2023.11.16 思考题 n维正态分布的性质 221900180 田永铭

Theorem ~~0.28~~ 0.29

证明：设 $X=(X_1,X_2\cdots X_n)^T$，$Y=(Y_1,Y_2\cdots Y_m)^T$. $\begin{pmatrix}X\\Y\end{pmatrix}\sim N\left(\begin{pmatrix}\mu_x\\\mu_y\end{pmatrix},\begin{pmatrix}\Sigma_{xx}&\Sigma_{xy}\\\Sigma_{yx}&\Sigma_{yy}\end{pmatrix}\right)$

① X边缘分布为 $X\sim N(\mu_x,\Sigma_{xx})$

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证明：由打洞法知：

$$\begin{bmatrix}I_n&0\\-\Sigma_{yx}\Sigma_{xx}^{-1}&I_m\end{bmatrix}\begin{bmatrix}\Sigma_{xx}&\Sigma_{xy}\\\Sigma_{yx}&\Sigma_{yy}\end{bmatrix}\begin{bmatrix}I_n&-\Sigma_{xx}^{-1}\Sigma_{xy}\\0&I_m\end{bmatrix}=\begin{bmatrix}\Sigma_{xx}&0\\0&\Sigma_{yy}-\Sigma_{yx}^T\Sigma_{xx}^{-1}\Sigma_{yx}\end{bmatrix}.$$

取 $A=\begin{bmatrix}I_n&0\\-\Sigma_{yx}\Sigma_{xx}^{-1}&I_m\end{bmatrix}$，~~这里~~ 作线性变换 $Z=A\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}X\\-\Sigma_{yx}\Sigma_{xx}^{-1}X+Y\end{bmatrix}$.

$A\Sigma A^T$ 则 由PPT结论：$Z\sim N(A\mu,A^T\Sigma A)$.

又 $\because A^T\Sigma A=\Lambda(\Sigma_{xx},\Sigma_{yy}-\Sigma_{yx}^T\Sigma_{xx}^{-1}\Sigma_{yx})$.

$\therefore X$ 与 $-\Sigma_{yx}\Sigma_{xx}^{-1}X+Y$ 独立，而 $\boxed{\text{直接可}}$ 看出：$X\sim N(\mu_x,\Sigma_{xx})$.

$\therefore$ 得证.

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② X,Y 独立 $\Leftrightarrow \Sigma_{xy}=0$.

证明：1° $\Leftarrow$ 若 $\Sigma_{xy}=0$，则 $f(X,Y)=\dfrac{1}{(2\pi)^{\frac{m+n}{2}}|\Sigma|^{\frac12}}\exp\left(-\frac12\begin{pmatrix}X-\mu_x\\Y-\mu_Y\end{pmatrix}^T\begin{bmatrix}\Sigma_{xx}&0\\0&\Sigma_{yy}\end{bmatrix}^{-1}\begin{pmatrix}X-\mu_x\\Y-\mu_y\end{pmatrix}\right)$

$$=\frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma_{xx}|^{\frac12}}\exp\left(-\frac12(X-\mu_x)^T\Sigma_{xx}^{-1}(X-\mu_x)\right)$$

$$\cdot\frac{1}{(2\pi)^{\frac{m}{2}}|\Sigma_{yy}|^{\frac12}}\exp\left(-\frac12(Y-\mu_Y)^T\Sigma_{yy}^{-1}(Y-\mu_Y)\right)$$

$=f_X(x)\cdot f_Y(y)$. $\therefore X,Y$ 独立.

2° $\Rightarrow$ 若 X,Y 独立，则 $f(X,Y)=\dfrac{1}{(2\pi)^{\frac{m+n}{2}}|\Sigma|^{\frac12}}\exp\left(-\frac12\begin{bmatrix}X-\mu_x\\Y-\mu_Y\end{bmatrix}^T\begin{bmatrix}\Sigma_{xx}&\Sigma_{xy}\\\Sigma_{yx}&\Sigma_{yy}\end{bmatrix}^{-1}\begin{bmatrix}X-\mu_x\\Y-\mu_Y\end{bmatrix}\right)$

$$=f_X(x)\cdot f_Y(y)=\frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma_{xx}|^{\frac12}}\exp\left(-\frac12(X-\mu_x)^T\Sigma_{xx}^{-1}(X-\mu_x)\right)$$

$$\cdot\frac{1}{(2\pi)^{\frac{m}{2}}|\Sigma_{yy}|^{\frac12}}\exp\left(-\frac12(Y-\mu_y)^T\Sigma_{yy}^{-1}(Y-\mu_y)\right)$$

$\therefore\Rightarrow\Sigma_{xy}=\Sigma_{yx}=0$. $\therefore$ 得证.

③ 求 $Y=y$ 条件下 X 的分布：$X|Y=y \sim N(\mu_x+\Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y), \Sigma_{xx}-\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})$.

证明：已知 $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_{m+n}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix}\right)$.

先有分块求逆公式：$\begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}^{-1} = \Sigma^{-1} = \begin{bmatrix} X^{-1} & -X^{-1}\Sigma_{xy}\Sigma_{yy}^{-1} \\ -\Sigma_{yy}^{-1}\Sigma_{yx}X^{-1} & \Sigma_{yy}^{-1}+\Sigma_{yy}^{-1}\Sigma_{yx}X^{-1}\Sigma_{xy}\Sigma_{yy}^{-1} \end{bmatrix}$

其中：$X=\Sigma_{xx}-\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}$.（可由线代知识"打洞法"证明）

$f(X,Y)=f(X|Y)\cdot f(Y)$

构造线性变换：$Z=\begin{bmatrix} X-\Sigma_{xy}\Sigma_{yy}^{-1}Y \\ Y \end{bmatrix}=AX$. 解之得：$A=\begin{bmatrix} E & -\Sigma_{xy}\Sigma_{yy}^{-1} \\ O & E \end{bmatrix}$.

由PPT结论：则 $Z \sim N(A\mu, A\Sigma A^T)$.

欲证：$X-\Sigma_{xy}\Sigma_{yy}^{-1}Y$ 与 Y 独立.

证明：$A\Sigma A^T=\begin{bmatrix} E & -\Sigma_{xy}\Sigma_{yy}^{-1} \\ O & E \end{bmatrix}\begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}\begin{bmatrix} E & O \\ -\Sigma_{xy}\Sigma_{yy}^{-1} & E \end{bmatrix}=\begin{bmatrix} X & O \\ O & \Sigma_{yy} \end{bmatrix}$.

X 定义同上. $\therefore A\Sigma A^T=\Lambda$，这表明 $X-\Sigma_{xy}\Sigma_{yy}^{-1}Y$ 与 Y 独立.

设 Z 密度函数为 $g(X-\Sigma_{xy}\Sigma_{yy}^{-1}Y, Y)=g_1(X-\Sigma_{xy}\Sigma_{yy}^{-1}Y)\cdot g_2(Y)=g_1(X-\Sigma_{xy}\Sigma_{yy}^{-1}Y)\cdot f_2(Y)$.

$Z=AX$，$\therefore |J|=\begin{vmatrix} Z_1'x & Z_1'Y \\ Z_2'x & Z_2'Y \end{vmatrix}=\begin{bmatrix} 1 & -\Sigma_{xy}\Sigma_{yy}^{-1} \\ 0 & 1 \end{bmatrix}=1$.

$\therefore f(X,Y)=g(AX)\cdot 1=g_1(X-\Sigma_{xy}\Sigma_{yy}^{-1}Y)\cdot f_2(Y)$.

$\therefore f(X|Y)=\dfrac{f_{1,2}(X,Y)}{f_2(Y)}=g_1(X-\Sigma_{xy}\Sigma_{yy}^{-1}Y)$

$$=\frac{1}{(2\pi)^{\frac{n}{2}}|X|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}(X-(\mu_x+\Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y)))^T X^{-1}(X-(\mu_x+\Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y)))\right]$$

即 $X|Y=y \sim N(\mu_x+\Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y), \Sigma_{xx}-\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})$.

（补证：$g_1(x)$ 很好求：$\sim N(\mu_x-\mu_y\Sigma_{xy}\Sigma_{yy}^{-1}, X)$.

$\therefore g_1(X-\Sigma_{xy}\Sigma_{yy}^{-1}Y)$ 代入后有关均值一项即：

$X-\Sigma_{xy}\Sigma_{yy}^{-1}Y-\mu_x+\mu_y\Sigma_{xy}\Sigma_{yy}^{-1}$

$=X-(\mu_x+\Sigma_{xy}\Sigma_{yy}^{-1}(Y-\mu_y))$. ∴得证）. 证毕