Differentiability (1)

- A fn is differentiable at a point $x=a$ if the $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists.

$h = x-a$

$x = h+a$

$h = x-a$

$a+x-a = f(x)$

we call this limit, the derivative of $f(x)$ and denote it as $f'(x)$.

$\Rightarrow$ A fn is differentiable at $x=a$ if it's derivative, $f'(x)$ exists,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}.$$

Example

$$f(x) = \begin{cases} x^2 & x<3 \\ 6x-9 & x \geq 3 \end{cases}$$

$$\lim_{x \to 3} \frac{f(x)-f(3)}{x-3}$$

$$\Rightarrow \lim_{x \to 3^-} \frac{f(x)-f(3)}{x-3} = \lim_{x \to 3^-} \frac{f(x)-9}{x-3}$$

$$= \lim_{x \to 3^-} \frac{x^2-9}{x-3}$$

$$= \lim_{x \to 3^-} \frac{(x+3)(x-3)}{x-3}$$

$$= \lim_{x \to 3^-} x+3 = 6$$

$$\lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{x \to 3^+} \frac{6x-9-9}{x-3}$$

$$= \lim_{x \to 3^+} \frac{6x-18}{x-3} = \lim_{x \to 3} \frac{6(x-3)}{x-3} = 6$$

$\Rightarrow$ it is differentiable.

is $f(x) = \begin{cases} x-1, & x<1 \\ (x-1)^2, & x \geq 1 \end{cases}$ differentiable at $x=1$

soln

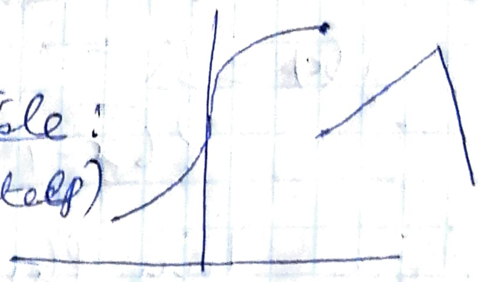
$\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$

$\lim_{x\to -1} \frac{x-1-0}{x-1} = \lim_{x\to^{-}1} \frac{\cancel{x-1}}{\cancel{x-1}}$

$= \lim_{x\to^{-}1} 1 = 1$

$\lim_{x\to^{-}1} \frac{(x-1)^2-0}{x-1} = \lim_{x\to^{-}1} x-1 = 0$

Not differentiable



when is f not differentiable:

1. Vertical tangent (too steep)
2. f(x) not continous,
3. "sharp" turn.

NB/ in all these $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ doesn't exist.

eg:

ex is f(x) continous and differentiable at $x=-1$

a) $f(x) = \begin{cases} -2x-1, & x<-1 \\ x^2+3, & x \geq -1 \end{cases}$ at $x=-1$

(Not)

b) $f(x) = \begin{cases} x^2+5x, & x<0 \\ 2x^2-3x, & x \geq 0 \end{cases}$ at $x=0$

continous
not differentiable

c) $f(x) = \begin{cases} 2x^2-x, & x<0 \\ -x, & x \geq 0 \end{cases}$ at $x=0$

cont.
~~not~~ diff.

diff

(3)

$$f(x) = \begin{cases} \frac{1}{2}x^2, & x < -4 \\ x^2+5x & x \geq -4 \end{cases}$$ at $x = -4$

not continuous
not differentiable

$$f(x) = \begin{cases} -3x+2, & x < 3 \\ x^2-5x-1, & x \geq 3 \end{cases}$$

$f(3) = -7$

$\lim_{x\to 3^-} (-3x+2) = -7$

$\lim_{x\to 3^+} (x^2-5x-1) = -7$ $\Big\}$ $\lim_{x\to 3} f(x) = \cancel{DNE}\ -7.$

$\Rightarrow$ ~~not~~ continuous.

diff.

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \text{exist}$$

$$\Rightarrow \lim_{x\to 3^-} \frac{f(x) - f(3)}{x-3} = \lim_{x\to 3^-} \frac{-3x+2+7}{x-3} =$$

$$= \lim_{x\to 3^-} \frac{-3x+9}{x-3}$$

$$= \lim_{x\to 3^-} \frac{-3(x-3)}{x-3} = -3.$$

$$\lim_{x\to 3^+} \frac{x^2-5x+6}{x-3} = \cancel{(\text{factored})} \lim_{x\to 3} \frac{(x-2)\cancel{(x-3)}}{\cancel{x-3}}$$

$$\lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = \cancel{DNE}\ 1.$$

not differentiable

$$f(x) = \begin{cases} x^2-6, & x < 1 \\ 7x^2-12x, & x \geq 1 \end{cases}$$ at $x = 1$

$f(1) = -5$ $\quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2-6 = -5$

$\lim_{x\to 1} 7x^2-12x = -5$ continuous.

diff. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-6+5}{x-1} = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = \lim_{x\to 1^-} x+1 = 2.$

$\lim_{x\to 1^+} \frac{7x^2-12x+5}{x-1} = \frac{(7x-5)\cancel{(x-1)}}{\cancel{x-1}} = 7x-5 = 2.$ diff

(4)

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at a.

proof:

if f is differentiable at a ~~then~~ then,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists.}$$

$$\Rightarrow \lim_{x \to a} f(x) - f(a) = \lim_{x \to a} (x-a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

$$= \lim_{x \to a} (x-a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= 0$$

hence $\lim_{x \to a} f(x) = f(a)$

This is the definition of continuity at $x = a$.

OB/ differentiability means Continuity. but Continuity doesn't mean differentiability

(5) Continuity

$$f(x) = \begin{cases} 2x+5 & x < -1 \\ x^2+2 & x > -1 \\ 5 & x = -1 \end{cases}$$

(1) $f(-1) = 5$ (defined)

(2) $\lim\limits_{x \to -1^-} f(x) = 2(-1)+5 = 3$

$\lim\limits_{x \to -1^+} f(x) = (-1)^2 + 2 = 3$

$\Rightarrow \lim\limits_{x \to -1} f(x) = 3$

(3) $\lim\limits_{x \to -1} f(x) \neq f(-1)$ fails 3

$\Rightarrow f(x)$ is discontinuos at $x = -1$

Type: hole (Removable) discontinuity.

differentiable ??

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to -1} \frac{f(x) - f(-1)}{x+1}$$

$\cancel{\lim\limits_{x \to -1^-} 2x+5 = 3}$

$\cancel{\lim\limits_{x \to -1^+} x^2+2 =}$

$(x+1)(x-3)$

$x^2 - 3x + x - 3$

$$\lim_{x \to -1^-} \frac{2x+5-5}{x+1} = \lim_{x \to -1^-} \frac{2x}{x+1} = DNE$$

$$\lim_{x \to -1^+} \frac{x^2+2-5}{x+1} = \lim_{x \to -1^+} \frac{x^2-3}{x+1} = DNE$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = -2 \Rightarrow \text{Not differentiable}$$