About Kostka numbers

1 What is Kostka Numbers

Kostka numbers は色々なところに出てくるので，色々定義がある．2 通りの定義を書いておく．

1.1 Semi-standard 盤の数としての定義

generalised Young tableau の定義から始めて，Kostka 数の定義をする．最後に重複度への応用も書いておく．

Definition shape λ, content $\mu = (\mu_1, \ldots, \mu_m)$ の **generalised Young tableau** とは, shape λ の Young 図形の各 box に，$i \in \mathbb{Z}_{>0}$ を μ_i 個書き込んだもの．特に，行に関して weakly increasing, 列に関して strictly increasing になっている generalised Young tableau を **semi-standard 盤**という．

Remark content μ は，partition である必要でなく，composition で十分．つまり，$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ はいらない．例えば，$\mu = (0, 2, 1)$ なども取れる．

Example $\lambda = (3, 2), \mu = (2, 2, 1)$．$\lambda = (3, 2)$ に $1, 1, 2, 2, 3$ を埋める．

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 3 & \\ \cline{1-2}\end{array},\ T_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array}$$

T_1 は単なる generalised Young tableau (列は strictly なので), T_2 は semi-standard.

Definition Kostka number $K_{\lambda\mu}$ とは，shapeλ, contentμ の semi-standard 盤の数.

Example $\lambda = (3, 2), \mu = (2, 2, 1)$ の半標準盤を列挙すると，

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \cline{1-2}\end{array}$$

なので，$K_{\lambda\mu} = 2$.

1.2 Schur 多項式の係数としての定義

Definition shape λ の semi-standard 盤全体を SSTAB と書く．このとき, Schur 多項式 $s_\lambda(x_1,\ldots,x_k)$ は次で定義される．

$$s_\lambda(x_1,\ldots,x_k) := \sum_{T\in SSTAB(\lambda,k)} x^T$$

Example $\lambda=(2,1)$ の半標準盤は,

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

で，これに対応して,

$$x_1^2x_2 \quad x_1^2x_3 \quad x_1x_2^2 \quad x_1x_2x_3 \quad x_1x_3x_2 \quad x_1x_3^2 \quad x_2^2x_3 \quad x_2x_3^2$$

が定まり,

$$s_\lambda(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

となる．

Proposition $s_\lambda = \sum_\beta K_{\lambda\beta}x^\beta$

1.3 応用

Proposition $M^\lambda = \bigoplus_{\lambda\lhd\mu} K_{\lambda\mu}S^\mu$

2 Survey

2.1 Wikipedia

In general, there are no nice formulas known for the Kostka numbers.

nice formula?

2.2 古典

それっぽいことは書いてなさそう，ただのメモ．無視していい．

2.2.1 sagan

$$K_{\lambda\mu}\neq 0 \Rightarrow \lambda \rhd \mu$$

2.2.2 Fulton

> **Exercise** If $r = (r_1, \cdots, r_n)$ and $c = (c_1, \cdots, c_n)$, show that the number of m by n matrices with nonnegative integers entries and row sums $r_1, \cdots, r_n$ and columns sums $c_1, \cdots, c_n$ is $\sum K_{\lambda r} K_{\lambda c}$, the sums taken over all partitions λ of $\sum r_i = \sum c_j$, where $K_{\lambda r}$ and $K_{\lambda c}$ are costka numbers. Show that the number of symmetric n by n matrices with nonnegative integer entries and row sums $r_1, \cdots, r_n$ is $\sum K_{\lambda r}$, the sum over all partitions *lambda* of $\sum r_i$

2.3 StackExchange

Kostka numbres についての survey は見つからない．組合せ論的表現論とか組合せ論の survey に少し書かれていたりするのかもしれない．現状 Kostka 数についての survey 的なものとしては，次の質問と回答がいい．回答者は UCLA の教員．リンクが豊富なので web で見たほうがいい．

Question Why is a general formula for Kostka numbers "unlikely" to exist?

In reference to Stanley's Enumerative Combinatorics Vol. 2: right after he has defined Kostka numbers (section 7.10), he mentions that it is unlikely that a general formula for $K_{\lambda\mu}$ exists, where $K_{\lambda\mu}$ is the number of semistandard Young tableaux of shape λ and type μ with $\lambda \vdash n$ and μ a weak composition of n. Why? In particular, is this an expression of something rigorous, and if so, what?

Answer This is a really good question, the kind of question I think about from time to time. The problem with this question is that it is so much imprecise, it is basically open ended. Here are some variations on the way to make question precise.

1) By a "general formula" you mean a product of some kind of factorials. This is rather uninteresting, since it's unclear what those factorials would be. Kostka numbers tend to be chaotic, so I am sure you can find relatively small partitions with annoying large prime factors. What do you do next? One can also ask about asymptotic results which don't allow this, in the flavor of de Bruijn (see Section 6). But again there are too many choices to consider, and none are really enlightening.

2) There is a formal notion of #P-completness, a computational complexity class, loosely corresponding to hard counting problems (see WP). It is known that Kostka numbers are #P-complete (see this paper). This means that computing Kostka numbers in full generality is just as hard as computing the number of 3-colorings in graphs. 3SAT solutions, etc.

3) Continuing with the theme "formula" as a polynomial algorithm. Such "formulas" do exist in special cases then. For example, if the number of rows in both partitions is fixed, Kostka numbers become the number of integer points in a finite dimensional polytope (see e.g. this nice presentation), which can be computed in polynomial time (see this book).

4) Alternatively, there is a rather weak notion of "formula" due to Wilf (see here, by subscription). Roughly, he asks for the algorithm which is asymptotically faster than trivial enumeration. But then one can use the "inverse Kostka numbers" which have their own combinatorial interpretation (see here), which are similar but perhaps slightly faster to compute. Since Wilf only asks for a little better than trivial bound, one can compute the whole matrix of Kostka numbers which has sub-exponential size $p(n)$, while Kostka numbers are exponential under mild conditions.

Hope this helps.

shareciteimprove this answer

`https://math.stackexchange.com/questions/17891-`
`/why-is-a-general-formula-for-kostka-numbers-unlikely-to-exist`

2.4 papers

2.4.1 A DETERMINANT-LIKE FORMULA FOR THE KOSTKA NUMBERS

これはそれっぽい，ちゃんと読んでないから全然わからない．2005.

> **Theorem** The number $K_{\alpha,\beta}$ of semistandard Young tableaux of shape α and of content β is given by $K_{\alpha,\beta} = \sum sgn(\sigma)\mu_\beta(\alpha - (k) + (\sigma(k))$.

2.4.2 AN ELEMENTARY METHOD FOR COMPUTING THE KOSTKA COEFFICIENTS

初等的な方法 (連立方程式・不等式を解く) で，いくつかの場合に具体的に計算し，公式を導出してる．各場合で同一の公式が出ているよう．ただ参考文献が少ない．Sagan と Fulton だけ．2010.

> Abstract. We present a simple and elementary method for com- puting the Kostka numbers. We use this method to give compact formulas for K π ,μ in some special cases.

2.4.3 On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients

"Thus, unless P = NP, which is widely disbelieved, there do not exist efficient algorithms that compute these numbers."

> Abstract. Kostka numbers and Littlewood-Richardson coefficients appear in combinatorics and representation theory. Interest in their computation stems from the fact that they are present in quantum mechanical computations since Wigner [15]. In

recent times, there have been a number of algorithms proposed to perform this task [1 - 3, 11, 12]. The issue of their computational complexity has received at-tention in the past, and was raised recently by E. Rassart in [11]. We prove that the problem of computing either quantity is #P-complete. Thus, unless P = NP, which is widely disbelieved, there do not exist efficient algorithms that compute these numbers.

2.4.4 Kostka Numbers and Littlewood-Richardson Coefficients: Distributed Computation

"Computing Kostka numbers and Littlewood-Richardson coefficients remains of great interest in combinatorics" $K_{N\lambda N\mu}$ に関する公式,hive を使った計算，maple 使用．perspective の節がある．

Abstract. Computing Kostka numbers and Littlewood-Richardson coefficients remains of great interest in combinatorics, and Brian G. Wybourne was among the first people to design software -SCHUR- for their computation. The efficiency of existing software -SCHUR, Stembridge package, LattE, Cochet's programs- is generally constrained by the lengths or weights of partitions. This work describes another method, based on the hives model, applying distributed computing tech- niques to the determination of generating polynomials for stretched Kostka numbers and stretched Littlewood-Richardson coefficients. This method can be used to "quickly" find such polynomials, with the help (of a predefined subset) of the available computers of the Local Area Network.

時期が分からないけど少なくとも 2005 年以降.

2.4.5 On a Formula for the Kostka Numbers

Abstract. From Kostant's multiplicity formula for general linear groups, one can derive a formula for the Kostka numbers. In this note we give a combinatorial proof of this formula.

2006. 上の 2.4.1 の別証明.

2.4.6 STRETCHED LITTLEWOOD-RICHARDSON AND COEFFICIENTS

$K_{N\lambda N\mu}$ に関する公式．$P(N) = K_{N\lambda N\mu}$ となる多項式 P を見つける問題は，特別な場合に限っては，示されている．conjecture も一つ乗ってる．

2.4.7 Polynomiality properties of the Kostka numbers and Littlewood-Richardson coefficients

StackExchange で"nice presentation"と紹介されているやつ，open problem も一つ書いている．ある分割の kostka 数は，格子点を数える問題に帰着するような話．sl の表現論が出てきたりする．面白そうだけどこれだけだとよくわからないとこが結構ある．この論文は Littlewood-Richardson coefficients の場合にだけ書かれている．hive という比較的新しい組合せ論の対象を使って求められるらしい．

2.4.8 COMBINATORIAL REPRESENTATION THEORY

組合せ論的表現論に関する survey. 1997 年で少し古いけど包括的.

2.4.9 The Ubiquitous Young Tableau

ヤング図形に関する survey. 1988 年，ヤング図形が登場する分野を網羅してる．リトルウッドリチャードソン係数は色々な場面で出て来る $\cdots$ みたいな文脈でよく参考文献にされている．

2.4.10 REPRESENTATION THEORY OF SYMMETRIC GROUPS AND RELATED HECKE ALGEBRAS

対称群の表現論に関連する話題の survey．2009 年，70 ページくらいの大作.