Latent Networks Models

Game of Thrones

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December 9, 2018

Overview

Weighted Network Model

Let Y_{ij} be the weight on edge $E_{ij} \in \mathbf{E}$.

$$Y_{ij} | \lambda_{ij} \overset{ind}{\sim} \text{Pois}(\lambda_{ij})$$

$$\lambda_{ij} \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$$

Then the log-likelihood for this model can be written as

$$l(\lambda, \alpha, \beta; Y) = \sum_{i<j} \Big\{ \log \lambda_{ij}(Y_{ij} + \alpha - 1) - \lambda_{ij}(1 + \beta)$$

$$- \log(Y_{ij}!) + \alpha \log(\beta) - \log \Gamma(\alpha) \Big\}$$

Weighted Network Model: E-Step

Taking an expectation of this log-likelihood given the data $\mathbf{Y}$ and parameters $\theta = (\alpha, \beta)$

$$Q(\theta; \theta^{(t)}) = \sum_{i<j} \Big\{ (Y_{ij} + \alpha - 1) \mathbb{E}_{\lambda_{ij}|Y_{ij},\theta^{(t)}} \big[\log \lambda_{ij} \big]$$

$$- (1 + \beta) \mathbb{E}_{\lambda_{ij}|Y_{ij},\theta^{(t)}} \big[\lambda_{ij} \big] - \log(Y_{ij}!) + \alpha \log(\beta) - \log \Gamma(\alpha) \Big\}$$

Seeing as $\lambda_{ij}|Y_{ij}, \theta \propto \mathsf{Gamma}(\alpha + Y_{ij}, \beta + 1)$ we can define

$$\pi_{ij} \equiv \mathbb{E}_{\lambda_{ij}|Y_{ij},\theta} \big[\lambda_{ij} \big] = \frac{\alpha + Y_{ij}}{1 + \beta}$$

$$\eta_{ij} \equiv \mathbb{E}_{\lambda_{ij}|Y_{ij},\theta} \big[\log \lambda_{ij} \big] = \log(1 + \beta) + \Psi(\alpha + Y_{ij})$$

Weighted Network Model: M-Step

Maximizing $Q(\theta, \theta^{(t)})$ with respect to θ, we see that $\theta^{(t+1)}$ must satisfy

$$\beta^{(t+1)} = \frac{\binom{n}{2}}{\sum_{i<j} \pi_{ij}} \alpha^{(t+1)}$$

$$\Psi(\alpha^{(t+1)}) = \frac{\sum_{i<j} \eta_{ij} + \binom{n}{2} \log(\beta^{(t+1)})}{\binom{n}{2}}$$

We first update $\beta^{(t+1)}$ using $\alpha^{(t)}$ then use Netwon-Raphson to attain $\theta^{(t+1)}$.

Weighted Network Model: Psuedo-Code

Algorithm 1: EM for simplified latent network weighted model

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1 <u>LNM EM</u> (G, tol);

   Input : Graph G

             Tolerance tol

   Output: Nuisance Parameters α^*, β^*

             Latent Mean Estimates $\hat{\lambda}$

             Latent Distance Estimates $\hat{d}$

2 Initialize $Q^{(0)}$ **repeat**

3   **E:** calculate $\pi^{(t)}$, $\eta^{(t)}$;

4   **M:** update $\beta^{(t+1)}$ using (β_W);

5   update $\alpha^{(t+1)}$ using (α_W);

6   calculate $Q(\theta, \theta^{(t+1)})$

7 **until** $|\frac{Q(\theta^{(t+1)}, \theta^{(t)}) - Q(\theta^{(t)}, \theta^{(t)})}{Q(\theta^{(t)}, \theta^{(t)})}| < tol$;

8 **return** α^*, β^*, $\hat{\lambda} = \pi^*$, $\hat{d} = \frac{1}{\pi^*}$; where α^*, β^*, π^* are converged values

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Weighted Network Model: Distance Estimates

Weighted Network Model: Distance Density

Weighted Network Model: λ Estimates

Weighted Network Model: Spectral Clustering