04. Convex Optimization Problems

By Yang Lin[1] (2023 秋季, @NJU)

Contents

[1]Institute: Nanjing University. Email: linyang@nju.edu.cn.

1 Optimization problem in standard form

$$
\begin{array}{ll}
\texttt{minimize} & f_0(x) \\
\texttt{subject to} & f_i(x) \leq 0, \;\; i = 1, 2, \ldots, m \\
 & h_i(x) = 0, \;\; i = 1, 2, \ldots, p
\end{array}
$$

$x \in \mathbf{R}^n$ is the optimization variable

$f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function

$f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$ are the inequality constraint functions

$h_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, p$ are the equality constraint functions

Optimal value:

$$p^* = \inf\{f_0(x) | f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, \;\; i = 1, \ldots, p\}$$

$p^* = \infty$ if problem is infeasible (no x satisfies the constraints)

$p^* = -\infty$ if problem is unbounded below

1 Optimization problem in standard form

Optimal and locally optimal points:

x is feasible if $x \in \mathbf{dom}\ f_0$ and it satisfies the constraints

Feasible x is optimal if $f_0(x) = p^*$; X_{opt} the set of optimal points

x is locally optimal if there is an $R > 0$ such that x is optimal for

$$
\begin{array}{ll}
\texttt{minimize (over } z\texttt{)} & f_0(z) \\
\texttt{subject to} & f_i(z) \leq 0, \ \ i = 1, 2, \ldots, m \\
 & h_i(z) = 0, \ \ i = 1, 2, \ldots, p \\
 & \|z - x\|_2 \leq R
\end{array}
$$

Examples (with $n = 1$, $m = p = 0$)

$f_0(x) = 1/x$, $\mathbf{dom}\ f_0 = \mathbf{R}_{++}$: $p^* = 0$, no optimal point

$f_0(x) = -\log x$, $\mathbf{dom}\ f_0 = \mathbf{R}_{++}$: $p^* = -\infty$

$f_0(x) = x \log x$, $\mathbf{dom}\ f_0 = \mathbf{R}_{++}$: $p^* = -1/e$, $x = 1/e$ is optimal

$f_0(x) = x^3 - 3x$, $p^* = -\infty$, local optimum at $x = 1$

1 Optimization problem in standard form

Implicit constraints

The standard form optimization problem has an implicit constraint

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\ f_i \cap \bigcap_{i=0}^{p} \mathbf{dom}\ h_i$$

We call $\mathcal{D}$ the domain of the problem

The constraints $f_i(x) \leq 0, h_i(x) = 0$ are the explicit constraints

A problem is unconstrained if it has no explicit constraints ($m = p = 0$)

Examples

$$\texttt{minimize}\ f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

1 Optimization problem in standard form

Feasibility problem

$$\begin{array}{ll} \texttt{find} & x \\ \texttt{subject to} & f_i(x) \leq 0, \;\; i = 1, 2, \ldots, m \\ & h_i(x) = 0, \;\; i = 1, 2, \ldots, p \end{array}$$

Can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{ll} \texttt{minimize} & 0 \\ \texttt{subject to} & f_i(x) \leq 0, \;\; i = 1, 2, \ldots, m \\ & h_i(x) = 0, \;\; i = 1, 2, \ldots, p \end{array}$$

$p^* = 0$, if constraints are feasible; any feasible x is optimal

$p^* = \infty$ if constraints are infeasible

2 Convex Optimization Problem

Standard form convex optimization problem

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \le 0, \;\; i = 1, 2, \ldots, m \\ & a_i^T x = b_i, \;\; i = 1, 2, \ldots, p \end{array}$$

$f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine

problem is *quasiconvex* if f_0 is quasiconvex (and $f_1, \ldots, f_m$ convex)

often written as

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \le 0, \;\; i = 1, 2, \ldots, m \\ & Ax = b, \;\; i = 1, 2, \ldots, p \end{array}$$

important property: feasible set of a convex optimization problem is convex

2 Convex Optimization Problem

Example

$$\begin{array}{ll} \texttt{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \texttt{subject to} & f_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1+x_2)^2 = 0 \end{array}$$

f_0 is convex; feasible set $\{(x_1, x_2) | x_1 = -x_2 \leq 0\}$ is convex

Not a convex problem (according to our definition): f_1 is not convex, h_1 is not affine

Equivalent (but not identical) to the convex problem

$$\begin{array}{ll} \texttt{minimize} & x_1^2 + x_2^2 \\ \texttt{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{array}$$

2 Convex Optimization Problem

Local and global optimal

Any locally optimal point of a convex problem is (globally) optimal

proof.

Suppose x is locally optimal and y is optimal with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

$$z \text{ feasible}, \ \|z - x\|_2 \leq R \Longrightarrow f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1-\theta)x$ with $\theta = R/(2\|y-x\|_2)$

$\|y-x\|_2 > R$, so $0 < \theta < 1/2$

z is a convex combination of two feasible points, hence also feasible

$\|z-x\|2 = R/2$ and

$$f_0(z) \leq \theta f_0(x) + (1-\theta) f_0(y) < f_0(x)$$

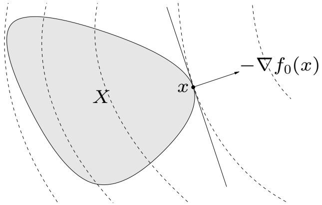
which contradicts our assumption that x is locally optimal

2 Convex Optimization Problem

Optimality criterion for differentiable f_0

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \text{ for all feasible } y$$



if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x

2 Convex Optimization Problem

Unconstrained problem: x is optimal if and only if

$$x \in \textbf{dom} f_0, \nabla f_0(x) = 0$$

Equality constrained problem:

$$\texttt{minimize}\ f_0(x) \quad \texttt{subject to}\ Ax = b$$

x is optimal if and only if there exists a ν such that

$$x \in \textbf{dom}\ f_0,\ \ Ax = b, \nabla f_0(x) + A^T\nu = 0$$

Minimization over nonnegative orthant:

$$\texttt{minimize}\ f_0(x) \quad \texttt{subject to}\ x \succeq 0$$

x is optimal if and only if

$$x \in \textbf{dom}\ f_0,\ \ x \succeq 0, \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$

2 Convex Optimization Problem

Equivalent convex problems

Two problems are (informally) equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa

Some common transformations that preserve convexity:

Eliminating equality constraints

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, \ \ i = 1, \ldots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } z\texttt{)} & f_0(Fz + x_0) \\ \texttt{subject to} & f_i(Fz + x_0) \leq 0, \ \ i = 1, \ldots, m \end{array}$$

where F and x_0 are such that

$$Ax = b \Longleftrightarrow x = Fz + x_0 \text{ for some } z$$

2 Convex Optimization Problem

Equivalent convex problems

Introducing equality constraints

$$\begin{array}{ll} \texttt{minimize} & f_0(A_0x+b_0) \\ \texttt{subject to} & f_i(A_ix+b_i) \leq 0, \;\; i=1,\ldots,m \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } x \; y_i\texttt{)} & f_0(y_0) \\ \texttt{subject to} & f_i(y_i) \leq 0, \;\; i=1,\ldots,m \\ & y_i = A_ix+b_i, \;\; i=0,1,\ldots,m \end{array}$$

2 Convex Optimization Problem

Equivalent convex problems

Introducing slack variables for linear inequalities

$$
\begin{array}{ll}
\texttt{minimize} & f_0(x) \\
\texttt{subject to} & a_i^T x \leq b_i, \ \ i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\texttt{minimize (over } x \ s\texttt{)} & f_0(x) \\
\texttt{subject to} & a_i^T x + s_i = b_i, \ \ i = 1, \ldots, m \\
 & s_i \geq 0, \ \ i = 1, \ldots, m
\end{array}
$$

2 Convex Optimization Problem

Equivalent convex problems

Epigraph form: standard form convex problem is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } x\ t\texttt{)} & t \\ \texttt{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & Ax = b \end{array}$$

Minimizing over some variables:

$$\begin{array}{ll} \texttt{minimize (over } x\texttt{)} & f_0(x_1, x_2) \\ \texttt{subject to} & f_i(x_1) \leq 0, \;\; i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } x_1\texttt{)} & \tilde{f}_0(x_1) \\ \texttt{subject to} & f_i(x_1) \leq 0, \;\; i = 1, \ldots, m \end{array}$$

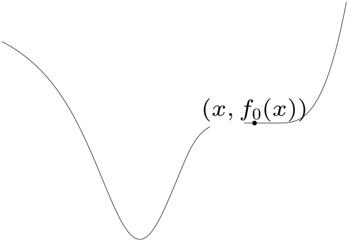
where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

3 Quasiconvex Optimization

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, \ \ i = 1, \ldots, m \\ & Ax = b \end{array}$$

with $f_0 : \boldsymbol{R}^n \to R$ quasiconvex, $f_1, \ldots, f_m$ convex

can have locally optimal points that are not (globally) optimal



3 Quasiconvex Optimization

Convex representation of sublevel sets of f_0

if f_0 is quasiconvex, there exists a family of functions ϕ_t such that:

1. $\phi_t(x)$ is convex in x for fixed t
2. t-sublevel set of f_0 is 0-sublevel set of ϕ_t, i.e.,

$$f_0(x) \leq t \Longleftrightarrow \phi_t(x) \leq 0$$

Example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with p convex, q concave, and $p(x) \geq 0$, $q(x) > 0$ on **dom** f_0

can take $\phi_t(x) = p(x) - tq(x)$:

1. for $t \geq 0$, ϕ_t convex in x
2. $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$

3 Quasiconvex Optimization

Quasiconvex optimization via convex feasibility problems

$$\phi_t(x) \le 0, \qquad f_i(x) \le 0, \ i = 1, ..., m, \qquad Ax = b$$

1. For fixed t, a convex feasibility problem in x
2. If feasible, we can conclude that $t \ge p^*$; if infeasible, $t \le p^*$

Bisection method for quasiconvex optimization

given $l \le p^\star$, $u \ge p^\star$, tolerance $\epsilon > 0$.
repeat
  1. $t := (l + u)/2$.
  2. Solve the convex feasibility problem (1).
  3. **if** (1) is feasible, $u := t$;   **else** $l := t$.
until $u - l \le \epsilon$.

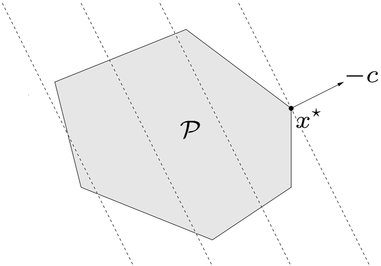
---

requires exactly $\lceil \log 2((u{-}l)/\epsilon) \rceil$ iterations (where u, l are initial values)

4 Linear Program (LP)

$$\begin{array}{ll} \texttt{minimize} & c^T x + d \\ \texttt{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

1. Convex problem with affine objective and constraint functions
2. Feasible set is a polyhedron



4 Linear Program (LP)

Examples

Diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

1. One unit of food j costs c_j, contains amount a_{ij} of nutrient i
2. Healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll} \texttt{minimize} & c^T x \\ \texttt{subject to} & Ax \succeq b, \qquad x \succeq 0 \end{array}$$

Piecewise-linear minimization

$$\texttt{minimize} \quad \max_{i=1,\ldots,m} \ (a_i^T x + b_i)$$

equivalent to an LP

$$\begin{array}{ll} \texttt{minimize} & t \\ \texttt{subject to} & a_i^T x + b_i \leq t, \ \ i = 1, \ldots, m \end{array}$$

4 Linear Program (LP)

Chebyshev center of a polyhedron

Chebyshev center of $\mathcal{P} = \{x | a_i^T x \leq b_i, \ i = 1, \ldots, m\}$ is center of largest inscribed ball $\mathcal{B} = \{x_c + u | \|u\|_2 \leq r\}$

1. $a_i^T x \leq b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c + u) | \|u\|_2 \leq r\} = a_i^T x_c + r\|a_i\|_2 \leq b_i$$

2. hence, x_c, r can be determined by solving the LP

$$\begin{array}{ll} \texttt{maximize} & r \\ \texttt{subject to} & a_i^T x_c + r\|a_i\|_2 \leq b_i, i = 1, \ldots, m \end{array}$$

5 (Generalized) Linear-fractional Program

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & Gx \preceq h, \qquad Ax = b \end{array}$$

linear-fractional program

$$f_0(x) = \frac{c^T x + d}{e^T x + f}, \qquad \textbf{dom } f_0(x) = \{x | e^T x + f > 0\}$$

a quasiconvex optimization problem; can be solved by bisection

also equivalent to the LP(variables y, z)

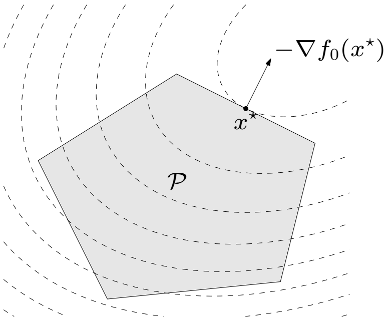
$$\begin{array}{ll} \texttt{minimize} & c^T y + dz \\ \texttt{subject to} & Gy \preceq hz \\ & Ay = bz \\ & e^T y + fz = 1 \\ & z \geq 0 \end{array}$$

5 Quadratic Program (QP)

$$\begin{array}{ll} \texttt{minimize} & (1/2)x^T P x + q^T x + r \\ \texttt{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

$P \in \boldsymbol{S}^n$, so objective is convex quadratic

minimize a convex quadratic function over a polyhedron



5 Quadratic Program (QP)

Examples: least-squares

$$\texttt{minimize} \quad \|Ax - b\|_2^2$$

analytical solution $x^* = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)

can add linear constraints, e.g., $l \succeq x \succeq u$

Linear program with random cost

$$\begin{array}{ll} \texttt{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E} c^T x + \gamma \mathbf{var}(c^T x) \\ \texttt{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

1. c is random vector with mean $\bar{c}$ and covariance Σ
2. hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
3. $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)