05. Duality

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1 Lagrange dual problem

Lagrangian: standard form problem (not necessarily convex)

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, \;\; i = 1, 2, \ldots, m \\ & h_i(x) = 0, \;\; i = 1, 2, \ldots, p \end{array}$$

Variable $x \in \mathbf{R}^n$, domain $\mathcal{D}$, optimal value p^*

Lagrangian: $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \rightarrow \mathbf{R}$, with $\mathbf{dom}\, L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

Weighted sum of objective and constraint functions

λ_i is Lagrange multiplier associated with $f_i(x) \leq 0$

ν_i is Lagrange multiplier associated with $h_i(x) = 0$

1 Lagrange dual problem

Lagrange dual function: $g : \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$,

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \end{aligned}$$

g is concave, can be $-\infty$ for some λ, ν

Lower bound property: if $\lambda \succeq 0$, then $g(\lambda, \nu) \leq p^*$

proof.

if $\tilde{x}$ is feasible and $\lambda \succeq 0$, then

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = g(\lambda, \nu)$$

Minimizing over all feasible $\tilde{x}$ gives $p^* \geq g(\lambda, \nu)$

1 Lagrange dual problem

Least-norm solution of linear equations

$$\begin{array}{ll} \texttt{minimize} & x^T x \\ \texttt{subject to} & Ax = b \end{array}$$

dual function:

1. Lagrangian is $L(x, \nu) = x^T x + \nu^T (Ax - b)$
2. To minimize L over x, set gradient equal to zero:

$$\nabla_x L(x, \nu) = 2x + A^T \nu = 0 \Longrightarrow x = -(1/2) A^T \nu$$

Plug into L to obtain g:

$$g(\nu) = L((-1/2) A^T \nu, \nu) = -\frac{1}{4} \nu^T A A^T \nu - b^T \nu$$

A concave function of ν

Lower bound property: $p^* \geq -(1/4)\nu^T A A^T \nu - b^T \nu$ for all ν

1 Lagrange dual problem

Standard form LP

$$\begin{array}{ll} \texttt{minimize} & c^T x \\ \texttt{subject to} & Ax = b, x \succeq 0 \end{array}$$

1. Lagrangian is

$$\begin{aligned} L(x, \lambda, \nu) &= c^T x + \nu^T (Ax - b) - \lambda^T x \\ &= -b^T \nu + (c + A^T \nu - \lambda)^T x \end{aligned}$$

2. L is affine in x, hence

$$g(\lambda, \nu) = L(x, \lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

g is linear on affine domain $\{(\lambda, \nu) | A^T \nu - \lambda + c = 0\}$, hence concave

Lower bound property: $p^* \geq -b^T \nu$ for all ν if $A^T \nu + c \succeq 0$

1 Lagrange dual problem

Equality constrained norm minimization

$$\begin{array}{ll} \texttt{minimize} & \|x\| \\ \texttt{subject to} & Ax = b \end{array}$$

dual function:

$$g(\nu) = \inf_x (\|x\| - \nu^T Ax + b^T \nu) = \begin{cases} b^T\nu & \|A^T\nu\|_* \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

where $\|\nu\|_* = \sup_{\|u\|\leq 1} u^T \nu$ is dual norm of $\|\cdot\|$

proof. follows from $\inf_x(\|x\| - y^T x) = 0$ if $\|y\|_* \leq 1, -\infty$, otherwise

1. if $\|y\|_* \leq 1$, then $\|x\| - y^T x \geq 0$ for all x, with equality if $x = 0$
2. if $\|y\|_* > 1$, choose $x = tu$ where $\|u\| \leq 1$, $u^T y = \|y\|_* > 1$:

$$\|x\| - y^T x = t(\|u\| - \|y\|_*) \to -\infty \text{ as } t \to \infty$$

Lower bound property: $p^* \geq b^T \nu$ if $\|A^T \nu\|_* \leq 1$

1 Lagrange dual problem

Two-way partitioning

$$\begin{array}{ll} \texttt{minimize} & x^T W x \\ \texttt{subject to} & x_i^2 = 1, \ i = 1, \ldots, n \end{array}$$

1. A non-convex problem; feasible set contains 2^n discrete points
2. interpretation: partition $\{1, \ldots, n\}$ in two sets; W_{ij} is cost of assigning i, j to the same set; $-W_{ij}$ is cost of assigning to different sets

dual function:

$$g(\nu) = \inf_x \left(x^T W x + \sum_i \nu_i (x_i^2 - 1) \right) = \inf_x x^T (W + \mathbf{diag}(\nu)) x - \mathbf{1}^T \nu$$

$$= \begin{cases} -\mathbf{1}^T \nu, & W + \mathbf{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Lower bound property: $p^* \geq -\mathbf{1}^T \nu$ if $W + \mathbf{diag}(\nu) \succeq 0$

Example: $\nu = -\lambda_{\min}(W)\mathbf{1}$ gives bound $p^* \geq n\lambda_{\min}(W)$

1 Lagrange dual problem

Lagrange dual and conjugate function

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & Ax \preceq b, \;\; Cx = d \end{array}$$

dual function:

$$\begin{aligned} g(\nu) &= \inf_{x \in \mathbf{dom}\ f_0} \left(f_0(x) + (A^T\lambda + C^T\nu)^T x - b^T\lambda - d^T\nu \right) \\ &= -f_0^*(-A^T\lambda - C^T\nu) - b^T\lambda - d^T\nu \end{aligned}$$

1. Recall definition of conjugate $f^*(y) = \sup_{x \in \mathbf{dom}\ f}(y^T x - f(x))$
2. Simplifies derivation of dual if conjugate of f_0 is known

Example: entropy maximization

$$f_0(x) = \sum_{i=1}^n x_i \log x_i, \qquad f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

1 Lagrange dual problem

The dual problem

$$\begin{array}{ll}\texttt{maximize} & g(\lambda,\nu)\\ \texttt{subject to} & \lambda \succeq 0\end{array}$$

1. Finds best lower bound on p^*, obtained from Lagrange dual function
2. A convex optimization problem; optimal value denoted d^*
3. λ, ν are dual feasible if $\lambda \succeq 0$, $(\lambda, \nu) \in \mathbf{dom}\, g$
4. Often simplified by making implicit constraint $(\lambda, \nu) \in \mathbf{dom}\ g$ explicit

Example: standard form LP and its dual (page 5–5)

$$\begin{array}{ll}\texttt{minimize} & c^T x\\ \texttt{subject to} & Ax = b\\ & x \succeq 0\end{array} \qquad \begin{array}{ll}\texttt{maximize} & -b^T\nu\\ \texttt{subject to} & A^T\nu + c \succeq 0\end{array}$$

2 Weak and strong duality

Weak duality: $d^* \leq p^*$

1. Always holds(for convex and nonconvex problems)
2. Can be used to find nontrivial lower bounds for difficult problems for example, solving the SDP

$$\begin{array}{ll} \texttt{maximize} & -\mathbf{1}^T\nu \\ \texttt{subject to} & W + \mathbf{diag}(\nu) \succeq 0 \end{array}$$

Gives a lower bound for the two-way partitioning problem on page 5–7

Stong duality: $d^* = p^*$

1. does not hold in general
2. (usually) holds for convex problems
3. conditions that guarantee strong duality in convex problems are called **constraint qualifications**

2 Weak and strong duality

Slater' s constraint qualification

Strong duality holds for a convex problem

$$\begin{array}{ll} \texttt{maximize} & -f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, \ \ i = 1, \dots, m \\ & Ax = b \end{array}$$

if it is strictly feasible, i.e.,

$$\exists x \in \textbf{int } \mathcal{D}: \ f_i(x) < 0, \ \ i = 1, \dots, m, \ \ Ax = b$$

1. also guarantees that the dual optimum is attained (if $p^* > -\infty$)

2. can be sharpened: e.g., can replace **int** $\mathcal{D}$ with **relint** $\mathcal{D}$ (interior relative to affine hull); linear inequalities do not need to hold with strict inequality, . . .

3.there exist many other types of constraint qualifications

2 Weak and strong duality

Inequality form LP

Primal problem

$$\begin{array}{ll} \texttt{minimize} & c^T x \\ \texttt{subject to} & Ax \preceq b \end{array}$$

Dual function

$$g(\lambda) = \inf_x \left((c + A^T\lambda)^T x - b^T\lambda \right) = \begin{cases} -b^T\lambda & A^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \tag{1}$$

Dual problem

$$\begin{array}{ll} \texttt{maximize} & -b^T\lambda \\ \texttt{subject to} & A^T\lambda + c = 0, \ \lambda \succeq 0 \end{array}$$

1. from Slater' s condition: $p^* = d^*$ if $A\tilde{x} \prec b$ for some $\tilde{x}$
2. in fact, $p^* = d^*$ except when primal and dual are infeasible

2 Weak and strong duality

Quadratic program

Primal problem (assume $P \in \mathbf{S}^n_{++}$)

$$\begin{array}{ll} \texttt{minimize} & x^T P x \\ \texttt{subject to} & Ax \preceq b \end{array}$$

Dual function

$$g(\lambda) = \inf_x \left(x^T P x + \lambda^T (Ax - b) \right) = -\frac{1}{4}\lambda^T A P^{-1} A^T \lambda - b^T \lambda \quad (2)$$

Dual problem

$$\begin{array}{ll} \texttt{maximize} & -(1/4)\lambda^T A P^{-1} A^T \lambda - b^T \lambda \\ \texttt{subject to} & \lambda \succeq 0 \end{array}$$

1. from Slater's condition: $p^* = d^*$ if $A\tilde{x} \prec b$ for some $\tilde{x}$
2. in fact, $p^* = d^*$ always holds

2 Weak and strong duality

A nonconvex problem with strong duality
Primal problem

$$\begin{array}{ll} \texttt{minimize} & x^T A x + 2 b^T x \\ \texttt{subject to} & x^T x \leq 1 \end{array}$$

$A \not\succeq 0$, hence nonconvex
Dual function

$$g(\lambda) = \inf_x \left(x^T (A + \lambda I) x + 2 b^T x - \lambda \right) \tag{3}$$

1. unbounded below if $A + \lambda I \not\succeq 0$ or if $A + \lambda I \succeq 0$ and $b \notin \mathcal{R}(A + \lambda I)$
2. minimized by $x = -(A + \lambda I)^{\dagger} b$ otherwise: $g(\lambda) = -b^T (A + \lambda I)^{\dagger} b - \lambda$

Dual problem and equivalent SDP:

$$\begin{array}{ll} \texttt{minimize} & -b^T (A + \lambda I)^{\dagger} b - \lambda \\ \texttt{s.t.} & A + \lambda I \succeq 0 \\ & b \in \mathcal{R}(A + \lambda I) \end{array} \qquad \begin{array}{ll} \texttt{maximize} & -t - \lambda \\ \texttt{s.t.} & \begin{bmatrix} A + \lambda I & b \\ b^T & t \end{bmatrix} \succeq 0 \end{array}$$

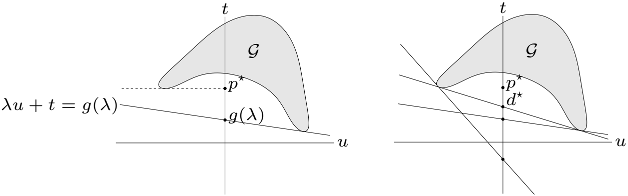
Strong duality although primal problem is not convex (not easy to show)

3 Geometric interpretation

For simplicity, consider problem with one constraint $f_1(x) \leq 0$

interpretation of dual function:

$$g(\lambda) = \inf_{(u,t)\in\mathcal{G}} (t + \lambda u), \quad \mathcal{G} = \{(f_1(x), f_0(x)) | x \in \mathcal{D}\}$$

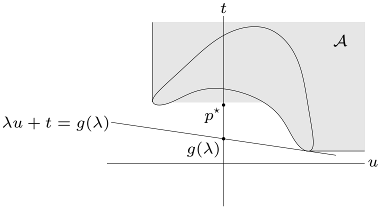


1. $\lambda u + t = g(\lambda)$ is (non-vertical) supporting hyperplane to $\mathcal{D}$
2. hyperplane intersects t-axis at $t = g(\lambda)$

3 Geometric interpretation

epigraph variation: same interpretation if $\mathcal{G}$ is replaced with

$$\mathcal{A} = \{(u,t) | f_1(x) \leq u, f_0(x) \leq t \text{ for some } x \in \mathcal{D}\}$$



Strong duality

1. Holds if there is a non-vertical supporting hyperplane to $\mathcal{A}$ at $(0, p^*)$
2. For convex problem, $\mathcal{A}$ is convex, hence has supp. hyperplane at $(0, p^*)$
3. Slater' s condition: if there exist $(\tilde{u}, \tilde{t}) \in \mathcal{A}$ with $\tilde{u} < 0$, then supporting hyperplanes at $(0, p^*)$ must be non-vertical

A proof of slater's condition

Assumptions: 1. contain non-empty interior point; 2. **rank** $A = p$. Define the following two exclusive sets:

$$\mathcal{A} = \{(u, v, t) | \exists x \in \mathcal{D}, f_i(x) \leq u_i, h_i(x) = v_i, f_0(x) \leq t\}$$
$$\mathcal{B} = \{(0, 0, s) \in \mathbf{R}^m \times \mathbf{R}^p \times \mathbf{R} | s < p^*\}$$

$\exists(\tilde{\lambda}, \tilde{\nu}, \mu) \neq 0$ and α, such that

$$(u, v, t) \in \mathcal{A} \Longrightarrow \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \geq \alpha$$
$$(u, v, t) \in \mathcal{B} \Longrightarrow \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \leq \alpha$$

We have 1. $\tilde{\lambda} \succeq 0$ and $\mu \geq 0$; 2. $\mu p^* \leq \alpha$

In other words, there exists feasible $x \in \mathcal{D}$

$$\sum \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T (Ax - b) + \mu f_0(x) \geq \alpha \geq \mu p^*$$

A proof of slater's condition (cont'd)

Case 1: $\mu > 0$
$L(x, \tilde{\lambda}/\mu, \tilde{\nu}/\mu) \geq p^*$, hence, $g(\tilde{\lambda}/\mu, \tilde{\nu}/\mu) \geq p^*$
$g(\tilde{\lambda}/\mu, \tilde{\nu}/\mu) = p^*$ proved

Case 2: $\mu = 0$

$$\sum \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T(Ax - b) \geq 0, \forall x \in \mathcal{D}$$

Assume $\tilde{x}$ satisfies the slater's conditions. Then

$$\sum \tilde{\lambda}_i f_i(\tilde{x}) \geq 0 \Longrightarrow \tilde{\lambda} = 0$$

As a result, $\tilde{\nu} \neq 0$. Consider the following facts: 1. $\tilde{\nu}^T(A\tilde{x} - b) = 0$, 2. $\tilde{x} \in \mathbf{int}\mathcal{D}$. There exists x, such that $\tilde{\nu}^T(Ax - b) < 0$ ($\mathbf{rank} A = p$, $\tilde{\nu}$ is p-dimension).

4 Optimality Conditions

Complementary slackness: assume strong duality holds, x^* is primal optimal, (λ^*, ν^*) is dual optimal

$$\begin{aligned} f_0(x^*) &= g(\lambda^*, \nu^*) = \inf \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right) \\ &\leq f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) \\ &\leq f_0(x^*) \end{aligned}$$

hence, the two inequalities hold with equality

1. x^* minimizes $L(x, \lambda^*, \nu^*)$
2. $\lambda_i^* f_i(x^*) = 0$ for $i = 1, \ldots, m$ (known as complementary slackness):

$$\lambda_i^* > 0 \Longrightarrow f_i(x^*) = 0, \;\; f_i(x^*) < 0 \Longrightarrow \lambda_i^* = 0$$

4 Optimality Conditions

Karush-Kuhn-Tucker(KKT) conditions

The following four conditions are called KKT conditions (for a problem with differentiable f_i, h_i):

1. primal constraints: $f_i(x) \leq 0, i = 1, \ldots, m$, $h_i(x) = 0$, $i = 1, \ldots, p$
2. dual constraints: $\lambda \succeq 0$
3. complementary slackness: $\lambda_i f_i(x) = 0, i = 1, \ldots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

from page 5–17: if strong duality holds and x, λ, ν are optimal, then they must satisfy the KKT conditions

4 Optimality Conditions

KKT conditions for convex problem

if $\tilde{x}, \tilde{\lambda}, \tilde{\mu}$ satisfy KKT for a convex problem, then they are optimal:

1. from complementary slackness: $f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
2. from 4th condition(and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$

if Slater’ s condition is satisfied:

x is optimal if and only if there exist λ, ν that satisfy KKT conditions

1. recall that Slater implies strong duality, and dual optimum is attained
2. generalizes optimality condition $\nabla f_0(x) = 0$ for unconstrained problem

5 Perturbation and sensitivity analysis

(unperturbed) optimization problem and its dual

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{array} \qquad \begin{array}{ll} \texttt{maximize} & g(\lambda, \nu) \\ \texttt{subject to} & \lambda \succeq 0 \\ & \end{array}$$

perturbed problem and its dual

$$\begin{array}{ll} \texttt{min.} & f_0(x) \\ \texttt{s.t.} & f_i(x) \leq u_i, i = 1, \ldots, m \\ & h_i(x) = v_i, i = 1, \ldots, p \end{array} \qquad \begin{array}{ll} \texttt{max.} & g(\lambda, \nu) - u^T\lambda - v^T\nu \\ \texttt{s.t.} & \lambda \succeq 0 \\ & \end{array}$$

1. x is primal variable; u, v are parameters
2. $p^*(u, v)$ is optimal value as a function of u, v
3. we are interested in information about $p\,(u, v)$ that we can obtain from the solution of the unperturbed problem and its dual

5 Perturbation and sensitivity analysis

global sensitivity result

assume strong duality holds for unperturbed problem, and that λ , ν are dual optimal for unperturbed problem

apply weak duality to perturbed problem:

$$\begin{aligned} p^*(u, v) &\geq g(\lambda^*, \nu^*) - u^T\lambda^* - v^T\nu^* \\ &= p^*(0, 0) - u^T\lambda^* - v^T\nu^* \end{aligned}$$

sensitivity interpretation

1. if λ_i^* large: p^* increases greatly if we tighten constraint i ($u_i < 0$)
2. if λ_i^* small: p^* does not decrease much if we loosen constraint i ($u_i > 0$)
3. if ν_i large and positive: p^* increases greatly if we take $v_i < 0$; if ν_i large and negative: p^* increases greatly if we take $v_i > 0$
4. if ν_i small and positive: p^* does not decrease much if we take $v_i > 0$; if ν_i^* small and negative: p^* does not decrease much if we take $v_i < 0$

5 Perturbation and sensitivity analysis

local sensitivity: if (in addition) $p(u, v)$ is differentiable at $(0, 0)$, then

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i}, \qquad \nu_i^* = -\frac{\partial p^*(0,0)}{\partial v_i}$$

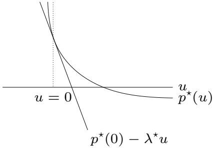
proof. (for $_i$): from global sensitivity result,

$$\frac{\partial p^*(0,0)}{\partial u_i} = \lim_{t \searrow 0} \frac{p^*(te_i, 0) - p^*(0,0)}{t} \geq -\lambda_i^*$$

$$\frac{\partial p^*(0,0)}{\partial u_i} = \lim_{t \nearrow 0} \frac{p^*(te_i, 0) - p^*(0,0)}{t} \leq -\lambda_i^*$$

hence, equality

$p(u)$ for a problem with one(inequality) constraint:



6 Examples

Duality and problem reformulations

1. equivalent formulations of a problem can lead to very different duals

2. reformulating the primal problem can be useful when the dual is difficult to derive, or uninteresting

common reformulations

1. introduce new variables and equality constraints

2. make explicit constraints implicit or vice-versa

3. transform objective or constraint functions e.g., replace $f_0(x)$ by $\phi(f_0(x))$ with ϕ convex, increasing

6 Examples

Introducing new variables and equality constraints

$$\texttt{minimize} \quad f_0(Ax+b)$$

1. dual function is constant: $g = \inf_x L(x) = \inf_x f_0(Ax+b) = p^*$
2. we have strong duality, but dual is quite useless

reformulated problem and its dual

$$\begin{array}{ll} \texttt{minimize} & f_0(y) \\ \texttt{subject to} & Ax+b-y=0 \end{array} \qquad \begin{array}{ll} \texttt{maximize} & b^T\nu - f_0^*(\nu) \\ \texttt{subject to} & A^T\nu = 0 \end{array}$$

dual function follows from

$$\begin{aligned} g(\nu) &= \inf_{x,y}(f_0(y) - \nu^T y + \nu^T Ax + b^T\nu) \\ &= \begin{cases} -f_0^*(\nu) + b^T\nu & A^T\nu = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

6 Examples

norm approximation problem: minimize $\|Ax - b\|$

$$\begin{array}{ll} \texttt{minimize} & \|y\| \\ \texttt{subject to} & y = Ax - b \end{array}$$

can look up conjugate of $\|\cdot\|$, or derive dual directly

$$\begin{aligned} g(\nu) &= \inf_{x,y}(\|y\| - \nu^T y - \nu^T Ax + b^T \nu) \\ &= \begin{cases} -b^T \nu + \inf_y(\|y\| + \nu^T y) & A^T \nu = 0 \\ -\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} -b^T \nu & A^T \nu = 0, \|\nu\|_* \leq 1 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

dual of norm approximation problem

$$\begin{array}{ll} \texttt{maximize} & b^T \nu \\ \texttt{subject to} & A^T \nu = 0, \|\nu\|_* \leq 1 \end{array}$$

6 Examples

Implicit constraints

LP with box constraints: primal and dual problem

`minimize`	$c^T x$	`maximize`	$-b^T\nu - \mathbf{1}^T\lambda_1 - \mathbf{1}^T\lambda_2$
`subject to`	$Ax + b = 0$	`subject to`	$c + A^T\nu + \lambda_1 - \lambda_2 = 0$
	$-\mathbf{1} \preceq x \preceq \mathbf{1}$		$\lambda_1 \succeq 0, \lambda_2 \succeq 0$

reformulation with box constraints made implicit

$$\begin{array}{ll} \texttt{minimize} & f_0(x) = \begin{cases} c^T x & -\mathbf{1} \preceq x \preceq \mathbf{1} \\ \infty & \text{otherwise} \end{cases} \\ \texttt{subject to} & Ax = b \end{array}$$

dual function

$$\begin{aligned} g(\nu) &= \inf_{-\mathbf{1} \preceq x \preceq \mathbf{1}} (c^T x + \nu^T(Ax - b)) \\ &= -b^T\nu - \|A^T\nu + c\|_1 \end{aligned}$$

dual problem: maximize $-b^T\nu - \|A^T\nu + c\|_1$