Semantic Theory
Week 4 – Type Theory

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First-order logic

First-order logic talks about:

- Individual objects
- Properties of and relations between individual objects
- Quantification over individual objects

Limitations of first-order logic

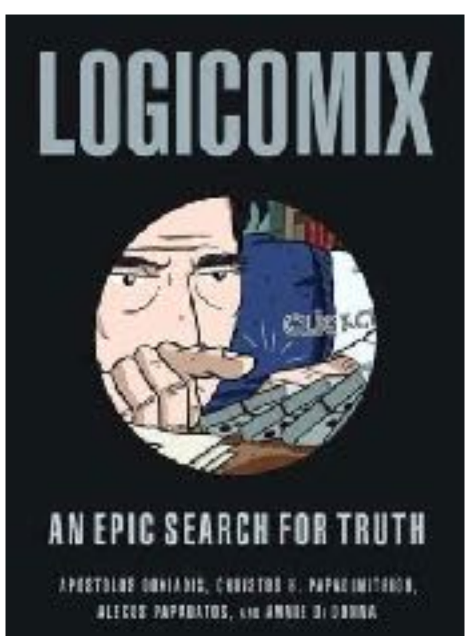
FOL is not expressive enough to capture all meanings that can be expressed by basic natural language expressions:

Jumbo is a <u>small</u> elephant.	(Predicate modifiers)
Blond is a <u>hair color.</u>	(Second-order predicates)
<u>Yesterday</u>, it rained.	(Non-logical sentence operators)
Bill and John have <u>the same</u> hair color.	(Higher-order quantification)

What logical system can we use to capture this diversity?

Bertrand Russell

Q: Does the barber shave himself?



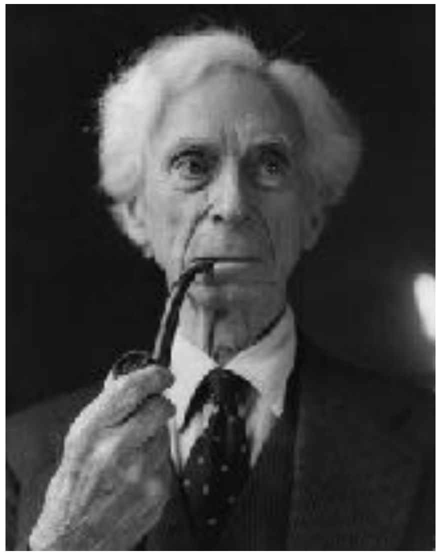
Russell's paradox

What if we extend the FOL interpretation of predicates, and interpret higher-order predicates as sets of sets of properties?

For every predicate P, we can define a set $\{x \mid P(x)\}$ containing all and only those entities for which P holds.

Then we can define a set $S = \{X \mid X \notin X\}$ representing the set of all sets that are not members of itself.

Q: does S belong to itself?



… we need a more restricted way of talking about *properties* and *relations between properties*!

Type Theory

Basic types:

- **e** – the type of individual terms (“entities”)
- **t** – the type of formulas (“truth-values”)

Complex types:

- If **σ**, **τ** are types, then ⟨**σ, τ**⟩ is a type
  (representing a functor expression that takes a **σ** type expression as argument and returns a type **τ** expression; sometimes written as: **(σ→ τ)**).

Types & Function Application

Types of first-order expressions:

- Individual constants (Luke, Death Star) : **e**
- One-place predicates (walk, jedi): ⟨**e, t**⟩
- Two-place predicates (fight, admire): ⟨**e,** ⟨**e, t**⟩⟩
- Three-place predicates (give, introduce): ⟨**e,** ⟨**e,** ⟨**e, t**⟩⟩⟩

Function application: Combining a functor of complex type with an appropriate argument, resulting in an expression of a less complex type: $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle(\boldsymbol{\alpha}) \mapsto \boldsymbol{\beta}$

- jedi'(luke') :: $\langle \mathbf{e}, \mathbf{t} \rangle(\mathbf{e}) \Longrightarrow \mathbf{t}$
- fight'(luke') :: $\langle \mathbf{e}, \langle \mathbf{e}, \mathbf{t} \rangle\rangle(\mathbf{e}) \Longrightarrow \langle \mathbf{e}, \mathbf{t} \rangle$

More examples of types

Types of higher-order expressions:

- Predicate modifiers (expensive, poor): ⟨⟨**e, t**⟩**,** ⟨**e, t**⟩⟩
- Second-order predicates (hair colour): ⟨⟨**e, t**⟩**, t**⟩
- Sentence operators (yesterday, possibly, unfortunately): ⟨**t, t**⟩
- Degree particles (very, too): ⟨⟨⟨**e, t**⟩**,** ⟨**e, t**⟩⟩**,** ⟨⟨**e, t**⟩**,** ⟨**e, t**⟩⟩⟩

Tip: If **σ**, **τ** are basic types, ⟨**σ, τ**⟩ can be abbreviated as **στ**. Thus, the type of predicate modifiers and second-order predicates can be more conveniently written as ⟨**et, et**⟩ and ⟨**et, t**⟩, respectively.

Type Theory — Vocabulary

Non-logical constants:

- For every type τ a (possibly empty) set of non-logical constants CON_τ (pairwise disjoint)

Variables:

- For every type τ an infinite set of variables VAR_τ (pairwise disjoint)

Logical symbols: $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $=$

Brackets: (,)

Type Theory — Syntax

For every type τ, the set of well-formed expressions WE_τ is defined as follows:

(i) $CON_\tau \subseteq WE_\tau$ and $VAR_\tau \subseteq WE_\tau$;

(ii) If $\alpha \in WE_{\langle\sigma, \tau\rangle}$, and $\beta \in WE_\sigma$, then $\alpha(\beta) \in WE_\tau$; (function application)

(iii) If A, B are in WE_t, then $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are in WE_t;

(iv) If A is in WE_t and x is a variable of arbitrary type, then $\forall xA$ and $\exists xA$ are in WE_t;

(v) If α, β are well-formed expressions of the same type, then $\alpha = \beta \in WE_t$;

(vi) Nothing else is a well-formed expression.

Function application

(ii) If $\alpha \in WE_{\langle\sigma,\tau\rangle}$, and $\beta \in WE_{\sigma}$, then $\alpha(\beta) \in WE_{\tau}$

"Luke is a talented jedi"

$$\dfrac{\text{luke'} :: e \qquad \dfrac{\text{jedi'} :: \langle e, t\rangle \qquad \text{talented'} :: \langle\langle e, t\rangle, \langle e, t\rangle\rangle}{\text{talented'(jedi')} :: \langle e, t\rangle}}{\text{talented'(jedi')(luke')} :: t}$$

Higher-order predicates

Higher-order quantification:

- *Leia has the same <u>hair colour</u> as Padmé*

$$\exists C\ (\underbrace{\text{hair\_colour}}_{\langle\langle e,t\rangle,t\rangle}(C) \wedge \underbrace{C}_{\langle e,t\rangle}(\underbrace{l'}_{e}) \wedge C(p'))$$

Higher-order equality:

- For p, q $\in$ CON_t, "p=q" expresses material equivalence: "p $\leftrightarrow$ q".
- For F, G $\in$ $CON_{\langle e,\ t\rangle}$, "F=G" expresses co-extensionality: "$\forall x(Fx \leftrightarrow Gx)$"
- For any formula ϕ of type *t*, ϕ=(x=x) is a representation of "ϕ is true".

Type inferencing: examples

1. Yoda$_e$ [is faster than Palpatine$_e$].

2. Yoda$_e$ [is much [faster than]] Palpatine$_e$.

3. [[Han Solo]$_e$ fights] [because [[the Dark Side]$_e$ is rising]].

4. Obi-Wan$_e$ [[told [Qui-Gon Jinn]$_e$] he will take [the Jedi-exam]$_e$].

5. Yoda$_e$ [[encouraged [Obi-Wan$_e$]] to take the exam].

6. Wookiee$_{\langle e,t \rangle}$ [[is a hairier [species] than] Ewok$_{\langle e,t \rangle}$].

Type Theory — Semantics [1]

Let **U** be a non-empty set of entities.

The domain of possible denotations $\mathbf{D_\tau}$ for every type $\boldsymbol{\tau}$ is given by:

- $D_e = U$
- $D_t = \{0,1\}$
- $D_{\langle\sigma,\tau\rangle}$ is the set of all functions from D_σ to D_τ

Expressions of type $\boldsymbol{\sigma}$ denote elements of $\mathbf{D_\sigma}$

Characteristic functions

Many natural language expressions have a type ⟨**σ, t**⟩

Expressions with type ⟨**σ, t**⟩ are functions mapping elements of type **σ** to truth values: {**0,1**}

Such functions with a range of {**0,1**} are called *characteristic functions*, because they uniquely specify a subset of their domain $\mathbf{D_\sigma}$

The characteristic function of set M in a domain U is the function F_M: U→{0,1} such that for all a ∈ U, $F_M(a) = 1$ iff a ∈ M.

NB: For first-order predicates, the FOL representation (using sets) and the type-theoretic representation (using characteristic functions) are equivalent.

Interpretation with characteristic functions: example

For M = ⟨U, V⟩, let U consist of five entities. For selected types, we have the following sets of possible denotations:

- $D_t = \{0,1\}$
- $D_e = U = \{e_1, e_2, e_3, e_4, e_5\}$
- $D_{<e,t>} = \left\{ \begin{bmatrix} e_1 \to 1 \\ e_2 \to 0 \\ e_3 \to 1 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix}, \begin{bmatrix} e_1 \to 1 \\ e_2 \to 1 \\ e_3 \to 0 \\ e_4 \to 1 \\ e_5 \to 1 \end{bmatrix}, \begin{bmatrix} e_1 \to 0 \\ e_2 \to 1 \\ e_3 \to 1 \\ e_4 \to 0 \\ e_5 \to 0 \end{bmatrix}, \ldots \right\}$

Alternative set notation: $D_{<e,t>} = \{\{e_1,e_3,e_5\},\{e_1,e_2,e_4,e_5\},\{e_2,e_3\},\ldots\}$

Type Theory — Semantics [2]

A model structure for a type theoretic language is a tuple $\mathbf{M} = \langle \mathbf{U}, \mathbf{V} \rangle$ such that:

- **U** is a non-empty domain of individuals
- **V** is an interpretation function, which assigns to every $\boldsymbol{\alpha} \in \mathbf{CON}_{\boldsymbol{\tau}}$ an element of $\mathbf{D}_{\boldsymbol{\tau}}$ (where $\boldsymbol{\tau}$ is an arbitrary type)

The variable assignment function g assigns to every typed variable $\mathbf{v} \in \mathbf{VAR}_{\boldsymbol{\tau}}$ an element of $\mathbf{D}_{\boldsymbol{\tau}}$

Type Theory — Interpretation

Given a model structure $M = \langle U, V \rangle$ and a variable assignment g:

$[\![\alpha]\!]^{M,g}$	$= V(\alpha)$	if α is a constant
	$= g(\alpha)$	if α is a variable
$[\![\alpha(\beta)]\!]^{M,g}$	$= [\![\alpha]\!]^{M,g}([\![\beta]\!]^{M,g})$	
$[\![\alpha = \beta]\!]^{M,g}$	$= 1$ iff	$[\![\alpha]\!]^{M,g} = [\![\beta]\!]^{M,g}$
$[\![\neg\phi]\!]^{M,g}$	$= 1$ iff	$[\![\phi]\!]^{M,g} = 0$
$[\![\phi \wedge \psi]\!]^{M,g}$	$= 1$ iff	$[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
$[\![\phi \vee \psi]\!]^{M,g}$	$= 1$ iff	$[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$

…

For any variable v of type σ:

$[\![\exists v\phi]\!]^{M,g}$	$= 1$ iff	there is a $d \in D_\sigma$ such that $[\![\phi]\!]^{M,g[v/d]} = 1$
$[\![\forall v\phi]\!]^{M,g}$	$= 1$ iff	for all $d \in D_\sigma$: $[\![\phi]\!]^{M,g[v/d]} = 1$

Interpretation: Example

Luke is a talented jedi

$$\dfrac{\text{luke'} :: e \qquad \dfrac{\text{jedi'} :: \langle e, t\rangle \qquad \text{talented'} :: \langle\langle e, t\rangle, \langle e, t\rangle\rangle}{\text{talented'(jedi')} :: \langle e, t\rangle}}{\text{talented'(jedi')(luke')} :: t}$$

$[\![\text{talented'(jedi')(luke')}]\!]^{M,g}$

$= [\![\text{talented'(jedi')}]\!]^{M,g} ([\![\text{luke'}]\!]^{M,g})$

$= [\![\text{talented'}]\!]^{M,g}([\![\text{jedi'}]\!]^{M,g}) ([\![\text{luke'}]\!]^{M,g})$

$= V_M(\text{talented'})(V_M(\text{jedi'}))(V_M(\text{luke'}))$

Interpretation: Example (cont.)

$[\![\textit{Luke is a talented jedi}]\!]^{M,g} = V_M(\text{talented'})(V_M(\text{jedi'}))(V_M(\text{luke'}))$

$V_M(\text{luke'}) = e_1 \ (\in \mathbf{D_e})$

$$V_M(\text{jedi'}) = \begin{bmatrix} e_1 \to 1 \\ e_2 \to 0 \\ e_3 \to 1 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix} (\in \mathbf{D}_{\langle \mathbf{e,t} \rangle}) \iff \{e_1, e_3, e_5\}$$

$$V_M(\text{talented'}) = \begin{bmatrix} \begin{bmatrix} e_1 \to 1 \\ e_2 \to 0 \\ e_3 \to 1 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix} \to \begin{bmatrix} e_1 \to 1 \\ e_2 \to 0 \\ e_3 \to 0 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix} \\ \begin{bmatrix} e_1 \to 0 \\ e_2 \to 0 \\ e_3 \to 1 \\ e_4 \to 1 \\ e_5 \to 1 \end{bmatrix} \to \begin{bmatrix} e_1 \to 0 \\ e_2 \to 0 \\ e_3 \to 0 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix} \\ \ldots \end{bmatrix} (\in \mathbf{D}_{\langle\langle \mathbf{e,t} \rangle \langle \mathbf{e,t} \rangle\rangle}) \iff \begin{bmatrix} \{e_1,e_3,e_5\} \to \{e_1,e_5\} \\ \{e_3,e_4,e_5\} \to \{e_5\} \\ \ldots \end{bmatrix}$$

$V_M(\text{talented'})(V_M(\text{jedi'}))$

$V_M(\text{talented'})(V_M(\text{jedi'}))(V_M(\text{luke'}))$

Defining the right model

Consider the following Model M:

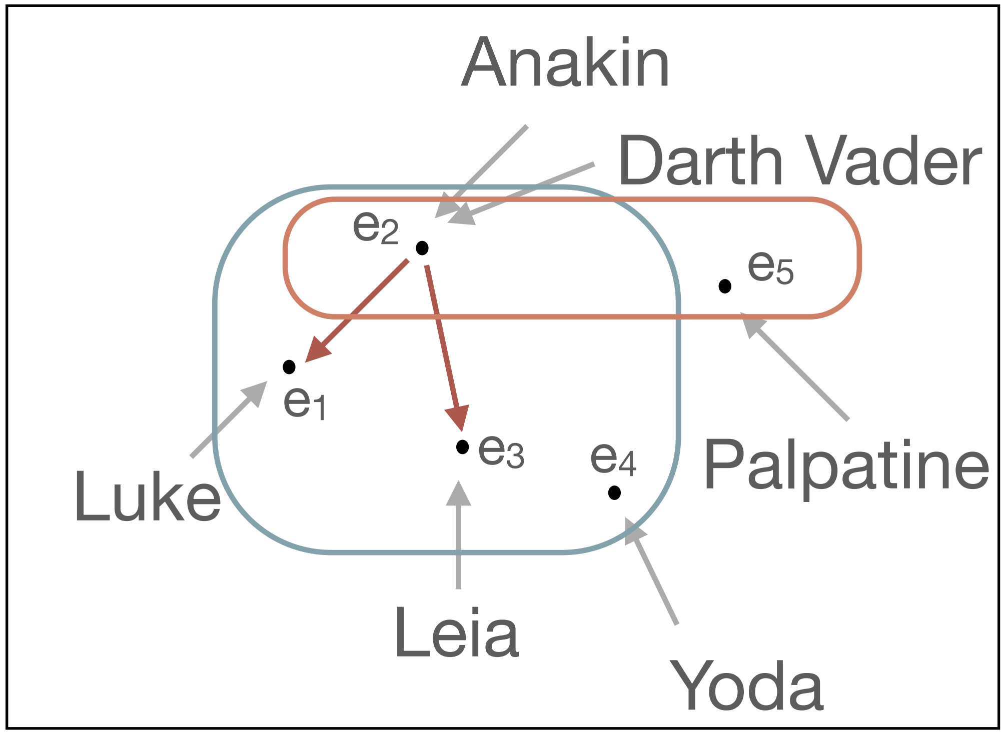
$D_e = U_M = \{e_1, e_2, e_3, e_4, e_5\}$

$V_M(\text{anakin'}_e) = V_M(\text{darth\_vader'}_e) = e_2$

$V_M(\text{jedi'}_{\langle e,t\rangle}) = \begin{bmatrix} e_1 \to 1 \\ e_2 \to 1 \\ e_3 \to 1 \\ e_4 \to 1 \\ e_5 \to 0 \end{bmatrix}$ $V_M(\text{dark\_sider'}_{\langle e,t\rangle}) = \begin{bmatrix} e_1 \to 0 \\ e_2 \to 1 \\ e_3 \to 0 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix}$

$$V_M(\text{powerful'}_{\langle\langle e,t\rangle\langle e,t\rangle\rangle}) = \begin{bmatrix} \begin{bmatrix} e_1 \to 1 \\ e_2 \to 1 \\ e_3 \to 1 \\ e_4 \to 1 \\ e_5 \to 0 \end{bmatrix} \to \begin{bmatrix} e_1 \to 0 \\ e_2 \to 1 \\ e_3 \to 0 \\ e_4 \to 1 \\ e_5 \to 0 \end{bmatrix} \\ \begin{bmatrix} e_1 \to 0 \\ e_2 \to 1 \\ e_3 \to 0 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix} \to \begin{bmatrix} e_1 \to 0 \\ e_2 \to 1 \\ e_3 \to 0 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix} \\ \ldots \end{bmatrix}$$

M:



--→ Powerful $X_{\langle e,t\rangle} \vDash X_{\langle e,t\rangle}$

Adjective classes & Meaning postulates

Some valid inferences in natural language:

- Bill is a poor piano player $\vDash$ Bill is a piano player
- Bill is a blond piano player $\vDash$ Bill is blond
- Bill is a former professor $\vDash$ Bill isn't a professor

➔ These entailments do not hold in type theory. Why?

Meaning postulates: restrictions on models which constrain the possible meaning of certain words

Adjective classes & Meaning postulates (cont.)

Restrictive or Subsective adjectives (“poor”)

- $[\![\text{poor N}]\!] \subseteq [\![\text{N}]\!]$
- Meaning postulate: $\forall G \forall x(\text{poor}(G)(x) \rightarrow G(x))$

Intersective adjectives (“blond”)

- $[\![\text{blond N}]\!] = [\![\text{blond}]\!] \cap [\![\text{N}]\!]$
- Meaning postlate: $\forall G \forall x(\text{blond}(G)(x) \rightarrow (\text{blond}^*(x) \wedge G(x))$
- NB: $\text{blond} \in WE\langle\langle e, t\rangle, \langle e, t\rangle\rangle \neq \text{blond}^* \in WE\langle e, t\rangle$

Privative adjectives (“former”)

- $[\![\text{former N}]\!] \cap [\![\text{N}]\!] = \varnothing$
- Meaning postlate: $\forall G \forall x(\text{former}(G)(x) \rightarrow \neg G(x))$

Background reading material

- Gamut: Logic, Language, and Meaning Vol II (Chapter 4)
- Winter: Elements of Formal Semantics (Chapter 3)
  http://www.phil.uu.nl/~yoad/efs/main.html