

1 Setup

We observe a random sample consisting of N_0 untreated observations and N_1 treated observations, such that $N_0 + N_1 = N$. Each observation i is characterized by potential outcomes $Y_i(1)$ and $Y_i(0)$ (treated and untreated outcome, respectively). Thus, $Y_i(0)$ is observed for control observations, and $Y_i(1)$ for treated observations.

For all observations, we observe both a scalar covariate X_i and a binary treatment indicator W_i ($W_i = 1$ if i received treatment, $W_i = 0$ otherwise). The vector of all observed outcomes, covariates, or treatment indicators is represented as \mathbf{Y} , \mathbf{X} , or \mathbf{W} respectively.

We are concerned with estimation of the average treatment effect on the treated (henceforth ATET):

$$\tau = \mathbb{E}[Y_i(1) - Y_i(0) | W_i = 1] \quad (1)$$

Since we are using a matching estimator approach, I may sometimes refer to “the matching matrix”. The matching matrix is a $N_0 \times N_1$ matrix whose ij ’th entry is 1 if the following holds (and 0 otherwise):

$$|X_j - X_i| \leq |X_j - X_k| \quad \forall k \neq i \text{ s.t. } W_k = 0$$

Intuitively, the ij ’th entry in this matrix is 1 if the j ’th treated unit is matched with the i ’th control unit, and 0 otherwise. The matching matrix is a function only of \mathbf{X} and \mathbf{W} , not \mathbf{Y} .

Identifying assumptions are:

1. Unconfoundedness - *For almost all x , $(Y_i(1), Y_i(0))$ is independent of W_i conditional on $X_i = x$.*
2. Overlap - *For some $c > 0$, and almost all x , $c \leq \mathbb{P}[W_i = 1 | X_i = x] \leq 1 - c$.*

For simplicity, we ignore the possibility of multiple control units matching to a single treatment unit, as this occurs with probability 0 in both the original sample and in the bootstrapped sample. Ignoring ties, we construct the estimated counterfactual outcome of interest:

$$\widehat{Y_i(0)} = \begin{cases} Y_k & \text{if } W_i = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Where Y_k is the unique (since we ignore ties) control unit that matches to unit i . Thus, the matching estimator of the ATET is:

$$\hat{\tau} = \frac{1}{N_1} \sum_{W_i=1} (Y_i - \widehat{Y_i(0)}) \quad (3)$$

Let:

$$K_i = \begin{cases} 0 & \text{if } W_i = 1 \\ \sum_{W_j=1} 1 \{i \text{ matches to } j\} & \text{if } W_i = 0 \end{cases} \quad (4)$$

When we ignore ties, K_i has a simple interpretation as the number of times unit i is used to construct $\widehat{Y_j(0)}$. We can now represent $\hat{\tau}$ as follows:

$$\hat{\tau} = \frac{1}{N_1} \sum_{i=1}^N (W_i - (1 - W_i)K_i) Y_i \quad (5)$$

It is easy to see that this is correct if one notes that this can be split up into $\frac{1}{N_1} \sum_{W_i=1} Y_i - \frac{1}{N_1} \sum_{W_i=0} K_i Y_i$. The first part gives the average treated outcome, and the latter part the average estimated counterfactual outcome.

2 Bootstrap Variance

The bootstrap algorithm is as follows:

1. Estimate $\hat{\tau}$ on the original sample.
2. Get residuals $\varepsilon_i = Y_i - \widehat{Y_i(0)} - \hat{\tau}$ for each treated observation i .
3. Draw, for each i , μ_i from a Bernoulli distribution with success probability $\frac{1}{2}$.
4. If $\mu_i \neq 1$, set it to -1 . This simply changes failures from being reported as 0 to being reported as -1 .
5. Create perturbed outcomes $Y_i^* = Y_i(0) + \hat{\tau} + \mu_i \varepsilon_i$.
6. Now estimate $\hat{\tau}_b$ on the bootstrapped sample $\{\mathbf{Y}^*, \mathbf{W}, \mathbf{X}\}$.
7. Repeat steps 1-6 for b bootstraps, and estimate the variance of $\hat{\tau}$ as the sample variance of $\hat{\tau}_b$ over the b bootstraps.

Note first that this algorithm does nothing to \mathbf{X} and \mathbf{W} . Thus, it does nothing to the matching matrix. For each treated unit i , the estimated counterfactual outcome $\widehat{Y_i(0)}$ is unchanged by this procedure.

From (5) it follows that:

$$\hat{\tau}_b = \frac{1}{N_1} \sum_{W_i=1} Y_i^* - \frac{1}{N_1} \sum_{W_i=0} K_i Y_i \quad (6)$$

$$\begin{aligned} &= \frac{1}{N_1} \sum_{W_i=1} (\widehat{Y_i(0)} + \mu_i \varepsilon_i + \hat{\tau}) - \frac{1}{N_1} \sum_{W_i=0} K_i Y_i \\ &= \hat{\tau} + \frac{1}{N_1} \sum_{W_i=1} \widehat{Y_i(0)} + \frac{1}{N_1} \sum_{W_i=1} \mu_i \varepsilon_i - \frac{1}{N_1} \sum_{W_i=0} K_i Y_i \end{aligned} \quad (7)$$

Note that the residuals ε_i are:

$$\begin{aligned} \varepsilon_i &= Y_i - \widehat{Y_i(0)} - \hat{\tau} \\ &= Y_i - \widehat{Y_i(0)} - \frac{1}{N_1} \sum_{W_i=1} Y_i - \frac{1}{N_1} \sum_{W_i=0} K_i Y_i \end{aligned}$$

Thus $\sum_{W_i=1} \varepsilon_i = 0$. This does not let us claim that $\sum_{W_i=1} \mu_i \varepsilon_i$ is 0, however.

Note also that by the logic of (5), it follows that:

$$\frac{1}{N_1} \left(\sum_{W_i=1} \widehat{Y_i(0)} - \sum_{W_i=0} K_i Y_i \right) = 0$$

Because both sums are adding up the same things. Thus, (7) becomes:

$$\hat{\tau}_b = \hat{\tau} + \frac{1}{N_1} \sum_{W_i=1} \mu_i \varepsilon_i \quad (8)$$

It should at this point be obvious that $\mathbb{E}^*[\hat{\tau}_b] = \hat{\tau}$, where \mathbb{E}^* is the expectation over the bootstrap (or the expectation conditioned on \mathbf{Y} , \mathbf{W} , and \mathbf{X}).

Thus, using similar notation:

$$\begin{aligned} \mathbb{V}^*(\hat{\tau}_b) &= \mathbb{E}^* \left[(\hat{\tau}_b - \hat{\tau})^2 \right] \\ &= \mathbb{E}^* \left[\frac{1}{N_1^2} \left(\sum_{W_i=1} \mu_i \varepsilon_i \right)^2 \right] \quad \text{by substituting (8) and distributing} \end{aligned}$$

Without loss of generality, assume that $W_i = 1$ for $i \in \{1, 2, \dots, N_1\}$ - that is, the first N_1 units are the treated units. Then:

$$\begin{aligned} \mathbb{V}^*(\hat{\tau}_b) &= \mathbb{E}^* \left[\frac{1}{N_1^2} (\mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \dots + \mu_{N_1} \varepsilon_{N_1})^2 \right] \\ &= \mathbb{E}^* \left[\frac{1}{N_1^2} (\mu_1^2 \varepsilon_1^2 + \mu_1 \varepsilon_1 \mu_2 \varepsilon_2 + \dots + \mu_1 \varepsilon_1 \mu_{N_1} \varepsilon_{N_1}) \right. \\ &\quad + \frac{1}{N_1^2} (\mu_2 \varepsilon_2 \mu_1 \varepsilon_1 + \mu_2^2 \varepsilon_2^2 + \dots + \mu_2 \varepsilon_2 \mu_{N_1} \varepsilon_{N_1}) \\ &\quad + \dots \\ &\quad \left. + \frac{1}{N_1^2} (\mu_{N_1} \varepsilon_{N_1} \mu_1 \varepsilon_1 + \mu_{N_1} \varepsilon_{N_1} \mu_2 \varepsilon_2 + \dots + \mu_{N_1}^2 \varepsilon_{N_1}^2) \right] \end{aligned} \quad (9)$$

To get further into this, consider first $\mu_i^2 \varepsilon_i^2$. Note that μ_i^2 is always 1, and thus:

$$\begin{aligned} \mu_i^2 \varepsilon_i^2 &= \varepsilon_i^2 \\ &= \left(Y_i - \widehat{Y_i(0)} - \hat{\tau} \right)^2 \\ &= Y_i^2 + \widehat{Y_i(0)}^2 + \hat{\tau}^2 + 2\hat{\tau} \widehat{Y_i(0)} - 2Y_i \widehat{Y_i(0)} - 2Y_i \hat{\tau} \end{aligned} \quad (10)$$

The other case that matters is $\mu_i \mu_j \varepsilon_i \varepsilon_j$. Note first that $\mu_i \mu_j$ is, just like μ_i , -1 or +1 with equal probability. For simplicity, let $\mu_{ij} = \mu_i \mu_j$. Thus:

$$\mu_i \mu_j \varepsilon_i \varepsilon_j = \mu_{ij} \left(Y_i - \widehat{Y_i(0)} - \hat{\tau} \right) \left(Y_j - \widehat{Y_j(0)} - \hat{\tau} \right) \quad (11)$$

Beyond this point I am no longer entirely confident of what I'm doing.

Under the bootstrap, ε_i is a known quantity, not a random variable. The only randomness present in (10) or (11) is that brought in by μ_i and μ_j . Conveniently, that randomness disappears by definition in (10), but not in (11). Thus, from (9), we can go to:

$$\begin{aligned} \mathbb{V}^*(\hat{\tau}_b) = & \frac{1}{N_1^2} \left(\mathbb{E}^* [\mu_1^2 \varepsilon_1^2] + \mathbb{E}^* [\mu_1 \mu_2 \varepsilon_1 \varepsilon_2] + \dots + \mathbb{E}^* [\mu_1 \mu_{N_1} \varepsilon_1 \varepsilon_{N_1}] \right. \\ & + \mathbb{E}^* [\mu_2 \mu_1 \varepsilon_2 \varepsilon_1] + \mathbb{E}^* [\mu_2^2 \varepsilon_2^2] + \dots + \mathbb{E}^* [\mu_2 \mu_{N_1} \varepsilon_2 \varepsilon_{N_1}] \\ & + \dots \\ & \left. + \mathbb{E}^* [\mu_{N_1} \mu_1 \varepsilon_{N_1} \varepsilon_1] + \mathbb{E}^* [\mu_{N_1} \mu_2 \varepsilon_{N_1} \varepsilon_2] + \dots + \mathbb{E}^* [\mu_{N_1}^2 \varepsilon_{N_1}^2] \right) \quad (12) \end{aligned}$$

And by taking $\varepsilon_i \varepsilon_j$ out of the expectation (which I *think* I can do), we get:

$$\mathbb{V}^*(\hat{\tau}_b) = \frac{1}{N_1^2} \left(\sum_{i=1}^{N_1} \varepsilon_i^2 \right) \quad (13)$$

This is because $\mathbb{E}^* [\mu_i \mu_j] = 0$ by design, while $\mathbb{E}^* [\mu_i^2] = 1$.

This is quite promising. If the step where I take $\varepsilon_i \varepsilon_j$ out of the expectations is valid, I am confident of this "proof".

Do I want to show that $\mathbb{V}^*(\hat{\tau}_b) \rightarrow_p \mathbb{V}(\hat{\tau}|\mathbf{X}, \mathbf{W}, \mathbf{Y})$ as N approaches infinity (holding the treatment ratio fixed)? Or do I want to show that $\frac{1}{B} \mathbb{V}^*(\hat{\tau}_b) \rightarrow_p \mathbb{V}(\hat{\tau}|\mathbf{X}, \mathbf{W})$ as B approaches infinity?

From Abadie & Imbens (2008):

$$\hat{\mathbb{V}}^{AI} = \frac{1}{N_1^2} \sum_{i=1}^N \left(Y_i - \widehat{Y_i(0)} - \hat{\tau} \right)^2 + \frac{1}{N_1^2} \sum_{i=1}^N (K_i^2 - K_i^{sq})^2 \hat{\sigma}^2(X_i, W_i) \quad (14)$$

The first term on the RHS of (14) is equivalent to (13). K_i is (since we are ignoring multiple matches) the number of times unit i is used as a match. K_i^{sq} is the same, since we are ignoring multiple matches, and the latter term is $K_i^2 - K_i$?