Optimizing Random Access for Information Freshness in Spatially Distributed Wireless Networks

by

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Abstract

We analyze Age of Information (AoI) in wireless networks where nodes use a spatially adaptive random access scheme to send status updates to a central base station. We show that the set of achievable AoI in this setting is convex, and design policies to minimize weighted sum, min-max, and proportionally fair AoI by setting transmission probabilities as a function of node locations. We show that under the capture model, when the spatial topology of the network is considered, AoI can be significantly improved, and we obtain tight performance bounds on weighted sum and min-max AoI. Finally, we design a policy where each node sets its transmission probability based only on its own distance from the base station, when it does not know the positions of other nodes, and show that it converges to the optimal proportionally fair policy as the size of the network goes to infinity.

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Chapter 1

Introduction

Wireless networks emerged as a primary way to enable internet connectivity and communication between individuals. As these networks have become more sophisticated, their capacity has increased to allow higher levels of data throughput. However, with the rise of connected devices and the Internet of Things (IoT), the purpose and design of some networks is changing. In contrast to networks of human users, IoT networks generally have lower data rates but stricter latency requirements. Instead of data intensive uses like video streaming, these networks commonly involve small update packets which are time sensitive.

Age of Information (AoI) was introduced to provide a metric for information freshness in networks [1] and measures the time elapsed since the last received update from a source was generated. Significant work has been done in designing scheduling policies to minimize AoI, and the results in these areas are quite mature [2–7].

All of the works above focus on centralized scheduling policies, and less work has been done to minimize AoI in random access channels, where nodes decide whether to access the channel in a completely decentralized way. In [8], the authors analyze AoI with slotted ALOHA and find the optimal attempt probabilities to minimize weighted sum AoI. In [9], the authors analyze random access networks with stochastic arrivals and optimize the transmission probabilities for both slotted ALOHA and CSMA. In [10], the authors analyze AoI in CSMA and optimize the back-off timers to minimize Age when updates are generated at will.

In [11, 12], the authors propose an AoI threshold below which nodes will never access the channel. When their AoI exceeds the threshold, they participate in standard slotted ALOHA, and otherwise remain silent to prioritize nodes with larger AoI. In [13], a hybrid between ALOHA and CSMA is proposed, where nodes use an Age threshold coupled with carrier sensing.

All of these works assume a collision interference model, meaning that if multiple nodes access the channel simultaneously none of them succeed. When nodes are distributed in space this is often an oversimplification. At the physical layer, wireless signals attenuate over distance, so transmissions from varying distances will be perceived differently at the receiver. When two nodes, one close by and one far away, transmit simultaneously, it is common for the farther node's transmission to be drowned out and for the closer node's transmission to succeed, contrary to what's predicted in the collision model. This leads to unfairness in the network, where farther nodes can see much larger AoI on average. This is especially true in CSMA-based protocols like 802.11 [14], where (i) sensing the channel may fail to detect farther nodes' transmissions, and (ii) successes cause nodes to transmit more frequently and failures cause them to transmit less frequently. This creates a positive feedback loop where closer nodes transmit more often, amplifying the spatial unfairness. When applied to human users, this phenomenon may not be catastrophic, but in an IoT network with critical time-sensitive updates, this unfairness can result in safety or reliability issues.

A commonly used model which accounts for spatial effects is the capture model, and some work has been done under this model analyzing random access networks in regards to throughput. In [15], the authors highlight the spatial unfairness in CSMA-based protocols specifically with multipacket reception, and design a protocol which counter-intuitively backs off after a success, sharing a similar flavor to threshold ALOHA.

In [16], the authors analyze what they call an adaptive spatial ALOHA protocol, which operates like slotted ALOHA but allows nodes to transmit with varying probabilities based on their spatial location. They optimize this protocol to maximize sum, max-min, and proportionally fair throughput, both with full topology information and without. They extend these results in [17], where they introduce the notion of a stopping set, or limited topology information. Following a similar line, [18] derives a spatial ALOHA policy to minimize sum AoI in a network of transmitter-receiver pairs using stochastic geometry and the same idea of stopping sets. In [19], the authors find a spatial distribution for the mean Peak AoI in a network of source-destination pairs.

In this thesis, we analyze AoI in a network of sources sending status updates to a central base station. These sources use a random access policy, which sets transmission probabilities as a function of node locations. We show that the set of achievable AoI in this setting is convex, and derive policies to minimize weighted sum, min-max, and proportionally fair AoI. We show that under the capture model, when node locations are considered, AoI can be significantly improved, and the spatial unfairness of traditional random access (c.f. Figure 3-3) can be reduced or eliminated. We derive tight performance bounds on weighted sum and min-max AoI, which also provide fairness guarantees. Finally, we design a topology-agnostic policy where each node sets its transmission probability based only on its own location, and show that it asymptotically converges to the optimal proportionally fair policy as the size of the network goes to infinity.

The thesis is organized as follows. In the remainder of this chapter, we introduce the system model and AoI more formally. In Chapter 2, we analyze the achievable AoI region under a random access policy and the capture model. In Chapter 3, we design random access policies using perfect topology information to minimize AoI, and derive performance bounds on these policies. Finally, in chapter 4, we derive a topology-agnostic policy and show its performance and convergent behavior as the size of the network goes to infinity.

1.1 System Model

We consider a network of N devices, each sampling some process and sending status updates wirelessly to a central base station. The base station acts as either a centralized monitor or controller for decision making within the system. Each device, or node, is located at some location in two dimensional space, and we assume that these locations are fixed over the time horizon of interest. We define the distance between node i and the base station as r_i and assume that nodes are located inside a circle of a fixed radius. Without loss of generality, we scale distances so that r_i takes values from 0 to 1. We refer to the vector of distances $\boldsymbol{r}$ as the *position vector*.

We assume that update packets have a fixed length, and that time is broken into discrete slots. The duration of a slot is the time required to send one packet. Nodes operate in a random access fashion, where node i attempts transmission according to a Bernoulli process with probability p_i, independent of other nodes and across time slots. The process for each node is stationary, so the vector $\boldsymbol{p}$ is fixed across time, but its components can vary between nodes. The network is saturated, meaning nodes sample their process in every time slot and always have an update to send.

As in many wireless systems, a single communication channel is shared by the network. When two or more nodes try to access the channel in the same time slot, their signals will interfere and one or more of the transmissions may fail. We focus on the case of single-packet reception, where the base station can receive a maximum of one packet in each time slot, but our model is more general and can be extended to multi-packet reception. Because nodes transmit according to a random process, the interference and success of transmissions is also a random process. Let τ_i be the success probability of node i in a given slot, and note that it is stationary under the random access model.

Under the capture model, a success occurs when the signal to interference plus noise ratio is larger than a known threshold θ. Therefore, τ_i is defined as the probability that this event occurs for node i. The threshold θ is set by the modulation and coding schemes used at the physical layer, and represents a fundamental trade-off between probability of success and channel capacity. This trade-off is beyond the scope of this thesis, and we assume a fixed and known value of θ.

Assuming that every node transmits at the same unit power level, the signal strength seen at the base station is a function of the signal attenuation over distance with roll-off parameter β, and Rayleigh fading modeled as a random variable $K^2 \sim Exp(1)$. We assume noise is negligible relative to interference, so the network operates in the interference limited regime, and we consider only the signal to interference (SIR) ratio.

Under these assumptions, the success probability of node i is the product of its attempt probability and conditional success probability, because each node transmits independently. This can be expressed as

$$\tau_i = p_i \cdot \mathbb{P}[SIR_i > \theta] = p_i \cdot \mathbb{P}\left[\frac{r_i^{-\beta} K_i^2}{\sum_{j \in I_i} r_j^{-\beta} K_j^2 V_j} > \theta\right], \tag{1.1}$$

where $V_j \sim Ber(p_j)$ is a random variable indicating whether node j transmits in the current time slot, and I_i is the set of nodes which interfere with node i, here assumed to be every other node in the network. Following the same procedure as [16], by conditioning on the Rayleigh fading and Bernoulli transmissions of interferers, and using the complementary CDF of the exponential distribution,

$$\begin{aligned}\mathbb{P}[SIR_i > \theta \mid V_j, K_j^2,\ \forall j \in I_i] &= e^{-\theta r_i^{\beta} \sum_{j \in I_i} (r_j^{-\beta} K_j^2 V_j)} && (1.2)\\ &= e^{-\theta r_i^{\beta}} \prod_{j \in I_i} e^{-r_j^{-\beta} K_j^2 V_j}. && (1.3)\end{aligned}$$

Averaging over the Rayleigh fading,

$$\mathbb{P}[SIR_i > \theta \mid V_j,\ \forall j \in I_i] = \prod_{j \in I_i} \frac{1}{1 + \theta r_i^{\beta} V_j r_j^{-\beta}}. \tag{1.4}$$

Now, averaging over the Bernoulli transmissions,

$$\mathbb{P}[SIR_i > \theta] = \prod_{j \in I_i} \left(1 - \frac{p_j}{1 + r_j^{\beta} / r_i^{\beta} \theta}\right). \tag{1.5}$$

Finally, by plugging this result back into (1.1),

$$\tau_i = p_i \prod_{j \in I_i} \left(1 - \frac{p_j}{1 + r_j^\beta / r_i^\beta \theta}\right) = p_i \prod_{j \in I_i} \left(1 - \frac{p_j}{1 + d_{ij}}\right), \tag{1.6}$$

where $d_{ij} \triangleq r_j^\beta / (r_i^\beta \theta)$.

Examining the terms in this expression, the ratio $1/(1 + d_{ij})$ can be interpreted as the amount node j will interfere with node i given that both nodes transmit. We refer to this as the *interference factor* from node j to node i. Note that this factor is not commutative, and in fact a large d_{ij} corresponds to a small d_{ji} for values of θ around 1.

The interference factor is mostly impacted by the nodes' respective distances. If i is significantly closer to the base station than j, the factor drops to almost 0, whereas it approaches 1 in the opposite extreme. As β increases, this shift becomes more abrupt. This fits with our understanding of signal attenuation over distance. As θ increases, the factor grows closer to 1 for a fixed set of distances, making it "easier" for node j to interfere when there is a larger SIR threshold. As θ approaches infinity, the interference factor converges to 1 for all pairs of nodes, and the capture model converges to the collision model.

1.2 Age of Information

We formally define the Age of Information of node i at time t as $A_i(t)$. This quantity measures the time elapsed since the last update received by the base station from node i was generated. Because nodes are saturated in our model, and we assume that the base station is only interested in the most recent status update, nodes operate using a Last Come First Served queue with preemption. In other words, each node samples its process and replaces the last update packet with this new sample at the beginning of each time slot. This guarantees that each successful update is generated at the beginning of the time slot in which it is received, and the AoI of node i evolves

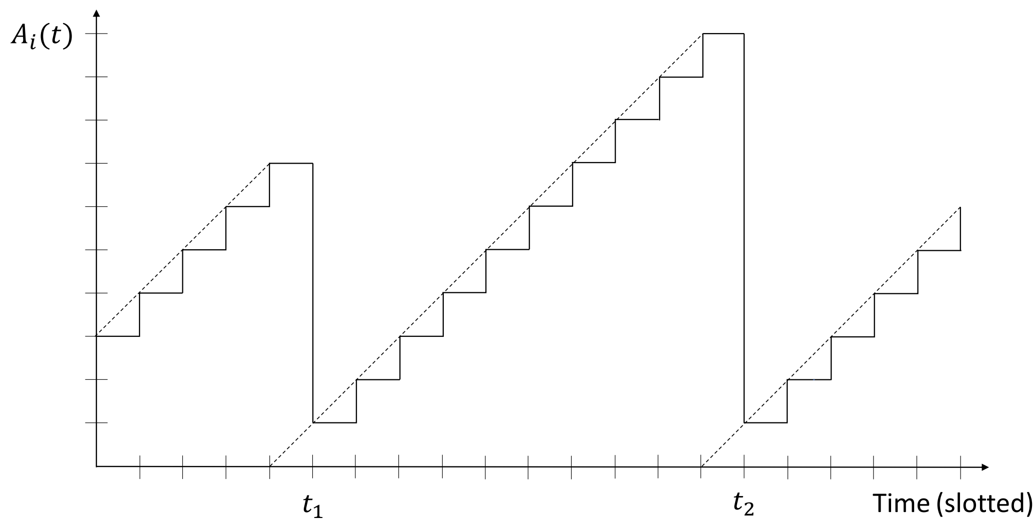

Figure 1-1: Evolution of the AoI of node i with successes at t_1 and t_2

as

$$A_i(t+1) = \begin{cases} A_i(t)+1, & \text{if } s_i(t) \neq 1 \\ 1, & \text{if } s_i(t) = 1 \end{cases} \tag{1.7}$$

where $s_i(t) = 1$ if node i successfully transmits in slot t.

It's important to note that AoI is not the same as delay. Consider a simple network with two nodes under our model. The first node transmits with probability $p_1 = 1$, and the second node transmits with probability $p_2 = 0$. Node 1 will succeed in every time slot, and the average delay in the network will be 1. However, the base station will never receive updates from node 2, so the Age of node 2 will grow without bound, driving the average AoI of the network to infinity.

The infinite time average expected AoI of node i is defined as

$$h_i \triangleq \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[A_i(t)]. \tag{1.8}$$

The success probability of each node i is iid Bernoulli with parameter τ_i, so the inter-arrival time X_i between successes is iid Geometric with parameter τ_i. In [3], the authors show that the AoI process is a Renewal process, and from the Renewal Reward theorem [20, Sec 5.7],

$$
\begin{aligned}
h_i &= \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]} + \frac{1}{2} = \frac{(2-\tau_i)\tau_i}{2\tau_i^2} + \frac{1}{2} = \frac{1}{\tau_i} \\
&= \frac{1}{p_i \prod_{j \in I_i} (1 - \frac{p_j}{1+d_{ij}})}.
\end{aligned}
\tag{1.9}
$$

This equation shows that h_i is uniquely determined by the position vector $\boldsymbol{r}$ and the transmission probabilities $\boldsymbol{p}$, which we call the *policy vector.* We define the vector function Φ such that

$$
\boldsymbol{h} = \Phi(\boldsymbol{p}, \boldsymbol{r}) \tag{1.10}
$$

and each element $h_i = \phi_i(\boldsymbol{p}, \boldsymbol{r})$ is given by (1.9).

We are interested in the behavior of $\boldsymbol{h}$, how it is affected by spatial diversity in the network, and how to design policies that minimize AoI in the presence of this diversity. We begin by examining the expression and deriving some intuition.

Observation 1. *h_i is monotonically decreasing in p_i and monotonically increasing in p_j, for all $j \in I_i$*

This is an intuitive result, since time average Age is the inverse of throughput, and the more frequently node i attempts transmission, the more often it will succeed. Likewise, the more often an interferer attempts transmission, the more it will block node i's attempts. Clearly then, increasing p_i will boost node i's performance at the expense of the other nodes in the network, and achieving good performance as a whole is a balancing act.

In slotted ALOHA, the optimal transmission probabilities are $1/N$ [21, Sec 3.2.4], achieving a total average throughput of $1/e$ as the number of nodes in the network goes to infinity. Slotted ALOHA is optimized for the collision interference model, but using the capture model allows us to take advantage of the spatial diversity in the network, and the fact that when multiple nodes transit simultaneously, one or more can still succeed based on the SIR threshold θ and the interference factors between the nodes. More precisely, when $\theta \leq 1$ and two nodes transmit in the same slot, one is always guaranteed to succeed, because the SIR of one node must be larger than the other. Under this model, we are able to design policies that are more efficient than slotted ALOHA, both in terms of throughput and AoI.

In this thesis we examine three different AoI metrics. The most obvious and commonly used metric in the literature is expected weighted sum AoI (EWSAoI), but we also examine min-max AoI (MMAoI) and proportionally fair AoI (PFAoI). Optimizing for each of these metrics produces slightly different policies, but in all three cases we want to prevent any single node's Age from growing too large, which results in some notion of fairness.

When a node is farther away from the base station relative to its interferers, its interference factor is smaller, and it is less likely to block another node's transmission. Therefore to achieve some notion of fairness in the network, we intuitively expect nodes farther from the base station to transmit more often with a lower conditional success probability, and nodes closer to the base station to transmit more frequently with a higher rate of success. We will see that this intuition holds when we begin designing policies. In the next chapter, however, we first examine the achievable AoI region to gain a better understanding of how efficient policies can perform.

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Chapter 2

Characterizing the AoI Achievability Region

2.1 Introduction

To gain a better understanding of the performance we can achieve, we first characterize the set of achievable AoI under our model. This line of thought is inspired by information theoretic capacity regions, and the achievable throughput regions studied in contexts including random access [22]. We will see that understanding the achievability region helps gain intuition and derive both optimal policies and performance bounds for AoI.

2.2 Achievability Region

We define the set of achievable $\boldsymbol{h}$ for a given position vector $\boldsymbol{r}$ as

$$H(\boldsymbol{r}) \triangleq \{\boldsymbol{h} \in \mathbb{R}^N \mid h_i = \frac{1}{p_i \prod_{j \in I_i} \left(1 - \frac{p_j}{1+d_{ij}}\right)}, 0 \leq p_i \leq 1,\ \forall\ i\}, \tag{2.1}$$

and the set of all possible $H(\boldsymbol{r})$ as

$$\mathcal{H} \triangleq \{H(\boldsymbol{r}) \mid 0 < r_i \leq 1,\ \forall\ i\}. \tag{2.2}$$

When discussing general results that hold for any achievable region and are not explicitly dependent on $\boldsymbol{r}$, we drop the index and refer to H as a generic set in $\mathcal{H}$ with arbitrary position vector.

Observation 2. *For a network of N nodes, H is an N-dimensional set in the positive orthant and lower bounded by* 1 *in every dimension. It asymptotically approaches* 1 *in each dimension as the other dimensions go to infinity.*

This observation is straightforward to see. We can never achieve an AoI of less than 1, so H clearly resides in the positive orthant with this lower bound. To see the asymptotic behavior, consider again the case of $N = 2$, with $p_1 = 1$ and $p_2 = 0$. As noted in the previous chapter, $h_1 = 1$ and h_2 approaches infinity for any position vector, so H asymptotically approaches the point $(1, \infty)$. If we switch the entries of $\boldsymbol{p}$, H approaches $(\infty, 1)$. This idea can easily be generalized to N dimensions.

We now turn to the non-asymptotic behavior of H. Intuitively, under an efficient policy, adjusting $\boldsymbol{p}$ to decrease the AoI of one node will increase the AoI of another. To formalize this notion, we define a set of fixed point equations mapping weights to a policy vector. Let

$$f_i(p_i) \triangleq \frac{\lambda_i}{p_i} - \sum_{j \in I_i} \frac{\lambda_j}{1 + d_{ji} - p_i} = 0, \ \forall\ i \tag{2.3}$$

for $p_i \in [0, 1]$ and where λ_i is the weight associated with p_i. For any set of weights $\boldsymbol{\lambda}$ in the positive orthant, one can use (2.3) to find the associated $\boldsymbol{p}$ vector. This set of functions plays an important role both in characterizing the boundary of H and designing efficient policies, as seen in the following proposition.

Proposition 1. *For any position vector $\boldsymbol{r}$, there exists a Pareto boundary $H^*(\boldsymbol{r})$ to the set $H(\boldsymbol{r})$. For any vector of weights $\boldsymbol{\lambda}$ in the positive orthant, let*

$$\tilde{p_i} = \min\{p_i, 1\}, \ \forall\ i, \tag{2.4}$$

where $\boldsymbol{p}$ is the unique solution to (2.3)*. The resulting AoI vector $\boldsymbol{h} = \Phi(\tilde{\boldsymbol{p}}, \boldsymbol{r}) \in H^*(\boldsymbol{r})$, i.e., it lies on the Pareto Boundary.*

Furthermore, for every $\boldsymbol{h}^* = \Phi(\boldsymbol{p}^*, \boldsymbol{r}) \in H^*(\boldsymbol{r})$*, there exists a vector* $\boldsymbol{\lambda}$ *in the positive orthant whose components sum to 1 and for which* (2.3) *yields the solution* $\boldsymbol{p}^*$ *when* $p_i^* < 1$ *for all* i*.*

Proof. This follows from [22], where the authors show that these results hold for throughput. Because of the inverse relationship between throughput and AoI, every point on the throughput Pareto boundary has a one-to-one mapping to a point in H. Recall from the definition of Pareto optimality that, starting from any point on the boundary, throughput can only be increased for one node by reducing it for at least one other. Therefore, starting from the mapped point in H, AoI can only be decreased for one node by increasing it for at least one other, so the mapped points also form a Pareto boundary in H.

It is easy to see that this relationship holds in both directions, so a point lies on the throughput Pareto boundary if and only if it has a one-to-one mapping that lies on the AoI Pareto boundary. Because the results were shown to hold for the throughput boundary, they must also hold for H^*. □

Corollary 1. *The equation* (2.3) *yields a value of* $p_i \geq 1$ *if and only if*

$$\lambda_i r_i^{\beta} \geq \sum_{j \in I_i} \lambda_j r_j^{\beta} \theta. \tag{2.5}$$

Proof. The condition $f_i(1) \geq 0$ is equivalent to (2.5), because

$$f_i(1) = \lambda_i - \sum_{j \in I_i} \frac{\lambda_j}{d_{ji}} = \lambda_i - \frac{1}{r_i^{\beta}} \sum_{j \in I_i} \lambda_j r_j^{\beta} \theta. \tag{2.6}$$

It is straightforward to see that f_i is a decreasing function of p_i. Therefore if $f_i(1) \geq 0$, there cannot exist a $p_i < 1$ which is the solution to (2.3). Similarly, if the solution to (2.3) is greater than or equal to 1, $f_i(1)$ must be positive. This completes the proof. □

From this result we see that for values of θ close to 1, a node will only transmit with probability 1 when it's significantly farther away than its interferers, and won't

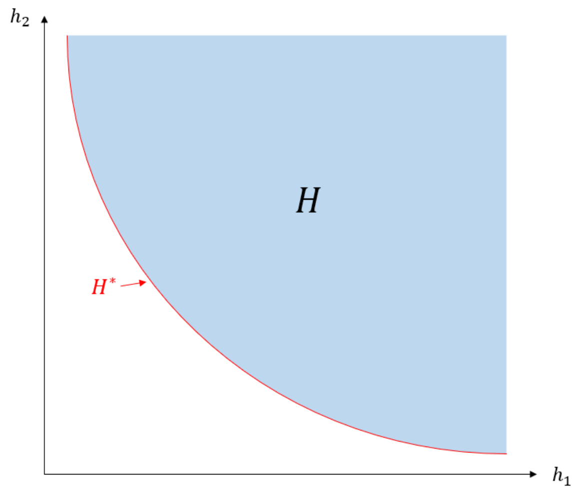

Figure 2-1: Example H region in two dimensions

itself present significant interference to other nodes. It can then transmit in every time slot without affecting the performance of other nodes in the network.

When θ is small and less than 1, i.e., in the case of multipacket reception, nodes are much more likely to transmit, so p_i will take the value 1 more frequently. Conversely when θ is large and we operate closer to the collision model, p_i will rarely equal 1.

Corollary 2. *For a given position vector $\boldsymbol{r}$, an AoI vector $\boldsymbol{h}$ is achievable, i.e., belongs to the set $H(\boldsymbol{r})$, if and only if it lies on or above the Pareto boundary $H^*(\boldsymbol{r})$.*

Proof. We first show the forward direction, that if some $\boldsymbol{h}$ is achievable then it lies on or above H^*. Assume it did not. Then it must lie below some $\boldsymbol{h}' \in H^*$ such that $h_i \leq h'_i$ for all i, so $\boldsymbol{h}'$ cannot be Pareto optimal, which is a contradiction.

To show the reverse direction, note that every point which lies above the Pareto boundary is in the interior of H, because H extends to infinity. Every point on the boundary is achievable from Proposition 1, and just as any point on the interior of the throughput region [23] is achievable, so is any point on the interior of H. Therefore, every point on or above H^* is achievable. $\square$

Although the achievability region for throughput in random access has a Pareto boundary, the region itself is non-convex [22]. This can easily be seen in two dimensions, where the points $(0, 1)$ and $(1, 0)$ are achievable, but any point on the diagonal in between is not due to collisions. Because of the inverse relationship between throughput and AoI, however, this does not preclude the achievable AoI region from being convex. We now show that it is, in fact, a convex set.

Theorem 1. *The set $H(\boldsymbol{r})$ is convex for any N and any position vector $\boldsymbol{r}$*

Proof. We start with a geometric proof in two dimensions, and then show that it holds for an arbitrary number of dimensions. Note that we cannot simply prove convexity by randomizing over policies, because random access policies are decentralized, and this randomization technique would require centralized coordination.

In two dimensions and for a fixed $\boldsymbol{r}$, the vector $\boldsymbol{h} = (h_1, h_2)$ is completely characterized by the vector $\boldsymbol{p} = (p_1, p_2)$. In particular, from (1.9) both h_1 and h_2 are convex functions of $\boldsymbol{p}$.

Consider any two points $\boldsymbol{h}^1 = (h_1^1, h_2^1)$ and $\boldsymbol{h}^2 = (h_1^2, h_2^2)$ on H^*. Each of these points is uniquely determined by some $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$. Furthermore, because ϕ_i is a continuous function of $\boldsymbol{p}$, there is a continuous curve traced between $\boldsymbol{h}^1$ and $\boldsymbol{h}^2$ by

$$\big(\phi_1(\lambda\boldsymbol{p}^1 + (1-\lambda)\boldsymbol{p}^2, \boldsymbol{r}),\ \phi_2(\lambda\boldsymbol{p}^1 + (1-\lambda)\boldsymbol{p}^2, \boldsymbol{r})\big), \tag{2.7}$$

for $0 \le \lambda \le 1$. Without loss of generality let $h_1^1 \le h_1^2$. Because H^* is a Pareto boundary (from Proposition 1), this implies $h_2^1 \ge h_2^2$, and the boundary is monotonically decreasing in the (h_1, h_2) plane. as shown in Figure 2-2. Because ϕ_i is convex in $\mathbf{p}$,

$$\begin{aligned} \phi_i(\lambda\boldsymbol{p}^1 + (1-\lambda)\boldsymbol{p}^2, \boldsymbol{r}) &\le \lambda\phi_i(\boldsymbol{p}^1, \boldsymbol{r}) + (1-\lambda)\phi_i(\boldsymbol{p}^2, \boldsymbol{r}) \\ &= \lambda h_i^1 + (1-\lambda)h_i^2, \end{aligned} \tag{2.8}$$

for $i = 1, 2$ and $0 \le \lambda \le 1$, so each point on the curve is element wise less than or equal to the convex combination of $\boldsymbol{h}^1$ and $\boldsymbol{h}^2$. The points on the curve are by definition achievable, so the achievable region must exist below the tangent connecting $\boldsymbol{h}^1$ and $\boldsymbol{h}^2$ and contain at least the points on the curve. This holds for any two points $\boldsymbol{h}^1$ and $\boldsymbol{h}^2$, so the set H must be convex in two dimensions. This argument is shown

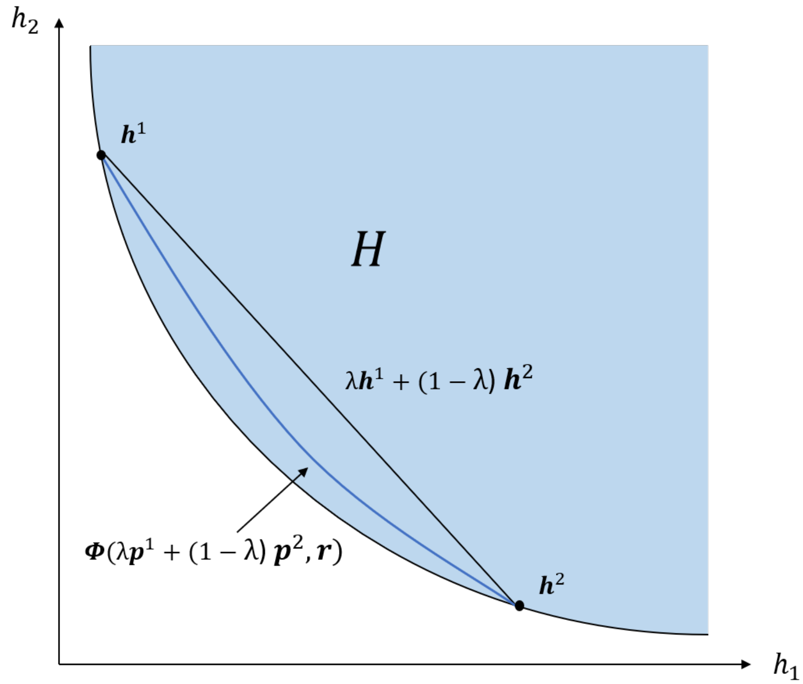

Figure 2-2: Curve traced between two points in H^* showing convexity of the region H

graphically in Figure 2-2.

We now show that this extends to an arbitrary number of dimensions N by showing the convexity of the vector function Φ, and relating H to the epigraph of this function. The convexity of a vector valued function is a natural generalization of the convexity of a scalar function [24], and holds if and only if

(i) the element-wise comparison

$$\Phi(\lambda \boldsymbol{p}^1 + (1-\lambda)\boldsymbol{p}^2, \boldsymbol{r}) \preceq \lambda \Phi(\boldsymbol{p}^1, \boldsymbol{r}) + (1-\lambda)\Phi(\boldsymbol{p}^2, \boldsymbol{r}) \tag{2.9}$$

is true for all $\boldsymbol{p}^1, \boldsymbol{p}^2$ in the domain of Φ and all $0 \leq \lambda \leq 1$, and

(ii) the domain of Φ is a convex set.

Here (i) is true by the convexity of ϕ_i, and (ii) holds because the domain of Φ is a cube in $\mathbb{R}^N$, where each element takes values from 0 to 1, which is easily seen to be convex.

The epigraph of the vector function Φ is defined in [24] as

$$(\text{epi } \Phi)(\boldsymbol{r}) \triangleq \{(\boldsymbol{p}, \boldsymbol{h}) \in \mathbb{R}^N \times \mathbb{R}^N \mid \boldsymbol{p} \in \text{dom } \Phi,\ \Phi(\boldsymbol{p}, \boldsymbol{r}) \preceq \boldsymbol{h}\}, \tag{2.10}$$

where the symbol $\preceq$ is the element-wise comparison of two vectors. In other words, the epigraph of Φ is the set of all $(\boldsymbol{p}, \boldsymbol{h})$ pairs such that $\boldsymbol{p}$ is in the domain of Φ, and $\boldsymbol{h}$ is element-wise greater than or equal to what can be achieved by $\boldsymbol{p}$. Consider some $\boldsymbol{p}'$ and its corresponding $\boldsymbol{h}' = \Phi(\boldsymbol{p}', \boldsymbol{r})$. Since $\boldsymbol{h}'$ is achievable, it must lie above $H^*(\boldsymbol{r})$ from Corollary 2. Because we can achieve any point above $H^*(\boldsymbol{r})$, it follows that any point element-wise greater than $\boldsymbol{h}'$ can be achieved by choosing a different $\boldsymbol{p}$, so any $\boldsymbol{h} \in \text{epi } \Phi$ is achievable. Likewise, any achievable $\boldsymbol{h}'$ is in the epigraph of Φ, because there exists a $\boldsymbol{p}'$ such that $\Phi(\boldsymbol{p}', \boldsymbol{r}) = \boldsymbol{h}'$.

Because Φ is a convex function, epi Φ is a convex set [24]. Furthermore, the projection of a convex set onto some of its coordinates is convex [25], so the projection of $(\text{epi } \Phi)(\boldsymbol{r})$ onto the set $\{\boldsymbol{h} \in \mathbb{R}^N\}$ is convex. Since a vector $\boldsymbol{h}$ is achievable if and only if it's in the epigraph of Φ, this projection is equal to $H(\boldsymbol{r})$. Therefore, the set $H(\boldsymbol{r})$ is convex. $\square$

In this chapter we showed the structure of the infinite time average AoI achievability region. We characterized its shape and asymptotic behavior, showed that it has a Pareto boundary, and that the set is convex. In the next chapters we turn our attention to the design of policy vectors $\boldsymbol{p}$, and use the intuition and results we derived here to find policies that minimize different metrics of AoI. We will see that the policies we derive lie on the Pareto boundary H^*, and the convexity of the set H proves useful in showing performance bounds on these policies.

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Chapter 3

Optimizing Age of Information with Perfect Topology Information

3.1 Introduction

In Chapter 2, we characterized the achievable region of time average AoI, and showed that it is convex and has a Pareto boundary. In this chapter we focus on designing policies to operate at specific points in that region. We are always interested in information freshness and keeping AoI small, so we expect every policy vector of interest to lie on the Pareto boundary. We will see that this is the case and that one can operate at different points on the boundary to achieve different objectives.

In this chapter, we assume perfect knowledge of the network topology, i.e., the position vector $\boldsymbol{r}$. Because interferer locations play a large role in the interference factor $1/(1 + d_{ij})$, and by extension play a large role in h_i, knowledge of $\boldsymbol{r}$ allows us to use spatial diversity to our advantage in designing policies.

3.2 Expected Weighted Sum AoI

We begin with the problem of minimizing the expected weighted sum AoI (EWSAoI) of the network over the policy vector $\mathbf{p}$,

$$\begin{aligned} &\min_{\boldsymbol{p}} \sum_{i=1}^{N} \alpha_i h_i \\ &\text{s.t. } 0 \leq p_i \leq 1, \ \forall\ i, \end{aligned} \tag{3.1}$$

where $\boldsymbol{\alpha} = (\alpha_i, \ldots, \alpha_N)$ is the set of positive weights guiding the minimization. This is perhaps the most natural optimization problem for information freshness, and appears commonly in the AoI literature [3] [6] [11]. In this section, we first find an optimal solution to the problem, and then show that this solution is a more general version of a result shown by Talak for a collision model [6].

It is important to note that while expected AoI is the inverse of expected throughput for an individual node, minimizing EWSAoI is not the same as maximizing expected weighted sum throughput. Consider the example from Chapter 1 of two nodes with $p_1 = 1$ and $p_2 = 0$. The sum throughput of this network is 1, which is optimal for single packet reception, but the expected AoI of node 2 grows without bound, driving the EWSAoI to infinity. This reinforces our motivation for studying AoI as opposed to throughput.

Rewriting the problem in terms of (1.9), it becomes

$$\begin{aligned} &\min_{\boldsymbol{p}} \sum_{i=1}^{N} \frac{\alpha_i}{p_i \prod_{j \in I_i} \left(1 - \frac{p_j}{1+d_{ij}}\right)} \\ &\text{s.t. } 0 \leq p_i \leq 1, \ \forall\ i. \end{aligned} \tag{3.2}$$

Note that this is a convex function in $\mathbf{p}$ minimized over a convex set. As a result, this is a convex optimization problem and can be solved using a number of algorithms. Nevertheless, to gain insight into the solution we derive a closed-form expression.

Theorem 2. *The solution to the EWSAoI minimization problem is given by*

$$p_i^{EWS} = \min\{\tilde{p}_i, 1\}, \ \forall\ i, \tag{3.3}$$

where $\tilde{p}_i$ is the solution to the fixed point equation

$$\frac{\alpha_i h_i}{\tilde{p}_i} - \sum_{j \in I_i} \frac{\alpha_j h_j}{1 + d_{ji} - \tilde{p}_i} = 0, \tag{3.4}$$

and where h_i is the resulting time average expected AoI of node i under this policy.

Before proving this result, we first note that the structure of the solution is the same as (2.3). Therefore, from Proposition 1, it must lie on the Pareto boundary of H for any positive vector of weights, as expected.

Proof. As noted previously, (3.2) is a convex optimization problem. Optimization theory tells us that the unconstrained problem has a unique solution, which can be found by setting the gradient equal to zero and solving for $\boldsymbol{p}$ [25]. If the solution satisfies the constraints, then it is also solves the constrained problem, otherwise the solution lies on the boundary of the constraint set.

In its current form, the product in the denominator of (3.2) makes it intractable to find a closed-form solution with this approach, even for small values of N. By moving each term in the sum into a separate constraint and taking the log, the equivalent problem

$$\begin{aligned} \min_{\boldsymbol{p}} \ & \sum_{i=1}^{N} \alpha_i h_i' \\ \text{s.t.} \ & \log h_i' \geq \log \frac{1}{p_i \prod_{j \neq i} \left(1 - \frac{p_j}{1+d_{ij}}\right)}, \ \forall\ i \\ & 0 \leq p_i \leq 1, \ \forall\ i. \end{aligned} \tag{3.5}$$

is derived. Note that this is equivalent because minimizing the sum in the objective will drive each constraint on $\boldsymbol{h'}$ to equality. The Lagrangian dual of this problem is given by

$$
\begin{aligned}
\min_{\boldsymbol{p}} \ & \sum_{i=1}^{N} \alpha_i h'_i + \sum_{i=1}^{N} \lambda_i \Bigg(\log \frac{1}{p_i \prod_{j \in I_i} \left(1 - \frac{p_j}{1+d_{ij}}\right)} - \log h'_i \Bigg) \\
\text{s.t. } & 0 \leq p_i \leq 1, \ \forall \ i \\
& \lambda_i \geq 0, \ \forall \ i,
\end{aligned} \tag{3.6}
$$

where $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers. This can be simplified further to

$$
\begin{aligned}
\min_{\boldsymbol{p}} \ & \sum_{i=1}^{N} \left(\alpha_i h'_i - \lambda_i \left(\log p_i + \sum_{j \in I_i} \log \left(1 - \frac{p_j}{1 + d_{ij}} \right) + \log h'_i \right) \right) \\
\text{s.t. } & 0 \leq p_i \leq 1, \ \forall \ i \\
& \lambda_i \geq 0, \ \forall \ i.
\end{aligned} \tag{3.7}
$$

Maximizing the solution to this problem over $\boldsymbol{\lambda}$ yields the solution to the dual problem, which is a lower bound on the solution to the primal. Because the primal problem is convex and the feasible set has a non-empty interior, Slater's condition is satisfied. Therefore, strong duality holds and this bound is tight [25, Sec 5.2], making the maximum of this problem over $\boldsymbol{\lambda}$ equivalent to the primal problem.

Because (3.7) is convex in $\boldsymbol{p}$ and $\boldsymbol{h'}$, and concave in $\boldsymbol{\lambda}$, the solution is found by taking the gradient with respect to each and setting it equal to zero, thereby maximizing over $\boldsymbol{\lambda}$ and minimizing over $\boldsymbol{p}$ and $\boldsymbol{h'}$. This gives the solution

$$
\begin{cases}
\dfrac{\lambda_i}{p_i} = \displaystyle\sum_{j \in I_i} \frac{\lambda_j}{1 + d_{ji} - p_i} \\
\lambda_i = \alpha_i h'_i \\
\log h'_i = - \log p_i - \displaystyle\sum_{j \in I_i} \log \left(1 - \frac{p_j}{1 + d_{ij}} \right)
\end{cases} \tag{3.8}
$$

for all i and when $p_i \in [0, 1]$. By taking the log of (1.9), it can immediately be seen that $h'_i = h_i$ under the resulting policy, and that the Lagrange multipliers are equal to the weighted expected AoI. Combining these three equations yields (3.4).

To see that this solution is unique, note that the left hand side of (3.4) is monotonically decreasing in p_i and that it goes to infinity as p_i goes to 0. If it becomes negative when $p_i = 1$, then by the Intermediate Value Theorem there exists a unique solution in the domain of p_i given by (3.4). Otherwise the minimum occurs at $p_i = 1$ by the monotonicity of (3.4). □

Corollary 3. *The optimal EWSAoI solution* (3.4) *is a more general version of the result found in [6] for the collision model. As θ goes to infinity, the capture model converges to the collision model and the optimal attempt probabilities become*

$$p_i = \frac{\alpha_i h_i}{\sum_{j=1}^{N} \alpha_j h_j}. \tag{3.9}$$

Proof. When θ goes to infinity, d_{ji} approaches 0 for any r_i and r_j, since distances are strictly positive and we disallow nodes from existing at the origin. Furthermore, recall that p_i is given by (3.4) provided the left hand side is negative when evaluated at $p_i = 1$. From Corollary 1, this is equivalent to the condition

$$\alpha_i h_i r_i^{\beta} \leq \sum_{j \in I_i} \alpha_j h_j r_j^{\beta} \theta, \tag{3.10}$$

which always holds as θ becomes large. Therefore, evaluating (3.4) as θ goes to infinity,

$$\frac{\alpha_i h_i}{p_i} = \lim_{\theta \to \infty} \sum_{j \in I_i} \frac{\alpha_j h_j}{1 + d_{ji} - p_i} = \sum_{j \in I_i} \frac{\alpha_j h_j}{1 - p_i}. \tag{3.11}$$

Rearranging terms,

$$\frac{1 - p_i}{p_i} = \frac{1}{\alpha_i h_i} \sum_{j \in I_i} \alpha_j h_j, \tag{3.12}$$

which further yields

$$\begin{aligned} \frac{1}{p_i} &= \frac{1}{\alpha_i h_i} \sum_{j \in I_i} \alpha_j h_j + 1 \\ &= \frac{1}{\alpha_i h_i} \left(\sum_{j \in I_i} \alpha_j h_j + \alpha_i h_i \right) \\ &= \frac{1}{\alpha_i h_i} \sum_{j=1}^{N} \alpha_j h_j. \end{aligned} \tag{3.13}$$

Finally, solving for p_i,

$$p_i = \frac{\alpha_i h_i}{\sum_{j=1}^{N} \alpha_j h_j} \tag{3.14}$$

for all i. This is the same result found in [6] for minimizing EWSAoI for distributed stationary policies under the collision model. $\square$

3.3 Min-Max Expected AoI

Minimizing EWSAoI achieves some notion of fairness in the network, because the optimization will not let any single node's AoI grow too large, but we may sometimes be interested in a clearer notion of fairness. Min-max fairness for AoI is equivalent to max-min fairness for throughput and ensures the most evenly minimized AoI across the network. In particular, min-max optimization is defined as minimizing the maximum AoI in the network over the vector $\boldsymbol{p}$,

$$\begin{aligned} &\min_{\mathbf{p}} \ \max \ \{h_i\} \\ &\text{s.t. } 0 \leq p_i \leq 1, \ \forall \ i. \end{aligned} \tag{3.15}$$

In terms of (1.9), this is equivalent to

$$\begin{aligned} &\min_{\mathbf{p}} \ \max \ \left\{ \frac{1}{p_i \prod_{j \in I_i} \left(1 - \frac{p_j}{1+d_{ij}}\right)} \right\} \\ &\text{s.t. } 0 \leq p_i \leq 1, \ \forall \ i. \end{aligned} \tag{3.16}$$

Similar to EWSAoI, this is a convex optimization problem and can be solved using a number of algorithms, but to gain insight into the solution we again derive a closed form expression.

Theorem 3. *The solution to the expected min-max AoI (MMAoI) problem is given by*

$$p_i^{MM} = \min\{\tilde{p}_i, 1\}, \ \forall \ i, \tag{3.17}$$

where $\tilde{p}_i$ is the solution to the fixed point equation

$$\frac{\lambda_i}{\tilde{p}_i} - \sum_{j \in I_i} \frac{\lambda_j}{1 + d_{ji} - \tilde{p}_i} = 0, \ \forall \ i, \tag{3.18}$$

and $\boldsymbol{\lambda}$ is such that the resulting vector $\boldsymbol{h^{MM}}$ has equal entries, and the sum

$$\sum_{i=1}^{N} \lambda_i = \log h_i^{MM} = \log h^{MM}, \ \forall \ i. \tag{3.19}$$

As with EWSAoI, the solution takes the form of (2.3), and so by Proposition 1, it lies on the Pareto boundary as expected. We also note that the resulting AoI is equal across all nodes, and is denoted by h^{MM}.

Proof. Using the well known technique for minimizing maxima, each h_i in (3.16) can be moved to the constraint set to form the equivalent problem

$$\begin{aligned} &\min_{\boldsymbol{p}} \ \alpha \\ &\text{s.t. } \alpha \geq \frac{1}{p_i \prod_{j \in I_i} \left(1 - \frac{p_j}{1+d_{ij}}\right)}, \ \forall \ i \\ &\qquad 0 \leq p_i \leq 1, \ \forall \ i. \end{aligned} \tag{3.20}$$

This problem is convex in $\boldsymbol{p}$, because the constraints form a convex set. As shown in [16], by taking the log of the constraints and setting $\tilde{\alpha} = -\log \alpha$, this problem is equivalent to

$$\begin{aligned} &\min_{\boldsymbol{p}} \ \alpha \\ &\text{s.t. } \tilde{\alpha} \leq \log p_i + \sum_{j \neq i} \log\left(1 - \frac{p_j}{1 + d_{ij}}\right), \ \forall \ i \\ &\qquad 0 \leq p_i \leq 1, \ \forall \ i. \end{aligned} \tag{3.21}$$

Because log is a monotonic function, the objective can be rewritten as maximizing over $\tilde{\alpha}$. Then recognizing that $\alpha \geq 1$, and therefore $\tilde{\alpha}$ is negative, it is equivalent to minimize its square,

$$
\begin{aligned}
&\min_{\boldsymbol{p}} \ \frac{1}{2}\tilde{\alpha}^2 \\
&\text{s.t. } \tilde{\alpha} \leq \log p_i + \sum_{j \neq i} \log\left(1 - \frac{p_j}{1 + d_{ij}}\right), \ \forall \ i \\
&\quad\ \ 0 \leq p_i \leq 1, \ \forall \ i.
\end{aligned} \tag{3.22}
$$

The Lagrangian dual of this problem is

$$
\begin{aligned}
&\min_{\boldsymbol{p}} \ \frac{1}{2}\tilde{\alpha}^2 + \sum_{i=1}^{N} \lambda_i \left(\tilde{\alpha} - \log p_i - \sum_{j \neq i} \log\left(1 - \frac{p_j}{1 + d_{ij}}\right)\right) \\
&\text{s.t. } 0 \leq p_i \leq 1, \ \forall \ i \\
&\quad\ \ \lambda_i \geq 0, \ \forall \ i.
\end{aligned} \tag{3.23}
$$

The duality gap is again zero, because the primal problem is convex and Slater's condition is satisfied [25]. Therefore, the maximum of the solution over $\boldsymbol{\lambda}$ is equivalent to the primal problem. Because (3.23) is convex in $\boldsymbol{p}$ and $\tilde{\alpha}$, and concave in $\boldsymbol{\lambda}$, taking the gradient with respect to each and setting it equal to zero yields the solution

$$
\begin{cases}
\dfrac{\lambda_i}{p_i} = \displaystyle\sum_{j \in I_i} \frac{\lambda_j}{1 + d_{ji} - p_i} & \text{(3.24a)} \\
\tilde{\alpha} + \displaystyle\sum_{i=1}^{N} \lambda_i = 0 & \text{(3.24b)} \\
\tilde{\alpha} = \log p_i + \displaystyle\sum_{j \in I_i} \log\left(1 - \frac{p_j}{1 + d_{ij}}\right) = -\log h_i & \text{(3.24c)}
\end{cases}
$$

for all i. Observe from (3.24c) that since $\tilde{\alpha}$ does not depend on i, the min-max solution achieves an equal expected AoI for all nodes.

Following the same approach as EWSAoI, the solution can be shown to be unique. It is easy to see that (3.18) is monotonically decreasing in p_i and goes to infinity as p_i goes to 0. If the left hand side evaluated at $p_i = 1$ is less than 0, then by the Intermediate Value Theorem there exists a unique solution in the domain $0 \leq p_i < 1$. Otherwise the minimum occurs at $p_i = 1$ by the monotonicity of (3.18). This completes the proof. $\square$

3.4 Proportionally Fair Expected AoI

The last metric we analyze is proportionally fair AoI (PFAoI). Proportional fairness is defined as the maximum sum log of throughput [26], so equivalently is the minimum sum log of AoI,

$$\begin{aligned} &\min_{\boldsymbol{p}} \ \sum_{i=1}^{N} \log h_i \\ &\text{s.t. } 0 \leq p_i \leq 1, \ \forall\, i, \end{aligned} \tag{3.25}$$

which can be rewritten in terms of (1.9) as

$$\begin{aligned} &\min_{\boldsymbol{p}} \ \sum_{i=1}^{N} \log \frac{1}{p_i \prod_{j\in I_i} \left(1 - \frac{p_j}{1+d_{ij}}\right)} \\ &\text{s.t. } 0 \leq p_i \leq 1, \ \forall\, i. \end{aligned} \tag{3.26}$$

In similar fashion to EWSAoI and MMAoI, we derive a closed-form solution to this problem, and in fact see that it takes a simpler form.

Theorem 4. *The solution to the PFAoI minimization is given by*

$$p_i^{PF} = \min\{\tilde{p}_i, 1\}, \ \forall\, i, \tag{3.27}$$

where $\tilde{p}_i$ is the solution to the fixed point equation

$$\frac{1}{\tilde{p}_i} - \sum_{j\in I_i} \frac{1}{1 + d_{ji} - \tilde{p}_i} = 0, \ \forall\, i. \tag{3.28}$$

Once again we note that the solution takes the form of (2.3) with weights equal to 1, and so lies on the Pareto boundary from Proposition 1.

Proof. The problem in (3.26) can be rewritten as

$$\begin{aligned} &\min_{\boldsymbol{p}} \sum_{i=1}^{N} \left(-\log p_i - \sum_{j \in I_i} \log \left(1 - \frac{p_j}{1+d_{ij}} \right) \right) \\ &\text{s.t. } 0 \leq p_i \leq 1, \ \forall\ i, \end{aligned} \tag{3.29}$$

which is convex due to the convexity of the equation in $\boldsymbol{p}$ and the convexity of the set. Therefore, the minimum can be found directly by taking the gradient and setting it equal to zero. This immediately gives the result (3.28).

Uniqueness follows along similar lines as EWSAoI and MMAoI. Once again, (3.28) is monotonically decreasing in p_i and approaches infinity as p_i goes to zero. If the left hand side of (3.28) is negative when evaluated at $p_i = 1$, then by the Intermediate Value Theorem there exists a unique solution in the domain $0 \leq p_i \leq 1$. Otherwise the minimum occurs at $p_i = 1$ by the monotonicity of (3.28). □

Not only does PFAoI have a closed-form solution similar to EWSAoI and MMAoI, it also has the advantage of being completely separable. This means the optimization can be performed in a distributed manner, with each node computing its own transmission probability and without sharing Lagrange multiplier values between nodes.

In networks where the achievable throughput region is convex, it can be shown that the average AoI under PFAoI has the additional property of being upper bounded by MMAoI. Since the throughput region is non-convex for random access networks [22], this is not necessarily true for our setting.

Nevertheless, we conjecture that PFAoI will have similar performance to EWSAoI with symmetric weights. From Theorem 2, the Lagrange multipliers in the EWSAoI solution with symmetric weights are equal to the resulting AoI of each node, and we know heuristically that when minimizing the sum, no single node's AoI will grow too large. As a result, the multipliers will be close to equal. Because scaling them all by a constant has no effect on the solution to (2.3), this is the same as if they were all close to 1, i.e., the PFAoI solution (3.28).

We will see through simulations that this is indeed the case, and while we provide no rigorous guarantees on performance, this heuristic argument combined with the separability of PFAoI makes it an appealing alternative to EWSAoI. We can, however, provide performance guarantees on EWSAoI and MMAoI, which we show next.

3.5 Performance Bounds

We have examined three policies which minimize different metrics of AoI. In this section, we compare the results of EWSAoI and MMAoI and bound the performance we can achieve under either policy.

We denote the solution to the EWSAoI problem with symmetric weights by $\boldsymbol{p}^S(\boldsymbol{r})$, and the corresponding AoI vector by $\boldsymbol{h}^S(\boldsymbol{r})$. Here the dependence on the position vector $\boldsymbol{r}$ is shown explicitly.

Next we highlight an assumption that we make use of in the remainder of the chapter.

Assumption 1. *The roll-off parameter $\beta = 2$, and the SIR threshold $\theta = 1$.*

Similar results can be found for other parameter values, but to simplify the analysis and to help illustrate the underlying ideas, we make use of these parameters.

Lemma 1. *For any N and position vector $\boldsymbol{r}$, and under Assumption 1,*

$$1 \leq \frac{1}{N^2} \sum_{i=1}^{N} h_i^S(\boldsymbol{r}). \tag{3.30}$$

Proof. Recall that since $\theta = 1$, the network operates in the single packet reception domain. This implies that the sum of expected throughputs is upper bounded by 1, and a relaxed version of the the EWSAoI problem with symmetric weights can be formed using this upper bound as a constraint,

$$\begin{aligned} \min \ & \sum_{i=1}^{N} h_i(\boldsymbol{r}) \\ \text{s.t.} \ & \sum_{i=1}^{N} \tau_i(\boldsymbol{r}) \leq 1. \end{aligned} \tag{3.31}$$

Using the inverse relationship between AoI and throughput, the solution is seen to be $h_i^* = N$ for all i by symmetry arguments. Since (3.31) is a relaxed version of the EWSAoI optimization with symmetric weights, this solution is a lower bound on the EWSAoI solution, and

$$N = \frac{1}{N}\sum_{i=1}^{N} h_i^*(\boldsymbol{r}) \leq \frac{1}{N}\sum_{i=1}^{N} h_i^S(\boldsymbol{r}). \tag{3.32}$$

Normalizing by N gives the result. □

Lemma 2. *For any N and position vector $\boldsymbol{r}$,*

$$\frac{1}{N^2}\sum_{i=1}^{N} h_i^S(\boldsymbol{r}) \leq \frac{1}{N} h^{MM}(\boldsymbol{r}). \tag{3.33}$$

Proof. This follows directly from the definition of EWSAoI with symmetric weights. This quantity is the minimum expected sum AoI, and so must be less than the expected sum AoI under any other policy, including min-max. Since the vector $\boldsymbol{h}^{MM}$ has equal entries from Theorem 3,

$$\sum_{i=1}^{N} h_i^S(\boldsymbol{r}) \leq \sum_{i=1}^{N} h_i^{MM}(\boldsymbol{r}) = N h^{MM}(\boldsymbol{r}), \tag{3.34}$$

and normalizing by N^2 gives the result. □

Using these two lemmas, we arrive at the main result of the chapter.

Theorem 5. *For any N and position vector $\boldsymbol{r}$, and under Assumption 1, the normalized AoI is bounded such that*

$$1 \leq \frac{1}{N^2}\sum_{i=1}^{N} h_i^S(\boldsymbol{r}) \leq \frac{1}{N} h^{MM}(\boldsymbol{r}) \leq \frac{e}{2}. \tag{3.35}$$

Proof. The two leftmost inequalities come from the lemmas above, so it remains only to show the upper bound. We provide a proof sketch here and present the full details in the appendix.

It can be shown that both EWSAoI and MMAoI are maximized when every node lies on a circle a fixed distance from the base station. Then the interference factor $1/(1+d_{ij}) = 1/2$ for every node pair, and Lagrange multiplier values are all equal, so both problem solutions reduce to the PFAoI solution. Then $p_i^S = p_i^{MM} = p_i^{PF} = 2/N$, and from (1.9),

$$h_i^S = h_i^{MM} = h_i^{PF} = \frac{N}{2\left(1-\frac{1}{N})\right)^{(N-1)}} \leq \frac{eN}{2} \tag{3.36}$$

for all i. Normalizing by N gives the result. $\square$

The performance bounds shown here for EWSAoI and MMAoI guarantee that adopting either of these two policies will result in a normalized average AoI of less than $e/2$. Given that the lower bound was shown to be 1, this is a relatively tight upper bound. These results also lead to the following corollary.

Corollary 4. *In a symmetric network, when all nodes are fixed on a circle around the base station and weights are equal, the optimal EWSAoI, MMAoI, and PFAoI policies are the same.*

Furthermore, this is the worst case performance for EWSAoI and MMAoI under any network topology. Introducing spatial diversity into the network improves average performance under either of these policies.

The bounds derived in this section provides tight guarantees on average AoI, and in the case of min-max, fairness guarantees that *every* node will lie below this bound. The driving motivation behind this work is the lack of fairness and the poor performance achieved by traditional random access policies in the presence of spatial diversity, but this result shows that when spatial diversity is built into the policy, it actually improves performance.

3.6 Numerical Results

In this section, we verify our results numerically and evaluate the performance of the three policies discussed in this chapter - expected weighted sum, min max, and

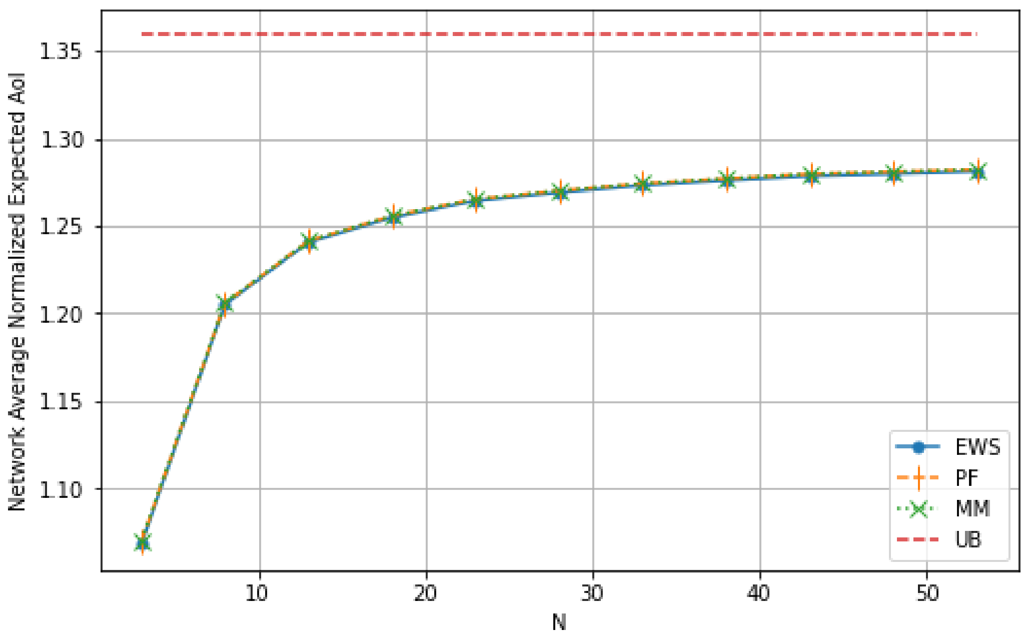

Figure 3-1: Normalized network average expected AoI for varying N under each policy, averaged over 100 random network topologies

proportionally fair AoI. We use normalized expected AoI as our comparison metric, defined as h_i/N. In figure 3-1, we show the network average normalized AoI as a function of N, averaged over 100 randomly generated network topologies. For each sample, the topology was generated and nodes were added incrementally to observe the change with N. From the results, we see that the three policies achieve ostensibly the same network average AoI, and that it approaches a constant value below the normalized EWSAoI and MMAoI upper bound (UB) of $e/2$.

Next we show that while the network average AoI is very close for all three policies, there is some variation across the network under EWS and PF, since they take advantage of spatial diversity to drive their objective functions down. We fix node i at varying distances from the base station and randomly generate 1000 different topologies for its interferers when $N = 50$. Under each topology, we find the optimal policy and plot both the transmission probabilities and the normalized AoI of node i, averaged over each topology.

The average transmission probabilities are shown in Figure 3-2. We see that the three policies have the same general shape, increasing with r_i as we expect, but with

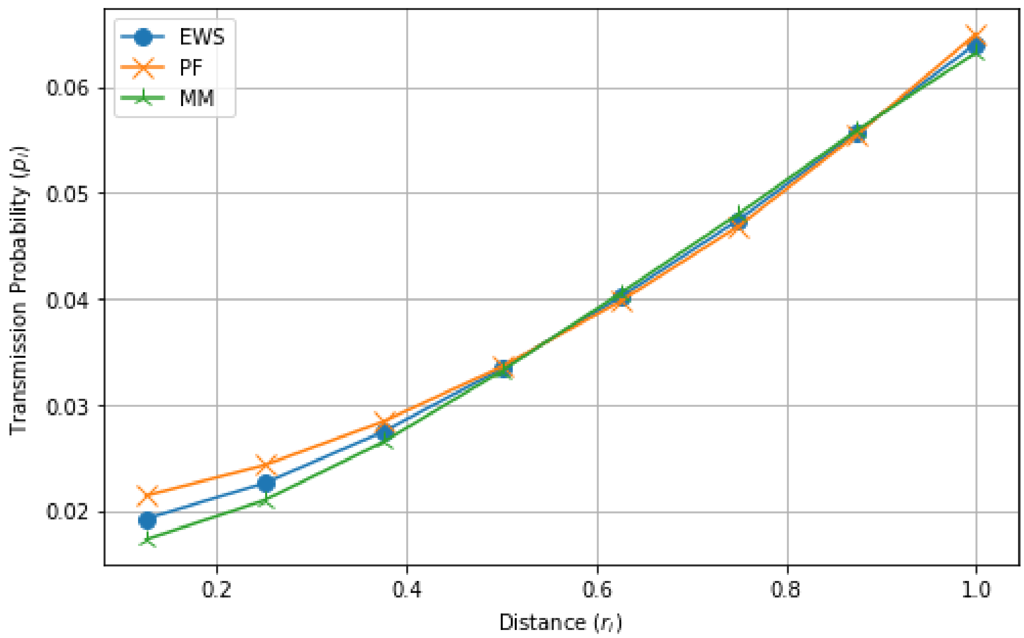

Figure 3-2: Transmission probabilities under each policy with varying position and $N = 50$, averaged over 1000 random topologies

small variations.

The normalized AoI of each policy is shown in Figure 3-3, and also compared to traditional slotted ALOHA. We see that although the network average AoI for the three policies is very close, there is some disparity across the network. The MM policy is flat as we expect, but the other policies have a concave shape peaking at about 2/3 the distance from the base station, so some unfairness still exists but is small. Intuitively, nodes close to the base station perform well because of their high signal strength, and nodes far from the base station perform well because of their large transmission probability. This creates the concave shape and causes nodes in the middle to perform the worst.

Nevertheless, the performance at each point still lies below the UB, showing that while the bound was only shown for network average EWSAoI and MMAoI, in practice it seems to hold for *each individual node* under any of the three policies. The plot also highlights the relative fairness of all three policies over slotted ALOHA, which sees a factor of 4 increase as r_i increases from 0.125 to 1.

As conjectured, the PF policy closely mimics the EWS policy and justifies its use

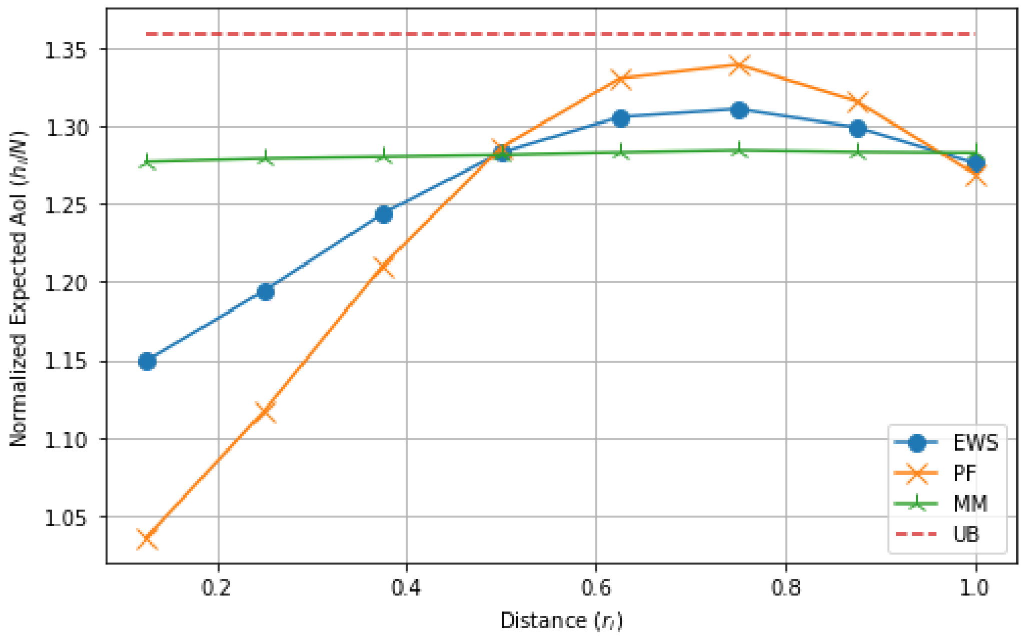

Figure 3-3: Normalized expected AoI of a single node under each policy with varying position and $N = 50$, averaged over 1000 random topologies

as a proxy. It has slightly more disparity as a function of distance, but the network average AoI is very close. We conclude that PF is a good proxy for minimizing sum or min-max AoI, while also easily implementable in a distributed system. In the next chapter, we examine the case where the network topology is unknown, and design a policy where each node sets its transmission probability based only on its own location.

Chapter 4

Optimizing Topology-Agnostic Age of Information

4.1 Introduction

In the previous chapter, we examined policies optimizing for three different metrics of AoI under the assumption of perfect topology information, where the entire position vector $\boldsymbol{r}$ is known. In practical settings, this may not be the case. It is realistic to assume that each node has knowledge of its own position through GPS or another technology, but may be blind to the locations of other nodes. Moreover, in a distributed setting, nodes must be able to compute their transmission probabilities independently.

Clearly, without location information one expects some loss in performance. Our goal is to design a policy which minimizes this loss, while setting transmission probabilities for each node based only on its own location. We refer to this class of policies as *topology-agnostic*, and focus on minimizing the loss to proportionally fair AoI, because the PFAoI policy (3.28) is completely decoupled and hence amenable to distributed implementation. Moreover, as we conjectured and showed through simulation results, the PF policy serves as a good proxy for minimizing EWSAoI. We will further show that we can derive a probability distribution on the optimal PFAoI transmission probability as the number of nodes goes to infinity, and show that it asymptotically converges to the topology-agnostic policy we derive.

4.2 Topology-Agnostic Proportionally Fair AoI

To model unknown interferer locations, we assume nodes are uniformly distributed in $\mathbb{R}^2$, and that the locations of interfering nodes are random variables. Because we are dealing with random variables, we use capital R_j to denote the distance of node j from the base station.

Because nodes are uniformly distributed over a circle of radius 1, this circle has an area equal to π, and the probability that a node lies within some region in space is the area of that region divided by π. Therefore the probability that node j lies within a circle of radius r_j, equivalent to saying it's distance from the base station is less than r_j, is

$$\mathbb{P}[R_j \leq r_j] = r_j^2, \ \forall \ j. \tag{4.1}$$

This is the CDF for the random variable R_j, and its pdf is given by its derivative

$$f_{R_j}(r_j) = 2r_j, \ \forall \ j. \tag{4.2}$$

We define the expected difference between our topology-agnostic (TA) and proportionally fair objectives as the optimality gap, which quantifies the loss incurred by not knowing $\boldsymbol{r}$. The optimal TA policy is then given by the solution to

$$\min_{\pi \in \Pi} \sum_{i=1}^{N} \mathbb{E}[\log h_i^{\pi}] - \sum_{i=1}^{N} \log h_i^{PF}, \tag{4.3}$$

where Π is the class of topology-agnostic policies. Here the expectation in each term of the sum is taken with respect to the locations of the interferers for that term, i.e., the expectation of h_i is taken with respect to R_j for all $j \in I_i$. In designing this policy, we again make use of Assumption 1, which we repeat here.

Assumption 1. *The roll-off parameter* $\beta = 2$, *and the SIR threshold* $\theta = 1$

The resulting policy is simple and elegant, as shown in the following theorem.

Theorem 6. *For large N, when nodes are blind to interferer locations, and under Assumption 1, the policy TA that minimizes the optimality gap* (4.3) *to proportionally fair AoI is given by*

$$p_i^{TA} \approx \frac{1}{(N-1)\big(1 - r_i^2 \log(1+\frac{1}{r_i^2})\big)}, \ \forall\ i. \tag{4.4}$$

Proof. Because (4.3) is equivalent to minimizing the first summation, we focus on this term and rewrite the problem in terms of (1.9),

$$\begin{aligned} \min_{\boldsymbol{p}} \ & \sum_{i=1}^{N} \mathbb{E}_{\boldsymbol{R}} \left[\log \frac{1}{p_i \prod_{j \in I_i} (1 - \frac{p_j}{1+D_{ij}})} \right] \\ \text{s.t. } & 0 \le p_i \le 1, \ \forall\ i, \end{aligned} \tag{4.5}$$

where $D_{ij} = R_j/(R_i\theta) = R_j/R_i$ when $\theta = 1$. Rearranging the objective function further,

$$\begin{aligned} & \sum_{i=1}^{N} \mathbb{E}_{\boldsymbol{R}} \Bigg[-\log p_i - \sum_{j \in I_i} \log \left(1 - \frac{p_j}{1 + R_j/R_i}\right) \Bigg] \\ & \qquad = \sum_{i=1}^{N} \mathbb{E}_{\boldsymbol{R}} \Bigg[-\log p_i - \sum_{j \in I_i} \log \left(1 - \frac{p_i}{1 + R_i/R_j}\right) \Bigg], \end{aligned} \tag{4.6}$$

where in the re-indexed sums on the right hand side, p_i only appears in one term of the outer sum, for each i.

Recall that topology-agnostic policies assume a distributed implementation, where each node has access to its own r_i. Thus, re-indexing the sum effectively decouples the optimization so that each node can set its p_i by minimizing a single term in the outer sum, independent of other nodes. Because each node has access to its location, the expectation can be conditioned on r_i in each term, and the problem becomes

$$\begin{aligned} \min_{\boldsymbol{p}} \ & \sum_{i=1}^{N} \mathbb{E}_{\boldsymbol{R}} \Bigg[-\log p_i - \sum_{j \in I_i} \log \left(1 - \frac{p_i}{1 + D_{ji}}\right) \mid R_i = r_i \Bigg] \\ \text{s.t. } & 0 \le p_i \le 1, \ \forall\ i. \end{aligned} \tag{4.7}$$

A single term of the sum with its expectation written explicitly is equal to

$$-\log p_i - \sum_{j \in I_i} \int_0^1 2r_j \log\left(1 - \frac{p_i}{1 + r_i^2/r_j^2}\right) dr_j. \tag{4.8}$$

This integral does not have a closed-form solution, but can be closely approximated using $\log(1-x) \approx -x$ for small x to eliminate the log in the integral. This is valid for large N, because p_i will be small and the denominator is greater than 1. Then

$$\begin{aligned}
\sum_{j \in I_i} \int_0^1 2r_j \log\left(1 - \frac{p_i}{1 + r_i^2/r_j^2}\right) dr_j &\approx -\sum_{j \in I_i} \int_0^1 2r_j \left(\frac{p_i}{1 + r_i^2/r_j^2}\right) dr_j \\
&= -\sum_{j \in I_i} p_i \left(1 - r_i^2 \log\left(1 + \frac{1}{r_i^2}\right)\right) \\
&= -(N-1) p_i \left(1 - r_i^2 \log\left(1 + \frac{1}{r_i^2}\right)\right).
\end{aligned} \tag{4.9}$$

Plugging this result back into (4.7), the optimization becomes

$$\begin{aligned}
&\min_{\boldsymbol{p}} \sum_{i=1}^{N} \left(-\log p_i + (N-1) p_i \left(1 - r_i^2 \log\left(1 + \frac{1}{r_i^2}\right)\right)\right) \\
&\text{s.t. } 0 \le p_i \le 1, \ \forall\, i.
\end{aligned} \tag{4.10}$$

This problem is convex in $\boldsymbol{p}$ and by setting the gradient equal to 0 and solving,

$$p_i = \frac{1}{(N-1)\big(1 - r_i^2 \log(1 + \frac{1}{r_i^2})\big)}, \ \forall\, i. \tag{4.11}$$

$\square$

Under this policy, each node sets its transmission probability based only on its own location and the number of nodes in the network. It has a clean, closed-form expression and doesn't require an algorithm to compute. In all these senses, it is a very attractive policy. The natural question is how much performance we lose compared to the optimal PF policy, i.e., how large is the optimality gap. This is the question examined in the following theorem.

Theorem 7. *For any position vector $\boldsymbol{r}$ and number of nodes N, and under Assumption 1, the normalized network average AoI under policy TA is bounded such that*

$$\frac{1}{N^2}\sum_{i=1}^{N} h_i^{TA} \leq e. \tag{4.12}$$

Proof. Recall that h_i is inversely proportional to the success probability of node i, and that success probability is the probability that the SIR of node i is greater than some threshold. It is therefore a decreasing function of the expected signal strength seen from each interferer.

Since nodes transmit independently with unit power and the expected Rayleigh fading is equal to 1, this is given by

$$\mathbb{E}[S_j] = \mathbb{E}[V_j] r_j^{-\beta} = p_j r_j^{-\beta}, \tag{4.13}$$

where S_j refers to the signal strength of node j. Under the TA policy and using the fact that $\beta = 2$ from Assumption 1, the transmission probability p_j is a function only of r_j and N. Specifically,

$$\mathbb{E}[S_j^{TA}] = \frac{r_j^{-2}}{(N-1)\left(1 - r_j^2 \log(1 + \frac{1}{r_j^2})\right)}. \tag{4.14}$$

This is a decreasing function of r_j on the interval $0 < r_j \leq 1$, so the expected interference seen from any node j following policy TA is maximized as it approaches the origin. Furthermore,

$$\lim_{r_j \to 0} p_j^{TA} = \lim_{r_j \to 0} \frac{1}{(N-1)\left(1 - r_j^2 \log(1 + \frac{1}{r_j^2})\right)} = \frac{1}{N-1}. \tag{4.15}$$

From this analysis we see that, for a fixed r_i in a network using the TA policy, h_i

is maximized when the interferers are arbitrarily close to the origin. In this case,

$$\begin{aligned}\lim_{r_j \to 0} h_i^{TA} &= \lim_{r_j \to 0} \frac{1}{p_i^{TA} \prod_{j \in I_i} \left(1 - \frac{p_j^{TA}}{1 + r_j^2/r_i^2}\right)} \\ &= \frac{(N-1)\big(1 - r_i^2 \log(1 + 1/r_i^2)\big)}{\big(1 - \frac{1}{N-1}\big)^{N-1}}\end{aligned} \tag{4.16}$$

for all $r_i >> r_j$, so $d_{ij} \to 0$. This is a decreasing function of r_i, and is maximized when r_i approaches 0 but the condition above still holds. Then

$$h_i^{TA} \to \frac{N-1}{\big(1 - \frac{1}{N-1}\big)^{N-1}}. \tag{4.17}$$

This is the maximum value h_i^{TA} can take for any node i, so is clearly an upper bound on the average h_i^{TA} in the network, and

$$\frac{1}{N} \sum_{i=1}^{N} h_i^{TA} \leq \frac{N-1}{\big(1 - \frac{1}{N-1}\big)^{N-1}} \leq (N-1)e \leq Ne. \tag{4.18}$$

Normalizing by N gives the result. □

Comparing this bound to the result in Theorem 5, we observe the following result.

Corollary 5. *The TA policy achieves a network average AoI within a factor of 2 of the worst case EWSAoI and MMAoI.*

Furthermore, while this provides a worst case bound on the performance of h_i^{TA}, practical network topologies yield much better performance, as we will see through simulations. Intuitively, the policy TA tries to average over the locations of other nodes, assuming a uniform distribution, to find the optimal transmission probabilities. The pathological case constructed in the proof occurs with arbitrarily low probability, and we expect the policy to perform much better for scenarios where nodes are more uniformly distributed.

4.3 Asymptotic Results and Convergence

As we show next, we can quantify how far the true optimal p_i^{PF} deviates from p_i^{TA} as N goes to infinity, by computing a probability distribution on p_i^{PF}, which is treated as a random variable for the remainder of this chapter.

Theorem 8. *When nodes are uniformly distributed in $\mathbb{R}^2$, and under Assumption 1, the inverse of the proportionally fair optimal p_i^{PF} converges to a truncated normally distributed random variable as N goes to infinity. Specifically,*

$$Z_i^{PF} = \frac{1}{p_i^{PF}} \to \mathcal{N}(N\mu_i, N\sigma_i^2) \tag{4.19}$$

on the interval $[1, \infty)$, with a probability mass spike at 1, and where

$$\mu_i = 1 - r_i^2 \log\left(1 + \frac{1}{r_i^2}\right), \tag{4.20}$$

$$\sigma_i^2 = 1 - \frac{1}{1 + r_i^2} - \left(r_i^2 \log\left(1 + \frac{1}{r_i^2}\right)\right)^2. \tag{4.21}$$

Proof. Recall from Theorem 4 that the PFAoI policy is given by the fixed point equation (3.28), which we repeat again here,

$$\frac{1}{p_i} - \sum_{j \in I_i} \frac{1}{1 + D_{ji} - p_i} = 0, \ \forall\ i,$$

when the resulting $p_i \leq 1$. Note that D_{ji} is now a random variable. From Corollary 1, this occurs if and only if

$$R_i^2 \leq \sum_{j \in I_i} R_j^2. \tag{4.22}$$

When this condition holds and as N goes to infinity, the value of p_i becomes small, so the fixed point equation can be approximated as

$$\frac{1}{p_i} \approx \sum_{j \in I_i} \frac{1}{1 + D_{ji}}, \ \forall\ i. \tag{4.23}$$

Because R_j is iid uniform in space, when conditioned on the value of r_i, each term in this sum becomes an iid random variable. Applying the Central Limit Theorem, as N goes to infinity the sum converges to a normal distribution,

$$Z_i^{PF} \to \mathcal{N}(N\mu_i, N\sigma_i^2),\ \forall\ i, \tag{4.24}$$

where μ_i and σ_i^2 are the mean and variance of the iid terms in the sum, conditioned on r_i. These can easily be computed as (4.20) and (4.21) respectively.

There is a non-zero probability that the condition (4.22) does not hold, and the normally distributed $1/p_i$ takes a value larger than 1. This clearly cannot happen since p_i is a probability, and as a result the normal distribution in (4.19) is only valid on the interval $[1, \infty)$. The remaining probability mass is contained in the spike

$$\mathbb{P}[Z_i^{PF} = 1] = \mathbb{P}\Bigg[R_i^2 \geq \sum_{j \in I_i} R_j^2 \mid R_i = r_i\Bigg], \tag{4.25}$$

from Corollary 1, where we condition on the value of r_i known to node i. Because N is large, this condition has negligible probability. Nevertheless, it gives us a valid probability distribution with the negligible mass that would have existed in the negative tail of the distribution occurring at this spike.

This subtlety is clearly necessary because probabilities cannot take values larger than 1. Furthermore, this ensures that the mean and variance of the distribution (4.19) are well defined as N goes to infinity, because the distribution only takes positive values. This completes the proof. □

Corollary 6. *When nodes are uniformly distributed in $\mathbb{R}^2$. and under Assumption 1, the proportionally fair optimal p_i^{PF} converges to a random variable with distribution*

$$\begin{cases} \dfrac{1}{\sqrt{2\pi N}\sigma_i p_i^2} \exp\{-\dfrac{(1/p_i - N\mu_i)^2}{2N\sigma_i^2}\}, & 0 \leq p_i^{PF} < 1 \\ \mathbb{P}[\sum\limits_{j \in I_i} R_j^2 \leq r_i^2], & p_i^{PF} = 1. \end{cases} \tag{4.26}$$

Proof. From (4.19), $1/p_i^{PF}$ is a normal random variable with pdf

$$f_{1/p_i^{PF}}(1/p_i^{PF}) = \frac{1}{\sqrt{2\pi N}\sigma_i} \exp\{-\frac{(1/p_i^{PF} - N\mu_i)^2}{2N\sigma_i^2}\} \tag{4.27}$$

on the interval $0 \leq p_i^{PF} \leq 1$. The distribution of p_i can be derived from this expression. Let $g(1/p_i^{PF}) = p_i^{PF}$ and its inverse function $g^{-1}(p_i^{PF}) = 1/p_i^{PF}$. Since g is a monotonic function, the pdf of p_i^{PF} is

$$\begin{aligned} f_{p_i^{PF}}(p_i^{PF}) &= f_{1/p_i^{PF}}(g^{-1}(p_i^{PF}))\left|\frac{dg^{-1}}{dp_i^{PF}}(p_i^{PF})\right| \\ &= \frac{1}{(p_i^{PF})^2} f_{1/p_i^{PF}}(1/p_i^{PF}) \\ &= \frac{1}{\sqrt{2\pi N}\sigma_i (p_i^{PF})^2} \exp\{-\frac{(1/p_i^{PF} - N\mu_i)^2}{2N\sigma_i^2}\}, \end{aligned} \tag{4.28}$$

for all $0 \leq p_i < 1$ [27, Sec 4.1]. Furthermore, there is a probability mass spike at $p_i^{PF} = 1$ corresponding to the spike in the distribution of $1/p_i^{PF}$. This takes probability equal to (4.25), as shown in the proof of Theorem 8, which completes the proof. □

Theorem 9. *As N becomes large, the random variable p_i^{PF} converges to p_i^{TA} such that*

$$\mathbb{P}\left[|p_i^{PF} - p_i^{TA}| \leq \frac{k}{N^{3/2}}\right] = 1 - \epsilon, \tag{4.29}$$

for some constant k and any $\epsilon > 0$. In other words, the TA policy p_i^{TA} is asymptotically optimal with arbitrarily high probability, and converges at a rate of $1/N^{3/2}$.

Proof. Because Z_i^{PF} is normally distributed according to (4.19), the event that it lies within m standard deviations of the mean is

$$|Z_i^{PF} - N\mu_i| \leq m\sqrt{N}\sigma_i, \tag{4.30}$$

and for any fixed value of m, the probability that this occurs can be written as

$$\mathbb{P}\big[|Z_i^{PF} - N\mu_i| \leq m\sqrt{N}\sigma_i\big] = 1 - \epsilon_m, \tag{4.31}$$

where ϵ_m decreases with increasing m. Substituting p_i^{PF} and solving for it in (4.30),

$$\frac{1}{N\mu_i + m\sqrt{N}\sigma_i} \leq p_i^{PF} \leq \frac{1}{N\mu_i - m\sqrt{N}\sigma_i}. \tag{4.32}$$

Now rearranging the lower bound to be in the form

$$\frac{1}{N\mu_i + m\sqrt{N}\sigma_i} = \frac{1}{N\mu_i} - p_i^L \approx p_i^{TA} - p_i^L, \tag{4.33}$$

and doing some algebraic manipulation,

$$\begin{aligned} p_i^L &= \frac{1}{N\mu_i} - \frac{1}{N\mu_i + m\sqrt{N}\sigma_i} \\ &= \frac{N\mu_i + m\sqrt{N}\sigma_i - N\mu_i}{N^2\mu_i^2 + mN^{3/2}\mu_i\sigma_i} \\ &= \frac{m\sigma_i}{N^{3/2}\mu_i^2 + mN\mu_i\sigma_i} \\ &\leq \frac{k}{N^{3/2}} \end{aligned} \tag{4.34}$$

for any finite m and some constant k as N goes to infinity. Following the same approach for the upper bound $p_i^{TA} + p_i^U$, one can see that p_i^U is on the same order. Therefore, the event

$$|p_i^{PF} - p_i^{TA}| \leq \frac{k}{N^{3/2}} \tag{4.35}$$

is equivalent to (4.30) for any finite m, and occurs with probability

$$\mathbb{P}\left[|p_i^{PF} - p_i^{TA}| \leq \frac{k}{N^{3/2}}\right] = 1 - \epsilon_m, \tag{4.36}$$

where ϵ_m becomes arbitrarily small as m increases. This completes the proof. □

We have shown that with high probability, the optimal proportionally fair policy converges to the policy TA at a rate of $1/N^{3/2}$. It is important to note that this is faster than p_i^{TA} converges to 0, which occurs at a rate of $1/N$.

Therefore, for large N, we expect the policy TA to achieve similar performance to the PFAoI policy (which uses all node locations), and we show through simulations

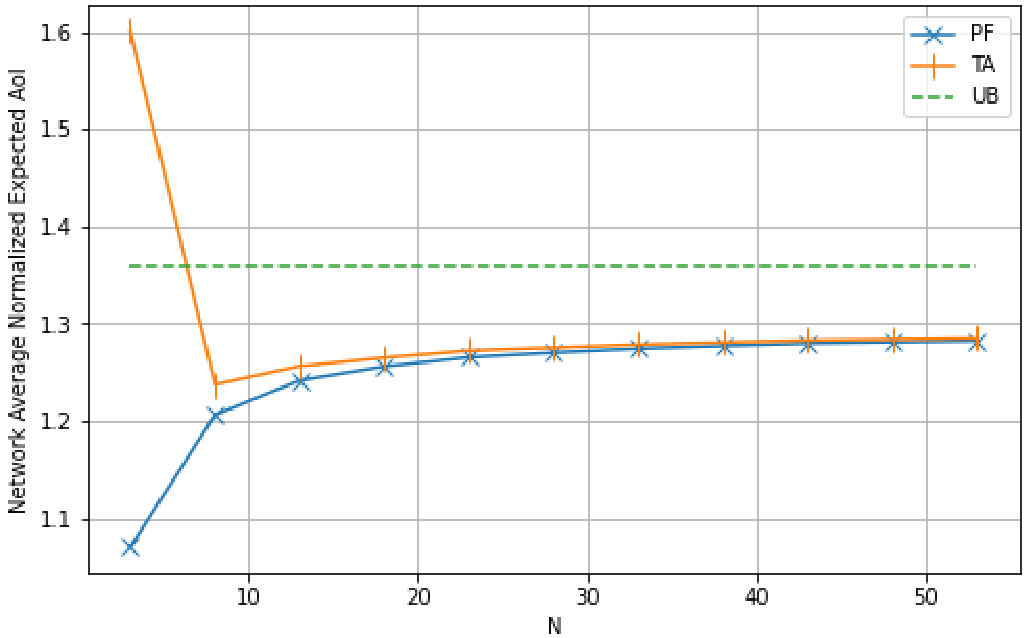

Figure 4-1: Convergence of the TA policy to the PFAoI policy in normalized network average AoI

in the next section that this is the case. This is a powerful result, which shows that a decoupled policy in which each node sets its transmission probability based *only on its own location* asymptotically achieves proportional fairness with arbitrarily high probability, and furthermore converges quickly. In the next section, we verify this convergence and the performance of the TA policy through simulations.

4.4 Numerical Results

In this section, we evaluate the topology-agnostic policy we derived and compare its performance to the PF policy from Chapter 3, again using normalized expected AoI as our comparison metric. In Figure 4-1 we show the normalized network average AoI as a function of N, averaged over 100 randomly generated topologies. As in Chapter 3, each of the topologies was fully generated and nodes were added incrementally to observe the change with N. The performance of TA is remarkably close to that of PF, even for small values of N.

These results show that, on average, the TA policy achieves a network average AoI

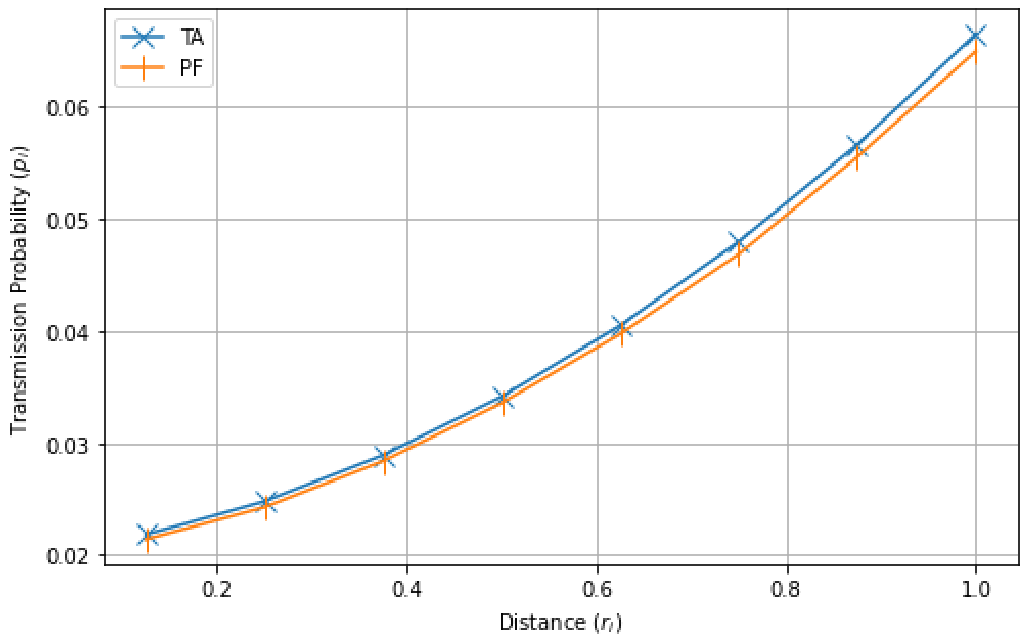

Figure 4-2: Transmission probabilities under TA and PF with varying position and $N = 50$, with the PF probabilities averaged over 1000 random network topologies

close to that of PFAoI, and well below the $e/2$ upper bound proved for EWSAoI and MMAoI, for values of N even below 10. In Theorem 7, the normalized network average AoI under TA was shown to be less than e, but we stated that for non-pathological cases it would perform much better. This verifies that claim.

Next, we examine the TA policy vector compared to that of PFAoI. Figure 4-2 shows the transmission probabilities of TA plotted as a function of r_i, and compared to those of PF, averaged over 1000 randomly generated network topologies when $N = 50$. Theorem 9 shows that as N goes to infinity, PF converges to TA at a relatively quick rate, and this demonstrates that convergence.

Finally, in Figure 4-3 we examine the normalized expected AoI under PF and TA as a function of r_i, again averaged over 1000 randomly generated topologies when $N = 50$. We also show the middle 95 percentile of topologies for TA in the shaded region. As expected, the average of the two policies is very close, but what is more shocking is that TA does not deviate from this performance by more than a small amount from the 2.5th to 97.5th percentile of results. Every point in the shaded region still ostensibly lies below the UB of $e/2$, showing that in the vast majority of

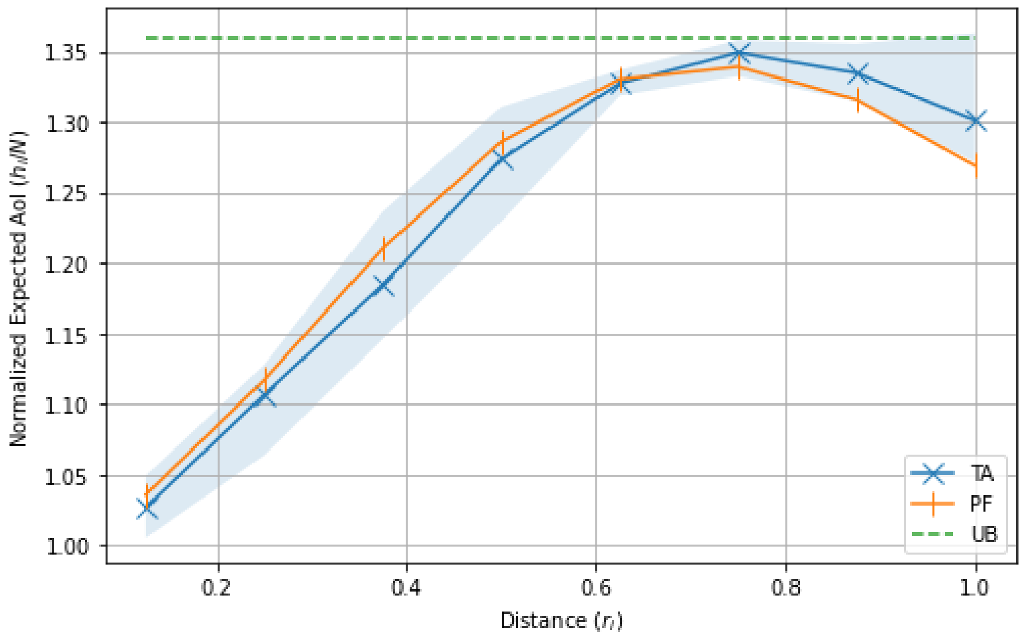

Figure 4-3: Normalized expected AoI of a single node under TA and PF with $N = 50$ and averaged over 1000 random network topologies. The shaded region represents the middle 95 percentile of performance under TA

cases, the TA policy achieves an AoI below this bound for every node in the network.

These simulations verify and outperform the results proven in this chapter for the topology-agnostic policy TA, and more than justify its use in minimizing both PFAoI and EWSAoI when the network topology is unknown.

4.5 Conclusion and Future Work

In this chapter, we extended the analysis of AoI in a random access spatially distributed network to the case where the network topology is unknown by the nodes, which only have access to their own location and the size of the network. We developed a topology-agnostic policy based only on this information, and showed that it converges to the optimal PF policy as the size of the network becomes large.

One future direction of work is implementing this policy in a real system and comparing its performance against 802.11 and other protocols. One challenge remaining before implementation is estimating the size of the network N.

A key assumption made in this thesis was that nodes are saturated and always have an update to send, which simplifies the analysis. Random access is more easily motivated when this is not the case, when nodes sample some changing process and only send updates when the process has actually changed. In this setting, one can use estimation error of a real-time system or the Age of Inforrect Information [28] as a metric to optimize for fresh *up to date* information at the base station, as opposed to simply fresh information. A second future direction is extending our analysis to this unsaturated setting.

Appendix A

Proofs

A.1 Proof of the Upper Bound in Theorem 5

It remains to be shown that

$$\frac{1}{N} h^{MM}(\boldsymbol{r}) \leq \frac{e}{2} \tag{A.1}$$

for any N and position vector $\boldsymbol{r}$. We first show that this holds asymptotically as N goes to infinity, and then extend the proof to hold for finite N. The basic proof outline is to show that the worst case position vector for min-max AoI is when all nodes are located on a circle a fixed distance away from the base station.

We start by examining the dual of the min-max optimization problem (3.23), which we repeat again here,

$$\begin{aligned} &\min_{\boldsymbol{p}} \frac{1}{2}\tilde{\alpha}^2 + \sum_{i=1}^{N} \lambda_i \left(\tilde{\alpha} - \log p_i - \sum_{j \neq i} \log\left(1 - \frac{p_j}{1 + d_{ij}}\right)\right) \\ &\text{s.t. } 0 \leq p_i \leq 1, \ \forall\, i. \end{aligned}$$

As shown in the proof of Theorem 3, strong duality holds and maximizing this problem over $\boldsymbol{\lambda}$ is equivalent to the primal min-max problem. Maximizing that solution over $\boldsymbol{r}$ is the solution to the worst case min-max optimization. Defining the objective of (3.23) as $\mathcal{L}(\tilde{\alpha}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda})$, and using the minmax inequality,

$$\begin{aligned} \max_{\boldsymbol{r}} \ \max_{\boldsymbol{\lambda}} \ \min_{\boldsymbol{p}} \ \mathcal{L}(\tilde{\alpha}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) &= \max_{\boldsymbol{\lambda}} \ \max_{\boldsymbol{r}} \ \min_{\boldsymbol{p}} \ \mathcal{L}(\tilde{\alpha}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) \\ &\leq \max_{\boldsymbol{\lambda}} \ \min_{\boldsymbol{p}} \ \max_{\boldsymbol{r}} \ \mathcal{L}(\tilde{\alpha}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}) \\ &\leq \min_{\boldsymbol{p}} \ \max_{\boldsymbol{\lambda}} \ \max_{\boldsymbol{r}} \ \mathcal{L}(\tilde{\alpha}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{\lambda}). \end{aligned} \tag{A.2}$$

Taking only the inner maximization of the right hand side over $\boldsymbol{r}$, we form an equivalent problem by rewriting $\mathcal{L}$ in its full form and dropping the terms that are independent of $\boldsymbol{r}$,

$$\max_{\boldsymbol{r}} \sum_{i=1}^{N} \lambda_i \big(-\sum_{j \in I_i} \log(1 - \frac{p_j}{1 + d_{ij}}) \big). \tag{A.3}$$

This can be rewritten as the minimum of the negative sum over all (i, j) pairs of nodes,

$$\min_{\boldsymbol{r}} \sum_{(i,j):i<j} \big(\lambda_i \log(1 - \frac{p_j}{1 + d_{ij}}) + \lambda_j \log(1 - \frac{p_i}{1 + d_{ji}}) \big). \tag{A.4}$$

We define the function $g(i, j)$ equal to the corresponding (i, j) term in this sum such that

$$g(i, j) \triangleq \lambda_i \log(1 - \frac{p_j}{1 + d_{ij}}) + \lambda_j \log(1 - \frac{p_i}{1 + d_{ji}}). \tag{A.5}$$

We now show that for each (i, j) pair, $g(i, j)$ is minimized for any vector $\boldsymbol{\lambda}$ when $r_i = r_j$. The function is not convex, but it is continuous and differentiable, so a necessary condition for a minimum is for the gradient to be zero or to be at a boundary point. Taking the derivative with respect to r_i and setting it equal to zero, there are two possible candidates for the minimum of $g(i, j)$ in terms of r_i,

$$r_i = 0, \ r_i = r_j \sqrt{\frac{p_i \lambda_j - p_j \lambda_i + p_i p_j \lambda_i}{p_j \lambda_i - p_i \lambda_j + p_i p_j \lambda_j}} = r_j \gamma. \tag{A.6}$$

Adding the other boundary point at $r_i = 1$, there are three candidate solutions in terms of r_i, and by symmetry the same three candidate solutions exist for r_j. The possible candidates in both coordinates are $(0, 0), (0, 1), (\gamma x, x), (1, 1),$ and (x, x), where x can take any value in the range 0 to 1.

To see why the last point is a candidate, observe that when $r_i = r_j$ at any point, nodes i and j will observe the same interference from the other nodes and their λ and p values will be equal. Then $\gamma = 1$, which yields the point (x, x).

We want to minimize the sum of g over all (i, j) pairs. If there exists a vector $\boldsymbol{r}$ such that g is minimized for all pairs, then the sum must also be minimized at that $\boldsymbol{r}$. It remains for us to examine the candidate minima and determine if such a position vector can be found.

For the points $(0, 0), (1, 1)$, and (x, x), the values of $d_{ij} = d_{ji} = 1$. Likewise at the point $(0, 1)$, it can be seen that $d_{ij} \to \infty$ and $d_{ji} = 0$.

At the point $(\gamma x, x)$, $d_{ij} = 1/\gamma^2$ and $d_{ji} = \gamma^2$. Evaluating g at these points,

$$g(i,j) = \begin{cases} \lambda_j \log(1 - \frac{p_i}{2}) + \lambda_i \log(1 - \frac{p_j}{2}), & (x,x) \\ \lambda_j \log(1 - p_i), & (0,1) \\ \lambda_j \log(1 - \frac{p_i}{1+\gamma^2}) + \lambda_i \log(1 - \frac{p_j}{1+1/\gamma^2}), & (\gamma x, x). \end{cases} \tag{A.7}$$

We now show the solution for $(\gamma x, x)$ converges to the solution for (x, x). As N goes to infinity, it can easily be verified from (3.18) that p_i and p_j both go to zero. λ_i and λ_j are both on the order of 1, since the sum of the entries in the λ vector is equal to the log min-max AoI, and from (3.18), the elements of the vector must all be on the same order or the values of $\boldsymbol{p}$ would vary wildly.

Armed with these approximations, one can see that the third term in both the numerator and denominator of γ goes to zero much quicker than the first two terms, since it contains the product of p_i and p_j. The expression then takes the form

$$\gamma = \sqrt{\frac{p_i\lambda_j - p_j\lambda_i + \delta}{p_j\lambda_i - p_i\lambda_j + \delta}} \tag{A.8}$$

as $\delta \to 0$. The inside of the radical must remain positive, so the values of $p_i\lambda_j$ and $p_j\lambda_i$ must converge. Then the value of γ converges to 1, and the solution for $(\gamma x, x)$ converges to the solution for (x, x).

We now show that this result also converges to the solution for $(0, 1)$. Since

$\log(1-x)$ converges to $-x$ as x approaches zero, the two asymptotic solutions can be rewritten using this result. Then since $p_i\lambda_j$ and $p_j\lambda_i$ converge,

$$\lambda_j \log(1-p_i) \to -\lambda_j p_i, \tag{A.9}$$

and furthermore,

$$\lambda_j \log(1-\frac{p_i}{2}) + \lambda_i \log(1-\frac{p_j}{2}) \to -\frac{1}{2}(\lambda_j p_i + \lambda_i p_j) \to -\lambda_j p_i, \tag{A.10}$$

and the solutions all converge to the same expression. From this analysis we conclude that as N goes to infinity, the function g is minimized at any point where $r_i = r_j$. We do not claim the uniqueness of this solution, since we showed other possible minima, but since the values of all possible minima converge, the vector $\boldsymbol{r}$ in which all entries are equal represents a global minimum to the function g for every (i,j) pair. We define this vector as $\boldsymbol{r}^*$.

Because every term in the sum (A.4) is minimized at the same vector $\boldsymbol{r^*}$, the sum must also be minimized at this vector. Plugging this result back into (A.2), the maximum over $\boldsymbol{\lambda}$ must occur when all values of $\boldsymbol{\lambda}$ are equal because of symmetry. Then, because scaling Lagrange multiplier values has no effect on (3.18), this is equivalent to when all Lagrange multipliers are equal to 1, which is the PF solution (3.28). With symmetric locations, the PF solution is

$$\frac{1}{p_i^{PF}} = \sum_{j\in I_i} \frac{1}{2-p_i^{PF}},\ \forall\ i \Rightarrow p_i^{PF} = \frac{2}{N},\ \forall\ i, \tag{A.11}$$

and from (A.2), the AoI under this vector $\boldsymbol{p}^{PF}$ and $\boldsymbol{r}^*$ is an upper bound on the worst case min-max solution. Therefore, as N goes to infinity,

$$\frac{1}{N}h^{MM}(\boldsymbol{r}) \le \frac{1}{2\prod_{j\in I_i}\left(1-\frac{1}{N}\right)} = \frac{1}{2\left(1-\frac{1}{N}\right)^{N-1}} \to \frac{e}{2}. \tag{A.12}$$

for any position vector $\mathbf{r}$.

We now show that this holds for finite N. Because (A.12) holds as $N \to \infty$, there

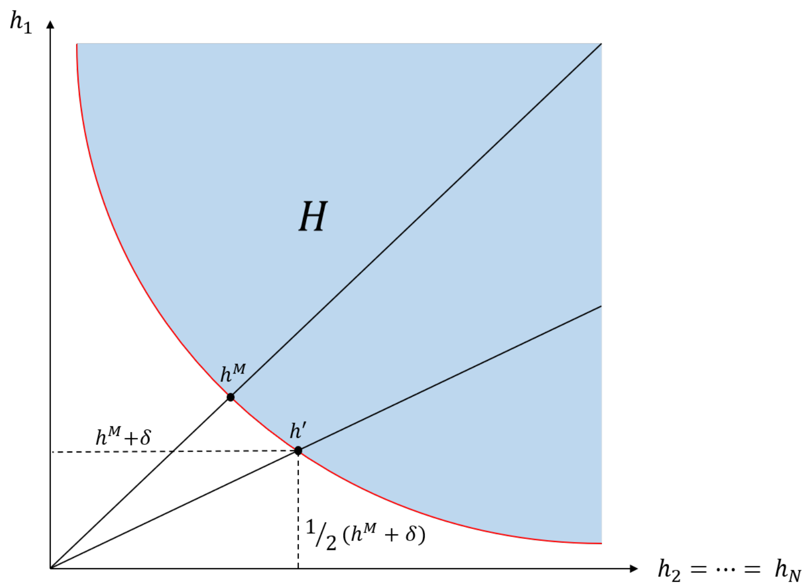

Figure A-1: h' after projecting onto the plane formed by h_1 and the vector with equal entries in the other $N-1$ dimensions

exists some N_0 where the bound holds for all $N \geq N_0$. For any $N < N_0$, consider a set of N nodes with arbitrary locations and normalized min-max AoI h^{MM}/N. If nodes can be added one by one until $N = N_0$, such that the normalized min-max AoI increases as each node is added, then

$$\frac{1}{N} h^{MM}(\boldsymbol{r}) \leq \frac{1}{N_0} h^{MM}(\boldsymbol{r}') \leq \frac{e}{2}, \tag{A.13}$$

where $\boldsymbol{r}$ is the initial position vector of N nodes, and $\boldsymbol{r}'$ is the same vector with nodes added in some fashion until there are N_0 nodes. We only need to show that nodes can be added so that this condition is satisfied.

We will make use of the Pareto boundary from Proposition 1. Given N nodes with position vector $\boldsymbol{r}$, operating at the min-max AoI point $\boldsymbol{h}^{MM}(\boldsymbol{r})$, choose a node arbitrarily and without loss of generality call it node 1. Decrease h_1 by moving along the boundary $H^*(\boldsymbol{r})$ in the direction of the diagonal of the other $N-1$ dimensions, until reaching the point $\boldsymbol{h}' = \big((h^{MM}+\delta)/2, (h^{MM}+\delta), \ldots, (h^{MM}+\delta)\big)$ for some $\delta > 0$.

The existence of this point on the boundary can be seen by projecting $H(\boldsymbol{r})$ onto the two dimensional plane formed by h_1 and the vector with equal entries in the other $N-1$ dimensions, as shown in Figure A-1. Because $H(\boldsymbol{r})$ is convex by Theorem 1, the resulting two dimensional curve $H^*(\boldsymbol{r})$ is continuous and monotonically decreasing, so it must reach the point $\boldsymbol{h}'$ for some $\delta > 0$.

For simplicity, we look at this problem in terms of throughput using the inverse relationship,

$$\frac{1}{h^{MM}+\delta} = \frac{1}{h^{MM}} - \delta' = \tau^{MM} - \delta', \tag{A.14}$$

for some $\delta' > 0$. Then, equivalent to the point $\boldsymbol{h}'$, we have $\boldsymbol{\tau}' = \big(2(\tau^{MM} - \delta'), \ldots, (\tau^{MM} - \delta')\big)$. Define the policy vector at this point as $\boldsymbol{p}'$. At $\boldsymbol{\tau}'$, the total network throughput is $(N+1)(\tau^{MM} - \delta')$.

Now, keeping $\mathbf{p}'$ fixed except for p_1', add an $(N+1)$st node k at the same location as node 1, and adjust the transmission probabilities of these two nodes to $p_1'' = p_k''$, such that their throughput at the new operating point $\tau_1'' = \tau_k'' = (\tau^{MM} - \delta')$ is half of τ_1'. Specifically,

$$\begin{aligned}
\tau_1'' = \tau_k'' &= p_1'' \prod_{j \in I_1'' \setminus k} \left(1 - \frac{p_j'}{1+d_{1j}}\right)\left(1 - \frac{p_k''}{2}\right) \\
&= \frac{1}{2} p_1' \prod_{j \in I_1'} \left(1 - \frac{p_j'}{1+d_{1j}}\right) \\
&= \frac{1}{2}\tau_1',
\end{aligned} \tag{A.15}$$

where I_1' is the interference set of node 1 before the addition of node k, and I_1'' is after the addition of node k. Therefore, $I_1'' \setminus k = I_1'$, and

$$\frac{1}{2}p_1' = p_1''\left(1 - \frac{p_k''}{2}\right) = p_1''\left(1 - \frac{p_1''}{2}\right). \tag{A.16}$$

The throughput of each other node i is then

$$
\begin{aligned}
\tau_i'' &= p_i' \prod_{j \in I_i'' \setminus 1,k} \left(1 - \frac{p_j'}{1+d_{ij}}\right)\left(1 - \frac{p_1''}{1+d_{i1}}\right)\left(1 - \frac{p_k''}{1+d_{ik}}\right) \\
&= p_i' \prod_{j \in I_i'' \setminus 1,k} \left(1 - \frac{p_j'}{1+d_{ij}}\right)\left(1 - \frac{p_1''}{1+d_{i1}}\right)^2 \\
&= p_i' \prod_{j \in I_i'' \setminus 1,k} \left(1 - \frac{p_j'}{1+d_{ij}}\right)\left(1 - 2\frac{p_1''}{1+d_{i1}} + \frac{(p_1'')^2}{(1+d_{i1})^2}\right) \\
&\leq p_i' \prod_{j \in I_i'' \setminus 1,k} \left(1 - \frac{p_j'}{1+d_{ij}}\right)\left(1 - 2\frac{p_1''}{1+d_{i1}} + \frac{(p_1'')^2}{1+d_{i1}}\right) \\
&= p_i' \prod_{j \in I_i'' \setminus 1,k} \left(1 - \frac{p_j'}{1+d_{ij}}\right)\left(1 - \frac{2p_1''(1-p_1''/2)}{1+d_{i1}}\right) \\
&= p_i' \prod_{j \in I_i' \setminus 1} \left(1 - \frac{p_j'}{1+d_{ij}}\right)\left(1 - \frac{p_1'}{1+d_{i1}}\right) \\
&= \tau^{MM} - \delta',
\end{aligned}
\tag{A.17}
$$

where the right hand side is the throughput after node 1 moved but before the addition of node k. We are left with $N-1$ nodes with throughput less than $\tau^{MM} - \delta'$, and two nodes with throughput equal to $\tau^{MM} - \delta'$. Therefore, after moving along the Pareto boundary to the new min-max point $(\tilde{\tau}^M, \ldots, \tilde{\tau}^M)$, $\tilde{\tau}^M \leq \tau^{MM} - \delta'$ by the definition of a Pareto boundary.

The total network throughput at this point is

$$
(N+1)\tilde{\tau}^M \leq (N+1)(\tau^{MM} - \delta') \leq (N+1)\tau^{MM}, \tag{A.18}
$$

and from the inverse relationship of AoI and throughput,

$$
\frac{1}{N+1}\tilde{h}^M \geq \frac{1}{N+1} h^{MM}. \tag{A.19}
$$

Therefore, the normalized min-max AoI increases with the addition of node k for any finite N. This completes the proof.

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Bibliography

[1] S. Kaul, R. Yates, and M. Gruteser, "Real-time status: How often should one update?" in *2012 Proceedings IEEE INFOCOM*, 2012, pp. 2731–2735.

[2] Y. Sun, E. Uysal-Biyikoglu, R. D. Yates, C. E. Koksal, and N. B. Shroff, "Update or wait: How to keep your data fresh," *IEEE Transactions on Information Theory*, vol. 63, no. 11, pp. 7492–7508, 2017.

[3] I. Kadota, A. Sinha, E. Uysal-Biyikoglu, R. Singh, and E. Modiano, "Scheduling policies for minimizing age of information in broadcast wireless networks," *IEEE/ACM Transactions on Networking*, vol. 26, no. 6, pp. 2637–2650, 2018.

[4] I. Kadota, A. Sinha, and E. Modiano, "Scheduling algorithms for optimizing age of information in wireless networks with throughput constraints," *IEEE/ACM Transactions on Networking*, vol. 27, no. 4, pp. 1359–1372, 2019.

[5] R. Talak, S. Karaman, and E. Modiano, "Optimizing information freshness in wireless networks under general interference constraints," *IEEE/ACM Transactions on Networking*, vol. 28, no. 1, pp. 15–28, 2020.

[6] ——, "Distributed scheduling algorithms for optimizing information freshness in wireless networks," in *2018 IEEE 19th International Workshop on Signal Processing Advances in Wireless Communications (SPAWC)*. IEEE, 2018, pp. 1–5.

[7] V. Tripathi and E. Modiano, "A whittle index approach to minimizing functions of age of information," in *2019 57th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*. IEEE, 2019, pp. 1160–1167.

[8] R. D. Yates and S. K. Kaul, "Status updates over unreliable multiaccess channels," in *2017 IEEE International Symposium on Information Theory (ISIT)*. IEEE, 2017, pp. 331–335.

[9] I. Kadota and E. Modiano, "Age of information in random access networks with stochastic arrivals," in *IEEE INFOCOM 2021 - IEEE Conference on Computer Communications*, 2021, pp. 1–10.

[10] A. Maatouk, M. Assaad, and A. Ephremides, "On the age of information in a csma environment," *IEEE/ACM Transactions on Networking*, vol. 28, no. 2, pp. 818–831, 2020.

[11] X. Chen, K. Gatsis, H. Hassani, and S. S. Bidokhti, “Age of information in random access channels,” *IEEE Transactions on Information Theory*, 2022.

[12] O. T. Yavascan and E. Uysal, “Analysis of slotted aloha with an age threshold,” *IEEE Journal on Selected Areas in Communications*, vol. 39, no. 5, pp. 1456–1470, 2021.

[13] M. Ahmetoglu, O. T. Yavascan, and E. Uysal, “Mista: An age-optimized slotted aloha protocol,” *IEEE Internet of Things Journal*, 2022.

[14] G. Bianchi, “Performance analysis of the ieee 802.11 distributed coordination function,” *IEEE Journal on selected areas in communications*, vol. 18, no. 3, pp. 535–547, 2000.

[15] G. D. Celik, G. Zussman, W. F. Khan, and E. Modiano, “Mac for networks with multipacket reception capability and spatially distributed nodes,” *IEEE Transactions on Mobile Computing*, vol. 9, no. 2, pp. 226–240, 2009.

[16] F. Baccelli and C. Singh, “Adaptive spatial aloha, fairness and stochastic geometry,” in *2013 11th International Symposium and Workshops on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt)*, 2013, pp. 7–14.

[17] F. Baccelli, B. Błaszczyszyn, and C. Singh, “Analysis of a proportionally fair and locally adaptive spatial aloha in poisson networks,” in *IEEE INFOCOM 2014-IEEE Conference on Computer Communications*. IEEE, 2014, pp. 2544–2552.

[18] H. H. Yang, A. Arafa, T. Q. Quek, and H. V. Poor, “Optimizing information freshness in wireless networks: A stochastic geometry approach,” *IEEE Transactions on Mobile Computing*, vol. 20, no. 6, pp. 2269–2280, 2020.

[19] P. D. Mankar, M. A. Abd-Elmagid, and H. S. Dhillon, “Spatial distribution of the mean peak age of information in wireless networks,” *IEEE Transactions on Wireless Communications*, vol. 20, no. 7, pp. 4465–4479, 2021.

[20] R. G. Gallager, *Stochastic processes: theory for applications*. Cambridge University Press, 2013.

[21] D. Bertsekas and R. Gallager, *Data networks*. Athena Scientific, 2021.

[22] P. Gupta and A. L. Stolyar, “Throughput region of random-access networks of general topology,” *IEEE Transactions on Information Theory*, vol. 58, no. 5, pp. 3016–3022, 2012.

[23] J. Massey and P. Mathys, “The collision channel without feedback,” *IEEE Transactions on Information Theory*, vol. 31, no. 2, pp. 192–204, 1985.

[24] J. Dattorro, *Convex optimization & Euclidean distance geometry*. Lulu. com, 2010.

[25] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex optimization.* Cambridge university press, 2004.

[26] F. Kelly, "Charging and rate control for elastic traffic," *European transactions on Telecommunications*, vol. 8, no. 1, pp. 33–37, 1997.

[27] D. Bertsekas and J. N. Tsitsiklis, *Introduction to probability.* Athena Scientific, 2008, vol. 1.

[28] A. Maatouk, S. Kriouile, M. Assaad, and A. Ephremides, "The age of incorrect information: A new performance metric for status updates," *IEEE/ACM Transactions on Networking*, vol. 28, no. 5, pp. 2215–2228, 2020.