Bootstrap Variance Derivation

Nik Julius

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Alternate Title: Just fire yourself at the algebra like an idiot for 6 hours and something will happen.

Note that I am defining τ to be the treatment effect *on the treated* for my convenience. We start from (13) in `bootstrap_variance.pdf`:

$$V^*(\hat{\tau}_b|X,W,Y) = \frac{1}{N_1^2}\sum_{W_i=1}\hat{\varepsilon}_i^2 \tag{1}$$

We want to show that $V^*(\hat{\tau}_b|X,W) \to_p V(\hat{\tau}|X,W)$. To get $V*(\hat{\tau}_b|X,W)$, we just take the expectation of (1) over all possible Y:

$$\begin{aligned} V^*(\hat{\tau}_b|X,W) &= E\left[\frac{1}{N_1^2}\sum_{W_i=1}\hat{\varepsilon}_i^2|X,W\right] \\ &= E\left[\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2|X,W\right] \end{aligned} \tag{2}$$

First, expand the square (and substitute for $\hat{\tau}$:

$$\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2 = \left(Y_i(1)-\widehat{Y_i(0)}-\frac{1}{N_1}\sum_{W_i=1}Y_i(1)+\frac{1}{N_1}\sum_{W_i=1}\widehat{Y_i(0)}\right)^2 \tag{3}$$

Let $\frac{1}{N_1}\sum_{W_i=1}Y_j(1)=\overline{Y_j(1)}$ and $\frac{1}{N_1}\sum_{W_j=1}\widehat{Y_i(0)}=\overline{\widehat{Y_j(0)}}$. Then (3) is:

$$
\begin{aligned}
&Y_i(1)^2 - Y_i(1)\widehat{Y_i(0)} - Y_i(1)\overline{Y_j(1)} + Y_i(1)\widehat{\overline{Y_j(0)}} \\
&-\widehat{Y_i(0)}Y_i(1) + \widehat{Y_i(0)}^2 + \widehat{Y_i(0)}\overline{Y_j(1)} - \widehat{Y_i(0)}\widehat{\overline{Y_j(0}} \\
&-\overline{Y_j(1)}Y_i(1) + \overline{Y_j(1)}\widehat{Y_i(0} + \overline{Y_j(1)}^2 - \overline{Y_j(1)\widehat{Y_j(0)}} \\
&+\widehat{\overline{Y_j(0)}}Y_i(1) - \widehat{\overline{Y_j(0)}}\widehat{Y_i(0)} - \overline{\widehat{Y_j(0)}Y_j(1)} + \widehat{\overline{Y_j(0)}}^2
\end{aligned} \tag{4}
$$

Cancelling terms leaves us with:

$$
\begin{aligned}
\left(Y_i(1) - \widehat{Y_i(0)} - \hat{\tau}\right)^2 = {} & Y_i(1)^2 + \widehat{Y_i(0)}^2 + \overline{Y_j(1)}^2 + \widehat{\overline{Y_j(0)}}^2 - 2Y_i(1)\widehat{Y_i(0)} - 2Y_i(1)\overline{Y_j(1)} \\
& + 2Y_i(1)\widehat{\overline{Y_j(0)}} + 2\widehat{Y_i(0)}\overline{Y_j(1)} - 2\widehat{Y_i(0)}\widehat{\overline{Y_j(0)}} - 2\overline{\widehat{Y_j(0)}Y_j(1)}
\end{aligned} \tag{5}
$$

Return to (2) and consider the inside of the expectation:

$$
\begin{aligned}
\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1) - \widehat{Y_i(0)} - \hat{\tau}\right)^2 = {} & \frac{1}{N_1^2}\sum_{W_i=1} Y_i(1)^2 + \frac{1}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}^2 + \frac{1}{N_1^2}\sum_{W_i=1}\overline{Y_j(1)}^2 + \frac{1}{N_1^2}\sum_{W_i=1}\widehat{\overline{Y_j(0)}}^2 \\
& - 2\frac{1}{N_1^2}\sum_{W_i=1} Y_i(1)\widehat{Y_i(0)} - 2\frac{1}{N_1^2}\sum_{W_i=1} Y_i(1)\overline{Y_j(1)} + 2\frac{1}{N_1^2}\sum_{W_i=1} Y_i(1)\widehat{\overline{Y_j(0)}} \\
& + 2\frac{1}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}\overline{Y_j(1)} - 2\frac{1}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}\widehat{\overline{Y_j(0)}} - 2\frac{1}{N_1^2}\sum_{W_i=1}\overline{\widehat{Y_j(0)}Y_j(1)}
\end{aligned} \tag{6}
$$

Recall the definition of $\overline{Y_j(1)}$ and $\widehat{\overline{Y_j(0)}}$. Note that by using one of the $\frac{1}{N_1}$ terms that we just distributed, we can make more of these terms (noting that $\overline{Y_i(1)} = \overline{Y_j(1)}$ and likewise for $\widehat{\overline{Y_j(0)}}$). This gives us:

$$
\begin{aligned}
\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1) - \widehat{Y_i(0)} - \hat{\tau}\right)^2 = {} & \frac{1}{N_1^2}\sum_{W_i=1} Y_i(1)^2 + \frac{1}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}^2 + \frac{1}{N_1^2}\sum_{W_i=1}\overline{Y_j(1)}^2 + \frac{1}{N_1^2}\sum_{W_i=1}\widehat{\overline{Y_j(0)}}^2 \\
& - 2\frac{1}{N_1^2}\sum_{W_i=1} Y_i(1)\widehat{Y_i(0)} - 2\frac{1}{N_1}\overline{Y_j(1)}^2 + 2\frac{1}{N_1}\overline{\widehat{Y_j(0)}Y_j(1)} \\
& + 2\frac{1}{N_1}\overline{Y_j(1)\widehat{Y_j(0)}} - 2\frac{1}{N_1}\widehat{\overline{Y_j(0)}}^2 - 2\frac{1}{N_1}\overline{\widehat{Y_j(0)}Y_j(1)}
\end{aligned} \tag{7}
$$

Further, note that $\overline{Y_j(1)}$ and $\widehat{\overline{Y_j(0)}}$ are constants, which gives us:

$$
\begin{aligned}
\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2 = &\frac{1}{N_1^2}\sum_{W_i=1}Y_i(1)^2+\frac{1}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}^2+\frac{1}{N_1}\overline{Y_j(1)}^2+\frac{1}{N_1}\overline{\widehat{Y_j(0)}}^2\\
&-2\frac{1}{N_1^2}\sum_{W_i=1}Y_i(1)\widehat{Y_i(0)}-2\frac{1}{N_1}\overline{Y_j(1)}^2+2\frac{1}{N_1}\overline{\widehat{Y_j(0)}Y_j(1)}\\
&+2\frac{1}{N_1}\overline{Y_j(1)\widehat{Y_j(0)}}-2\frac{1}{N_1}\overline{\widehat{Y_j(0)}}^2-2\frac{1}{N_1}\overline{\widehat{Y_j(0)}Y_j(1)}
\end{aligned}
\tag{8}
$$

Cancelling terms gives us:

$$
\begin{aligned}
\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2 = &\frac{1}{N_1^2}\sum_{W_i=1}Y_i(1)^2+\frac{1}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}^2-\frac{2}{N_1^2}\sum_{W_i=1}Y_i(1)\widehat{Y_i(0)}\\
&-\frac{1}{N_1}\overline{Y_j(1)}^2+\frac{2}{N_1}\overline{Y_j(1)\widehat{Y_j(0)}}-\frac{1}{N_1}\overline{\widehat{Y_j(0)}}^2
\end{aligned}
\tag{9}
$$

There is a tantalizing glimpse of further cancellation - if the 5th term in (9) wasn't multiplied by $\frac{1}{N_1}$, it would partially cancel with the 3rd term (it would leave only cross products of $Y_i(1)\widehat{Y_j(0)}$ where $i \neq j$). I have been unable to find a mistake leading to that $\frac{1}{N_1}$ being there, though, so I'm fairly sure this is just reality teasing me.

I don't really know how to move on from here without using the things we know about the DGP, so thats how I move forward. We know that $Y_i(1)=\tau$ always, and $Y_i(0)\sim N(0,1)$. Also, note that we've been working inside the expectation up to now. I transition to considering the entirety of (2) now:

$$
\begin{aligned}
E\left[\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2|X,W\right] = &E\left[\frac{1}{N_1^2}\tau^2+\frac{1}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}^2-\frac{2\tau}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}\right.\\
&\left.-\frac{1}{N_1}\tau^2+\frac{2\tau}{N_1}\overline{\widehat{Y_j(0)}}-\frac{1}{N_1}\overline{\widehat{Y_j(0)}}^2|X,W\right]
\end{aligned}
\tag{10}
$$

Again using the definition of $\overline{\widehat{Y_j(0)}}$ note that the 3rd term is equivalent to the 5th term. Cancelling leaves us with:

$$
E\left[\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2|X,W\right] = E\left[\frac{1}{N_1^2}\tau^2+\frac{1}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}^2-\frac{1}{N_1}\tau^2-\frac{1}{N_1}\overline{\widehat{Y_j(0)}}^2|X,W\right] \tag{11}
$$

Re-arranging and distributing the expectation, we arrive at:

$$E\left[\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2|X,W\right]=\frac{1}{N_1}\left(\frac{\tau^2}{N_1}-\tau^2\right)+\frac{1}{N_1^2}E\left[\sum_{W_i=1}\widehat{Y_i(0)}^2|X,W\right]-\frac{1}{N_1}E\left[\widehat{\overline{Y_j(0)}}^2|X,W\right]$$

At this point it is necessary to change to the K_iY_i notation used in A& I. Note that $\sum_{W_i=1}\widehat{Y}_i^2 = \sum_{W_i=0}K_i^2Y_i^2$ (this may be wrong?). In the next step I use this on the 3rd term, use the definition of $\widehat{\overline{Y_j(0)}}$ on the 4th term, and then use the same equality again on that 4th term:

$$E\left[\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2|X,W\right]=\frac{1}{N_1}\left(\frac{\tau^2}{N_1}-\tau^2\right)+\frac{1}{N_1^2}\sum_{W_i=0}K_i^2E[Y_i^2|X,W] - \frac{1}{N_1^3}\sum_{W_i=0}K_i^2E[Y_i^2|X,W] \tag{12}$$

Note that going from the 4th term in the previous equation to the 4th term in (12) is fairly involved, and makes use of the fact that $Y_i(0)$ are iid $N(0,1)$.

Using that distribution of $Y_i(0)$, we know that $E[Y_i^2|X,W]=1$. Thus, (12) becomes:

$$E\left[\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2|X,W\right]=\frac{1}{N_1}\left(\frac{\tau^2}{N_1}-\tau^2\right)+\left(\frac{1}{N_1^2}-\frac{1}{N_1^3}\right)\sum_{W_i=0}K_i^2 \tag{13}$$

This is as far as I've gone. This is *infuriating* because the lower bound of (13) is easily seen to be negative (indeed, for *any* value of $\sum_{W_i=0}K_i^2$, there is a value of τ that makes (13) negative).

However, if we ignore the first term, the last term looks extremely promising. It differs from the correct variance only in the inclusion of the $-\frac{1}{N_1^3}$ factor, and that would probably explain a lot of the underestimation seen in my simulations.

1 To do:

1. Check if (13) matches the variances from the simulations.

2. The above almost certainly will come out false. Check if (13) *without the term involving* τ matches the variances from the simulations.

2 I DID IT

Look at (10). I have converted $\frac{1}{N_1^2}\sum_{W_i=1} Y_i(1)^2$ into $\frac{1}{N_1^2}\tau^2$, but this ignores the sum. The first term in (9) consists of N_1 τ^2 terms added together. Thus, (10) should be:

$$E\left[\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2|X,W\right] = E\left[\frac{1}{N_1}\tau^2+\frac{1}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}^2-\frac{2\tau}{N_1^2}\sum_{W_i=1}\widehat{Y_i(0)}\right.$$
$$\left.-\frac{1}{N_1}\tau^2+\frac{2\tau}{N_1}\overline{\widehat{Y_j(0)}}-\frac{1}{N_1}\overline{\widehat{Y_j(0)}}^2|X,W\right] \tag{14}$$

This means (13) should be:

$$E\left[\frac{1}{N_1^2}\sum_{W_i=1}\left(Y_i(1)-\widehat{Y_i(0)}-\hat{\tau}\right)^2|X,W\right] = \left(\frac{1}{N_1^2}-\frac{1}{N_1^3}\right)\sum_{W_i=0}K_i^2 \tag{15}$$

This is **much** better.