Finally, Equation A22 can be solved for the resultant force Q to give

$$Q = \left[-F_v \sin\alpha - F_h \cos\alpha - \frac{\bar{c}}{F} \Delta x \sec\alpha + (F_v \cos\alpha - F_h \sin\alpha + u\Delta x \sec\alpha) \frac{\tan\bar{\phi}}{F} \right] m_\alpha \qquad \text{(A23)}$$

where

$$m_\alpha = \frac{1}{\cos(\alpha - \theta) + \sin(\alpha - \theta) \frac{\tan\bar{\phi}}{F}} \qquad \text{(A24)}$$

15. Expression for y_Q. The line of action of the resultant force Q is expressed by the coordinate y_Q located on the line of action at a point directly above the center of the base of the slice. The coordinate y_Q is shown in Figure A2. Summing moments about the center of the base of the slice and noting that the sum must be equal to zero for equilibrium gives

$$-Q \cos\theta (y_Q - y_b) + M_o = 0 \qquad \text{(A25)}$$

where, as shown earlier by Equation A3, M_o represents the moment about a point on the center of the base of the slice due to all of the known forces (KW, P, R, etc.). Equation A25 is solved for y_Q to give

$$y_Q = y_b + \frac{M_o}{Q \cos\theta} \qquad \text{(A26)}$$

Solution of equilibrium equations

16. The expressions for Q (Equation A23) and y_Q (Equation A26) are substituted into the equations of equilibrium (Equations A15 and A16) to produce two equations in two unknowns (F and θ) which must be satisfied to satisfy static equilibrium. The solution to Equations A15 and A16 for the factor of safety and side-force inclination is accomplished using an iterative procedure based on Newton's method for solving two equations in two unknowns.

For assumed values of the factor of safety F_0 and side-force inclination θ_0, it is convenient to write the two equilibrium equations in the form

$$R_1 = \Sigma Q_0 \tag{A27}$$

and

$$R_2 = \Sigma Q_0 \left(x_b \sin \theta_0 - y_{Q_0} \cos \theta_0 \right) \tag{A28}$$

where

Q_0 = value of Q based on the assumed values

F_0, θ_0, R_1, R_2 = force and moment imbalances, respectively, based on the assumed values F_0 and θ_0

Application of Newton's method to find the roots to Equations A27 and A28 corresponding to $R_1 = R_2 = 0$, gives the following for the new estimates for F and θ based on the assumed values:

$$F_1 = F_0 + \Delta F \tag{A29}$$

and

$$\theta_1 = \theta_0 + \Delta\theta \tag{A30}$$

where ΔF and $\Delta\theta$ represent adjustments to the assumed values of F and θ, respectively, to be used for the next iteration. The expressions for ΔF and $\Delta\theta$ are as follows:

$$\Delta F = \frac{R_1 \dfrac{\partial R_2}{\partial \theta} - R_2 \dfrac{\partial R_1}{\partial \theta}}{\dfrac{\partial R_1}{\partial \theta}\dfrac{\partial R_2}{\partial F} - \dfrac{\partial R_1}{\partial F}\dfrac{\partial R_2}{\partial \theta}} \tag{A31}$$

$$\Delta \theta = \frac{R_2 \dfrac{\partial R_1}{\partial F} - R_1 \dfrac{\partial R_2}{\partial F}}{\dfrac{\partial R_1}{\partial \theta}\dfrac{\partial R_2}{\partial F} - \dfrac{\partial R_1}{\partial F}\dfrac{\partial R_2}{\partial \theta}} \tag{A32}$$

In the computer program UTEXAS2, Equations A31 and A32 are used to compute the values of ΔF and $\Delta\theta$, respectively, up to the point in the iterative solution where the respective values become less than 0.5 and 0.15 radians. Once the values become less than these limits ($\Delta F = 0.5$, $\Delta\theta = 0.15$ radians), an "extended" form of Newton's method is used based on the following two equations:

$$R_1 + \Delta F \frac{\partial R_1}{\partial F} + \Delta\theta \frac{\partial R_1}{\partial \theta} + \frac{1}{2} \Delta F^2 \frac{\partial^2 R_1}{\partial F^2} + \Delta F \Delta\theta \frac{\partial^2 R_1}{\partial F \partial \theta} + \frac{1}{2} \Delta\theta^2 \frac{\partial^2 R_1}{\partial \theta^2} = 0 \tag{A33}$$

$$R_2 + \Delta F \frac{\partial R_2}{\partial F} + \Delta\theta \frac{\partial R_2}{\partial \theta} + \frac{1}{2} \Delta F^2 \frac{\partial^2 R_2}{\partial F^2} + \Delta F \Delta\theta \frac{\partial^2 R_2}{\partial F \partial \theta} + \frac{1}{2} \Delta\theta^2 \frac{\partial^2 R_2}{\partial \theta^2} = 0 \tag{A34}$$

Equations A33 and A34 are derived from Taylor (1937) series expansions including the second-order terms. Estimates of new trial values are obtained by solving these two equations simultaneously for ΔF and $\Delta\theta$.

17. The partial derivatives of R_1 and R_2 in Equations A31 through A34 are obtained from Equations A27 and A28 and are as follows:

$$\frac{\partial R_1}{\partial F} = \Sigma \frac{\partial Q}{\partial F} \tag{A35}$$

$$\frac{\partial R_1}{\partial \theta} = \Sigma \frac{\partial Q}{\partial \theta} \tag{A36}$$

$$\frac{\partial^2 R_1}{\partial F^2} = \Sigma \frac{\partial^2 Q}{\partial F^2} \tag{A37}$$

$$\frac{\partial^2 R_1}{\partial F \partial \theta} = \Sigma \frac{\partial^2 Q}{\partial F \partial \theta} \tag{A38}$$

$$\frac{\partial^2 R_1}{\partial \theta^2} = \Sigma \frac{\partial^2 Q}{\partial \theta^2} \tag{A39}$$

$$\frac{\partial R_2}{\partial F} = \Sigma \frac{\partial Q}{\partial F} \left(x_b \sin \theta_0 - y_{Q_0} \cos \theta_0\right) - \Sigma Q_0 \left(\frac{\partial y_Q}{\partial F} \cos \theta_0\right) \tag{A40}$$

$$\frac{\partial R_2}{\partial \theta} = \Sigma \frac{\partial Q}{\partial \theta} x_b \left(\sin \theta_0 - y_{Q_0} \cos \theta_0 \right) + \Sigma Q_0 \left(x_b \cos \theta_0 + y_{Q_0} \sin \theta_0 \right.$$

$$\left. - \frac{\partial y_Q}{\partial \theta} \cos \theta_0 \right) \qquad \text{(A41)}$$

$$\frac{\partial^2 R_2}{\partial F^2} = \Sigma \frac{\partial^2 Q}{\partial F^2} \left(x_b \sin \theta_0 - y_{Q_0} \cos \theta_0 \right) - 2\Sigma \frac{\partial Q}{\partial F} \left(\frac{\partial y_Q}{\partial F} \cos \theta_0 \right)$$

$$- \Sigma Q_0 \left(\frac{\partial^2 y_Q}{\partial F^2} \cos \theta_0 \right) \qquad \text{(A42)}$$

$$\frac{\partial^2 R_2}{\partial F \partial \theta} = \Sigma \frac{\partial^2 Q}{\partial F \partial \theta} \left(x_b \sin \theta_0 - y_{Q_0} \cos \theta_0 \right)$$

$$+ \Sigma \frac{\partial Q}{\partial F} \left(x_b \cos \theta_0 + y_Q \sin \theta_0 - \frac{\partial y_Q}{\partial \theta} \cos \theta_0 \right) - \Sigma \frac{\partial Q}{\partial \theta} \left(\frac{\partial y_Q}{\partial F} \cos \theta_0 \right)$$

$$- \Sigma Q_0 \left(\frac{\partial^2 y_Q}{\partial F \partial \theta} \cos \theta_0 - \frac{\partial y_Q}{\partial F} \sin \theta_0 \right) \qquad \text{(A43)}$$

$$\frac{\partial^2 R_2}{\partial \theta^2} = \Sigma \frac{\partial^2 Q}{\partial \theta^2} (x_b \sin \theta_0 - y_Q \cos \theta_0)$$

$$+ 2\Sigma \frac{\partial Q}{\partial \theta} \left(x_b \cos \theta_0 + y_{Q_0} \sin \theta_0 - \frac{\partial y_Q}{\partial \theta} \cos \theta_0 \right)$$

$$- \Sigma Q \left(x_b \sin \theta_0 - y_{Q_0} \cos \theta_0 - 2 \frac{\partial y_Q}{\partial \theta_0} \sin \theta_0 + \frac{\partial^2 y_Q}{\partial \theta^2} \cos \theta \right) \qquad \text{(A44)}$$

18. In evaluating the various partial derivatives of Q in Equations A35 through A44, it is convenient to write the expression for Q (Equation A23) as

$$Q = \frac{C_1 + \frac{C_2}{F}}{C_3 + \frac{C_4}{F}} \tag{A45}$$

where

$$C_1 = -F_v \sin \alpha - F_h \cos \alpha \tag{A46}$$

$$C_2 = -\bar{c}\Delta x \sec \alpha + (F_v \cos \alpha - F_h \sin \alpha + u\Delta x \sec \alpha) \tan \bar{\phi} \tag{A47}$$

$$C_3 = \cos (\alpha - \theta) \tag{A48}$$

$$C_4 = \sin (\alpha - \theta) \tan \phi \tag{A49}$$

Then

$$\frac{\partial Q}{\partial F} = \frac{-1}{\left(C_3 + \frac{C_4}{F}\right)^2} \left[\left(C_3 + \frac{C_4}{F}\right) \frac{C_2}{F^2} - \left(C_1 + \frac{C_2}{F}\right) \frac{C_4}{F^2}\right] \tag{A50}$$

$$\frac{\partial Q}{\partial \theta} = \frac{-1}{\left(C_3 + \frac{C_4}{F}\right)^2} \left(C_1 + \frac{C_2}{F}\right)\left(\frac{\partial C_3}{\partial \theta} + \frac{1}{F}\frac{\partial C_4}{\partial \theta}\right) \tag{A51}$$

$$\begin{aligned}\frac{\partial^2 Q}{\partial F^2} = \frac{1}{\left(C_3 + \frac{C_4}{F}\right)^3} &\left\{\left(C_3 + \frac{C_4}{F}\right)\left[2\left(C_3 + \frac{C_4}{F}\right)\frac{C_2}{F^3} - 2\left(C_1 + \frac{C_2}{F}\right)\frac{C_4}{F^3}\right]\right. \\ &\left. - 2\frac{C_4}{F^2}\left[\left(C_3 + \frac{C_4}{F}\right)\frac{C_2}{F^2} - \left(C_1 + \frac{C_2}{F}\right)\frac{C_4}{F^2}\right]\right\}\end{aligned} \tag{A52}$$

$$\begin{aligned}\frac{\partial^2 Q}{\partial F \partial \theta} = \frac{-1}{\left(C_3 + \frac{C_4}{F}\right)^3} &\left\{\left(C_3 + \frac{C_4}{F}\right)\left[\frac{C_2}{F^2}\left(\frac{\partial C_3}{\partial \theta} + \frac{1}{F}\frac{\partial C_4}{\partial \theta}\right) - \left(C_1 + \frac{C_2}{F}\right)\frac{1}{F^2}\frac{\partial C_4}{\partial \theta}\right]\right. \\ &\left. - 2\left(\frac{\partial C_3}{\partial \theta} + \frac{1}{F}\frac{\partial C_4}{\partial \theta}\right)\left[\left(C_3 + \frac{C_4}{F}\right)\frac{C_2}{F^2} - \left(C_1 + \frac{C_2}{F}\right)\frac{C_4}{F^2}\right]\right\}\end{aligned} \tag{A53}$$

$$\frac{\partial^2 Q}{\partial \theta^2} = \frac{-1}{\left(C_3 + \frac{C_4}{F}\right)^3} \left[\left(C_3 + \frac{C_4}{F}\right) \left(C_1 + \frac{C_2}{F}\right)\left(\frac{\partial^2 C_3}{\partial \theta^2} + \frac{1}{F} \frac{\partial^2 C_4}{\partial \theta^2}\right) - 2\left(C_1 + \frac{C_2}{F}\right) \left(\frac{\partial C_3}{\partial \theta} + \frac{1}{F} \frac{\partial C_4}{\partial \theta}\right)^2\right] \quad \text{(A54)}$$

where

$$\frac{\partial C_3}{\partial \theta} = \sin (\alpha - \theta) \quad \text{(A55)}$$

$$\frac{\partial C_4}{\partial \theta} = -\cos (\alpha - \theta) \tan \phi \quad \text{(A56)}$$

$$\frac{\partial^2 C_3}{\partial \theta^2} = -\cos (\alpha - \theta) \quad \text{(A57)}$$

$$\frac{\partial^2 C_4}{\partial \theta^2} = -\sin (\alpha - \theta) \tan \phi \quad \text{(A58)}$$

19. Expressions for the various partial derivatives of the variable y_Q in Equations A40 through A44 are as follows:

$$\frac{\partial y_Q}{\partial F} = \frac{-1}{\left(Q_0 \cos \theta_0\right)^2} M_o \frac{\partial Q}{\partial F} \cos \theta_0 \quad \text{(A59)}$$

$$\frac{\partial y_Q}{\partial \theta} = \frac{-1}{\left(Q_0 \cos \theta_0\right)^2} M_o\left(\frac{\partial Q}{\partial \theta} \cos \theta_0 - Q_0 \sin \theta_0\right) \quad \text{(A60)}$$

$$\frac{\partial^2 y_Q}{\partial F^2} = \frac{-1}{Q^2 \cos \theta} M_o \left[\frac{\partial^2 Q}{\partial F^2} - \frac{2}{Q} \left(\frac{\partial Q}{\partial F}\right)^2\right] \quad \text{(A61)}$$

$$\frac{\partial^2 y_Q}{\partial F \partial \theta} = \frac{-1}{Q_0^2 \cos \theta_0} M_o \left(\frac{\partial^2 Q}{\partial F \partial \theta} + \frac{\partial Q}{\partial F} \tan \theta - 2 \frac{\partial Q}{\partial F} \frac{\partial Q}{\partial \theta} \frac{1}{Q_0} \right) \quad \text{(A62)}$$

$$\frac{\partial^2 y_Q}{\partial \theta^2} = \frac{-1}{Q_0^2 \cos \theta_0} M_o \left[2 \frac{\partial^2 Q}{\partial \theta^2} \tan \theta_0 - \frac{\partial^2 Q}{\partial \theta^2} + Q_0 + \frac{2}{Q_0} \left(\frac{\partial Q}{\partial \theta} - Q_0 \tan \theta_0 \right)^2 \right] \quad \text{(A63)}$$

Solution for remaining unknowns

20. Once the values of the factor of safety and side-force inclination are determined which satisfy the equilibrium Equations A15 and A16, the remaining unknowns are calculated. The remaining unknowns consist of the normal force on the base of the slice N, the side force Z between slices, and the locations of the side forces y_t. The normal forces are calculated from Equation A18 which was derived by summing forces in a direction perpendicular to the base of each slice. The value of the force Q in Equation A18 is calculated from Equation A23. Although the shear force is not actually considered an "unknown" (it is known if N and F are known), the shear force can only be calculated once F is found; the shear force is calculated from Equation A21.

21. Side forces Z are calculated slice-by-slice, beginning with the first, leftmost slice. In general, for the i^{th} slice,

$$Z_{i+1} = Z_i - Q_i \quad \text{(A64)}$$

where

Z_i = the side force on the left of the slice

Z_{i+1} = the side force on the right of the slice

Q_i = the resultant of the side forces on each side of the slice

The side forces are shown in Figure A3. The resultant of the side forces Q_i is calculated for each slice from Equation A23 once values for the factor of safety and side-force inclination have been determined. The side forces are then calculated for each slice beginning with the first slice. For the first slice, the side force on the left of the slice Z_1 must be zero and, thus,