221900180 田永铭 计算方法作业 11

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P199 第七章

13. 追赶法：

解：$A=\begin{pmatrix}2&-1&0&0&0\\-1&2&-1&0&0\\0&-1&2&-1&0\\0&0&-1&2&-1\\0&0&0&-1&2\end{pmatrix}$，$b=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$.　　$Ax=b$，$LUx=b$，$Ly=b$，$Ux=y$.

①令 $A=LU$，则：$\alpha_1=b_1=2$，$\beta_1=\frac{c_1}{\alpha_1}=\frac{-1}{2}=-\frac{1}{2}$；

$\alpha_2=b_2-a_2\beta_1=2+1\times(-\frac{1}{2})=\frac{3}{2}$，$\beta_2=\frac{c_2}{b_2-a_2\beta_1}=\frac{-1}{2+1\times(-\frac{1}{2})}=-\frac{2}{3}$，

$\alpha_3=b_3-a_3\beta_2=2+1\times(-\frac{2}{3})=\frac{4}{3}$，$\beta_3=\frac{c_3}{\alpha_3}=\frac{-1}{\frac{4}{3}}=-\frac{3}{4}$，

$\alpha_4=b_4-a_4\beta_3=2+1\times(-\frac{3}{4})=\frac{5}{4}$，$\beta_4=\frac{c_4}{\alpha_4}=\frac{-1}{\frac{5}{4}}=-\frac{4}{5}$.

②解 $Ly=b$：$y_1=\frac{1}{2}$，$y_2=\frac{0+1\times\frac{1}{2}}{\alpha_2}=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}$，$y_3=\frac{0+1\times\frac{1}{3}}{\alpha_3}=\frac{\frac{1}{3}}{\frac{4}{3}}=\frac{1}{4}$，$y_4=\frac{1}{5}$. $y_5=\frac{1}{6}$.

③解 $Ux=y$：$\cancel{x_4=y_4=\frac{1}{5}}$ $x_5=y_5=\frac{1}{6}$，$x_4=y_4-\beta_4x_5=\frac{1}{5}+\frac{4}{5}\times\frac{1}{6}=\frac{1}{3}$，$x_3=\frac{1}{2}$，$x_2=\frac{2}{3}$，$x_1=\frac{5}{6}$

~~综上~~：解得：$x=\begin{pmatrix}\frac{5}{6}\\\frac{2}{3}\\\frac{1}{2}\\\frac{1}{3}\\\frac{1}{6}\end{pmatrix}$.

15. LU分解：

解：①假设A能分解，则 $A=LU=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}1&2&3\\2&4&1\\4&6&7\end{bmatrix}$.

由公式：$u_{11}=1$，$u_{12}=2$，$u_{13}=3$，$l_{21}=2$ $l_{31}=4$，$u_{22}=0$.

但 $a_{32}=l_{32}u_{22}+l_{31}u_{12}=0+4\times2=8$ 与 $a_{32}=6$ 矛盾了.

∴A不能LU分解.

又∵ $|A|\neq0$，∴A非奇异，∴∃ P，s.t. $PA=LU$. 事实上换1,2行即可.

∴ A不能分解，但换行后可以.

②假设 $B=LU$，则 $\begin{bmatrix}1&1&1\\2&2&1\\3&3&1\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$.

则 $u_{11}=u_{12}=u_{13}=1$，$l_{21}=2$，$l_{31}=3$，$u_{22}=0$.

因 $l_{31}u_{12}+l_{32}u_{22}=3$，∴ $3=3+l_{32}u_{22}$. ∴ l_{32} 可为任意实数.

∴B可分解，且分解方式不唯一

③ C的各阶顺序主子式分别为 1，1，1. 均不为0.

∴定理保证了C能分解，且分解唯一.

18. 求A的各范数.

解：① 行范数：$\|A\|_\infty = \max\{|0.6|+|0.5|, |0.1|+|0.3|\} = 1.1$.

② 列范数：$\|A\|_1 = \max\{|0.6|+|0.1|, |0.5|+|0.3|\} = 0.8$.

③ 2-范数：$A^TA = \begin{bmatrix} 0.6 & 0.1 \\ 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} 0.6 & 0.6 \\ 0.1 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.37 & 0.33 \\ 0.33 & 0.34 \end{bmatrix}$.

$$|\lambda E - A^TA| = \begin{vmatrix} \lambda-0.37 & -0.33 \\ -0.33 & \lambda-0.34 \end{vmatrix} = \lambda^2 - 0.71\lambda + 0.0169 = 0.$$

$\therefore \lambda_1 = 0.6853, \lambda_2 = 0.02466$.

$\therefore \lambda_{max}(A^TA) = 0.685$.

$\therefore \|A\|_2 = \sqrt{0.685} \approx 0.827$（中途若全保留则为0.825）.

④ F-范数：$\|A\|_F = \sqrt{0.6^2+0.5^2+0.1^2+0.3^2} = 0.8426$.

补充题：

1. Doolittle分解 $\begin{bmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 5 \\ -5 \end{bmatrix}$.

解：令 $A=LU$，$L = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}$，$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ & u_{22} & u_{23} & u_{24} \\ & & u_{33} & u_{34} \\ & & & u_{44} \end{bmatrix}$.

① 利用Doolittle分解公式得分解后的L、U矩阵~~结~~合起来形如：

$$\begin{bmatrix} 6 & 2 & 1 & -1 \\ \frac{1}{3} & \frac{10}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{5} & \frac{37}{10} & -\frac{9}{10} \\ -\frac{1}{6} & \frac{1}{10} & -\frac{9}{37} & \frac{191}{74} \end{bmatrix}. \quad \therefore A = LU = \begin{bmatrix} 1 & & & \\ \frac{1}{3} & 1 & & \\ \frac{1}{6} & \frac{1}{5} & 1 & \\ -\frac{1}{6} & \frac{1}{10} & -\frac{9}{37} & 1 \end{bmatrix}\begin{bmatrix} 6 & 2 & 1 & -1 \\ & \frac{10}{3} & \frac{2}{3} & \frac{1}{3} \\ & & \frac{37}{10} & -\frac{9}{10} \\ & & & \frac{191}{74} \end{bmatrix}.$$

$A=LU$，$LUx=b$，$Ux=y$，$Ly=b$.

② 解 $Ly=b$：$\begin{bmatrix} 1 & & & \\ \frac{1}{3} & 1 & & \\ \frac{1}{6} & \frac{1}{5} & 1 & \\ -\frac{1}{6} & \frac{1}{10} & -\frac{9}{37} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 5 \\ -5 \end{bmatrix}$，$\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ \frac{23}{5} \\ -\frac{191}{74} \end{bmatrix}$.

③ 解 $Ux=y$：$\begin{bmatrix} 6 & 2 & 1 & -1 \\ & \frac{10}{3} & \frac{2}{3} & \frac{1}{3} \\ & & \frac{37}{10} & -\frac{9}{10} \\ & & & \frac{191}{74} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ \frac{23}{5} \\ -\frac{191}{74} \end{bmatrix}$，得：$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

综上：解为 $X = [1 \quad -1 \quad 1 \quad -1]^T$.

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2. Cholesky分解 $\underbrace{\begin{bmatrix} 3 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 0 & 12 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}}_{b}$.

解: ① 首先, $A^T = A$, 且 $A \succ 0$ (正定). ∴ 可Cholesky分解.

令 $A = LL^T = \begin{bmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ & l_{22} & l_{32} \\ & & l_{33} \end{bmatrix}$.

$\therefore l_{11} = a_{11}^{\frac{1}{2}} = \sqrt{3}$. $l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}\right) / l_{jj}$. $l_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2\right)^{\frac{1}{2}}$.

$\therefore L = \begin{bmatrix} \sqrt{3} & & \\ \frac{2}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \\ \sqrt{3} & -\sqrt{6} & \sqrt{3} \end{bmatrix}$. $\therefore A = LL^T = \begin{bmatrix} \sqrt{3} & & \\ \frac{2}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \\ \sqrt{3} & -\sqrt{6} & \sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \sqrt{3} \\ & \frac{\sqrt{2}}{\sqrt{3}} & -\sqrt{6} \\ & & \sqrt{3} \end{bmatrix}$.

$Ax = b$, $\therefore LL^T x = b$. $L^T x = y$, $Ly = b$.

② 解 $Ly = b$: $\begin{bmatrix} \sqrt{3} & & \\ \frac{2}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \\ \sqrt{3} & -\sqrt{6} & \sqrt{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$, $\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$.

③ 解 $L^T x = y$: $\begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \sqrt{3} \\ & \frac{\sqrt{2}}{\sqrt{3}} & -\sqrt{6} \\ & & \sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$, $\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$.

∴ 解为 $x = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}^T$.

3. 追赶法. $\underbrace{\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 \\ 8 \\ 8 \\ 6 \end{bmatrix}}_{b}$.

解: 令 $A = LU$.

① $\alpha_1 = b_1 = 1$, $\beta_1 = \frac{c_1}{\alpha_1} = \frac{2}{1} = 2$;

$\alpha_2 = b_2 - a_2\beta_1 = 1 - 2\times 2 = -3$, $\beta_2 = \frac{c_2}{\alpha_2} = \frac{1}{-3} = -\frac{1}{3}$;

$\alpha_3 = b_3 - a_3\beta_2 = 2 - 1\times(-\frac{1}{3}) = \frac{7}{3}$, $\beta_3 = \frac{c_3}{\alpha_3} = \frac{1}{\frac{7}{3}} = \frac{3}{7}$;

$\alpha_4 = b_4 - a_4\beta_3 = 2 - 1\times\frac{3}{7} = \frac{11}{7}$. $\cancel{\beta_4 = \frac{c_4}{\alpha_4}}$

$\therefore A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 1 & \frac{7}{3} & 0 \\ 0 & 0 & 1 & \frac{11}{7} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$. $LUx = b$, $Ly = b$, $Ux = y$.

② 解 $Ly = b$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 1 & \frac{7}{3} & 0 \\ 0 & 0 & 1 & \frac{11}{7} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 8 \\ 6 \end{bmatrix}$, $\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ \frac{4}{3} \\ \frac{20}{7} \\ 2 \end{bmatrix}$.

③ 解 $Ux=y$:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ \frac{4}{3} \\ \frac{20}{7} \\ 2 \end{bmatrix}.$$

$$\therefore \cancel{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{19}{6} \\ \frac{17}{12} \\ 2 \\ 2 \end{bmatrix}}. \qquad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

∴ 解为 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$.