strength available (S) on the slip surface to the total shear strength mobilized (S_m) in order to maintain equilibrium:

In applying this definition to the method of slices, the value of F is the same in each slice. This condition implies mutual support between adjacent slices and therefore implies also the existence of inter-slice forces.

In the paper already mentioned, Bishop described a rigorous method of analysis which satisfied both force and moment conditions of equilibrium and which took full account of the inter-slice forces. Using this method, a solution was obtained by successive approximation, an initial value for F being obtained using an expression which had been simplified by assuming that the inter-slice forces were horizontal. This simplified expression satisfies the conditions of equilibrium in respect of moments but not in respect of forces. Obviously then this expression could not be regarded as rigorous, yet it was found that the value for F given by it was a very close approximation to the final value obtained using the rigorous method. Furthermore, it has been shown by Sevaldson (1957) and by Skempton (1961) that the simplified method gives a factor of safety close to unity when applied to cases where failure has actually taken place.

The evidence supporting Bishop's simplified method is substantial, yet no explanation has ever been given to explain how an expression which does not satisfy one of the basic conditions of equilibrium should, nevertheless, give accurate results. The original purpose of the work described in this Paper was to obtain further information on this point and also to find out whether the accuracy of the simplified method was affected either by the height or slope of the embankment or by the soil properties. This is dealt with in Part 1.

The scope of the investigation was later extended to obtain charts applicable to analysis in terms of effective stress which indicate the position of the critical slip circle and also stability charts similar to those published by Taylor (1948) for total stress analysis. In obtaining these charts it was assumed that the soil forming the embankment and that lying beneath it are uniform and have similar properties, and that the position of the critical slip circle is determined by the height and slope of the embankment and the properties of the soil and not by the presence of the hard stratum. These charts and a description of their derivation is given in Part 2.

Part 1

In application, the rigorous analysis proposed by Bishop was extremely lengthy and, as the intention was to examine a large number of problems extending over a wide range of soil properties in embankments of various slopes, it was necessary to derive an alternative method which would satisfy both force and moment equilibrium conditions and which would take the inter-slice forces into account. The derivation of the expressions used in the method is given below.

Figure 1(a) shows a section through an embankment of height H and slope β . Elsewhere in this Paper the slope of an embankment will be given in the form cot β : 1. In the same figure, a circular slip surface and a typical slice of mean height h and width b are shown. An enlarged

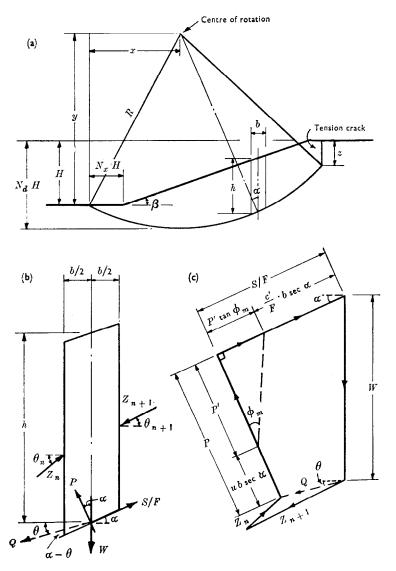


Fig. 1(a), (b), (c). Dimensions of slip surface and forces on a slice

sketch of the slice with the forces acting upon it is given in Fig. 1(b). The five forces can be described as follows:

- (a) the weight (W);
- (b) the total reaction (P) normal to the base of the slice; this force will have two components:
 - (i) the force (P') due to the effective or inter-granular stress;
 - (ii) the force ($ub \sec \alpha$) due to the pore pressure (u);

thus

(c) the mobilized shear force $\left(S_{\rm m}=\frac{S}{F}\right)$, where $S=c'b\sec\alpha+P'\tan\phi'$, i.e.

(d) the inter-slice forces (Z_n) and (Z_{n+1}) ; for equilibrium, the resultant (Q) of these two forces must pass through the point of intersection of the three other forces.

If the presence of the inter-slice forces is ignored, the three remaining forces are, of course, concurrent and in this case both conditions of equilibrium can be satisfied either by resolving or by taking moments. In either case the following expression is obtained for the factor of safety of the embankment:

$$F = \frac{1}{\Sigma[W \sin \alpha]} \Sigma[c'b \sec \alpha + \tan \phi' (W \cos \alpha - ub \sec \alpha)] \qquad . \qquad . \qquad (4)$$

The value for F given by this expression is appreciably less than that obtained when the effect of the inter-slice forces is taken into account.

Returning to the consideration of the inter-slice forces, these forces in a fully rigorous solution would be separated into two components like force P. One of these components would be derived from effective stress and the other from pore-pressure. In this analysis, for the sake of simplicity, the total force is used.

By resolving normal and parallel to the base of the slice the five forces shown in Figs 1(b) and 1(c), the following expression is obtained for the resultant (Q) of the two inter-slice forces:

$$Q = \frac{\frac{c'b}{F}\sec\alpha + \frac{\tan\phi'}{F}(W\cos\alpha - ub\sec\alpha) - W\sin\alpha}{\cos(\alpha - \theta)\left[1 + \frac{\tan\phi'}{F}\tan(\alpha - \theta)\right]} \qquad . \qquad . \qquad (5)$$

In this expression, u is the mean pore-pressure on the base of the slice of weight W. If the soil is assumed to be uniform and of density γ , the weight of a slice of mean height h and width b can be written:

$$W = \gamma bh$$
.

Furthermore, assuming a homogeneous pore-pressure distribution as proposed by Bishop and Morgenstern (1960), the mean pore-pressure on the base of the slice can be written:

$$u = r_{\nu} \gamma h$$

where r_{μ} is a pore-pressure coefficient.

Making these assumptions, equation (5) can now be transposed and re-written in a dimensionless form as follows:

$$Q = \gamma H b \left[\frac{\frac{c'}{F\gamma H} + \frac{1}{2} \frac{h}{H} \frac{\tan \phi'}{F} (1 - 2r_u + \cos 2\alpha) - \frac{1}{2} \frac{h}{H} \sin 2\alpha}{\cos \alpha \cos (\alpha - \theta) \left[1 + \frac{\tan \phi'}{F} \tan (\alpha - \theta) \right]} \right]$$
(6)

Now if the external forces on the embankment are in equilibrium, the vectorial sum of the inter-slice forces must be zero. In other words, the sum of the horizontal components of the inter-slice forces must be zero and the sum of their vertical components must also be zero, i.e.

Furthermore, if the sum of the moments of the external forces about the centre of rotation is zero, the sum of the moments of the inter-slice forces about the centre of rotation must also be zero:

$$\Sigma[QR\cos\left(\alpha-\theta\right)]=0.$$

And since the slip surface is assumed to be circular, the radius of curvature (R) is constant and:

In a given problem, there are thus three equations to be solved: two in respect of forces (7a, 7b) and one in respect of moments (8). Values of F and of θ must be found which satisfy all three equations and it must be noted that although, for a given slice, the value of θ will be the same in each equation, the inter-slice forces will not necessarily be parallel throughout.

SLOPE OF THE INTER-SLICE FORCES

It has been shown by Morgenstern and Price (1965) that, within limits, the variation in θ can be assumed arbitrarily. They also found, however, that the range of values obtained for F for different types of distribution in θ was quite small. The limiting factor which controls the variation in θ is that soil is able to withstand only a small intensity of tensile stress. Consequently, the point of application of an inter-slice force must not be far outside the middle third of the vertical boundary on which the force acts.

If it can be assumed that the inter-slice forces are parallel (i.e. that θ is constant throughout), equations (7a) and (7b) become identical:

In this case, there are only two equations to solve, (7) and (8), and the solution is therefore greatly simplified.

The result of assuming that the inter-slice forces are parallel was checked in a few trial cases in which the procedure was as follows.

- (1) A circular slip surface was chosen arbitrarily, the area inside it divided into vertical strips of equal width and the mean height (h) and base slope (α) of each slice determined graphically.
- (2) Several values of θ were chosen and, for each, the value of F was found which would satisfy both equations (7) and (8). The values of F obtained using the force equilibrium equation (7) are designated F_t , and those obtained using the moment equilibrium equation (8) as $F_{\rm m}$. The value of the factor of safety obtained using the moment equation and taking θ as zero is designated $F_{\rm mo}$.
- (3) A curve was then plotted showing the relationship between $F_{\rm f}$ and θ and, on the same graph, a second curve was plotted showing the relationship between F_m and θ . The type of graph resulting is shown in Fig. 2(b). The intersection of the two curves gives the value of the factor of safety (F_i) which satisfies both equations (7) and (8) and the corresponding slope (θ_i) of the inter-slice forces.
- (4) The values of F_1 and θ_1 were then substituted in equation (6) to obtain the values of the resultants of the inter-slice forces. Hence, working from the first slice to the last, the values of the inter-slice forces themselves were obtained.
- (5) Then, working again from the first slice to the last, the points of action of the interslice forces were found by taking moments about the middle of the base of each slice in turn. The positions of the points of action were then marked on the section of the embankment.

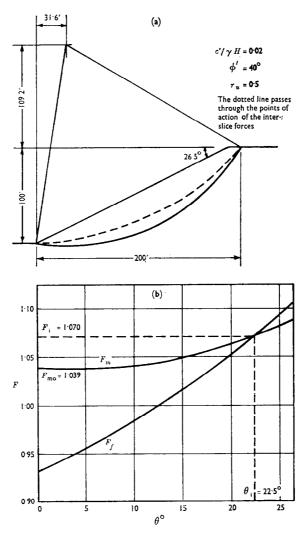


Fig. 2. Variation of $F_{\rm m}$ and $F_{\rm f}$ with θ

The results of one of the problems solved in this way are shown in Fig. 2. Several points emerged from the solution of this problem:

- (a) the value of θ_1 was less than the slope of the embankment (β) ;
- (b) variation in the value of θ affected the values of $F_{\rm f}$ to a very much greater extent than those of $F_{\rm m}$; in fact when θ was less than $\theta_{\rm i}$, the variation in $F_{\rm m}$ was very small indeed; in consequence there was not much difference between the values of $F_{\rm mo}$ and $F_{\rm i}$;
- (c) the line passing through the points of action of the inter-slice forces passed very close to the lower 'third-point' on each of the inter-slice boundaries; this implied an approximately triangular pressure distribution on these boundaries which was an acceptable result.

Each of these three observations was confirmed in one or two other trial problems solved using the above routine and in which the inter-slice forces were assumed to be parallel. The same assumption was therefore made in the remaining work described in this Paper. It may be that, in some of the cases considered, this assumption was less valid than it was found to be in the trial problems but, from the work of Morgenstern and Price (1965), it would seem that this would be unlikely to cause much error in the factors of safety obtained.

The relationship between $F_{\rm m}$ and θ shown in Fig. 2(b)—which is quite typical of the results obtained throughout—gives a clear indication of the basis of the accuracy of Bishop's simplified method. This method relies on a single equation which satisfies the condition of equilibrium in respect of moments, and the inter-slice forces are assumed to be horizontal. In other words, the value obtained for the factor of safety is $F_{\rm mo}$; in fact if θ is taken as zero in equations (5) and (8), an equation identical with that of Bishop's can be obtained. Evidently, the basis of the accuracy of his simplified method lies in the insensitivity of the moment equation to variation in θ when this angle is not greater than $\theta_{\rm i}$.

The next step was to investigate whether the accuracy of Bishop's simplified method is affected by the slope or height of the embankment or by the soil properties. Since the investigation was intended to cover a wide range of these variables, the first two parts of the routine of the solution described above were programmed for solution on the Atlas computer.

INITIAL COMPUTER PROGRAM

A program was devised which worked in two phases.

(1) The mean height (h) and the base slope (α) of each slice were determined from data giving the height (H) and slope (β) of the embankment, the position of the slip circle and the number of slices to be used.

The position of the slip surface was defined by the co-ordinates of the centre of rotation ((x, y) in Fig. 1(a)), the coefficient N_x and the depth of the tension crack (z), which are also shown in the figure.

(2) The values of h and α obtained in the first phase of the program were then used to compute the values of F which would satisfy equations (7) and (8) for arbitrarily chosen values of θ varying from zero to the slope of the embankment.

The data required for the program to complete the second phase of its operation consisted of the soil properties: c', ϕ' , γ and the pore-pressure coefficient r_u .

(3) The corresponding values of the factors of safety and θ were then plotted as shown in Fig. 2(b) to obtain the values of F_i and θ_i .

The first use to which this program was put was to determine the number of slices which must be taken in order to achieve a required standard of accuracy.

Number of slices

The effect on the factor of safety of using different numbers of slices was checked for one case in which the slip circle passed through the toe, and for another in which the slip circle passed beneath the toe. The latter case is illustrated in Fig. 3 and, in this problem, values of $\frac{c'}{F\gamma H}$ and ϕ' have been chosen which are consistent with a critical slip circle in this position. The results obtained for this problem taking the number of slices as 8, 16, 32, 64, and 128 are shown in Table 1.

It will be seen from this table that the value of the factor of safety increases as the number of slices increases but that there is little to be gained in accuracy by taking the number of slices as greater than 32. A similar result was obtained when the slip surface was assumed to pass through the toe and the number of slices was therefore standardized at 32.

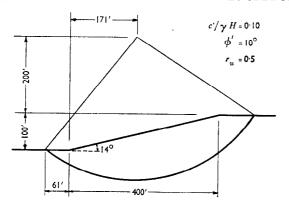


Table 1									
Number of slices Factor	8	16	32	64	128				
safety	1.233	1.245	1.249	1.251	1.251				
Percent- ageerror	1.36	0.48	0.16	0	0				

Fig. 3. and Table 1. Effect on F of varying the number of slices

RESULTS OBTAINED WITH INITIAL COMPUTER PROGRAM

The initial computer program was used to solve a large number of problems in which the relationships between F_t , F_m and θ were determined in respect of arbitrarily chosen slip circles in embankments of different slopes and with varying soil properties.

The results obtained with this initial program gave a clear indication that, using Bishop's simplified method, the error increases slightly as the slope of the embankment and the values of r_u and ϕ' increase and as $c'/\gamma H$ decreases. However, as it would have been purely coincidental if the slip circles chosen had been the critical ones, the scope of the investigation was now extended to confirming (or otherwise) these findings for critical cases.

Part 2 CRITICAL SLIP CIRCLES

Contours showing the variation in factor of safety at different centres of rotation have been published by Bishop and Morgenstern (1960) and by Sevaldson (1957). The contours are roughly elliptical with the major axis approximately at right angles to the surface of the slope and several times as large as the minor axis. In analyses in terms of effective stress, the centre of the critical slip circle is usually located close to the perpendicular bisector of the slope and normally on the 'uphill' side of it.

In order to identify the position of the critical slip circle, a rectangular grid was set out as shown in Fig. 4, one side of the grid coinciding with the perpendicular bisector. The initial program was modified so that it would deal with more than one slip surface at a time and the co-ordinates of each intersection were calculated by hand and taken as centres of slip circles. The program was later modified again so that, when the co-ordinates of the lower corner of the grid were

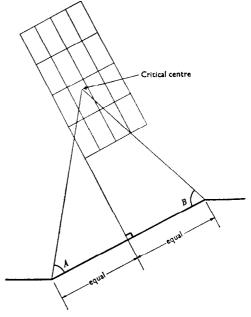


Fig. 4. Grid search pattern

stated together with the lengths of the sides of the grid, the co-ordinates of the intersections were calculated by the computer. Using this program it was possible to determine the values of $F_{\rm mo}$ and $F_{\rm i}$ for slip circles centred at each grid intersection and hence, by graphical interpolation, to find the position of the critical slip circle for each of the two values of the factor of safety.

Taking the value of the pore-pressure factor (r_u) as 0.5, the stability of embankments, varying in slope from $1\frac{1}{2}$:1 to 4:1 in soils of different properties, was investigated using this program; 20 cases were considered altogether. In each case 100 slip circles were investigated, of which 25 passed through the toe, the remainder passing beneath it.

The results obtained from this series showed that the critical circles for F_{mo} and F_{1} were invariably so close as to be almost coincident. Though some useful information was obtained, it was decided that the program should be modified so as to define the position of the critical circle with greater precision. The program developed was of the minimization type.

MINIMIZATION PROGRAM

The position of a slip circle can be defined by the co-ordinates of its centre and by its radius. In this Paper, the radius has been defined by the position of the lower end of the slip circle—by the value of the coefficient $N_{\mathbf{x}}$ as shown in Fig. 1. In the minimization program, the computer determined the co-ordinates of the centre of rotation and the value of $N_{\mathbf{x}}$ for which the value of $F_{\mathbf{mo}}$ was a minimum. To work in conjunction with this program, another program was devised which would compute the value of $F_{\mathbf{1}}$ for a given slip circle (which could be the critical circle from the other program), the position of the centre of the slip circle in terms of the angles A and B as shown in Fig. 4 and other parameters useful in assessing the results. The use of this pair of programs to solve a given problem implied the assumption that the critical circles for $F_{\mathbf{mo}}$ and $F_{\mathbf{1}}$ would coincide. This was indicated from the results obtained using the grid program, but a further check was made as follows.

Using the grid program it was found that the biggest difference between F_{mo} and F_1 had occurred in the case of the steepest slope considered (1½:1), where the value of ϕ' was high (40°) and the value of the parameter $\frac{c'}{\gamma H}$ was fairly low (0.05). This difference between the two values of the factor of safety implied a correspondingly large separation of the critical slip circles.

The same problem was tackled using the new pair of programs. First the minimization program identified the position of the critical circle for F_{mo} . Then, using the second program, the value of F_1 was found for this slip circle and also for others whose centres formed a grid pattern around its centre. The results are summarized in Table 2. In this table, the first line of figures refers to the critical circle for F_{mo} and the second line to the critical circle for F_1 . Both of the critical circles pass through the toe of the embankment and the angles defining their centres differ only by about $\frac{1}{2}$ °. It was evident therefore that, for values of r_u not exceeding 0.5, little inaccuracy would result if it were assumed that the two critical circles (for F_{mo} and F_1) were coincident. This assumption was made in the remaining

Table 2. Position of critical circles for F_i and F_{mo}

Slope
$$1\frac{1}{2}:1$$
; $\frac{c'}{\gamma H} = 0.05$; $\phi' = 40^{\circ}$; $r_u = 0.5$

Α°	B°	$N_{\mathbf{x}}$	$F_{ m mo}$	$F_{\mathbf{i}}$
56·3	65·2	0	1.018	1·039
56·9	65·7		1.018	1·038

work, the main purpose of which was to obtain charts giving the position of critical slip circles and also stability charts which would give the slope of an embankment when the mobilized angle of shearing resistance (ϕ'_{m}) and the stability number $(\frac{c'}{F\gamma H})$ were given.

POSITION OF CRITICAL CIRCLES

The factors which influence the position of the critical circle in a given case are: the slope of the embankment, the depth to hard stratum, the soil properties ϕ' , $\frac{c'}{\gamma H}$ and the pore-pressure which is represented here in the form of the coefficient r_u . In this investigation, it has been assumed that the depth to hard stratum was sufficiently great for the position of the critical circle to be determined by the values of the other four factors.

However, the number of factors can be reduced to three since the position of the critical circle is unique for any given value of the parameter $\gamma H \tan \phi'/c'$. Possibly the easiest way to demonstrate this is to consider a particular problem which has been solved by finding the factors of safety at the intersections of a grid as shown in Fig. 4. Suppose that the lowest factor of safety is, say, 1·2 and that this occurs at a particular point P. If the same problem were solved with the shear strength parameters c' and $\tan \phi'$ increased by 100%, then the factor of safety at each grid point would also be doubled, but the lowest value (now 2·4) would remain at P.

The factors which determine the position of the critical slip circle are therefore: the slope, the pore-pressure coefficient (r_u) and the parameter γH tan ϕ'/c' . With the position of the centre of the critical circle defined by angles A and B as shown in Fig. 4, it is therefore a simple matter to devise a series of charts, for different values of r_u , in which the position of the centre of the critical circle can be determined from the values of the slope and γH tan ϕ'/c' .

TENSION CRACKS

It should be noted that in order to retain the dimensionless form of analysis, the effect of tension cracks at the top of the embankment has been neglected. In some trial cases the upper end of the slip circle was assumed to terminate at the bottom of a tension crack of a depth (z) given by the expression:

$$z = \frac{2c'}{\gamma} \sqrt{\frac{1 + \sin \phi'}{1 - \sin \phi'}} \qquad (9)$$

When these problems were repeated omitting the tension crack, a value was obtained for the factor of safety which was less than that obtained when the tension crack was assumed to be present. It was also found that the difference between the two values of F increased as the value of the parameter c'/γ increased. However, the effect of water pressure in the tension crack was not taken into account in these trial cases and it may be that more work is required on this point before it can be assumed that a conservative result will always be obtained if the effect of shrinkage cracks is ignored.

CRITICAL SLIP CIRCLE AND STABILITY CHARTS

In order to obtain information from which critical circle and stability charts could be constructed, a series of 92 problems were considered and, in each, the critical slip circle was found. Three values were taken for r_u : 0, 0.25 and 0.5. For each value, embankments with slopes of $1\frac{1}{2}$:1, 2:1, 3:1 and 4:1 were considered and for each slope values of ϕ' and $\frac{c'}{\gamma H}$ were

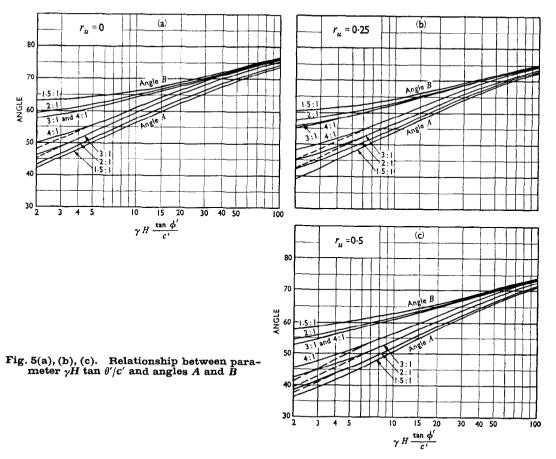
chosen such that the relationship between the mobilized angle of shearing resistance (ϕ'_m) and the stability number $(\frac{c'}{F\nu H})$ was obtained.

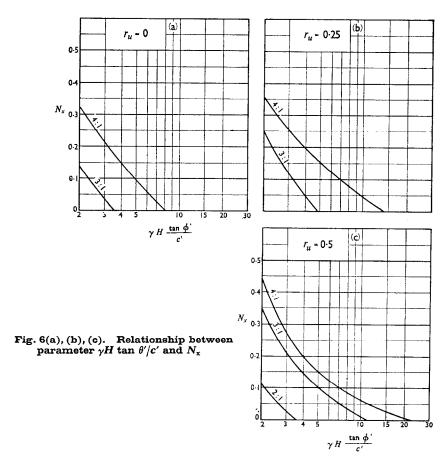
The results obtained confirmed those which had been obtained for the non-critical cases and gave a clear picture of the influence of different factors on the position of the critical circle. Before discussing the critical circle and stability charts, the conclusions regarding the difference between the two factors of safety will be stated.

Factors affecting the difference between Fmo and Fi

Firstly it must be pointed out that the difference between the two factors of safety exceeded 1% in only 7 out of the 92 cases considered. In each of these 7 cases, r_u had the maximum value considered (0.5). The greatest difference between F_{mo} and F_1 was about 4% and occurred in a case where the values of the slope, of r_u and of ϕ' were high $(1\frac{1}{2}:1,0.5)$ and 30° respectively) and the value of $\frac{c'}{\gamma H}$ low (0.006). When r_u was taken as 0.25 and 0, the difference never exceeded $\frac{1}{2}\%$.

For values of r_u of 0.5 and 0.25, the value of F_1 was never less than that of F_{mo} . On the other hand, with r_u equal to zero, F_1 was never greater than F_{mo} but the difference never exceeded 0.3%.



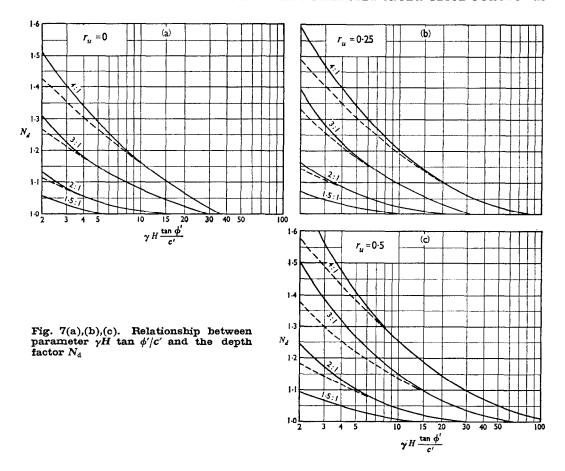


Critical circle charts

From the results given by the 92 cases considered, charts have been plotted for each value of r_u , giving the centre of the critical circle in terms of angles A and B. These charts are shown in Figs 5(a), 5(b) and 5(c). In these figures, the dashed branch lines at the lower end of the curves for angle A give the value of this angle for the critical toe circle; the heavier lines give the value for the most critical circle—whether passing through or beneath the toe. In fact it was found that, in general, the error in the factor of safety due to assuming that the critical circle will pass through the toe is quite small (less than 2% in the worst case examined).

If it is not assumed that the slip circle passes through the toe, other information is required (in addition to the position of its centre) to fix its position. This information is given in the form of the coefficient $N_{\mathbf{x}}$ in the charts shown in Figs 6(a), 6(b) and 6(c). When multiplied by the height of the embankment, $N_{\mathbf{x}}$ gives the distance of the lower end of the slip circle from the toe as shown in Fig. 1(a). Charts for the depth factor $(N_{\mathbf{d}})$ are given similarly in Figs 7(a), 7(b) and 7(c). Again, the dashed branch lines apply to the critical toe circle.

It will be observed in Figs 5(a), (b) and (c) that angle B decreases as the slope decreases and, in fact, that the curves for slopes of 3:1 and 4:1 are almost coincident. For slopes flatter than 4:1, however, it was found that angle B had begun to increase again. If the curves for angle B were represented in three dimensions (angle B, the slope and γH tan ϕ'/c') a 'trough'



shaped figure would result, with the values for slopes steeper than about 4:1 on one side and those for the flatter slopes on the other. This effect was not found in the case of angle A, in which the value of the angle continued to increase as the slope became flatter.

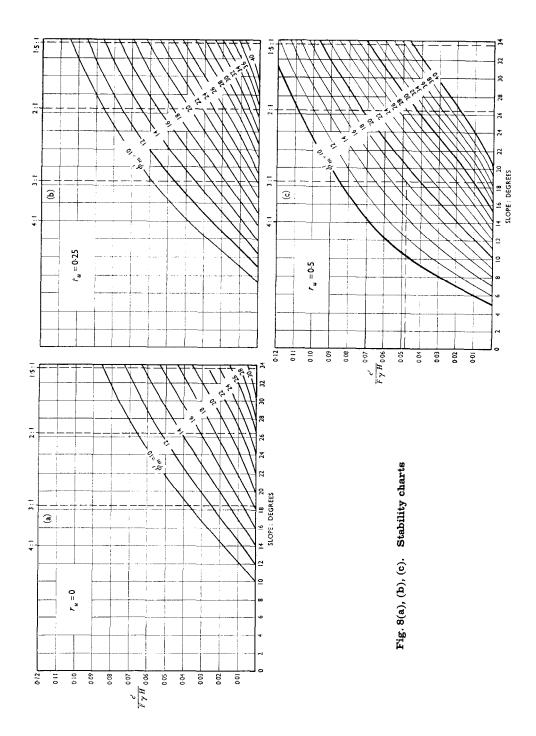
STABILITY CHARTS

For each of the three values of r_u , the relationship between the mobilized angle of shearing resistance (ϕ'_m) and the stability number $\left(\frac{c'}{F\gamma H}\right)$ had been obtained for four different slopes. From the curves given by these relationships, the stability charts shown in Figs 8(a), 8(b) and 8(c) were obtained using a graphical method. In order to define the shape of the curves in the stability charts for $r_u = 0.5$ some additional cases were considered at slopes flatter than 4:1.

The stability charts give the *critical* slope (F=1) when the height of an embankment and the soil properties are known. If, instead of the critical slope, one having a specified factor of safety is required the process is still very simple and is explained in the following example.

Example

To find the slope corresponding to a factor of safety of 1.5 for an embankment 100 ft high in a soil whose properties are as follows: c' = 870 lb/sq. ft; $\gamma = 120$ lb/cu. ft; $\phi' = 26^\circ$; $r_u = 0.5$.



The value of the stability number:

$$S_{n} = \frac{c'}{F\gamma H} = \frac{870}{1.5 \times 120 \times 100}$$

$$S_{n} = 0.048.$$

$$\tan \phi'_{m} = \frac{\tan \phi'}{F} = \frac{0.488}{1.5} = 0.325$$

$$\phi'_{m} = 18^{\circ}.$$

Referring to Fig. 8(c), for which $r_u = 0.5$, the slope corresponding to a stability number of 0.048 and ϕ'_m of 18° is 3:1.

For values of r_u intermediate between 0 and 0.25 and between 0.25 and 0.5, the slope (in degrees) can be obtained, sufficiently accurately for practical purposes, by linear interpolation. This point is illustrated in Fig. 9 which shows the relationship between the slope and r_u for two different cases.

When a safe slope is required for an embankment of a given height in a soil of given properties, these charts provide a very rapid solution. On the other hand, trial and error is necessary to find the factor of safety when the slope is given. Furthermore, if in a particular case the depth to hard stratum implies a depth factor less than that given in Figs 7(a), (b) and (c), then the stability charts will lead to a conservative solution.

The charts given in Figs 8(a), (b) and (c) may be compared with those given by Bishop and Morgenstern (1960) and have complementary advantages and disadvantages. Using their charts, the selection of a suitable slope, given the other parameters, calls for trial and error. On the other hand, the calculation of the factor of safety when the slope is known is simple and direct. It should be noted however that, in a case where the critical circle has a depth factor in between those for the charts given by Bishop and Morgenstern, the factor of safety given by these charts will be slightly too high.

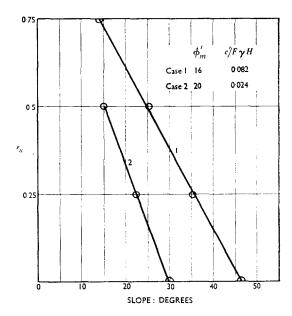


Fig. 9. Relationship between r_u and slope

SUMMARY

- (1) A method of analysis has been described for determining the stability of an embankment. The analysis is in terms of effective stress, satisfies the conditions of equilibrium for both forces and moments and assumes that the inter-slice forces are parallel.
- (2) The results given by this method have been compared with those given by Bishop's simplified method which relies solely on a moment equilibrium equation and assumes the interslice forces to be horizontal. Cases have been considered in which the soil properties varied over a wide range and for embankments with slopes varying between 1½:1 and 4:1.

The results of the comparison have shown that, though the accuracy of Bishop's simplified method decreases slightly as the slope of the embankment, r_u and ϕ' increase and as the parameter $c'/\gamma H$ decreases, the error was less than 1% in most of the cases considered. The worst combination of these factors resulted in an error in the factor of safety of 4%.

- (3) The reason for the accuracy of the results given by Bishop's simplified method lies in the insensitivity of the moment equation to the slope of the inter-slice forces.
- (4) In embankments with flat slopes and in which the parameter γH tan ϕ'/c' is small, the critical circle may pass beneath the toe. However, if it is assumed that the critical circle actually passes through the toe, the resulting error in the factor of safety is not likely to exceed 2% for values of r_n not greater than 0.5.
- (5) The method of analysis has been programmed for solution on the Atlas computer and the program has been used to obtain a set of charts giving the position of the critical circles and also a set of stability charts.

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