A Comprehensive Review on Quadratic Optimization

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Abstract: Recently, the research of quadratic programming problem, a kind of important problems which plays a vital role in non-linear programming problems, is in full swing. There are many new developments in this area, and a comprehensive review article is highly needed. So this article gives an overview of quadratic optimization problems in recent years, pointing out the hot spots, existing problems and future directions with respect to this field.

Keywords: optimization; quadratic programming; comprehensive review; development

1 Introduction

Quadratic programming problem is widely used in mathematics, computer science, finance and so on(see Matilda Nordman's work[1] and Liyang Wang's work[2]). Recently, there is a growing amount of literature on quadratic programming problems, exploring new models, algorithms and applications. However, the review papers in this field are too old, failing to satisfy the need of the rapid development of quadratic programming. So this review paper aims to provide a concise overview on the recent researches on quadratic optimization, showing what work has been done, explaining the modern research methods and indicating possible future directions in this field.

Specifically, the rest of this paper is organized as follows. Section 2 will show the contributions of the article. Section 3 talks about the recent researches on quadratic programming. Section 4 is the core of the article, discussing the research hotspots, existing problems and future directions in the field of quadratic optimization. Section 5 is a simple conclusion.

2 Contributions

There is an imbalance between the rapid growth of quadratic programming research and the lack of relevant review papers. A review paper is being needed to summarize the work already done and give out the future directions.

Unlike the existing review papers, which talk about the literatures on quadratic programming many years ago, my review paper focuses on the literatures in this field just in recent 10 years. What's better, I will make statistics and create charts, giving out useful conclusions which have not been shown in many review papers.

3 Overview and analysis

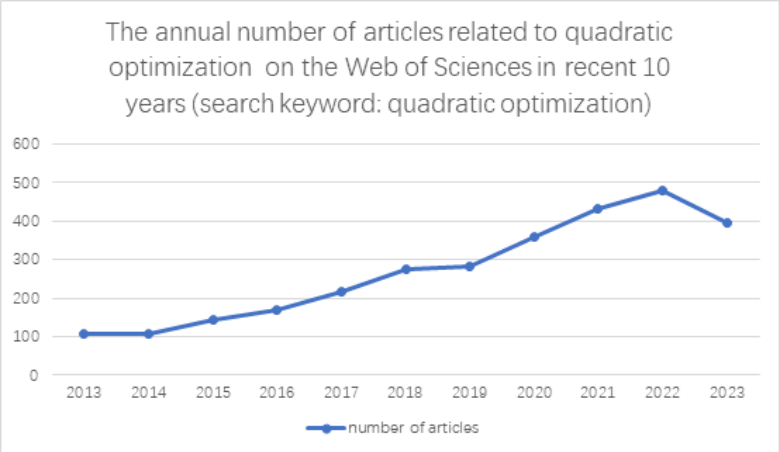
3.1 The development of the research on quadratic optimization problem

As quadratic programming is very important in many fields, many scholars have carried out continuous research on its related theories and methods. For quadratic programming problems constrained by 0,1, Lu and Guo[3] gave a new convex quadratic deformation, which could enable continuous relaxation bounds to be obtained. For general quadratic programming problems, Thoai[4] used the Lagrange duality problem to give the lower bound of the problem. There were also researches on KKT conditions and quadratic programming problems with all kinds of constraints. And in recent years, Zhu, Zhao and Liu[5], by means of differential flow, have presented a Lagrange dual method for solving concave quadratic programming problems with box constrains. Park and O'Leary[6] have proposed a polynomial complexity algorithm for solving semi-definite optimization problems. In 2021, Immanuel M. Bomze[7] made great efforts on the robust quadratic optimization. We can see that an increasing number of research have been made in the field of quadratic optimization recently.

3.2 The research of quadratic optimization is becoming more and more hot and important

I made statistics on the researches related to quadratic optimization and the result was like this.

Figure 1:



Intuitively, the overall number of articles in the field of quadratic optimization is increasing year by year.

From Newton to Lagrange, Gauss, great brains have laid a solid foundation in the study of quadratic optimization problems. As scientific and technological advancements have unfolded, modern scientists continue to follow in their footsteps, making significant contributions to our understanding and resolution of quadratic optimization problems, such as Stephen Boyd, Robert Vanderbei, John N. Tsitsiklis, in contemporary times. By reading the articles they wrote, we can see that the research of quadratic optimization is becoming more and more hot and important. It has various useful applications, attracting scientists to make efforts.

3.3 The recent articles tend to delve into the specific fields and applications of quadratic optimization

Comparing the articles published during the recent years with those published earlier in this field, we can find that the recent articles prefer to delve into the specific fields and applications of quadratic optimization, rather than shallowly talk about convex quadratic optimization and non-convex quadratic optimization, or

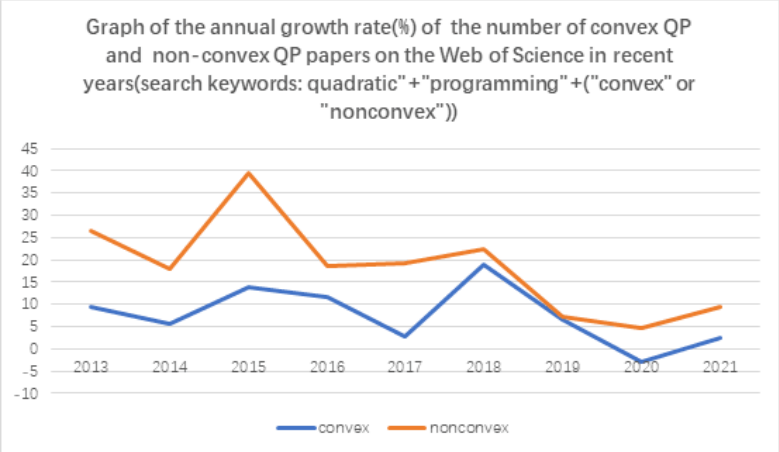
big models like QP, QCQP, SOCP. They mostly focus on a subtle and concrete model, push forward a useful algorithm, and explore applications in specific areas, like what Tao Rena and Peng Zhao[8] did. Though harder,it is this kind of researches that can really make contributions in a certain field.

4 Discussion and synthesis

4.1 Research hotspots

4.1.1 From convex to non-convex

Figure 2:



As Figure-2 shows, in recent years, the annual growth rate of the number of papers about non-convex QP is always higher than that of convex QP.

In recent years, people have conducted in-depth and extensive research on the problem of quadratic convex programming, and there are many related algorithms and solutions. However, Non-convex quadratic program with quadratic constraints is worthy of study. On the one hand, this is because the kind of problems have many applications in practical problems, such as, heat exchanger

engineering design, financial optimization, image processing, management science, etc.[9] On the other hand, most of the quadratic programming questions extracted from real life are not convex and should be calculated. And it seems that scientists have realized this point and have made effort to study non-convex quadratic programming recently.

4.1.2 Exploring ways to study NP-Hard problems

In general sense, quadratic programming problems are NP-Hard, but it doesn't mean that we are helpless about this. Sungwoo Park[6] presented an infeasible primal-dual interior point method for semidefinite quadratic optimization problems, making use of constraint reduction. He shows that the algorithm is globally convergent and has polynomial complexity. To tackle the NP-Hard problem (which is called as uni-modular quadratic programming (UQP)), Mojtaba Soltanalian[10] devised and studied several computational approaches that were really useful. In 2022, Samuel Burer[11] studied semidefinite programming (SDP) relaxations for the NP-hard problem of globally optimizing a quadratic function over the Stiefel manifold. He introduced a strengthened relaxation based on two recent ideas in the literatures. Besides, more ways to study NP-Hard problems are now being explored.

4.1.3 Exploring concrete quadratic optimization methods for specific problems

There are several ways to solve QP problems, and so the choice of the method depends of the characteristic of the problem. There is subclass of QP, for example: continuous QP, discrete QP, stochastic QP, continuous and discrete control variables (QPCD) and etc.[12] There may not exists a one-size-fit-all algorithm for all quadratic optimization problems, so exploring concrete quadratic optimiza-

tion methods for specific problems is highly needed. A successful example is that Fahdzyana[13] studied sequential quadratic programming and refined its application to optimizing strategy for integrated system design. The research is detailed, and highly successful. There are more concrete practical problems lacking suitable models and algorithms. For years ahead, such work need to be done continuously.

4.2 Existing problems and future directions

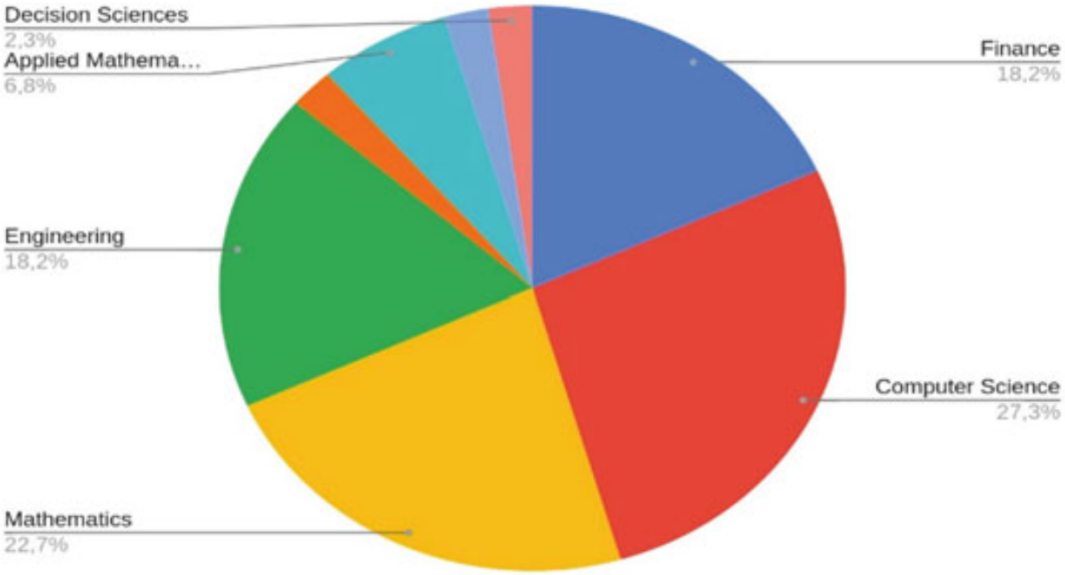
4.2.1 Research on quasi-convex QP is lacking

In 2023, Yagi[12] searched for articles on quasi-convex quadratic programming, but very few information was found in this field. Therefore, it represents a great research opportunity for researchers. He also discovered that there are not efficient algorithms to solve quasi-convex QP and therefore, it may be a field to be studied in the future. I did the same job as Yagi and found the same result. Stephen Boyd once said, "the watershed of optimization problems is not linear and non-linear, but convex and non-convex." Non-convex problems are hard to study, but quasi-convex problems, a special case of them, are relatively easy to study and has application value in various aspects. So quasi-convex problems need to be study in the future.

4.2.2 The application of quadratic optimization in some fields is waiting to be explored

Thanks to Yagi, who made a statistical chart about research areas of literatures that study quadratic optimization, We can get an intuitive sense of what scientists have done in different fields. It looks like this:

Figure 3: Applications and research areas of literatures that study quadratic optimization



From Figure 3, we can conclude that computer science, engineering and mathematics are major components of these studies. However, the applications of quadratic programming may not be limited to these. Nordman[1] creatively used quadratic programming to explore healthy and climate-friendly diets for Danish adults. Rashid[14] optimized drug combinations against multiple myeloma using a quadratic phenotypic optimization platform. These are the potential of quadratic optimization. The application of quadratic optimization in more fields is waiting to be explored.

4.2.3 The research methods of quadratic optimization need innovation

The current state-of-the-art research methods in quadratic optimization have undoubtedly made significant contributions to various fields. However, as technology advances and problem complexities increase, there is a growing need for

innovative approaches in this domain. Traditional methods may face challenges in handling large-scale problems, incorporating real-world constraints, and adapting to dynamic scenarios. Therefore, the next frontier in quadratic optimization research lies in the exploration and development of innovative methodologies. One possible avenue for innovation is the integration of machine learning techniques into quadratic optimization algorithms. Will the artificial neural networks help? Can we use the quadratic optimization pruning algorithm to optimize just as raised in some articles? To achieve big breakthrough in this field, innovation is indispensable. Honestly, the future of quadratic programming lies mostly here.

5 Conclusions

This paper presents the research status in the field of quadratic optimization in recent years, pointing out the hotspots and giving the future directions. Hopefully, aiming to serve as a pivotal resource this article, this article can provide a concise yet insightful overview of recent research endeavours in quadratic optimization. By illuminating the intricacies of this crucial field, it endeavours to deepen our comprehension of quadratic optimization problems and pave the way for potential advancements in the future.

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