Continuity

A fn $f(x)$ is continuous at a point $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

Test for continuity

(1) The fn is defined at $x = a$.

i.e $f(a)$ equals a real number.

(2) The limit of the fn as x approaches a exists

i.e $\lim_{x \to a} f(x)$ exists i.e $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

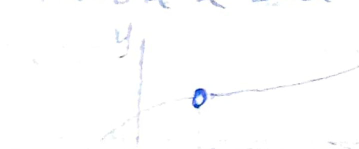
(3) The limit of the fn as x approaches a is equal to the function value at $x = a$.

i.e $\lim_{x \to a} f(x) = f(a)$

N/B If any of the three fails, then the fn is discontinuous.

Types of Discontinuities

(1) Removable (point) discontinuity.
- The graph has a hole at a single ~~value~~ x-value.
- Satisfies condition 1 & 2 but fails condition 3

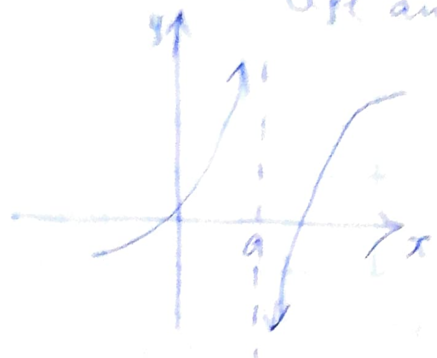


Can be made continuous by appropriately defining or redefined $f(c)$.

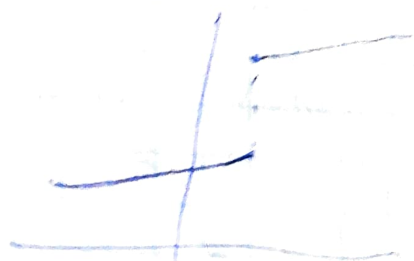
(2)

2) Infinite discontinuity
- the fn goes toward positive or negative infinity.
- fails condition 1: How? ~~the finite left and the limits that are not equal~~
~~i.e. $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$~~
Have infinite left and right limits

3)



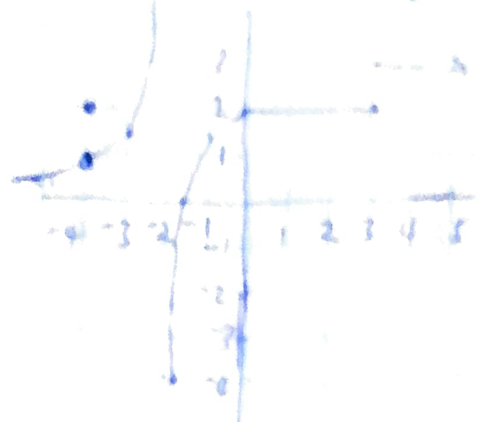
③ Jump discontinuity:
- graphs jumps from one place to another.



- Fails condition 2: How?
have finite left and right limits that are not equal

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Examples using graphs. (3)



a) is f(x) continuous at $x=0$?

check:

1. is the fn defined at $x=0$? — Yes $f(0)=2$.

2. Does the limit of the fn as x approaches $x=0$ exist?

$$\left.\begin{aligned}\lim_{x\to 0^-} f(x) &= 2\\ \lim_{x\to 0^+} f(x) &= 2\end{aligned}\right\} \Rightarrow \lim_{x\to 0} f(x) = 2.\ (\text{Yes})$$

3. Does the limit of $f(x)$ as $x\to 0$ equal to function value?

$$\lim_{x\to 0} f(x) = f(0) = 2\ (\text{yes})$$

Since the 3 conditions are met, the fn is continuous at $x=0$.

b) is f(x) continuous at $x=-4$?

soln:

(1) $f(-4) = 2$.

(2) $\lim_{x\to -4^-} f(x) = 1$, $\lim_{x\to -4^+} f(x) = 1 \Rightarrow \lim_{x\to -4} f(x) = 1$

(3) $\lim_{x\to -4} f(x) \neq f(-4)$

$\Rightarrow$ Not continuous. (fails 3 $\Rightarrow$ removable [illegible]

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Is $f(x)$ continuous at $x = -2$?

(1) $f(-2) =$ undefined.

(2) $\lim_{x \to -2^-} f(x) = DNE$, $\lim_{x \to -2^+} f(x) = DNE$ $\Bigg\}$ $\lim_{x \to -2} f(x) = DNE$

(3) conclude not continuous.
fails condition 2 [infinite discontinuity].

Examples with eqns

1. $f(x) = \dfrac{x^2 - x - 2}{x+1}$

is the fn continuous at i) $x = 2$ ii) $x = -1$?

Soln

i) (1) check whether $f(x)$ is defined at $x = 2$

i.e. $f(2) = \dfrac{0}{4} = 0$ (yes)

(2) check:

$\lim_{x \to 2} f(x) = ?$

$\Rightarrow \lim_{x \to 2^-} f(x) = 0$, $\lim_{x \to 2^+} f(x) = 0$ $\Bigg\}$ $\Rightarrow \lim_{x \to 2} f(x) = 0$ exist.

(3) check:

$\lim_{x \to 2} f(x) = f(2) = 0$ (yes)

$\Rightarrow$ it is continuous at $x = 2$.

ii) at $x = -1$

(1) $f(-1) = \dfrac{0}{0}$ doesnt exist (not defined)

(2) $\lim_{x \to -1^-} f(x) = -3$, $\lim_{x \to -1^+} f(x) = -3$

$\Rightarrow \lim_{x \to -1} f(x) = -3$ exists.

(3) $\lim_{x \to -1} f(x) \neq f(-1)$

$\Rightarrow$ discontinuous

Examples

1) $g(x) = \begin{cases} x^2+3x, & x \le 0 \\ 2, & 0 < x \le 2 \\ 3x^2, & x > 2 \end{cases}$

is $g(x)$ continuous at $x = 2$?

Soln

(1) $g(2) = 2$

(2) $\lim\limits_{x \to 2^-} g(x) = 2 \quad \lim\limits_{x \to 2^+} g(x) = 12.$

$\lim\limits_{x \to 2^-} g(x) \ne \lim\limits_{x \to 2^+} g(x) \Rightarrow \lim\limits_{x \to 2} f(x)$ DNE.

$\Rightarrow$ Jump continuity.

Ex: at $x = 0$??

① $g(0) = 0$

② $\lim\limits_{x \to 0} g(x) = 0$

③ $\lim\limits_{x \to 0} g(x) = g(0)$

$\Rightarrow$ continous.

Try Find the values of a and b so that

$h(x) = \begin{cases} ax+5, & x \le -1 \\ -2x^2-9 & -1 < x < 5 \\ 3x+b & , x > 5. \end{cases}$

is continous everywhere

Soln.

we need to check if the fn is continous at $x = -1$ and $x = 5$.

at $x = -1$

1. $f(-1) = -a+5$

2. $\lim\limits_{x \to -1^-} f(x) = -a+5 \;\Big|\; \lim\limits_{x \to -1^+} f(x) = -11$

in order for the limit to exist and equal to the function value, we need

$$-a+5=-11$$
$$\Rightarrow a=16$$

for $x=5$

$$h(5)=-59.$$

$$\lim_{x\to 5^-} f(x)=-59, \quad \lim_{x\to 5^+} f(x)=15+b$$

Then $\quad -59=15+b.$

$$\Rightarrow b=-74.$$

ex

1) $f(x)=\sqrt{4-x^2}, \quad x=1.$

2. $f(x)=\begin{cases} x+2 & x<2 \\ x^2-2 & 2\le x<3 \\ 2x+5 & x\ge 3 \end{cases}$

i) at $x=2$

ii) at $x=3$

3. $f(x)=\begin{cases} 2x+5 & x<-1 \\ x^2+ & x>-1 \\ 5 & x=-1 \end{cases}$

at $x=-1$

4) $f(x)=\begin{cases} \sqrt{x+2} & x<2 \\ x^2-2 & 2\le x<3 \\ 2x+5 & x\ge 3 \end{cases}$

i) at $x=2.$

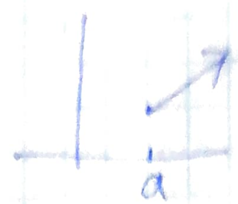
ii) at $x=3$

Continuity on an interval

f is continuous from the right of a,

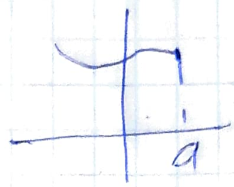
$\Rightarrow \lim_{x \to a^+} f(x) = f(a)$

i.e



f is continuous from the left of a

$\Rightarrow \lim_{x \to a^-} f(x) = f(a)$

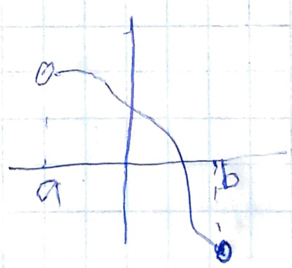


Continuity on an ~~inter~~ open interval

f is cont. on open interval (a,b) if f is continuous for all $x_0 \in (a,b)$.

i.e $\lim_{x \to x_0} f(x) = f(x_0)$ for all $x_0 \in (a,b)$

graph



Continuity on a half-closed interval.

f is cont. on $(a,b]$ if its continuous on (a,b) and it's cont. from the left of b.