Latent Networks Models

Game of Thrones

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Overview

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3. Expectaction Maximimation
   - Unweighted Network Model
   - Weighted Network Model
4. Markov Chain Monte Carlo
5. Model Comparison
6. Conclusion

Data

1. Data origin: A Song of Ice and Fire · A storm of Swords
2. Data form: 352 pairs of characters and the number of interaction between them

We start our analysis by constructing a Weighted Network G:

i. Characters → nodes $N_V(G) = 107$
ii. Interactions → edges $N_E(G) = 352$
iii. numbers of interaction → Weights on the edge

Form a Network

a Q: How to handle the sparsity of the network?

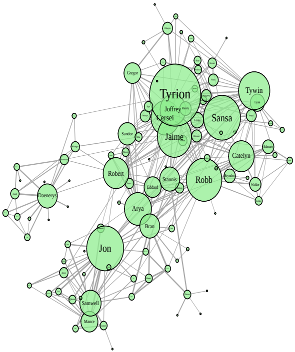
b A: Take a subnetwork of G, cut off at 100 interactions, call it G'

c $N_V(G') = 24$, $N_E(G') = 102$

d Use adjacency matrix A to represent G':

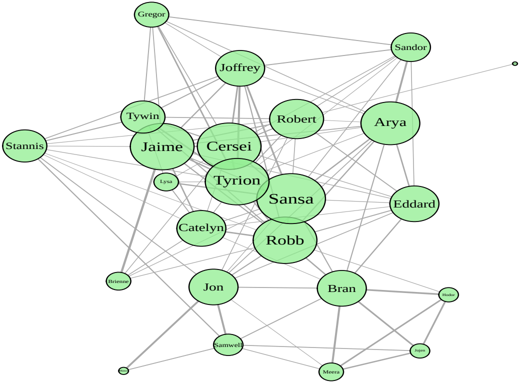
- i $a(i,j) =$ number of interactions between ith and jth character
- ii i.e., $a(i,j) = 0$ if there's no interaction between the two people

Unfiltered Network



Filtered Network

Filtered Network



Latent Network Model

Using Hoff's work, we model the presence of an edge given our latent variables as

$$\text{logit}\ \mathbb{P}(Y_{ij} = 1|Z) = ||Z_i - Z_j|| + \epsilon_{ij}$$

where $||Z_i - Z_j||$ is the latent distance between nodes i and j and

$$Z_i \overset{ind}{\sim} \sum_{k=1}^{G} \lambda_k \text{MVN}_d(\mu_k, \sigma_k^2 I_d)$$

Latent Network Model: Priors

$$
\begin{aligned}
Y_{ij}|Z_i, Z_j &\overset{ind}{\sim} \text{Bern}\left[\text{logit}^{-1}(\|Z_i - Z_j\|)\right] \\
Z_i|K_i = k_i &\overset{ind}{\sim} MVN(\mu_{k_i}, \sigma^2_{k_i} I_d) \\
K &\overset{iid}{\sim} \text{Multinoulli}(G, \lambda) \\
\lambda_k &\overset{iid}{\sim} \frac{1}{G} \\
\mu_k &\overset{iid}{\sim} \text{MVN}_d(0, I_2) \\
\sigma^2_k &\overset{iid}{\sim} \text{Inv}\chi^2_1
\end{aligned}
$$

Latent Network Model: Likelihood

$$
\begin{aligned}
\mathcal{L}(Z,\theta;Y) &= \prod_{i<j} \mathbb{P}(Y_{ij}|Z_i,Z_j)\mathbb{P}(Z_i|K_i,\mu_{k_i},\sigma^2_{k_i})\mathbb{P}(Z_j|K_j,\mu_{k_j},\sigma^2_j) \\
&\qquad \times \mathbb{P}(K_i|\lambda_i)\mathbb{P}(\lambda_i)\mathbb{P}(\mu_{k_i})\mathbb{P}(\sigma^2_{k_i})\mathbb{P}(K_j)\mathbb{P}(\mu_{k_j})\mathbb{P}(\sigma^2_{k_j}) \\
&\propto \prod_{i<j} \Big(\text{logit}^{-1}(\|Z_i - Z_j\|)\Big)^{Y_{ij}} \Big(1 - \text{logit}^{-1}(\|Z_i - Z_j\|)\Big)^{1-Y_{ij}} \\
&\times \frac{1}{(\sigma^2_{k_i})^{1/2}} \exp\Big\{ -\frac{1}{2\sigma^2_{k_i}}(Z_i - \mu_{k_i})^T(Z_i - \mu_{k_i})\Big\} \; \frac{1}{(\sigma^2_{k_j})^{1/2}} \\
&\times \exp\Big\{ -\frac{1}{2\sigma^2_{k_j}}(Z_j - \mu_{k_j})^T(Z_j - \mu_{k_j})\Big\} \exp\Big\{ -\frac{1}{2}\mu^T_{k_i}\mu_{k_i}\Big\} \\
&\times \exp\Big\{ -\frac{1}{2}\mu^T_{k_j}\mu_{k_j}\Big\} \times \frac{1}{(\sigma^2_{k_i})^2} \exp\Big\{ -\frac{1}{\sigma^2_{k_i}}\Big\} \frac{1}{(\sigma^2_{k_j})^2} \exp\Big\{ -\frac{1}{\sigma^2_{k_j}}\Big\} \\
&\times \lambda_i \times \lambda_j
\end{aligned}
$$

Unweighted Network Model

Let Y_{ij} indicate whether there is an edge E_{ij} between nodes i and j.

$$Y_{ij}|p_{ij} \overset{ind}{\sim} \text{Bern}(p_{ij})$$
$$p_{ij} \overset{iid}{\sim} \text{Beta}(\alpha, \beta)$$

where $p_{ij} \equiv 2 - 2 * \text{logit}^{-1}(d_{ij})$.

Then the log-likelihood for this model can be written as

$$l(p, \alpha, \beta; Y) = \sum_{i<j} Y_{ij} \log\left(\frac{p_{ij}}{1-p_{ij}}\right) + \log(1-p_{ij}) + \log\Gamma(\alpha+\beta) - \log\Gamma(\alpha) - \log\Gamma(\beta) + (\alpha-1)\log p_{ij} + (\beta-1)\log(1-p_{ij})$$

Unweighted Network Model: E-Step

We can take the expectation of the log-likelihood given the data $\mathbf{Y}$ and parameters $\theta = (\alpha, \beta)$

$$\begin{aligned} Q(\theta; \theta^{(t)}) = \sum_{i<j} &(Y_{ij} + \alpha - 1)\mathbb{E}_{p_{ij}|Y_{ij},\theta^{(t)}}[\log p_{ij}] \\ &+ (\beta - Y_{ij})\mathbb{E}_{p_{ij}|Y_{ij},\theta^{(t)}}\big[\log(1 - p_{ij})\big] + \log\Gamma(\alpha+\beta) \\ &- \log\Gamma(\alpha) - \log\Gamma(\beta) \end{aligned}$$

Since $p_{ij}|Y_{ij}, \theta \propto Beta(\alpha + Y_{ij}, \beta + 1 - Y_{ij})$, we define the following

$$\pi_{ij} \equiv \mathbb{E}_{p_{ij}|Y_{ij},\theta}\big[\log p_{ij}\big] = \Psi\Big(\alpha + Y_{ij}\Big) - \Psi\Big(\alpha + \beta + 1\Big)$$

$$\eta_{ij} \equiv \mathbb{E}_{p_{ij}|Y_{ij},\theta}\big[\log(1 - p_{ij})\big] = \Psi\Big(\beta + 1 - Y_{ij}\Big) - \Psi\Big(\alpha + \beta + 1\Big)$$

Unweighted Network Model: M-Step

To maximize, $Q(\theta, \theta^{(t)})$ with respect to θ, we see that $\theta^{(t+1)}$ must satisfy the following:

$$\Psi\left(\alpha^{(t+1)} + \beta^{(t)}\right) - \Psi(\alpha^{(t+1)}) = -\frac{\sum_{i<n} \mathbb{E}_{p_{ij}|Y_{ij},\theta^{(t)}}\left[\log p_{ij}\right]}{\binom{n}{2}} \qquad (\alpha_U)$$

$$\Psi\left(\alpha^{(t+1)} + \beta^{(t+1)}\right) - \Psi(\beta^{(t+1)}) = -\frac{\sum_{i<n} \mathbb{E}_{p_{ij}|Y_{ij},\theta^{(t)}}\left[\log(1 - p_{ij})\right]}{\binom{n}{2}} \qquad (\beta_U)$$

Here, we use Newton-Raphson to obtain both $\alpha^{(t+1)}$ and $\beta^{(t+1)}$.

Unweighted Network Model: Pseudocode

Algorithm 1: EM for simplified latent network unweighted model

LNM EM (G, tol);

Input : Graph G

Tolerance tol

Output: Nuisance Parameters α^*, β^*

Latent Probability Estimates $\hat{p}$

Latent Distance Estimates $\hat{d}$

Initialize $Q^{(0)}$ **repeat**

  E: calculate $\pi^{(t)}$, $\eta^{(t)}$;

  M: update $\alpha^{(t+1)}$ using (α_U);

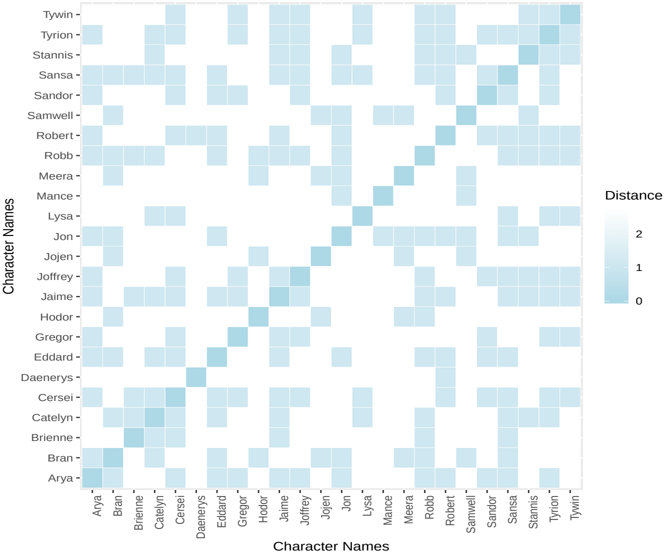
  update $\beta^{(t+1)}$ using (β_U);

  calculate $Q(\theta, \theta^{(t+1)})$

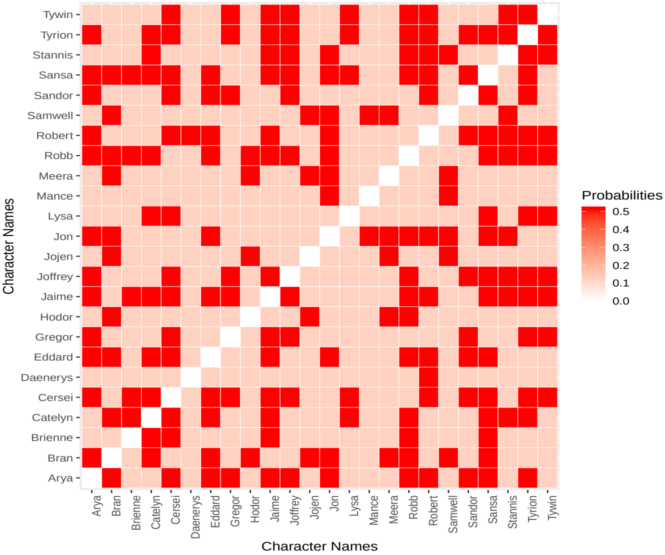
until $\left|\frac{Q(\theta^{(t+1)},\theta^{(t)})-Q(\theta^{(t)},\theta^{(t)})}{Q(\theta^{(t)},\theta^{(t)})}\right| < tol$;

return α^*, β^*, $\hat{p} = e^{\pi^*}$, $\hat{d} = logit^{-1}(1 - \frac{e^{\pi^*}}{2})$; where α^*, β^*, π^* are converged values

Unweighted Network Model: Distances



Unweighted Network Model: Probabilities



Weighted Network Model

Let Y_{ij} be the weight on edge $E_{ij} \in \mathbf{E}$.

$$
\begin{aligned}
Y_{ij}|\lambda_{ij} &\overset{ind}{\sim} \text{Pois}(\lambda_{ij}) \\
\lambda_{ij} &\overset{iid}{\sim} \text{Gamma}(\alpha, \beta)
\end{aligned}
$$

Then the log-likelihood for this model can be written as

$$
\begin{aligned}
l(\lambda, \alpha, \beta; Y) = \sum_{i<j} \Big\{ &\log \lambda_{ij} \big(Y_{ij} + \alpha - 1 \big) - \lambda_{ij}(1 + \beta) \\
&- \log(Y_{ij}!) + \alpha \log(\beta) - \log \Gamma(\alpha) \Big\}
\end{aligned}
$$

Weighted Network Model: E-Step

Taking an expectation of this log-likelihood given the data $\mathbf{Y}$ and parameters $\theta = (\alpha, \beta)$

$$Q(\theta; \theta^{(t)}) = \sum_{i<j} \Big\{ (Y_{ij} + \alpha - 1) \mathbb{E}_{\lambda_{ij}|Y_{ij}, \theta^{(t)}} \big[\log \lambda_{ij} \big] \\ - (1 + \beta) \mathbb{E}_{\lambda_{ij}|Y_{ij}, \theta^{(t)}} \big[\lambda_{ij} \big] - \log(Y_{ij}!) + \alpha \log(\beta) - \log \Gamma(\alpha) \Big\}$$

Seeing as $\lambda_{ij}|Y_{ij}, \theta \propto \text{Gamma}(\alpha + Y_{ij}, \beta + 1)$ we can define

$$\pi_{ij} \equiv \mathbb{E}_{\lambda_{ij}|Y_{ij}, \theta} \big[\lambda_{ij} \big] = \frac{\alpha + Y_{ij}}{1 + \beta}$$

$$\eta_{ij} \equiv \mathbb{E}_{\lambda_{ij}|Y_{ij}, \theta} \big[\log \lambda_{ij} \big] = \log(1 + \beta) + \Psi(\alpha + Y_{ij})$$

Weighted Network Model: M-Step

Maximizing $Q(\theta, \theta^{(t)})$ with respect to θ, we see that $\theta^{(t+1)}$ must satisfy

$$\beta^{(t+1)} = \frac{\binom{n}{2}}{\sum_{i<j} \pi_{ij}} \alpha^{(t+1)}$$

$$\Psi(\alpha^{(t+1)}) = \frac{\sum_{i<j} \eta_{ij} + \binom{n}{2} \log(\beta^{(t+1)})}{\binom{n}{2}}$$

We first update $\beta^{(t+1)}$ using $\alpha^{(t)}$ then use Netwon-Raphson to attain $\theta^{(t+1)}$.

Weighted Network Model: Psuedocode

Algorithm 2: EM for simplified latent network weighted model

LNM EM (G, tol);

Input : Graph G

Tolerance tol

Output: Nuisance Parameters α^*, β^*

Latent Mean Estimates $\hat{\lambda}$

Latent Distance Estimates $\hat{d}$

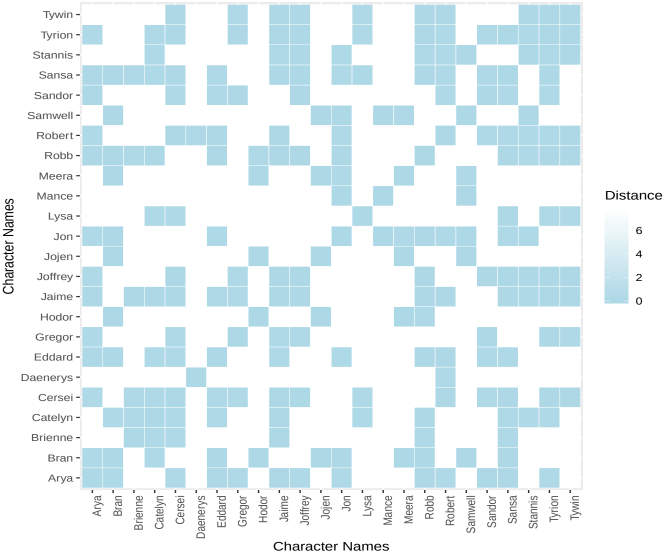
Initialize $Q^{(0)}$ **repeat**

- **E:** calculate $\pi^{(t)}$, $\eta^{(t)}$;
- **M:** update $\beta^{(t+1)}$ using (β_W);
- update $\alpha^{(t+1)}$ using (α_W);
- calculate $Q(\theta, \theta^{(t+1)})$

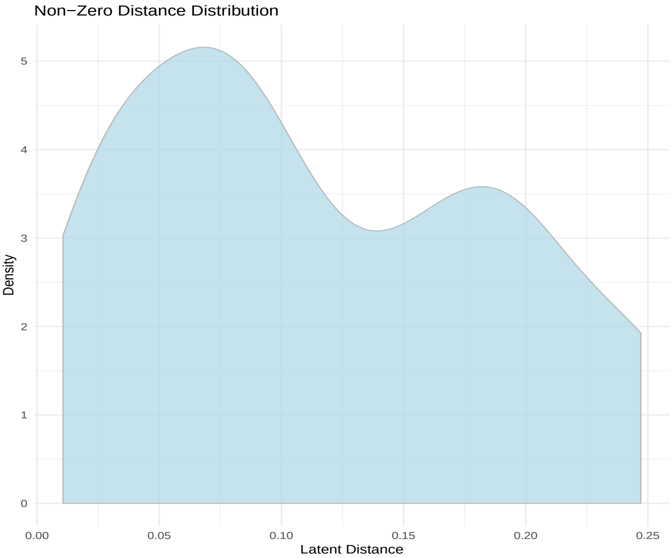
until $\left|\frac{Q(\theta^{(t+1)},\theta^{(t)})-Q(\theta^{(t)},\theta^{(t)})}{Q(\theta^{(t)},\theta^{(t)})}\right| < tol$;

return α^*, β^*, $\hat{\lambda} = \pi^*$, $\hat{d} = \frac{1}{\pi^*}$; where α^*, β^*, π^* are converged values

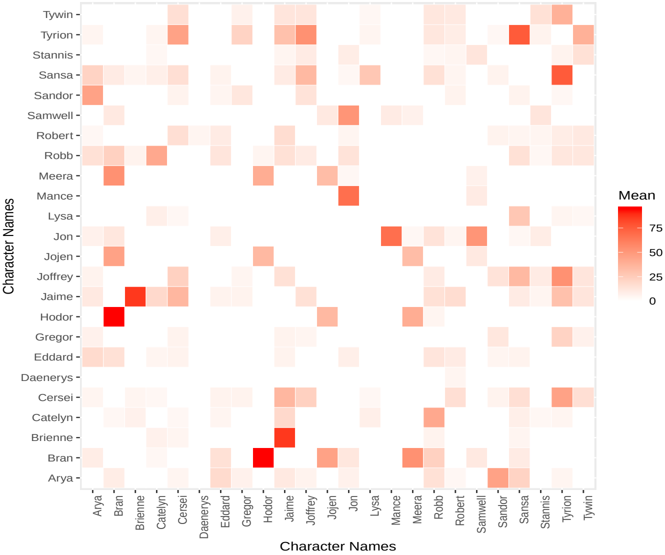
Weighted Network Model: Distance Estimates



Weighted Network Model: Distance Density



Weighted Network Model: λ Estimates



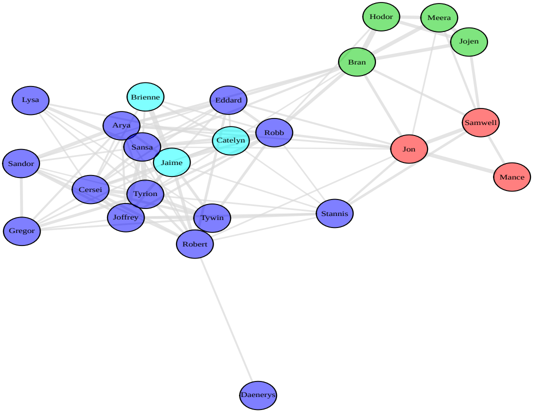
Weighted Network Model: Inference

1. Now that we have estimates for $\hat{\lambda}_{ij}$, how can we use these estimates to infer communities in the network?
2. One idea: Spectral Clustering
3. Interpret these estimates as smooth estimates of the weighted - Adjacency matrix

$$\widehat{\Lambda} = \begin{bmatrix} \hat{\lambda}_{11} & \hat{\lambda}_{12} & \dots & \hat{\lambda}_{1n} \\ \hat{\lambda}_{21} & \hat{\lambda}_{22} & \dots & \hat{\lambda}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\lambda}_{n1} & \hat{\lambda}_{n2} & \dots & \hat{\lambda}_{nn} \end{bmatrix}$$

4. Spectral Clustering Visualization

Weighted Network Model: Spectral Clustering



Latent Network Model

$$
\begin{aligned}
Y_{ij} | Z_i, Z_j &\overset{ind}{\sim} \text{Bern}\left[\text{logit}^{-1}(\|Z_i - Z_j\|)\right] \\
Z_i | K_i = k_i &\overset{ind}{\sim} MVN(\mu_{k_i}, \sigma^2_{k_i} I_d) \\
K &\overset{iid}{\sim} \text{Multinoulli}(G, \lambda) \\
\lambda_k &\overset{iid}{\sim} \frac{1}{G} \\
\mu_k &\overset{iid}{\sim} \text{MVN}_d(0, I_2) \\
\sigma^2_k &\overset{iid}{\sim} \text{Inv}\chi^2_1
\end{aligned}
$$

Conditionals: μ

$$f_{\mu_k|\theta^{(t)},Y}(\mu_k|\theta^{(t)},Y) \propto \prod_{k_i=k} \exp\left\{-\frac{1}{2\sigma^2_{k_i}}(Z_i-\mu_{k_i})^T(Z_i-\mu_{k_i})\right\}\exp\left\{-\frac{1}{2}\mu^T_{k_i}\mu_{k_i}\right\}$$

$$\propto \exp\left\{\sum_{i=1}^{N_v}\mathbb{I}\{k_i=k\}\left[-\frac{(\sigma^2_{k_i}+1)}{2\sigma^t_{k_i}}\left(\mu_{k_i}-\frac{Z_i}{(\sigma^2_{k_i}+1)}\right)^T\left(\mu_{k_i}-\frac{Z_i}{(\sigma^2_{k_i}+1)}\right)\right.\right.$$

Thus for all $k \in \{1,\dots,G\}$, we arrive at the following distribution for $\mu_k|\theta^{(t)},Y$:

$$\mu_k|\theta^{(t)},Y \sim f_{MVN_d}\left(\sum_{i=1}^{N_v}\mathbb{I}\{k_i=k\}\frac{Z_i^{(t)}}{(\sigma^2_{k_i})^{(t)}+1},\sum_{i=1}^{N_v}\mathbb{I}\{k_i=k\}\frac{(\sigma^2_{k_i})^{(t)}}{(\sigma^2_{k_i})^{(t)}+1}I_2\right)$$

Conditionals: σ^2

$$f_{\sigma^2_{k_i}|\theta,Y}(\sigma^2_{k_i}|\theta,Y) \propto \prod_{K_i=k} \sigma^{2\ -\frac{1}{2}}_{K_i} \exp\left\{\frac{1}{2\sigma_{k_i}}(Z_i-\mu_{k_i})^T(Z_i-\mu_{k_i})\right\}(\sigma^2_{k_i})^{-\frac{c}{2}-1}\exp\left\{-\frac{1}{2\sigma^2_{k_i}}\right\}$$
$$\propto (\sigma^2_{k_i})^{\left((\frac{-c-1}{2}ng-ng+1)-1\right)} \exp\left\{-\frac{1}{2\sigma^2_{k_i}}\sum_{K_i=k}\left((Z_i-\mu_{k_i})^T(Z_i-\mu_{k_i})+1\right)\right\}$$

Thus for all $k \in \{1,\ldots,G\}$, we arrive at the following distribution for $\sigma^2_k|\theta^{(t)}, Y$:

$$\sigma^2_k|\theta^{(t)}, Y \sim \text{Inv}\Gamma(\frac{c}{2},\frac{1}{2}) \overset{D}{=} \tau^2\ \nu\ \text{Inv}\chi^2_c$$

where $n_g = \sum \mathbb{I}_{\{k_i=K\}}$ and $SS_g + n_g = \sum_{K_i=k}\left((Z_i-\mu_{k_i})^T(Z_i-\mu_{k_i})+1\right)$ and

$$\nu_{post} = (c+1)n_g + 2(n_g-1)$$
$$\tau^2_{post} = \frac{SS_g+n_g}{(c+1)n_g+2(n_g-1)},$$

Conditionals: Group K

$$\mathbb{P}(K_i = k|\theta, Y) \propto \lambda_k f_{MVN_d(\mu_k, \sigma_k^2)}(Z_i)$$

Since K is Multinoulli, we arrive at the following probability by recognizing they must normalize to unity:

$$\begin{aligned}\mathbb{P}(K_i = k|\theta, Y) &= \frac{\lambda_k f_{MVN_d(\mu_k, \sigma_k^2)}(Z_i)}{\sum_{g=1}^{G} \lambda_g f_{MVN_d(\mu_g, \sigma_g^2)}(Z_i)} \\ &= \frac{f_{MVN_d(\mu_k, \sigma_k^2)}(Z_i)}{\sum_{g=1}^{G} f_{MVN_d(\mu_g, \sigma_g^2)}(Z_i)} \qquad (\lambda^{(t)})\end{aligned}$$

Conditionals: Latent variable Z

$$f_{Z_i|\theta,Y}(Z_i|\theta, Y) \propto \prod_{j \neq i} \Big(\text{logit}^{-1}\big(\|Z_i - Z_j\|\big)\Big)^{Y_{ij}} \Big(1 - \text{logit}^{-1}\big(\|Z_i - Z_j\|\big)\Big)^{1-Y_{ij}}$$

$$\exp\Big\{ - \frac{1}{2\sigma^2_{k_i}}(Z_i - \mu_{k_i})^T(Z_i - \mu_{k_i})\Big\}$$

Do not know how to sample directly from this distribution, hence MH step.
Symmetric proposal (deviating from Hoff):

$$q(Z_*|\theta^{(t)}, Y) \sim MVN_2(0, I_2)$$

$$R(Z^*, Z^{(t)}) = \frac{f_{Z|\theta,Y}(Z^*|\theta^{(t)}, Y)q(Z^{(t)}|\theta^{(t)}, Y)}{f_{Z|\theta,Y}(Z^{(t)}|\theta^{(t)}, Y)q(Z^*|\theta^{(t)}, Y)}$$

$$= \frac{f_{Z|\theta,Y}(Z^*|\theta^{(t)}, Y)}{f_{Z|\theta,Y}(Z^{(t)}|\theta^{(t)}, Y)}$$

MCMC Pseudocode

Algorithm 3: Gibbs sampler for latent network model

LNM MCMC (G, N_k, d, ns);

Input : Graph G
Number of groups N_k
Dimension of Latent Variable d
Number of samples ns

Output: Posterior $p(Z|Y, \theta)$

Initialize parameters $\mu^{(0)}, {\sigma^2}^{(0)}, \lambda^{(0)}, K^{(0)}, Z^{(0)}$;

for $t = 2, \ldots, ns$ **do**

  for $k = 1, \ldots, N_k$ **do**

    sample $\mu_k|\theta^{(t)}, Y \sim$
    $MVN_d\left(\sum_{i=1}^{N_v} \mathbb{I}\{k_i = k\} \frac{Z_i^{(t-1)}}{(\sigma_{k_i}^2)^{(t-1)}+1}, \sum_{i=1}^{N_v} \mathbb{I}\{k_i = k\} \frac{(\sigma_{k_i}^2)^{(t-1)}}{(\sigma_{k_i}^2)^{(t-1)}+1} I_d\right)$;

  end

  for $k = 1, \ldots, N_k$ **do**

    sample $\sigma_k^2|\theta^{(t)}, Y \sim \Big(1 + \sum_{i=1}^{N_v} \mathbb{I}\{k_i =$
    $k\}(Z_i^{(t-1)} - \mu_k^{(t)})^T(Z_i^{(t-1)} - \mu_k^{(t)})\Big) \mathrm{Inv}\chi^2_{1+d\sum_{i=1}^{N_V} \mathbb{I}\{k_i=k\}}$;

  end

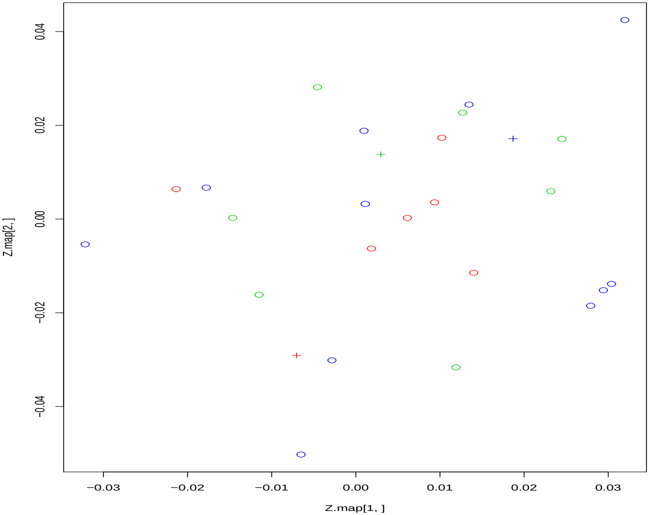
  for $i = 1, \ldots, N_v$ **do**

    sample $K_i \sim \mathrm{Multinoulli}(G, \lambda^{(t)})$;

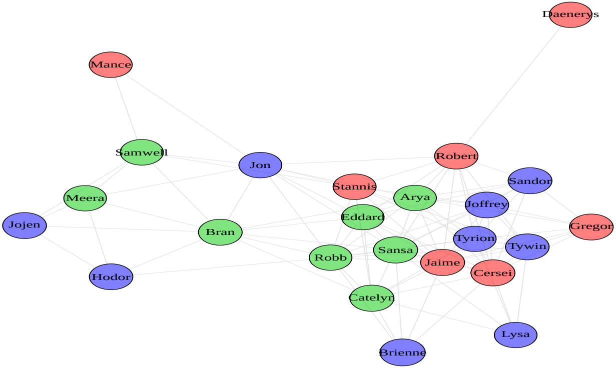
  end

for $i = 1, \ldots, N_v$ **do**
  sample $K_i \sim \text{Multinoulli}(G, \lambda^{(t)})$;
end
for $i = 1, \ldots, N_v$ **do**
  sample $Z_i^* \sim MVN_d(0, I_d)$;
  $R(Z_i^*, Z_i^{(t)}) = \min\left(1, \frac{f_{Z|\theta,Y}(Z_i^*|\theta^{(t)}, Y, Z_{[-1]})}{f_{Z|\theta,Y}(Z_i^{(t)}|\theta^{(t)}, Z_{[-1]})}\right)$;
  sample $U \sim \mathcal{U}(0, 1)$;
  if $U \leq R(Z_i^*, Z_i^{(t)})$ **then**
    $Z_i^{(t+1)} = Z_i^*$;
  else
    $Z_i^{*(t+1)} = Z_i^{(t)}$
  end
end
end
end

MCMC: Mean MAP Estimates



MCMC: Clustering



EM VS MCMC

a. results from EM using weighted network has more information than the unweighted

b. both EM algorithms converge much faster than the MCMC for this model

c. large room for the MCMC to improve :

    i. block-update covariate coefficients with the scale of latent space positions
    ii. once the distributions of the variables are known, we can perform further analysis such as regression

Conclusion

1. The information we can draw from both EM and MCMC is very interpretable. We can group characters by their geographical location, or personal relation; we can also detect if a character stands out (i.e. Daenerys).
2. This project had used many subjects covered in class Newton- Raphson, EM, MCMC, Graphical model, lse, and sampling – the material from class is very useful!