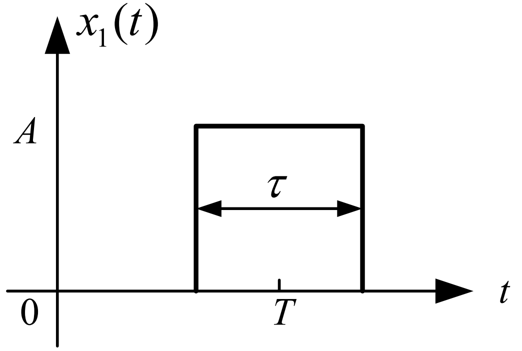
南京大学 人工智能学院
School of Artificial Intelligence, Nanjing University

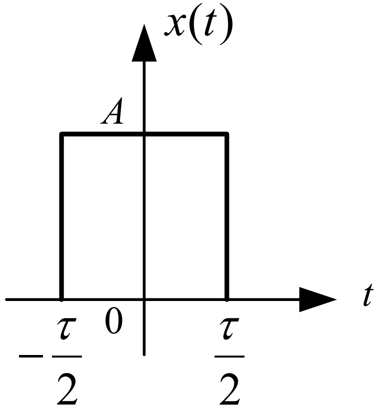
05 信号的傅里叶变换

傅里叶分析方法如何适用于一般的连续信号

傅里叶变换计算

求图示延时矩形脉冲信号$x_1(t)$的频谱函数$X_1(j\omega)$





傅里叶变换计算

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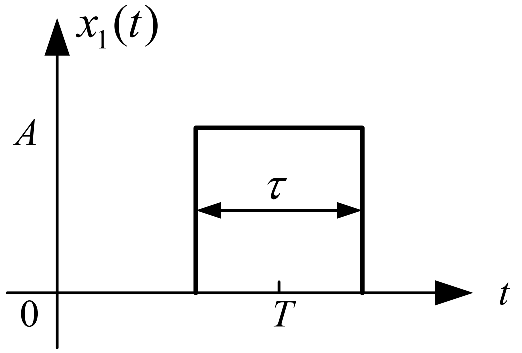
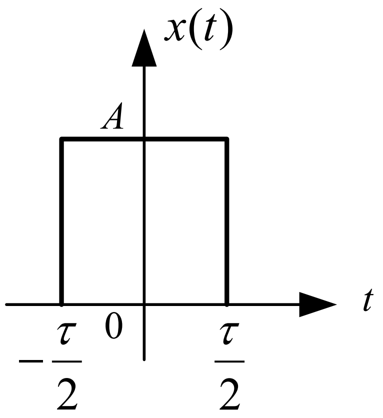




- 无延时且宽度为τ的矩形脉冲信号$x(t)$如右图，其对应的频谱函数为

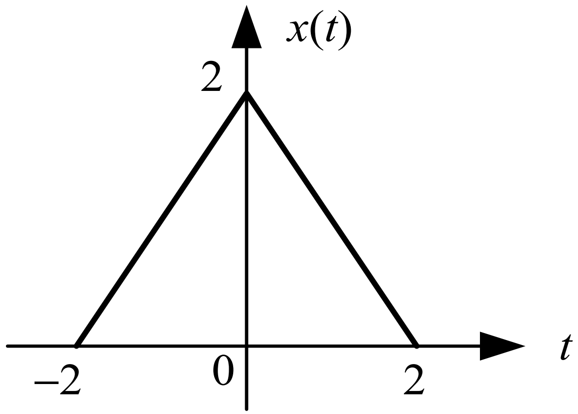
$$X(\mathrm{j}\omega) = A\tau \cdot \mathrm{Sa}(\frac{\omega\tau}{2})$$

- 因为$x_1(t) = x(t - T)$，由延时特性可得

$$X_1(\mathrm{j}\omega) = X(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega T} = A\tau \cdot \mathrm{Sa}(\frac{\omega\tau}{2})\mathrm{e}^{-\mathrm{j}\omega T}$$

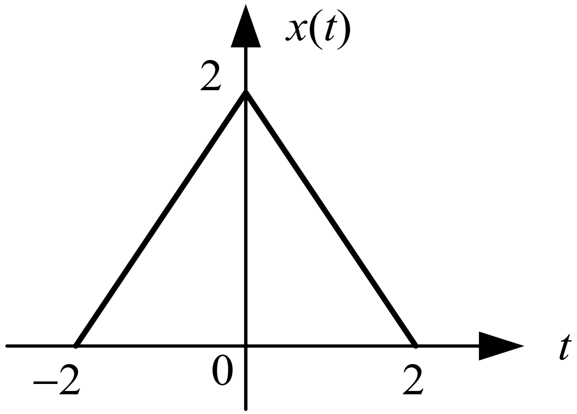
时域卷积特性

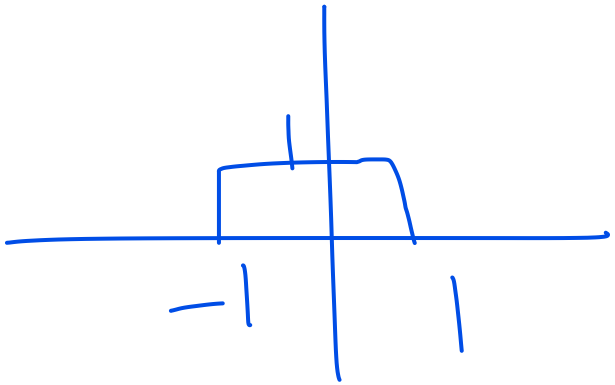
求如图所示信号的频谱



时域卷积特性

求如图所示信号的频谱





- 已知

$$x(t) = p_2(t) * p_2(t)$$

$$p_2(t) \longleftrightarrow 2\mathrm{Sa}(\omega)$$

- 由于

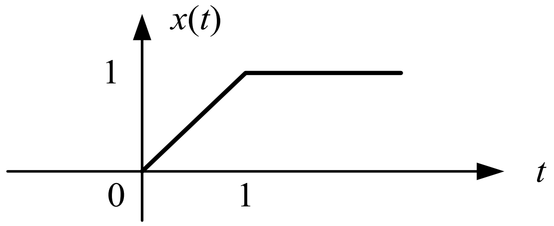
$$x_1(t) * x_2(t) \overset{\mathcal{F}}{\leftrightarrow} X_1(j\omega)X_2(j\omega)$$

- 因此

$$X(\mathrm{j}\omega) = 4\mathrm{Sa}^2(\omega)$$

时域积分特性

- 求图示信号$x(t)$的频谱函数



时域积分特性

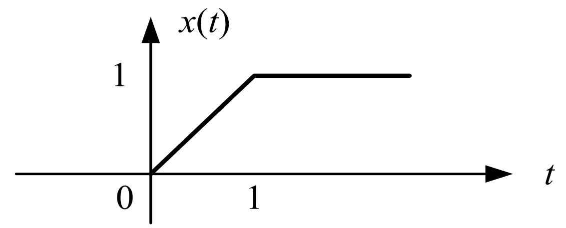
- 若$x(t)\overset{\mathcal{F}}{\leftrightarrow}X(j\omega)$，则$\int_{-\infty}^{t}x(\tau)\,d\tau\overset{\mathcal{F}}{\leftrightarrow}\frac{X(j\omega)}{j\omega}+\pi X(0)\delta(\omega)$
- 求图示信号$x(t)$的频谱函数

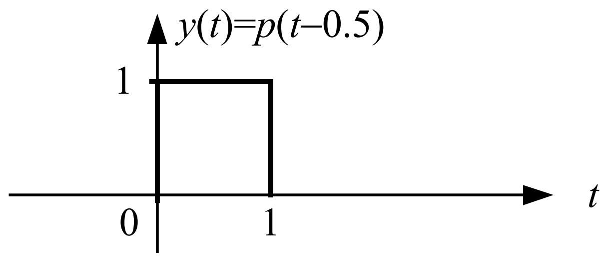
$$x(t)=\int_{-\infty}^{t}p(t-0.5)\,\mathrm{d}t=\int_{-\infty}^{t}y(t)\,\mathrm{d}t$$

  - 由于 $p(t-0.5)\overset{\mathcal{F}}{\leftrightarrow}Y(j\omega)=Sa(0.5\omega)e^{-j0.5\omega}$
  - 利用时域积分特性，可得

$$X(\mathrm{j}\omega)=\frac{1}{\mathrm{j}\omega}Y(\mathrm{j}\omega)+\pi Y(0)\delta(\omega)$$

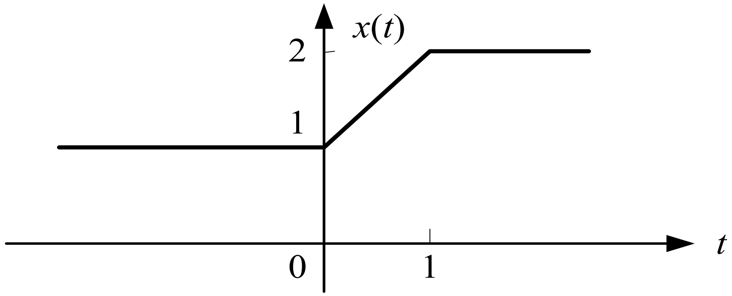
$$=\frac{1}{\mathrm{j}\omega}\mathrm{Sa}(0.5\omega)\mathrm{e}^{-\mathrm{j}0.5\omega}+\pi\delta(\omega)$$

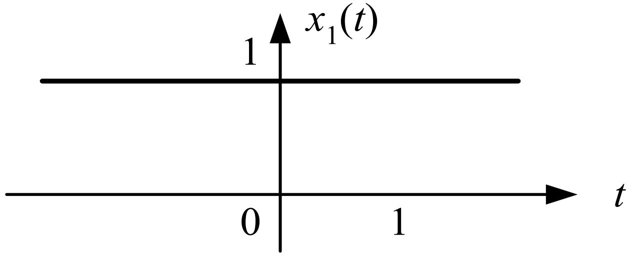


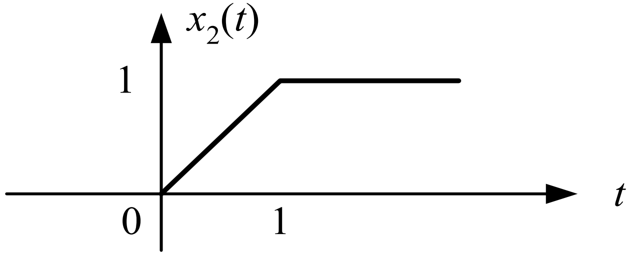


时域积分特性

- 若$x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$，则$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathcal{F}}{\leftrightarrow} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$
- 求图示信号$x(t)$的频谱函数
  - 将$x(t)$表示为$x_1(t) + x_2(t)$







  - 即

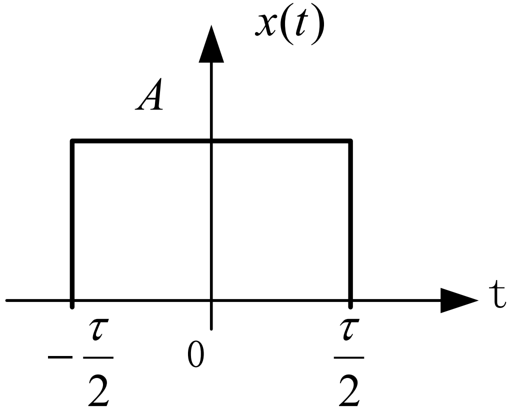
$$x(t) = 1 + \int_{-\infty}^{t} p(t-0.5)\,\mathrm{d}t$$

$$X(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega}\mathrm{Sa}(0.5\omega)\mathrm{e}^{-\mathrm{j}0.5\omega} + 3\pi\delta(\omega)$$

时域微分特性

定义！ 答近帘式 远乐！

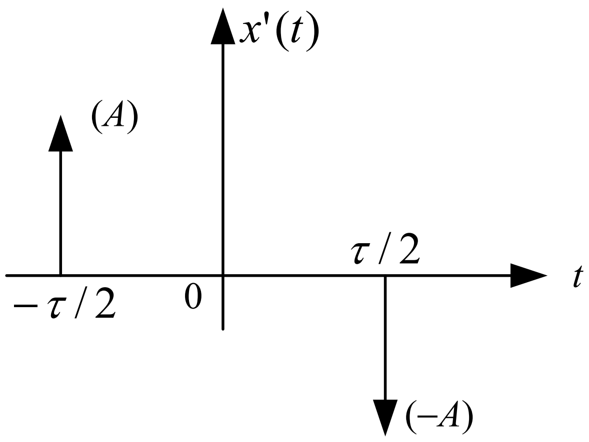
- 若$x(t)\overset{\mathcal{F}}{\leftrightarrow}X(j\omega)$，则$\frac{\mathrm{d}^n x(\mathrm{t})}{\mathrm{d}t^n}\overset{\mathcal{F}}{\leftrightarrow}(j\omega)^n\cdot X(j\omega)$



- 求矩形脉冲信号的频谱函数
  - 由于$x'(t)=A\delta(t+\frac{\tau}{2})-A\delta(t-\frac{\tau}{2})$
  - 由上式利用时域微分特性，得

$$\mathcal{F}[x'(t)]=A\mathrm{e}^{\mathrm{j}\omega\frac{\tau}{2}}-A\mathrm{e}^{-\mathrm{j}\omega\frac{\tau}{2}}=A\cdot 2\mathrm{j}\sin(\omega\frac{\tau}{2})$$

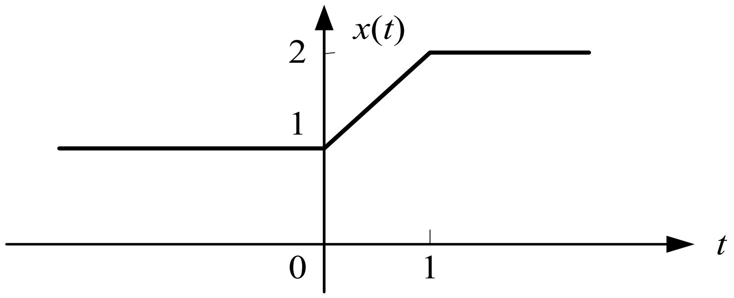
$$\mathcal{F}[x'(t)]=(\mathrm{j}\omega)X(\mathrm{j}\omega)=A\cdot 2\mathrm{j}\sin(\omega\frac{\tau}{2})$$



  - 因此$X(\mathrm{j}\omega)=\frac{2A}{\omega}\sin(\omega\frac{\tau}{2})=A\tau\mathrm{Sa}(\frac{\omega\tau}{2})$

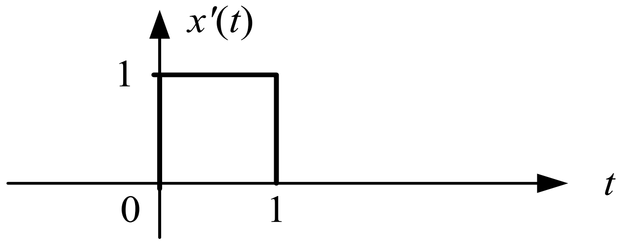
时域微分特性

- 若$x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$，则$\dfrac{\mathrm{d}^n x(\mathrm{t})}{\mathrm{d}t^n} \overset{\mathcal{F}}{\leftrightarrow} (j\omega)^n \cdot X(j\omega)$



- 求矩形脉冲信号的频谱函数
  - 由于

$$x'(t) = p(t-0.5) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{j\omega} Sa(0.5\omega) e^{-0.5\omega}$$



  - 由上式利用时域微分特性，得

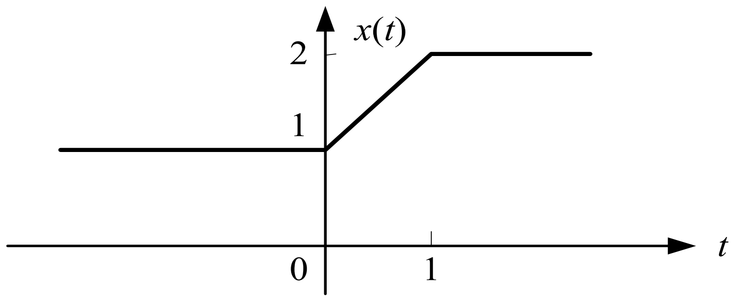
$$X(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega}\mathrm{Sa}(0.5\omega)\mathrm{e}^{-\mathrm{j}0.5\omega} \neq \frac{1}{\mathrm{j}\omega}\mathrm{Sa}(0.5\omega)\mathrm{e}^{-\mathrm{j}0.5\omega} + 3\pi\delta(\omega)$$

  - 信号的时域微分，使信号中的**直流分量丢失**

（修正的）时域微分特性

- 若$x'(t) = x_1(t)$，且$x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$， $x_1(t) \overset{\mathcal{F}}{\leftrightarrow} X_1(j\omega)$，则

$$X(\mathrm{j}\omega) = \pi[x(\infty) + x(-\infty)]\delta(\omega) + \frac{X_1(\mathrm{j}\omega)}{\mathrm{j}\omega}$$

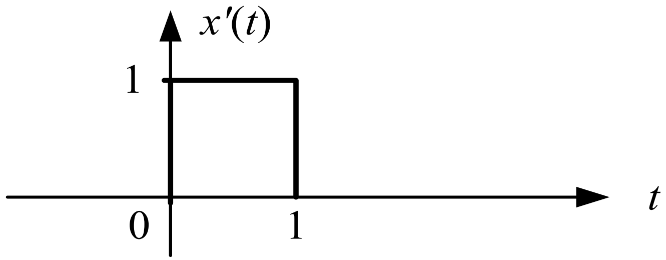


- 求矩形脉冲信号的频谱函数$x'(t) = p(t - 0.5) = x_1(t)$
  - 由于

$$x'(t) = p(t - 0.5) = x_1(t) \overset{\mathcal{F}}{\leftrightarrow} X_1(j\omega) = \frac{1}{j\omega} Sa(0.5\omega)e^{-0.5\omega}$$

  - 利用修正的微分特性，可得

$$X(\mathrm{j}\omega) = \pi[x(\infty) + x(-\infty)]\delta(\omega) + \frac{X_1(\mathrm{j}\omega)}{\mathrm{j}\omega} = 3\pi\delta(\omega) + \frac{1}{\mathrm{j}\omega}\mathrm{Sa}(\frac{\omega}{2})\mathrm{e}^{-\mathrm{j}\frac{\omega}{2}}$$



时域卷积特性

计算$y(t)=\int_{-2}^{t}\mathrm{e}^{-2\tau}\cdot\mathrm{e}^{-5(t-\tau)}\mathrm{d}\tau$，$(t>-2)$的频谱$Y(j\omega)$

时域卷积特性

计算$y(t)=\int_{-2}^{t}\mathrm{e}^{-2\tau}\cdot\mathrm{e}^{-5(t-\tau)}\mathrm{d}\tau$，$(t>-2)$的频谱$Y(j\omega)$

- 由于

$$y(t)=\int_{-2}^{t}\mathrm{e}^{-2\tau}\cdot\mathrm{e}^{-5(t-\tau)}\mathrm{d}\tau=\mathrm{e}^{-2t}u(t+2)*\mathrm{e}^{-5t}u(t)$$

- 利用卷积特性可得

$$Y(\mathrm{j}\omega)=\mathcal{F}[\mathrm{e}^{-2t}u(t+2)]\mathcal{F}[\mathrm{e}^{-5t}u(t)]$$

$$=\frac{\mathrm{e}^{4}\mathrm{e}^{\mathrm{j}2\omega}}{\mathrm{j}\omega+2}\frac{1}{\mathrm{j}\omega+5}=\frac{\mathrm{e}^{\mathrm{j}2\omega+4}}{(\mathrm{j}\omega)^{2}+7\mathrm{j}\omega+10}$$

非周期信号的能量谱密度

计算$\int_{-\infty}^{\infty}(\frac{\sin t}{t})^2\mathrm{d}t$

非周期信号的能量谱密度

计算$\int_{-\infty}^{\infty}(\frac{\sin t}{t})^2\mathrm{d}t$

- 由于

$$\mathcal{F}\{\frac{\sin t}{t}\}=\pi p_2(\omega)$$

- 根据Parseval能量守恒定律，可得

$$\int_{-\infty}^{\infty}(\frac{\sin t}{t})^2\,\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\pi p_2(\omega)|^2\mathrm{d}\omega=\frac{1}{2\pi}\int_{-1}^{1}\pi^2\mathrm{d}\omega=\pi$$