221900180 田永铭 计算方法作业 12

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第7章 18.解: $\|A\|_\infty=1.1$, $\|A\|_1=0.8$, $\|A\|_2=0.825$, $\|A\|_F=0.8426$.

这题上次作业已做过，这里直接给答案！

31. ~~解~~

证明：

(1) ∵ A 对称正定，∴ $A^T=A$, $A\succ 0$. 又有 $A=LDL^T=W^TW$，其中 $W=D^{\frac{1}{2}}L^T$.

$$\therefore cond(A)_2=\frac{|\lambda(A)|_{max}}{|\lambda(A)|_{min}}=\sqrt{\frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}}=\sqrt{\frac{\lambda_{max}(W^TW)^2}{\lambda_{min}(W^TW)^2}}=\sqrt{\left(\frac{\lambda_{max}(W^TW)}{\lambda_{min}(W^TW)}\right)^2}$$

$$=[cond(W)_2]^2.\ \square.$$

(2) 由(1): $cond(A)_2=[cond(W)_2]^2$，∴欲证 $cond(A)_2=cond(W^T)_2\,cond(W)_2$，

只需证: $cond(W)_2=cond(W^T)_2$.

而 $cond(W^T)_2=\sqrt{\frac{\lambda_{max}(WW^T)}{\lambda_{min}(WW^T)}}=\sqrt{\frac{\lambda_{max}(W^TW)}{\lambda_{min}(W^TW)}}=cond(W)_2$.

∴得证. □.

32. 解: ① ∵ $A=\begin{pmatrix}100 & 99\\ 99 & 98\end{pmatrix}$，∴ $A^{-1}=\begin{pmatrix}-98 & 99\\ 99 & -100\end{pmatrix}$.

∴ $\|A\|_\infty=199$, $\|A^{-1}\|_\infty=199$.

∴ $cond(A)_\infty=\|A^{-1}\|_\infty\|A\|_\infty=199^2=39601$.

~~② 再求 $cond(A)_2$:~~

~~$A^TA=\begin{pmatrix}100 & 99\\ 99 & 98\end{pmatrix}\begin{pmatrix}100 & 99\\ 99 & 98\end{pmatrix}=\begin{pmatrix}19801 & 19602\\ 19602 & 19405\end{pmatrix}$.~~

~~令 $|\lambda E-A^TA|=\begin{vmatrix}\lambda-19801 & -19602\\ -19602 & \lambda-19405\end{vmatrix}=0$，则 $\lambda_1=39205.99997$, $\lambda_2=0.0000255006$.~~

∵ A 为对称阵，∴ $cond(A)_2=\frac{|\lambda|_{max}}{|\lambda|_{min}}$

令 $|\lambda E-A|=\begin{vmatrix}\lambda-100 & -99\\ -99 & \lambda-98\end{vmatrix}=(\lambda-100)(\lambda-98)-99^2=0$，∴ $\lambda_1=198.0050$, $\lambda_2=-5.05038\times10^{-3}$.

∴ $cond(A)_2=\frac{198.0050}{5.05038\times10^{-3}}=39205.96\approx 39206$.

34. 证明:

$\mathrm{cond}(AB)=\|(AB)^{-1}\|\,\|AB\|$

(由相容性) $\le \|A^{-1}\|\,\|B^{-1}\|\,\|A\|\,\|B\| = (\|A^{-1}\|\,\|A\|)(\|B^{-1}\|\,\|B\|) = \mathrm{cond}(A)\,\mathrm{cond}(B)$.

证毕!

习题第8章

1. 解:(1) $\because 5>2+1,\ 4>1+2,\ 10>2+3$, $\therefore$ 系数矩阵严格对角占优,由定理知Jacobi法、Gauss-Seidel法均收敛.

(2) ① Jacobi:

$$\begin{cases} x_1^{(k+1)} = -\frac{2}{5}x_2^{(k)} - \frac{1}{5}x_3^{(k)} - \frac{12}{5}, \\ x_2^{(k+1)} = \frac{1}{4}x_1^{(k)} - \frac{1}{2}x_3^{(k)} + 5, \\ x_3^{(k+1)} = -\frac{1}{5}x_1^{(k)} - \frac{3}{10}x_2^{(k)} + \frac{3}{10}. \end{cases}$$

取 $X^{(0)}=(0,0,0)^T$.

利用计算机求解,得: $X^{(18)}=(-3.9999964, 2.9999739, 1.9999999)^T$,

此时迭代18次, $\|X^{(18)}-X^{(17)}\|_\infty \approx 0.41\times10^{-4} < 10^{-4}$.

② G-S:

$$\begin{cases} x_1^{(k+1)} = -\frac{2}{5}x_2^{(k)} - \frac{1}{5}x_3^{(k)} - \frac{12}{5} \\ x_2^{(k+1)} = \frac{1}{4}x_1^{(k+1)} - \frac{1}{2}x_3^{(k)} + 5 \\ x_3^{(k+1)} = -\frac{1}{5}x_1^{(k+1)} - \frac{3}{10}x_2^{(k+1)} + \frac{3}{10}, \end{cases}$$

取 $X^{(0)}=(0,0,0)^T$.

利用计算机求解,得: $X^{(8)}=(-4.000036, 2.999985, 2.000003)^T$,

此时迭代8次, $\|X^{(8)}-X^{(7)}\|_\infty \approx 0.41\times10^{-4} < 10^{-4}$.

5. (1) ① Jacobi: $B=D^{-1}(L+U)=\begin{bmatrix} 0 & -0.4 & -0.4 \\ -0.4 & 0 & -0.8 \\ -0.4 & 0.8 & 0 \end{bmatrix}$.

$|\lambda E-B|=(\lambda-0.8)(\lambda^2+0.8\lambda-0.32)$. $\rho(B)=\lambda_{max}\approx1.0928>1$, $\therefore$ 不收敛.

② G-S: $G=(D-L)^{-1}U=\begin{bmatrix} 0 & -0.4 & -0.4 \\ 0 & 0.16 & -0.64 \\ 0 & 0.032 & 0.672 \end{bmatrix}$. $\|G\|_\infty=0.8$.

$\therefore \rho(G)\le\|G\|_\infty=0.8<1$, 由定理知: G-S 收敛.

(2) ① Jacobi: $B=D^{-1}(L+U)=\begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$. $|\lambda E-B|=\lambda^3$, $\rho(B)=0<1$. $\therefore$ 收敛.

② G-S: ~~$|\lambda E$~~ $G=(D-L)^{-1}U=\begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$. $|\lambda E-G|=\lambda(\lambda-2)^2$. $\rho(G)=2>1$, $\therefore$ 不收敛.

8. (1) $B=D^{-1}(L+U)=\begin{pmatrix} 0 & 0 & \frac14 & \frac14 \\ 0 & 0 & \frac14 & \frac14 \\ \frac14 & \frac14 & 0 & 0 \\ \frac14 & \frac14 & 0 & 0 \end{pmatrix}$. $|\lambda E-B|=$ ~~$\lambda^3(\lambda-0.25)=0$~~ $\lambda^4-0.25\lambda^2$. ~~$\therefore\lambda_1=\lambda_2=\lambda_3=0,\lambda_4=0.25$~~

$\therefore \lambda_1=\lambda_2=0, \lambda_3=-0.5, \lambda_4=0.5$. $\therefore \rho(B)=\lambda_{max}=\lambda_4=0.5$.

(2) $G=(D-L)^{-1}U=\begin{pmatrix} 0 & 0 & \frac14 & \frac14 \\ 0 & 0 & \frac14 & \frac14 \\ 0 & 0 & \frac18 & \frac18 \\ 0 & 0 & \frac18 & \frac18 \end{pmatrix}$.

$|\lambda E-G|=\lambda^3(\lambda-0.25)$. $\therefore \lambda_1=\lambda_2=\lambda_3=0, \lambda_4=0.25$.

$\therefore \rho(G)=\lambda_{max}=\lambda_4=0.25$.

(3) $\because \rho(B)=0.5<1, \rho(G)=0.25<1, 0.25<0.5$,

$\therefore$ Jacobi法与Gauss-Seidel法均收敛,且G-S法收敛会更快一点.