Seminar1

2019/1/10

1 Group Representations

1.1 Fundamental Concepts

G: group with identity ϵ. $|G| < \infty$
$\mathfrak{S}_n$: the **symmetric group** of order n.
$\pi \in \mathfrak{S}_n$: **permutation**

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

cyclic notation: $\pi^p(i) = i \Rightarrow \pi = (i, \pi(i), \pi^2(i), \ldots, \pi^{p-1}(i))$. In this case, π is called **p-cycle** or **cycle of length p**. **cycle type** or simply **type**, of π:

$$(1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$$

where m_k is the number of cycle of length k in π.

Example 1. $\pi = (1, 2, 3)(4)(5) \Rightarrow (1^2, 2^0, 3^1, 4^0, 5^0)$.

Proposition 2. $\pi \in \mathfrak{S}_n \Rightarrow \pi = \pi_1\pi_2 \cdots \pi_k$, where π_i is a cycle.

A 1-cycle of π is called a **fixedpoint**. Also an involution is a permutation such that $\pi^2 = \epsilon$.

Proposition 3. π is involution $\Longleftrightarrow$ π's cycle have length 1 or 2.

Another way to give the cycle type is as a partition. $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a **partiton** of n :

- $\sum_{i=1}^{k} \lambda_i = n$
- $\lambda_i \geq \lambda_{i+1}$

Example 4. $\pi = (1,2,3)(4)(5) \Rightarrow \lambda_\pi = (3,1,1)$.

In any group G, g and h are **conjugate** if

$$g = khk^{-1} \quad \exists k \in G$$

Also

$$K_g := \{h \in G \mid g = khk^{-1} \quad \exists k \in G\}.$$

Returning to $\mathfrak{S}_n$, we take $\pi = (i_1, i_2, \ldots, i_l) \cdots (i_m, i_{m+1}, \ldots, i_n)$ then the following is holds.

$$\sigma\pi\sigma^{-1} = (\sigma(i_1), \ldots, \sigma(i_l)) \cdots (\sigma(i_m), \ldots, \sigma(i_m)) \quad \forall \sigma \in \mathfrak{S}_n$$

Therefore

$$h \in K_g \iff \text{the type of } g = \text{the type of } h.$$

Now we can compute the size of conjugate class. **Centlizer**:

$$Z_g := \{h \in G \mid hgh^-1 = g\}$$

Proposition 5.

$$|K_g| = \frac{|G|}{|Z_g|}$$

Proposition 6. $\lambda = (1^{m_1}, \ldots, n^{m_n})$, $g \in G$, the type of $g = \lambda$. Then $|Z_g|$ depends only on λ i.e.,

$$z_\lambda := |Z_g| = 1^{m_1} m_1! \cdots n^{m_n} m_n!.$$

In the symmetric group,

$$k_\lambda = \frac{n!}{z_\lambda} = \frac{n!}{1^{m_1} m_1! \cdots n^{m_n} m_n!}$$

where $k_\lambda = |K_\lambda|$.

transposition: $\tau = (i, j)$. The transpositions generate $\mathfrak{S}_n$, in fact, $s_i := (i, i+1)$ generate the symmetric group. s_i is called **adjacent transpositions**.

Proposition 7. $\forall$ cicle is written as a product of transpositions.

$\pi = \pi_1 \pi_2 \cdots \pi_k \quad (\pi_i \text{is transposition})$

$$sgn\pi := (-1)^k$$

This is well-defined hom.

Proposition 8.

$$sgn(\pi\sigma) = sgn(\pi)sgn(\sigma)$$

1.2 Matrix Representation

full complex matrix algebra of degree d :

$$\mathrm{Mat}_d := \{d \times d \text{ matrix with entries in } \mathbb{C}\}$$

The **complex general linear group** of degree d :

$$\mathrm{GL}_d := \{X \in \mathrm{Mat}_d \mid \exists X^{-1}\}$$

Definition 9. A **matrix Representationation** of a group G is a group hom

$$X : G \to \mathrm{GL}_d$$

$\iff \forall X(g)$ satisy

- $X(\epsilon) = I$
- $X(gh) = X(g)X(h)$

Also $\deg X = d$.

Note that $X(g^{-1}) = X(g)^{-1}$.

Example 10. The **trivial representation** 1_G

$$g \mapsto 1$$

Example 11. C_n: cyclic group.

$$C_n := \{g, g^2, \ldots, g^n = \epsilon\}$$

If

$$g \mapsto c \in \mathbb{C}$$

Then $g^k \mapsto c^k$, in paticular $c^n = X(g^n) = X(\epsilon) = I$. If we take c such that $c^n = 1$, then a matrix representation of degree 1 is determined.

$C_4 := \{\epsilon, g, g^2, g^3\}$. $x^4 = 1 \Rightarrow x = 1, i, -i, -1$.

Another representation of C_4.

$$X(g) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

This is combination of above rep.

Example 12. $\pi \mapsto sgn\pi$ is the **sign representation**.

$\pi \mapsto (x_{ij})_{ij}$ is the **defining representation** where

$$x_{ij} = \begin{cases} 1 & \pi(i) = j \\ 0 \text{otherwize} \end{cases}$$

Also $X(\pi)$ is called a **permutation matrix**.

1.3 G-Modules and the Group Algebras

$V\colon$ vector space over $\mathbb{C}$, $\dim V < \infty$. **General linear group** of V

$$\mathrm{GL}(V) := \{f\colon V \to V \mid \exists f^{-1}\}$$

$\dim V = d \Rightarrow \mathrm{GL}(V) \cong \mathrm{GL}_d$ as group.

Definition 13.

$$\begin{aligned} V\colon G\text{-}\mathbf{module} &\overset{\text{def}}{\iff} \exists\rho\colon G \to \mathrm{GL}(V) hom \\ &\iff \exists \text{action of } G \text{ on } V \text{ with linarity} \end{aligned}$$

G-module $V = V$ **carries** a representation of G. And

$$\rho\colon G \to \mathrm{GL}(V) \iff X(g)$$

G acts on $S \Rightarrow \mathbb{C}S$ is G-module.

Definition 14. G acts on $S \Rightarrow \mathbb{C}S$ is called **permutation representation** associated with S. Also the element of S form a basis for $\mathbb{C}S$, they are called the **stadard basis**.

Example 15. $G = \mathfrak{S}_n, S = \{1, 2, \ldots, n\}$.

$$\mathbb{C}S := \{c_1 1 + c_2 2 + \cdots c_n n \mid c_i \in \mathbb{C}\}$$

$\pi \in \mathfrak{S}_n$,

$$\pi(\sum c_i i) = \sum c_i \pi(i)$$

select a basis and compute matrix ...

Example 16. the (left) **regular representation**. G acts on G. $G = \{g_1, \ldots, g_n\}$ then

$$\mathbb{C}G = \mathbb{C}[G] = \textbf{group algebra} \text{ of } G$$

C_4:

$$\mathbb{C}[C_4] = \{c_1\epsilon + c_2 g + c_3 g^2 + c_4 g^3 \mid c_i \in \mathbb{C}\}$$

take the standard basis,

$$X(g^2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

They are all permutation matrix and distinct. In general, the regular representation gives an embeding of G into the symmetric group on $|G|$ elements.

Note that G acts on $V \Rightarrow G$ acts on $\mathbb{C}[G]$. Then

$$\exists \text{rep of } G \iff \exists \text{rep of } \mathbb{C}[G]$$

Example 17. $G \geq H$ the (left) **coset representation** of G with respect to H. Let $g_1, \ldots, g_k$ be a transversal for H, namely

$$\mathcal{H} = \{g_1 H, \ldots, g_k H\}$$

is a complete set of disjoint left cosets for H in G.

$$\mathbb{C}\mathcal{H} = \{\sum c_i g_i H \mid c_i \in \mathbb{C}\}$$
$$g(\sum c_i g_i H) = \sum c_i (g g_i) H$$

$H = G \Rightarrow \mathbb{C}\mathcal{H} \cong 1_G$. $H = \{\epsilon\} \Rightarrow \mathbb{C}\mathcal{H} \cong \mathbb{C}[G]$.

$G = \mathfrak{S}_3, H = \{\epsilon, (2,3)\}$

$$\mathcal{H} = \{H, (1,2)H, (1,3)H\}$$

$$\mathbb{C}\mathcal{H} = \{c_1 H + c_2(1,2)H + c_3(1,3)H \mid c_i \in \mathbb{C}\}$$

$$X((1,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$